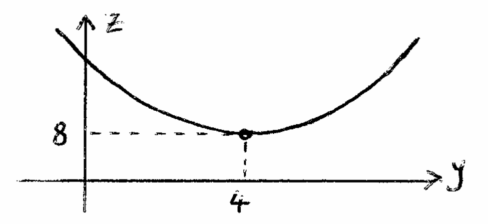
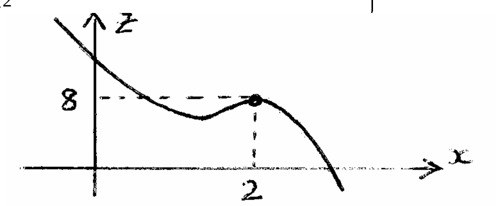
1 (i)	$\mathbf{d}_K = \begin{pmatrix} 8 \\ -1 \\ -14 \end{pmatrix} \times \begin{pmatrix} 6 \\ 2 \\ -5 \end{pmatrix} = \begin{pmatrix} 33 \\ -44 \\ 22 \end{pmatrix}$ $\left[= 11 \begin{pmatrix} 3 \\ -4 \\ 2 \end{pmatrix} \right]$ $\mathbf{d}_L = \begin{pmatrix} 8 \\ -1 \\ -14 \end{pmatrix} \times \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 15 \\ -20 \\ 10 \end{pmatrix}$ $\left[= 5 \begin{pmatrix} 3 \\ -4 \\ 2 \end{pmatrix} \right]$ Hence *K* and *L* are parallel For a point on *K*, $z = 0,\ x = 3,\ y = 4$ i.e. $(3,\ 4,\ 0)$ For a point on *L*, $z = 0,\ x = 6,\ y = 28$ i.e. $(6,\ 28,\ 0)$	M1* A1* A1 M1*A1* A1*	Finding direction of *K* or *L* One direction correct ** These marks can be earned anywhere in the question* Correctly shown Finding one point on *K* or *L* *or* $(6,\ 0,\ 2)$ *or* $(0,\ 8,\ -2)$ *etc* *Or* $(27,\ 0,\ 14)$ *or* $(0,\ 36,\ -4)$ *etc*
	$\left[\begin{pmatrix} 6 \\ 28 \\ 0 \end{pmatrix} - \begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix} \right] \times \begin{pmatrix} 3 \\ -4 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 24 \\ 0 \end{pmatrix} \times \begin{pmatrix} 3 \\ -4 \\ 2 \end{pmatrix} = \begin{pmatrix} 48 \\ -6 \\ -84 \end{pmatrix}$ Distance is $\dfrac{\sqrt{48^2 + 6^2 + 84^2}}{\sqrt{3^2 + 4^2 + 2^2}} = \dfrac{\sqrt{9396}}{\sqrt{29}} = 18$	M1 M1 A1 **9**	For $(\mathbf{b} - \mathbf{a}) \times \mathbf{d}$ Correct method for finding distance
	OR $\begin{pmatrix} 6 + 3\lambda - 3 \\ 28 - 4\lambda - 4 \\ 2\lambda \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -4 \\ 2 \end{pmatrix} = 0$ M1 $-87 + 29\lambda = 0, \quad \lambda = 3$ M1 Distance is $\sqrt{12^2 + 12^2 + 6^2} = 18$ A1		For $(\mathbf{b} + \lambda\mathbf{d} - \mathbf{a}).\mathbf{d} = 0$ Finding λ, and the magnitude
(ii)	Distance from $(3, 4, 0)$ to *R* is $\left\lvert \dfrac{2 \times 3 + 4 - 0 - 40}{\sqrt{2^2 + 1^2 + 1^2}} \right\rvert$ $= \dfrac{30}{\sqrt{6}} = \dfrac{30\sqrt{6}}{6} = 5\sqrt{6}$	M1A1 ft A1 ag **3**	
(iii)	*K*, *M* intersect if $1 + 5\lambda = 3 + 3\mu$ (1) $-4 - 4\lambda = 4 - 4\mu$ (2) $3\lambda = 2\mu$ (3) Solving (2) and (3): $\lambda = 4,\ \mu = 6$ Check in (1): LHS $= 1 + 20 = 21$, RHS $= 3 + 18 = 21$ Hence *K*, *M* intersect, at $(21,\ -20,\ 12)$	M1 A1 ft M1M1 M1A1 A1 **7**	At least 2 eqns, different parameters Two equations correct Intersection correctly shown *Can be awarded after M1A1M1M0M0*
	OR *M* meets *P* when M1 $8(1 + 5\lambda) - (-4 - 4\lambda) - 14(3\lambda) = 20$ A1 *M* meets *Q* when $6(1 + 5\lambda) + 2(-4 - 4\lambda) - 5(3\lambda) = 26$ A1 Both equations have solution $\lambda = 4$ A1 Point is on *P*, *Q* and *M*; hence on *K* and *M* M2 Point of intersection is $(21,\ -20,\ 12)$ A1		Intersection of *M* with **both** *P* and *Q*

