4757 (FP3) Further Applications of Advanced Mathematics

1 (i)	$\overrightarrow{AB}\times\overrightarrow{AC}=\begin{pmatrix}6\\8\\5\end{pmatrix}\times\begin{pmatrix}10\\-5\\1\end{pmatrix}=\begin{pmatrix}33\\44\\-110\end{pmatrix}$	B2	*Ignore subsequent working* Give B1 for one element correct SC1 for minus the correct vector
	ABC is $3x+4y-10z=-9+20-20$ $3x+4y-10z+9=0$	M1 A1 **4**	For $3x+4y-10z$ Accept $33x+44y-110z=-99$ etc
(ii)	Distance is $\dfrac{3\times5+4\times4-10\times8+9}{\sqrt{3^2+4^2+10^2}}$ $=(-)\ \dfrac{40}{\sqrt{125}}\quad\left(=\dfrac{8}{\sqrt{5}}\right)$	M1 A1 ft A1 **3**	Using distance formula (or other complete method) *Condone negative answer* *Accept a.r.t. 3.58*
(iii)	$\overrightarrow{AB}\times\overrightarrow{CD}=\begin{pmatrix}6\\8\\5\end{pmatrix}\times\begin{pmatrix}-2\\4\\5\end{pmatrix}=\begin{pmatrix}20\\-40\\40\end{pmatrix}\quad\left[=20\begin{pmatrix}1\\-2\\2\end{pmatrix}\right]$	M1 A1	Evaluating $\overrightarrow{AB}\times\overrightarrow{CD}$ *or* method for finding end-points of common perp PQ *or* P $(\frac{3}{2}, 11, \frac{23}{4})$ & Q $(\frac{71}{18}, \frac{55}{9}, \frac{383}{36})$ *or* $\overrightarrow{PQ}=(\frac{22}{9}, -\frac{44}{9}, \frac{44}{9})$
	Distance is $\overrightarrow{AC}\cdot\hat{\mathbf{n}}=\dfrac{\begin{pmatrix}10\\-5\\1\end{pmatrix}\cdot\begin{pmatrix}1\\-2\\2\end{pmatrix}}{\sqrt{1^2+2^2+2^2}}$ $=\dfrac{22}{3}$	M1 A1 **4**	
(iv)	Volume is $\frac{1}{6}(\overrightarrow{AB}\times\overrightarrow{AC})\cdot\overrightarrow{AD}$	M1 A1	Scalar triple product
	$=\dfrac{1}{6}\begin{pmatrix}33\\44\\-110\end{pmatrix}\cdot\begin{pmatrix}8\\-1\\6\end{pmatrix}$ $=(-)\ \dfrac{220}{3}$	M1 A1 **4**	 *Accept a.r.t. 73.3*
(v)	E is $(-3+10\lambda,\ 5-5\lambda,\ 2+\lambda)$ $3(-3+10\lambda)-2(2+\lambda)+5=0$ $\lambda=\dfrac{2}{7}$	M1 A1	
	F is $(-3+8\mu,\ 5-\mu,\ 2+6\mu)$ $3(-3+8\mu)-2(2+6\mu)+5=0$ $\mu=\dfrac{2}{3}$	M1 A1	
	Since $0<\lambda<1$, E is between A and C Since $0<\mu<1$, F is between A and D	B1 **5**	

