1 (i)	$\begin{pmatrix}-1\\4\\3\end{pmatrix}\times\begin{pmatrix}-k\\4\\k+2\end{pmatrix}=\begin{pmatrix}4k-4\\2-2k\\4k-4\end{pmatrix}$ [$=2(k-1)\begin{pmatrix}2\\-1\\2\end{pmatrix}$]	B1 M1 A2	**4**	$\overrightarrow{AB}$ and $\overrightarrow{CD}$ *(Condone* $\overrightarrow{BA}$ *and* $\overrightarrow{DC}$ *)* Evaluating vector product Give A1 ft for one element correct
(ii)(*A*)	$k=1$	B1	**1**	
(*B*)	$\overrightarrow{CA}\times\overrightarrow{AB}=\begin{pmatrix}-3\\-8\\4\end{pmatrix}\times\begin{pmatrix}-1\\4\\3\end{pmatrix}=\begin{pmatrix}-40\\5\\-20\end{pmatrix}$ Distance is $\dfrac{\lvert\overrightarrow{CA}\times\overrightarrow{AB}\rvert}{\lvert\overrightarrow{AB}\rvert}=\dfrac{45}{\sqrt{26}}$ (≈ 8.825)	M1 M1 A1 M1 M1 A1	**6**	For appropriate vector product Evaluation *Dependent on previous M1* Method for finding shortest distance *Dependent on first M1* Calculating magnitudes *Dependent on previous M1* Accept 8.82 to 8.83
	OR $\overrightarrow{CP}.\overrightarrow{AB}=\begin{pmatrix}-2-\lambda-1\\-3+4\lambda-5\\2+3\lambda+2\end{pmatrix}\cdot\begin{pmatrix}-1\\4\\3\end{pmatrix}=0$ M2A1 $\overrightarrow{CP}=\dfrac{1}{26}\begin{pmatrix}-95\\-140\\155\end{pmatrix}$ Distance is $\dfrac{\sqrt{52650}}{26}$ M1 M1A1			 Finding $\overrightarrow{CP}$ *Dependent on previous M1* *Dependent on previous M1*
(*C*)	Normal vector is $\overrightarrow{CA}\times\overrightarrow{AB}=\begin{pmatrix}-40\\5\\-20\end{pmatrix}=-5\begin{pmatrix}8\\-1\\4\end{pmatrix}$ Equation of plane is $8x-y+4z=-16+3+8$ $8x-y+4z+5=0$	M1 M1 A1	**3**	 *Dependent on previous M1* Allow $-40x+5y-20z=25$ etc
(iii)	$\dfrac{\overrightarrow{AC}.(\overrightarrow{AB}\times\overrightarrow{CD})}{\lvert\overrightarrow{AB}\times\overrightarrow{CD}\rvert}=\dfrac{\begin{pmatrix}k+2\\8\\-4\end{pmatrix}\cdot\begin{pmatrix}2\\-1\\2\end{pmatrix}(2k-2)}{3(2k-2)}$ Shortest distance is $\left\lvert\dfrac{2k-12}{3}\right\rvert$	M1 M1 A1 ft A1	**4**	For $\overrightarrow{AC}.(\overrightarrow{AB}\times\overrightarrow{CD})$ Fully correct method *(evaluation not required) Dependent on previous M1* Correct evaluated expression for distance ft from (i) Simplified answer *Modulus not required*

(iv)	Intersect when $k=6$ $-2-\lambda=6-6\mu$ $-3+4\lambda=5+4\mu$ $2+3\lambda=-2+8\mu$ Solving, $\lambda=4,\ \mu=2$ Point of intersection is $(-6,\ 13,\ 14)$	B1 ft M1 A1 ft M1 A1 A1 **6**	 Forming at least two equations Two correct equations Solving to obtain λ or μ *Dependent on previous M1* One value correct
	$-2-\lambda=k-k\mu$ M1 OR $-3+4\lambda=5+4\mu$ A1 $2+3\lambda=-2+(k+2)\mu$ Solving, $k=6$ M1A1 $\lambda=4,\ \mu=2$ A1 Point of intersection is $(-6,\ 13,\ 14)$ A1		Forming three equations All equations correct *Dependent on previous M1* One value correct