(iv)	$\left[\begin{pmatrix}6\\28\\0\end{pmatrix}-\begin{pmatrix}1\\-4\\0\end{pmatrix}\right]\cdot\left[\begin{pmatrix}3\\-4\\2\end{pmatrix}\times\begin{pmatrix}5\\-4\\3\end{pmatrix}\right]=\begin{pmatrix}5\\32\\0\end{pmatrix}\cdot\begin{pmatrix}-4\\1\\8\end{pmatrix}=12$ Distance is $\dfrac{12}{\sqrt{4^2+1^2+8^2}}=\dfrac{12}{9}=\dfrac{4}{3}$	M1A1 ft M1 A1 ft A1 **5**	For evaluating $\mathbf{d}_L \times \mathbf{d}_M$ For $(\mathbf{b}-\mathbf{c}).(\mathbf{d}_L \times \mathbf{d}_M)$ Numerical expression for distance

2 (i)	$\frac{\partial z}{\partial x} = y^2 - 8xy - 6x^2 + 54x - 36$ $\frac{\partial z}{\partial y} = 2xy - 4x^2$	B2 B1 **3**	Give B1 for 3 terms correct
(ii)	At stationary points, $\frac{\partial z}{\partial x} = 0$ and $\frac{\partial z}{\partial y} = 0$ When $x = 0$, $y^2 - 36 = 0$ $y = \pm 6$; points $(0, 6, 20)$ and $(0, -6, 20)$ When $y = 2x$, $4x^2 - 16x^2 - 6x^2 + 54x - 36 = 0$ $-18x^2 + 54x - 36 = 0$ $x = 1, 2$ Points $(1, 2, 5)$ and $(2, 4, 8)$	M1 M1 A1A1 M1 M1A1 A1 **8**	If A0, give A1 for $y = \pm 6$ or $y = 2, 4$ *A0 if any extra points given*
(iii)	When $x = 2$, $z = 2y^2 - 16y + 40$  When $y = 4$, $z = -2x^3 + 11x^2 - 20x + 20$ $\left(\frac{d^2z}{dx^2} = -12x + 22 = -2 \text{ when } x = 2 \right)$  The point is a minimum on one section and a maximum on the other; so it is neither a maximum nor a minimum	B1 B1 B1 B1 B1 B1 **6**	'Upright' parabola $(2, 4, 8)$ identified as a minimum (in the first quadrant) 'Negative cubic' curve $(2, 4, 8)$ identified as a stationary point Fully correct (unambiguous minimum and maximum)
(iv)	Require $\frac{\partial z}{\partial x} = -36$ and $\frac{\partial z}{\partial y} = 0$ When $x = 0$, $y^2 - 36 = -36$ $y = 0$; point $(0, 0, 20)$ When $y = 2x$, $4x^2 - 16x^2 - 6x^2 + 54x - 36 = -36$ $-18x^2 + 54x = 0$ $x = 0, 3$ $x = 0$ gives $(0, 0, 20)$ same as above $x = 3$ gives $(3, 6, -7)$	M1 M1 A1 M1 M1 A1 A1 **7**	$\frac{\partial z}{\partial x} = 36$ can earn all M marks Solving to obtain *x* (or *y*) *or stating 'no roots' if appropriate (e.g. when +36 has been used)*