(vi)	$V_{\text{ABEF}} = \frac{1}{6}(\overrightarrow{\text{AB}} \times \overrightarrow{\text{AE}}) \,.\, \overrightarrow{\text{AF}}$ $= \frac{1}{6}\lambda\mu(\overrightarrow{\text{AB}} \times \overrightarrow{\text{AC}}) \,.\, \overrightarrow{\text{AD}}$ $= \lambda\mu V_{\text{ABCD}}$ $= \frac{4}{21} V_{\text{ABCD}}$ Ratio of volumes is $\frac{4}{21} : \frac{17}{21}$ $= 4 : 17$	M1 A1 M1 A1 ag **4**	($13\frac{61}{63}$) *ft if numerical* Finding ratio of volumes of two parts SC1 for 4 : 17 deduced from $\frac{4}{21}$ without working
2 (i)	$\frac{\partial g}{\partial x} = 6z - 2(x + 2y + 3z) = -2x - 4y$ $\frac{\partial g}{\partial y} = -4(x + 2y + 3z)$ $\frac{\partial g}{\partial z} = 6x - 6(x + 2y + 3z) = -12y - 18z$	M1 A1 A1 A1 **4**	Partial differentiation *Any correct form, ISW*
(ii)	At P, $\frac{\partial g}{\partial x} = 16,\ \frac{\partial g}{\partial y} = -4,\ \frac{\partial g}{\partial z} = 36$ Normal line is $\mathbf{r} = \begin{pmatrix} 7 \\ -7.5 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ -1 \\ 9 \end{pmatrix}$	M1 A1 A1 ft **3**	Evaluating partial derivatives at P All correct *Condone omission of* ' **r =** '
(iii)	$\delta g \approx 16\,\delta x - 4\,\delta y + 36\,\delta z$ If $\overrightarrow{\text{PQ}} = \lambda \begin{pmatrix} 4 \\ -1 \\ 9 \end{pmatrix}$, $\delta g \approx 16(4\lambda) - 4(-\lambda) + 36(9\lambda) \quad (= 392\lambda)$ $h = \delta g$, so $h \approx 392\lambda$ $\overrightarrow{\text{PQ}} \approx \frac{h}{392}\begin{pmatrix} 4 \\ -1 \\ 9 \end{pmatrix}$, so $\mathbf{n} = \frac{1}{392}\begin{pmatrix} 4 \\ -1 \\ 9 \end{pmatrix}$	M1 M1 A1 ft M1 A1 **5**	*Alternative:* *M3 for substituting* $x = 7 + 4\lambda$, ... *into* $g = 125 + h$ *and neglecting* λ^2 *A1 ft for linear equation in* λ *and* h *A1 for* **n** *correct*
(iv)	Require $\frac{\partial g}{\partial x} = \frac{\partial g}{\partial y} = 0$ $-2x - 4y = 0$ and $x + 2y + 3z = 0$ $x + 2y = 0$ and $z = 0$ $g(x, y, z) = 0 - 0^2 = 0 \neq 125$ Hence there is no such point on *S*	M1 M1 M1 A1 **4**	 Useful manipulation using both eqns Showing there is no such point on *S* Fully correct proof
(v)	Require $\frac{\partial g}{\partial z} = 0$ and $\frac{\partial g}{\partial y} = 5\frac{\partial g}{\partial x}$ $-4x - 8y - 12z = 5(-2x - 4y)$	M1 M1 M1	 Implied by $\frac{\partial g}{\partial x} = \lambda,\ \frac{\partial g}{\partial y} = 5\lambda$ *This M1 can be awarded for* $-2x - 4y = 1$ and $-4x - 8y - 12z = 5$