2 (i)	Normal vector is $\begin{pmatrix} 2x-4y \\ -4x+6y \\ -4z \end{pmatrix}$	M1 A1 A1 A1 **4**	Partial differentiation Condone $\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \lambda \begin{pmatrix} 2x-4y \\ -4x+6y \\ -4z \end{pmatrix}$ *For 4 marks the normal must appear as a vector (isw)*
(ii)	At Q normal vector is $\begin{pmatrix} 18 \\ -44 \\ -4 \end{pmatrix}$ Tangent plane is $18x-44y-4z = 306-176-4 = 126$ $9x-22y-2z = 63$	M1 M1 M1 A1 **4**	For $18x-44y-4z$ *Dependent on previous M1* Using Q to find constant Accept any correct form
(iii)	$18\delta x - 44\delta y - 4\delta z \approx 0$ $18h - 44p - 4(-h) \approx 0$ $p \approx \frac{1}{2}h$	M1 A1 ft M1 A1 **4**	For $18\delta x - 44\delta y - 4\delta z$ *If left in terms of x, y, z: M1A0M1A0*
	OR $9(17+h) - 22(4+p) - 2(1-h) \approx 63$ M2A1 ft $p \approx \frac{1}{2}h$ A1		
	OR $(17+h)^2 - 4(17+h)(4+p) + ... = 0$ $-44p + 22h \approx 0$ M2A1 $p \approx \frac{1}{2}h$ A1		Neglecting second order terms
	OR $p = \dfrac{4h+44 \pm \sqrt{28h^2+88h+1936}}{6}$ M2A1 $p \approx \frac{1}{2}h$ A1		
(iv)	Normal parallel to *z*-axis requires $2x-4y=0$ and $-4x+6y=0$ $x=y=0$; then $-2z^2-63=0$ No solutions; hence no such points	M1A1 ft M1 A1 (ag) **4**	Correctly shown
	OR $2x-4y = -4x+6y$, so $y = \frac{3}{5}x$ $-\frac{8}{25}x^2 - 2z^2 - 63 = 0$, hence no points M2A2		Similarly if only $2x-4y=0$ used
(v)	$2x-4y = 5\lambda$ $-4x+6y = -6\lambda$ $-4z = 2\lambda$ $x = -\frac{3}{2}\lambda,\ y = -2\lambda,\ z = -\frac{1}{2}\lambda$ Substituting into equation of surface $\frac{9}{4}\lambda^2 - 12\lambda^2 + 12\lambda^2 - \frac{1}{2}\lambda^2 - 63 = 0$ $\lambda = \pm 6$	M1A1 ft M1 M1 M1 M1	Obtaining *x, y, z* in terms of λ or $x = 3z,\ y = 4z$ Obtaining a value of λ (or equivalent)

	Point $(-9,\ -12,\ -3)$ gives $k = -45 + 72 - 6 = 21$ Point $(9,\ 12,\ 3)$ gives $k = 45 - 72 + 6 = -21$	A1 A1 **8**	Using a point to find *k* *If* $\lambda = 1$ *is assumed:* *M0M1M0M0M1*