3 (i)	$1+\left(\frac{dy}{dx}\right)^2 = 1+\left(x-\frac{1}{4x}\right)^2$	M1	
	$= 1+x^2-\frac{1}{2}+\frac{1}{16x^2} = x^2+\frac{1}{2}+\frac{1}{16x^2}$		
	$= \left(x+\frac{1}{4x}\right)^2$	A1	
	Arc length is $\int_1^a \left(x+\frac{1}{4x}\right)dx$	M1	For $\int\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$
	$= \left[\frac{1}{2}x^2+\frac{1}{4}\ln x\right]_1^a$	M1	
	$= \frac{1}{2}a^2+\frac{1}{4}\ln a-\frac{1}{2}$	A1 ag **5**	
(ii)	Curved surface area is $\int 2\pi x\,ds$	M1	
	$= \int_1^4 2\pi x\left(x+\frac{1}{4x}\right)dx$	A1 ft	Any correct integral form (including limits)
	$= 2\pi\left[\frac{1}{3}x^3+\frac{1}{4}x\right]_1^4$	M1 A1	for $\frac{1}{3}x^3+\frac{1}{4}x$
	$= \frac{87\pi}{2}$ (≈ 137)	A1 **5**	
(iii)	$\rho = \frac{\left(1+\left(\frac{dy}{dx}\right)^2\right)^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} = \frac{\left(a+\frac{1}{4a}\right)^3}{1+\frac{1}{4a^2}}$	B1 B1	any form, in terms of x or a any form, in terms of x or a
	$= \frac{a\left(a+\frac{1}{4a}\right)^3}{a+\frac{1}{4a}} = a\left(a+\frac{1}{4a}\right)^2$	M1 A1 A1 ag **5**	Formula for ρ or κ ρ or κ correct, in any form, in terms of x or a
(iv)	At $(1, \frac{1}{2})$, $\rho = (\frac{5}{4})^2 = \frac{25}{16}$		
	$\frac{dy}{dx} = 1-\frac{1}{4} = \frac{3}{4}$, so $\hat{\mathbf{n}} = \begin{pmatrix}-\frac{3}{5}\\ \frac{4}{5}\end{pmatrix}$	M1 A1	Finding gradient Correct normal vector (not necessarily unit vector); may be in terms of x
	$\mathbf{c} = \begin{pmatrix}1\\ \frac{1}{2}\end{pmatrix} + \frac{25}{16}\begin{pmatrix}-\frac{3}{5}\\ \frac{4}{5}\end{pmatrix}$	M1	OR *M2A1 for obtaining equation of normal line at a general point and differentiating partially*
	Centre of curvature is $\left(\frac{1}{16}, \frac{7}{4}\right)$	A1A1 **5**	

(v)	Differentiating partially w.r.t. p $0 = x^2 - 2p\ln x$ $p = \dfrac{x^2}{2\ln x}$ and $y = \dfrac{x^4}{2\ln x} - \dfrac{x^4}{4\ln x}$ $y = \dfrac{x^4}{4\ln x}$	M1 A1 M1 A1 **4**	

4 (i)	By Lagrange's theorem, a proper subgroup has order 2 or 5 A group of prime order is cyclic Hence every proper subgroup is cyclic	M1 A1 M1 A1 **4**	Using Lagrange *(need not be mentioned explicitly)* or equivalent For completion
(ii)	e.g. $2^2=4,\ 2^3=8,\ 2^4=5,\ 2^5=10,$ $2^6=9,\ 2^7=7,\ 2^8=3,\ 2^9=6,\ 2^{10}=1$ 2 has order 10, hence M is cyclic	M1 A1 A1 A1 **4**	Considering order of an element Identifying an element of order 10 (2, 6, 7 or 8) Fully justified For conclusion (can be awarded after M1A1A0)
(iii)	$\{1,\ 10\}$ $\{1,\ 3,\ 4,\ 5,\ 9\}$	B1 B2 **3**	Ignore $\{1\}$ and M Deduct 1 mark (from B1B2) for each (proper) subgroup given in excess of 2
(iv)	E is the identity A, C, G, I are rotations B, D, F, H, J are reflections	B1 M1 A1 A1 **4**	Considering elements of order 2 *(or equivalent) Implied by four of B, D, F, H, J in the same set* Give A1 if one element is in the wrong set; or if two elements are interchanged
(v)	P and M are not isomorphic M is abelian, P is non-abelian	B1 B1 **2**	Valid reason e.g. M has one element of order 2 P has more than one
(vi)	| | A | B | C | D | E | F | G | H | I | J | | Order | 5 | 2 | 5 | 2 | 1 | 2 | 5 | 2 | 5 | 2 |	B3 **3**	Give B2 for 7 correct B1 for 4 correct
(vii)	 $\{E, B\}, \{E, D\}, \{E, F\}, \{E, H\}, \{E, J\}$ $\{E, A, C, G, I\}$	 M1 A1 ft B2 cao **4**	Ignore $\{E\}$ and P *Subgroups of order 2* Using elements of order 2 (allow two errors/omissions) Correct or ft. A0 if any others given *Subgroups of order greater than 2* Deduct 1 mark (from B2) for each extra subgroup given