	$y=-\frac{3}{2}z$ and $x=5z$ $6(5z)z-(5z)^2=125$ $z=\pm 5$ Points are $(25,\ -7.5,\ 5)$ and $(-25,\ 7.5,\ -5)$	A1 M1 M1 A1 A1 ft **8**	*or* $z=-\frac{2}{3}y$ and $x=-\frac{10}{3}y$ *or* $y=-\frac{3}{10}x$ and $z=\frac{1}{5}x$ *or* $x=-\frac{5}{4}\lambda,\ y=\frac{3}{8}\lambda,\ z=-\frac{1}{4}\lambda$ *or* $x:y:z=10:-3:2$ Substituting into $g(x,y,z)=125$ Obtaining one value of *x, y, z or* λ *Dependent on previous M1* *ft is minus the other point, provided all M marks have been earned*
3 (i)	$\dot{x}^2+\dot{y}^2=(24t^2)^2+(18t-8t^3)^2$ $=576t^4+324t^2-288t^4+64t^6$ $=324t^2+288t^4+64t^6$ $=(18t+8t^3)^2$ Arc length is $\int_0^2(18t+8t^3)\,dt$ $=\left[\ 9t^2+2t^4\ \right]_0^2$ $=68$	B1 M1 A1 ag M1 A1 A1 **6**	*Note* $\int_0^2(18+8t^3)\,dt\ =\left[\ 18t+2t^4\ \right]_0^2=68$ *earns M1A0A0*
(ii)	Curved surface area is $\int 2\pi y\,ds$ $=\int_0^2 2\pi(9t^2-2t^4)(18t+8t^3)\,dt$ $=\int_0^2 \pi(324t^3+72t^5-32t^7)\,dt$ $=\pi\left[\ 81t^4+12t^6-4t^8\ \right]_0^2$ $=1040\pi\quad(\approx 3267)$	M1 M1 A1 M1 M1 A1 **6**	Using $ds=(18t+8t^3)\,dt$ Correct integral expression including limits *(may be implied by later work)*
(iii)	$\kappa=\frac{\dot{x}\ddot{y}-\ddot{x}\dot{y}}{(\dot{x}^2+\dot{y}^2)^{\frac{3}{2}}}=\frac{(24t^2)(18-24t^2)-(48t)(18t-8t^3)}{(18t+8t^3)^3}$ $=\frac{48t^2(9-12t^2-18+8t^2)}{8t^3(9+4t^2)^3}=\frac{-48t^2(9+4t^2)}{8t^3(9+4t^2)^3}$ $=\frac{-6}{t(4t^2+9)^2}$	M1 A1A1 M1 A1 ag **5**	Using formula for κ (or ρ) For numerator and denominator Simplifying the numerator

(iv)	When $t=1,\ x=8,\ y=7,\ \kappa=-\frac{6}{169}$ $\rho=(-)\dfrac{169}{6}$ $\dfrac{dy}{dx}=\dfrac{\dot{y}}{\dot{x}}=\dfrac{18t-8t^3}{24t^2}=\dfrac{10}{24}$ $\hat{\mathbf{n}}=\begin{pmatrix}5/13\\-12/13\end{pmatrix}$ $\mathbf{c}=\begin{pmatrix}8\\7\end{pmatrix}+\frac{169}{6}\begin{pmatrix}5/13\\-12/13\end{pmatrix}$ Centre of curvature is $(18\frac{5}{6},\ -19)$	B1 M1 M1 A1 M1 A1A1 **7**	Finding gradient (or tangent vector) Finding direction of the normal Correct unit normal (either direction)
4 (i)	*Commutative:* $x*y=y*x$ (for all *x, y*) *Associative:* $(x*y)*z=x*(y*z)$ (for all *x, y, z*)	B1 B2 **3**	Accept e.g. 'Order does not matter' Give B1 for a partial explanation, e.g. 'Position of brackets does not matter'
(ii)	$2(x+\frac{1}{2})(y+\frac{1}{2})-\frac{1}{2}=2xy+x+y+\frac{1}{2}-\frac{1}{2}$ $=2xy+x+y=x*y$	B1 ag **1**	*Intermediate step required*
(iii)(*A*)	If $x, y\in S$ then $x>-\frac{1}{2}$ and $y>-\frac{1}{2}$ $x+\frac{1}{2}>0$ and $y+\frac{1}{2}>0$, so $2(x+\frac{1}{2})(y+\frac{1}{2})>0$ $2(x+\frac{1}{2})(y+\frac{1}{2})-\frac{1}{2}>-\frac{1}{2}$, so $x*y\in S$	M1 A1 A1 **3**	
(*B*)	0 is the identity since $0*x=0+x+0=x$ If $x\in S$ and $x*y=0$ then $2xy+x+y=0$ $y=\dfrac{-x}{2x+1}$ $y+\frac{1}{2}=\dfrac{1}{2(2x+1)}>0$ (since $x>-\frac{1}{2}$) so $y\in S$ *S* is closed and associative; there is an identity; and every element of *S* has an inverse in *S*	B1 B1 M1 A1 M1 A1 **6**	or $2(x+\frac{1}{2})(y+\frac{1}{2})-\frac{1}{2}=0$ or $y+\frac{1}{2}=\dfrac{1}{4(x+\frac{1}{2})}$ *Dependent on M1A1M1*
(iv)	If $x*x=0,\ 2x^2+x+x=0$ $x=0$ or -1 0 is the identity (and has order 1) -1 is not in *S*	M1 A1 A1 **3**	