3 (i)	$\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2=(6t^2-6)^2+(12t)^2$	M1A1	
	$=36t^4+72t^2+36$		
	$=36(t^2+1)^2$	A1	
	Arc length is $\int_0^1 6(t^2+1)\,dt$	M1	Using $\int\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$
	$=\left[\;2t^3+6t\;\right]_0^1$	A1	For $2t^3+6t$
	$=8$	A1 **6**	
(ii)		M1	Using $\int ...y\,ds$ (in terms of t) with 'ds' the same as in (i)
	Curved surface area is $\int 2\pi\,y\,ds=\int_0^1 2\pi(6t^2)6(t^2+1)\,dt$	A1	Any correct integral form in terms of t *(limits required)*
	$=\pi\left[\;\frac{72}{5}t^5+24t^3\;\right]_0^1$	M1 A1	Integration For $\pi(\frac{72}{5}t^5+24t^3)$
	$=\frac{192\pi}{5}\quad(\approx 120.6)$	A1 **5**	
(iii)	$\frac{dy}{dx}=\frac{12t}{6t^2-6}\quad\left(=\frac{2t}{t^2-1}\right)$	M1 A1	Method of differentiation
	Equation of normal is $y-6t^2=\frac{1-t^2}{2t}(x-2t^3+6t)$	M1	
	$y-6t^2=\frac{1}{2}\left(\frac{1}{t}-t\right)x-t^2(1-t^2)+3(1-t^2)$ $y=\frac{1}{2}\left(\frac{1}{t}-t\right)x+2t^2+t^4+3$	A1 (ag) **4**	*At least one intermediate step required* Correctly obtained
(iv)	Differentiating partially with respect to t	M1	
	$0=\frac{1}{2}\left(-\frac{1}{t^2}-1\right)x+4t+4t^3$	A2	Give A1 if just one error or omission
	$\frac{1}{2t^2}(1+t^2)x=4t(1+t^2)$		
	$x=8t^3$	M1	For obtaining $ax=bt^3$
	$t=\frac{1}{2}x^{\frac{1}{3}}$, so $y=\frac{1}{2}(2x^{-\frac{1}{3}}-\frac{1}{2}x^{\frac{1}{3}})x+\frac{1}{2}x^{\frac{2}{3}}+\frac{1}{16}x^{\frac{4}{3}}+3$	M1	Eliminating t
	$y=\frac{3}{2}x^{\frac{2}{3}}-\frac{3}{16}x^{\frac{4}{3}}+3$	A1 **6**	

(v)	P lies on the envelope of the normals Hence $a = \frac{3}{2} \times 64^{\frac{2}{3}} - \frac{3}{16} \times 64^{\frac{4}{3}} + 3$ $= -21$	M1 M1 A1 **3**	Or a fully correct method for finding the centre of curvature at a general pt [$(8t^3,\ 6t^2 - 3t^4 + 3)$] Or $t = 2$ and $a = 6 \times 2^2 - 3 \times 2^4 + 3$

<table>
<tr><td>4 (i)</td><td>
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<tr><th></th><th>I</th><th>J</th><th>K</th><th>L</th><th>–I</th><th>–J</th><th>–K</th><th>–L</th></tr>
<tr><td>I</td><td>I</td><td>J</td><td>K</td><td>L</td><td>–I</td><td>–J</td><td>–K</td><td>–L</td></tr>
<tr><td>J</td><td>J</td><td>–I</td><td>L</td><td>–K</td><td>–J</td><td>I</td><td>–L</td><td>K</td></tr>
<tr><td>K</td><td>K</td><td>–L</td><td>–I</td><td>J</td><td>–K</td><td>L</td><td>I</td><td>–J</td></tr>
<tr><td>L</td><td>L</td><td>K</td><td>–J</td><td>–I</td><td>–L</td><td>–K</td><td>J</td><td>I</td></tr>
<tr><td>–I</td><td>–I</td><td>–J</td><td>–K</td><td>–L</td><td>I</td><td>J</td><td>K</td><td>L</td></tr>
<tr><td>–J</td><td>–J</td><td>I</td><td>–L</td><td>K</td><td>J</td><td>–I</td><td>L</td><td>–K</td></tr>
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<tr><td>–L</td><td>–L</td><td>–K</td><td>J</td><td>I</td><td>L</td><td>K</td><td>–J</td><td>–I</td></tr>
</table>
</td><td>B6
6</td><td>Give B5 for 30 (bold) entries correct
Give B4 for 24 (bold) entries correct
Give B3 for 18 (bold) entries correct
Give B2 for 12 (bold) entries correct
Give B1 for 6 (bold) entries correct</td></tr>
<tr><td>(ii)</td><td>
<table>
<tr><td>Element</td><td>I</td><td>J</td><td>K</td><td>L</td><td>–I</td><td>–J</td><td>–K</td><td>–L</td></tr>
<tr><td>Inverse</td><td>I</td><td>–J</td><td>–K</td><td>–L</td><td>–I</td><td>J</td><td>K</td><td>L</td></tr>
</table>
</td><td>B3
3</td><td>Give B2 for six correct
Give B1 for three correct</td></tr>
<tr><td>(iii)</td><td>
<table>
<tr><td>Element</td><td>I</td><td>J</td><td>K</td><td>L</td><td>–I</td><td>–J</td><td>–K</td><td>–L</td></tr>
<tr><td>Order</td><td>1</td><td>4</td><td>4</td><td>4</td><td>2</td><td>4</td><td>4</td><td>4</td></tr>
</table>
</td><td>B3
3</td><td>Give B2 for six correct
Give B1 for three correct</td></tr>
<tr><td>(iv)</td><td>Only two elements of G do not have order 4; so any subgroup of order 4 must contain an element of order 4
A subgroup of order 4 is cyclic if it contains an element of order 4
Hence any subgroup of order 4 is cyclic</td><td>M1A1