Pre-multiplication by transition matrix

5 (i)	$\mathbf{P}=\begin{pmatrix}0&0&0.4&0.3\\0&0&0.6&0.7\\0.8&0.1&0&0\\0.2&0.9&0&0\end{pmatrix}$	B2	**2**	Give B1 for two columns correct
(ii)	$\mathbf{P}^4=\begin{pmatrix}0.3366&0.3317&0&0\\0.6634&0.6683&0&0\\0&0&0.3366&0.3317\\0&0&0.6634&0.6683\end{pmatrix}$ $\mathbf{P}^7=\begin{pmatrix}0&0&0.3334&0.3333\\0&0&0.6666&0.6667\\0.3335&0.3333&0&0\\0.6665&0.6667&0&0\end{pmatrix}$	B2 B2	**4**	Give B1 for two non-zero elements correct to at least 2dp Give B1 for two non-zero elements correct to at least 2dp
(iii)	$\mathbf{P}^7\begin{pmatrix}0.4\\0.3\\0.2\\0.1\end{pmatrix}=\begin{pmatrix}0.1000\\0.2000\\0.2334\\0.4666\end{pmatrix}$ P(8th letter is C) $=0.233$	M1 A1	**2**	Using $\mathbf{P}^7$ (or $\mathbf{P}^8$) and initial probs
(iv)	$0.1000\times0.3366\ +\ 0.2000\times0.6683$ $+\ 0.2334\times0.3366\ +\ 0.4666\times0.6683$ $=0.558$	M1 M1 A1 ft A1	**4**	Using probabilities for 8th letter Using diagonal elements from $\mathbf{P}^4$
(v)(*A*) (*B*)	$\mathbf{P}^n\begin{pmatrix}0.4\\0.3\\0.2\\0.1\end{pmatrix}\approx\begin{pmatrix}1/3&1/3&0&0\\2/3&2/3&0&0\\0&0&1/3&1/3\\0&0&2/3&2/3\end{pmatrix}\begin{pmatrix}0.4\\0.3\\0.2\\0.1\end{pmatrix}=\begin{pmatrix}0.2333\\0.4667\\0.1\\0.2\end{pmatrix}$ P($(n+1)$ th letter is A) $=0.233$ $\mathbf{P}^n\begin{pmatrix}0.4\\0.3\\0.2\\0.1\end{pmatrix}\approx\begin{pmatrix}0&0&1/3&1/3\\0&0&2/3&2/3\\1/3&1/3&0&0\\2/3&2/3&0&0\end{pmatrix}\begin{pmatrix}0.4\\0.3\\0.2\\0.1\end{pmatrix}=\begin{pmatrix}0.1\\0.2\\0.2333\\0.4667\end{pmatrix}$ P($(n+1)$ th letter is A) $=0.1$	M1 A1 M1 A1	**4**	Approximating $\mathbf{P}^n$ when n is large and even Approximating $\mathbf{P}^n$ when n is large and odd
(vi)	$\mathbf{Q}=\begin{pmatrix}0&0&0.4&0.3\\0&0&0.6&0.6\\0.8&0.1&0&0\\0.2&0.9&0&0.1\end{pmatrix}$	B1	**1**	

(vii)	$\mathbf{Q}^n \to \begin{pmatrix} 0.1721 & 0.1721 & 0.1721 & 0.1721 \\ 0.3105 & 0.3105 & 0.3105 & 0.3105 \\ 0.1687 & 0.1687 & 0.1687 & 0.1687 \\ 0.3487 & 0.3487 & 0.3487 & 0.3487 \end{pmatrix}$ Probabilities are 0.172, 0.310, 0.169, 0.349	M1 M1 A2 **4**	Considering $\mathbf{Q}^n$ for large n OR at least two eqns for equilib probs Probabilities from equal columns OR solving to obtain equilib probs Give A1 for two correct
(viii)	$0.3487 \times 0.1 \times 0.1$ $= 0.0035$	M1M1 A1 **3**	Using 0.3487 and 0.1