(v)	$4*6=48+4+6=58$ $=56+2=7\times 8+2$ So $4\circ 6=2$	B1 B1 ag **2**	
(vi)	Element: 0, 1, 2, 4, 5, 6 Order: 1, 6, 6, 3, 3, 2	B3 **3**	Give B2 for 4 correct B1 for 2 correct
(vii)	$\{0\}$, G $\{0,\ 6\}$ $\{0,\ 4,\ 5\}$	B1 B1 B1 **3**	*Condone omission of G* If more than 2 non-trivial subgroups are given, deduct 1 mark (from final B1B1) for each non-trivial subgroup in excess of 2

Pre-multiplication by transition matrix

5 (i)	$\mathbf{P} = \begin{pmatrix} 0.1 & 0.7 & 0.1 \\ 0.4 & 0.2 & 0.6 \\ 0.5 & 0.1 & 0.3 \end{pmatrix}$	B2 **2**	Give B1 for two columns correct
(ii)	$\mathbf{P}^6 \begin{pmatrix} 1/3 \\ 1/3 \\ 1/3 \end{pmatrix} = \begin{pmatrix} 0.328864 \\ 0.381536 \\ 0.2896 \end{pmatrix}$ P(*B* used on 7th day) $= 0.3815$	M1 M1 M1 A1 **4**	Using $\mathbf{P}^6$ (or $\mathbf{P}^7$) For matrix of initial probabilities For evaluating matrix product *Accept 0.381 to 0.382*
(iii)	$0.328864 \times 0.1 + 0.381536 \times 0.2 + 0.2896 \times 0.3$ $= 0.1961$	M1 M1 A1 **3**	Using diagonal elements from **P** Correct method *Accept a.r.t. 0.196*
(iv)	$\mathbf{P}^3 = \begin{pmatrix} 0.352 & 0.328 & 0.304 \\ 0.364 & 0.404 & 0.372 \\ 0.284 & 0.268 & 0.324 \end{pmatrix}$ $0.328864 \times 0.352 + 0.381536 \times 0.404 + 0.2896 \times 0.324$ $= 0.3637$	M1 M1 M1 A1 **4**	For evaluating $\mathbf{P}^3$ Using diagonal elements from $\mathbf{P}^3$ Correct method *Accept a.r.t. 0.364*
(v)	$\mathbf{Q} = \begin{pmatrix} 0.3289 & 0.3289 & 0.3289 \\ 0.3816 & 0.3816 & 0.3816 \\ 0.2895 & 0.2895 & 0.2895 \end{pmatrix}$ 0.3289, 0.3816, 0.2895 are the long-run probabilities for the routes *A, B, C*	B1 B1 B1 B1 **4**	*Deduct 1 if not given as a (3×3) matrix* *Deduct 1 if not 4 dp* *Accept 'equilibrium probabilities'*
(vi)	$\begin{pmatrix} 0.1 & 0.7 & a \\ 0.4 & 0.2 & b \\ 0.5 & 0.1 & c \end{pmatrix} \begin{pmatrix} 0.4 \\ 0.2 \\ 0.4 \end{pmatrix} = \begin{pmatrix} 0.4 \\ 0.2 \\ 0.4 \end{pmatrix}$ $0.04 + 0.14 + 0.4a = 0.4$, so $a = 0.55$ $0.16 + 0.04 + 0.4b = 0.2$, so $b = 0$ $0.2 + 0.02 + 0.4c = 0.4$, so $c = 0.45$ After *C*, routes *A, B, C* will be used with probabilities 0.55, 0, 0.45	M1 M1 A2 **4**	Obtaining a value for *a, b* or *c* Give A1 for one correct
(vii)	$0.4 \times 0.1 + 0.2 \times 0.2 + 0.4 \times 0.45$ $= 0.26$	M1 M1 A1 **3**	Using long-run probs 0.4, 0.2, 0.4 Using diag elements from new matrix