B1
A1
4</td><td>

(may be implied)
For completion</td></tr>
<tr><td></td><td>OR If a group of order 4 is not cyclic,
it contains three elements of order 2
B1
G has only one element of order 2; so this cannot occur M1A1
So any subgroup of order 4 is cyclic A1</td><td></td><td></td></tr>
<tr><td>(v)</td><td>$\{\mathbf{I},\ -\mathbf{I}\}$
$\{\mathbf{I},\ \mathbf{J},\ -\mathbf{I},\ -\mathbf{J}\}$
$\{\mathbf{I},\ \mathbf{K},\ -\mathbf{I},\ -\mathbf{K}\}$
$\{\mathbf{I},\ \mathbf{L},\ -\mathbf{I},\ -\mathbf{L}\}$</td><td>B1
B1
B1
B1
B1
5</td><td>For $\{\mathbf{I},\ -\mathbf{I}\}$, at least one correct subgroup of order 4, and no wrong subgroups. This mark is lost if G or $\{\mathbf{I}\}$ is included</td></tr>
</table>

(vi)	The symmetry group has 5 elements of order 2 (4 reflections and rotation through $180°$) *G* has only one element of order 2, hence *G* is not isomorphic to the symmetry group	M1 A1 A1 **3**	Considering elements of order 2 (or self-inverse elements) Identification of at least two elements of order 2 in the symmetry group For completion

Pre-multiplication by transition matrix

5 (i)	$\mathbf{P}=\begin{pmatrix}0.8&0.1&0\\0.2&0.6&0.15\\0&0.3&0.85\end{pmatrix}$	B1B1B1 **3**	For the three columns
(ii)	$\mathbf{P}^7\begin{pmatrix}0.6\\0.4\\0\end{pmatrix}=\begin{pmatrix}0.3204&0.1545&0.0927\\0.3089&0.2895&0.2780\\0.3706&0.5560&0.6293\end{pmatrix}\begin{pmatrix}0.6\\0.4\\0\end{pmatrix}=\begin{pmatrix}0.254\\0.301\\0.445\end{pmatrix}$ Division 3 is the most likely	M1 M1 A1 M1 A1 A1 **6**	Considering $\mathbf{P}^7$ (or $\mathbf{P}^8$ or $\mathbf{P}^6$) Evaluating a power of **P** For $\mathbf{P}^7$ (Allow ± 0.001 throughout) Evaluation of probabilities One probability correct Correctly determined
(iii)	$\mathbf{P}^n\to\begin{pmatrix}0.1429&0.1429&0.1429\\0.2857&0.2857&0.2857\\0.5714&0.5714&0.5714\end{pmatrix}$ Equilibrium probabilities are 0.143, 0.286, 0.571	M1 M1 A1 **3**	Considering powers of **P** Obtaining limit *Must be accurate to 3 dp if given as decimals*
	OR $\mathbf{P}\begin{pmatrix}p\\q\\r\end{pmatrix}=\begin{pmatrix}p\\q\\r\end{pmatrix}\Rightarrow\begin{matrix}0.8p+0.1q=p\\0.2p+0.6q+0.15r=q\\0.3q+0.85r=r\end{matrix}$ M1 $q=2p,\ r=2q=4p$ and $p+q+r=1$ M1 $p=\frac{1}{7},\ q=\frac{2}{7},\ r=\frac{4}{7}$ A1		Obtaining at least two equations Solving (must use $p+q+r=1$)
(iv)	$\mathbf{Q}=\begin{pmatrix}0.8&0.1&0&0\\0.2&0.6&0.15&0\\0&0.3&0.75&0\\0&0&0.1&1\end{pmatrix}$	B1 B1 B1 **3**	Third column Fourth column Fully correct
(v)	$\mathbf{Q}^5\begin{pmatrix}0\\1\\0\\0\end{pmatrix}=\begin{pmatrix}0.4122&0.1566&0.0592&0\\0.3131&0.2767&0.2052&0\\0.2369&0.4105&0.4030&0\\0.0378&0.1563&0.3326&1\end{pmatrix}\begin{pmatrix}0\\1\\0\\0\end{pmatrix}$ $=\begin{pmatrix}0.1566\\0.2767\\0.4105\\0.1563\end{pmatrix}$ P(still in league) $=1-0.1563$ $=0.844$	M1 M1 A1 M1 A1 ft **5**	Considering $\mathbf{Q}^5$ (or $\mathbf{Q}^6$ or $\mathbf{Q}^4$) Evaluating a power of **Q** For 0.1563 (Allow 0.156 ± 0.001) For $1-a_{4,2}$ ft dependent on M1M1M1
(vi)	P(out of league) is element $a_{4,2}$ in $\mathbf{Q}^n$ When $n=15$, $a_{4,2}=0.4849$ When $n=16$, $a_{4,2}=0.5094$ First year is 2031	M1 M1 A1 A1 **4**	Considering $\mathbf{Q}^n$ for at least two more values of *n* Considering $a_{4,2}$ *Dep on previous M1* For $n=16$ *SR* With no working, $n=16$ stated B3 2031 stated B4