Post-multiplication by transition matrix

5 (i)	$\mathbf{P} = \begin{pmatrix} 0 & 0 & 0.8 & 0.2 \\ 0 & 0 & 0.1 & 0.9 \\ 0.4 & 0.6 & 0 & 0 \\ 0.3 & 0.7 & 0 & 0 \end{pmatrix}$	B2	**2**	Give B1 for two rows correct
(ii)	$\mathbf{P}^4 = \begin{pmatrix} 0.3366 & 0.6634 & 0 & 0 \\ 0.3317 & 0.6683 & 0 & 0 \\ 0 & 0 & 0.3366 & 0.6634 \\ 0 & 0 & 0.3317 & 0.6683 \end{pmatrix}$ $\mathbf{P}^7 = \begin{pmatrix} 0 & 0 & 0.3335 & 0.6665 \\ 0 & 0 & 0.3333 & 0.6667 \\ 0.3334 & 0.6666 & 0 & 0 \\ 0.3333 & 0.6667 & 0 & 0 \end{pmatrix}$	B2 B2	**4**	Give B1 for two non-zero elements correct to at least 2dp Give B1 for two non-zero elements correct to at least 2dp
(iii)	$(0.4 \quad 0.3 \quad 0.2 \quad 0.1)\mathbf{P}^7$ $= (0.1000 \quad 0.2000 \quad 0.2334 \quad 0.4666)$ P(8th letter is C) $= 0.233$	M1 A1	**2**	Using $\mathbf{P}^7$ (or $\mathbf{P}^8$) and initial probs
(iv)	$0.1000 \times 0.3366 + 0.2000 \times 0.6683$ $+ 0.2334 \times 0.3366 + 0.4666 \times 0.6683$ $= 0.558$	M1 M1A1 ft A1	**4**	Using probabilities for 8th letter Using diagonal elements from $\mathbf{P}^4$
(v)(*A*) (*B*)	$\mathbf{u}\,\mathbf{P}^n \approx (0.4 \quad 0.3 \quad 0.2 \quad 0.1)\begin{pmatrix} \frac{1}{3} & \frac{2}{3} & 0 & 0 \\ \frac{1}{3} & \frac{2}{3} & 0 & 0 \\ 0 & 0 & \frac{1}{3} & \frac{2}{3} \\ 0 & 0 & \frac{1}{3} & \frac{2}{3} \end{pmatrix}$ $= (0.2333 \quad 0.4667 \quad 0.1 \quad 0.2)$ P($(n+1)$ th letter is A) $= 0.233$ $\mathbf{u}\,\mathbf{P}^n \approx (0.4 \quad 0.3 \quad 0.2 \quad 0.1)\begin{pmatrix} 0 & 0 & \frac{1}{3} & \frac{2}{3} \\ 0 & 0 & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} & 0 & 0 \\ \frac{1}{3} & \frac{2}{3} & 0 & 0 \end{pmatrix}$ $= (0.1 \quad 0.2 \quad 0.2333 \quad 0.4667)$ P($(n+1)$ th letter is A) $= 0.1$	M1 A1 M1 A1	**4**	Approximating $\mathbf{P}^n$ when n is large and even Approximating $\mathbf{P}^n$ when n is large and odd
(vi)	$\mathbf{Q} = \begin{pmatrix} 0 & 0 & 0.8 & 0.2 \\ 0 & 0 & 0.1 & 0.9 \\ 0.4 & 0.6 & 0 & 0 \\ 0.3 & 0.6 & 0 & 0.1 \end{pmatrix}$	B1	**1**	

(vii)	$\mathbf{Q}^n \to \begin{pmatrix} 0.1721 & 0.3105 & 0.1687 & 0.3487 \\ 0.1721 & 0.3105 & 0.1687 & 0.3487 \\ 0.1721 & 0.3105 & 0.1687 & 0.3487 \\ 0.1721 & 0.3105 & 0.1687 & 0.3487 \end{pmatrix}$ Probabilities are 0.172, 0.310, 0.169, 0.349	M1 M1 A2 **4**	Considering $\mathbf{Q}^n$ for large n OR at least two eqns for equilib probs Probabilities from equal rows OR solving to obtain equilib probs Give A1 for two correct
(viii)	$0.3487 \times 0.1 \times 0.1$ $= 0.0035$	M1M1 A1 **3**	Using 0.3487 and 0.1