Post-multiplication by transition matrix

5 (i)	$\mathbf{P}=\begin{pmatrix}0.1 & 0.4 & 0.5\\ 0.7 & 0.2 & 0.1\\ 0.1 & 0.6 & 0.3\end{pmatrix}$	B2	**2**	Give B1 for two rows correct
(ii)	$\begin{pmatrix}\frac{1}{3} & \frac{1}{3} & \frac{1}{3}\end{pmatrix}\mathbf{P}^6=\begin{pmatrix}0.328864 & 0.381536 & 0.2896\end{pmatrix}$ P(*B* used on 7th day) $=0.3815$	M1 M1 M1 A1	**4**	Using $\mathbf{P}^6$ (or $\mathbf{P}^7$) For matrix of initial probabilities For evaluating matrix product *Accept 0.381 to 0.382*
(iii)	$0.328864\times0.1+0.381536\times0.2+0.2896\times0.3$ $=0.1961$	M1 M1 A1	**3**	Using diagonal elements from **P** Correct method *Accept a.r.t. 0.196*
(iv)	$\mathbf{P}^3=\begin{pmatrix}0.352 & 0.364 & 0.284\\ 0.328 & 0.404 & 0.268\\ 0.304 & 0.372 & 0.324\end{pmatrix}$ $0.328864\times0.352+0.381536\times0.404+0.2896\times0.324$ $=0.3637$	M1 M1 M1 A1	**4**	For evaluating $\mathbf{P}^3$ Using diagonal elements from $\mathbf{P}^3$ Correct method *Accept a.r.t. 0.364*
(v)	$\mathbf{Q}=\begin{pmatrix}0.3289 & 0.3816 & 0.2895\\ 0.3289 & 0.3816 & 0.2895\\ 0.3289 & 0.3816 & 0.2895\end{pmatrix}$ 0.3289, 0.3816, 0.2895 are the long-run probabilities for the routes *A, B, C*	B1B1B1 B1	**4**	*Deduct 1 if not given as a (3×3) matrix* *Deduct 1 if not 4 dp* *Accept 'equilibrium probabilities'*
(vi)	$\begin{pmatrix}0.4 & 0.2 & 0.4\end{pmatrix}\begin{pmatrix}0.1 & 0.4 & 0.5\\ 0.7 & 0.2 & 0.1\\ a & b & c\end{pmatrix}=\begin{pmatrix}0.4 & 0.2 & 0.4\end{pmatrix}$ $0.04+0.14+0.4a=0.4$, so $a=0.55$ $0.16+0.04+0.4b=0.2$, so $b=0$ $0.2+0.02+0.4c=0.4$, so $c=0.45$ After *C*, routes *A, B, C* will be used with probabilities 0.55, 0, 0.45	M1 M1 A2	**4**	 Obtaining a value for *a, b* or *c* Give A1 for one correct
(vii)	$0.4\times0.1+0.2\times0.2+0.4\times0.45$ $=0.26$	M1 M1 A1	**3**	Using long-run probs 0.4, 0.2, 0.4 Using diag elements from new matrix