Post-multiplication by transition matrix

5 (i)	$\mathbf{P}=\begin{pmatrix}0.8 & 0.2 & 0\\ 0.1 & 0.6 & 0.3\\ 0 & 0.15 & 0.85\end{pmatrix}$	B1B1B1 **3**	For the three rows
(ii)	$(0.6 \quad 0.4 \quad 0)\mathbf{P}^7$ $=(0.6 \quad 0.4 \quad 0)\begin{pmatrix}0.3204 & 0.3089 & 0.3706\\ 0.1545 & 0.2895 & 0.5560\\ 0.0927 & 0.2780 & 0.6293\end{pmatrix}$ $=(0.254 \quad 0.301 \quad 0.445)$ Division 3 is the most likely	M1 M1 A1 M1 A1 A1 **6**	Considering $\mathbf{P}^7$ (or $\mathbf{P}^8$ or $\mathbf{P}^6$) Evaluating a power of **P** For $\mathbf{P}^7$ (Allow ± 0.001 throughout) Evaluation of probabilities One probability correct Correctly determined
(iii)	$\mathbf{P}^n \to \begin{pmatrix}0.1429 & 0.2857 & 0.5714\\ 0.1429 & 0.2857 & 0.5714\\ 0.1429 & 0.2857 & 0.5714\end{pmatrix}$ Equilibrium probabilities are 0.143, 0.286, 0.571	M1 M1 A1 **3**	Considering powers of **P** Obtaining limit *Must be accurate to 3 dp if given as decimals*
	OR $(p \quad q \quad r)\mathbf{P}=(p \quad q \quad r)$ $0.8p+0.1q=p$ $0.2p+0.6q+0.15r=q$ M1 $0.3q+0.85r=r$ $q=2p,\ r=2q=4p$ and $p+q+r=1$ M1 $p=\frac{1}{7},\ q=\frac{2}{7},\ r=\frac{4}{7}$ A1		 Obtaining at least two equations Solving (must use $p+q+r=1$)
(iv)	$\mathbf{Q}=\begin{pmatrix}0.8 & 0.2 & 0 & 0\\ 0.1 & 0.6 & 0.3 & 0\\ 0 & 0.15 & 0.75 & 0.1\\ 0 & 0 & 0 & 1\end{pmatrix}$	B1 B1 B1 **3**	Third row Fourth row Fully correct
(v)	$(0 \quad 1 \quad 0 \quad 0)\mathbf{Q}^5$ $=(0 \quad 1 \quad 0 \quad 0)\begin{pmatrix}0.4122 & 0.3131 & 0.2369 & 0.0378\\ 0.1566 & 0.2767 & 0.4105 & 0.1563\\ 0.0592 & 0.2052 & 0.4030 & 0.3326\\ 0 & 0 & 0 & 1\end{pmatrix}$ $=(0.1566 \quad 0.2767 \quad 0.4105 \quad 0.1563)$ P(still in league) $=1-0.1563$ $=0.844$	M1 M1 A1 M1 A1 ft **5**	Considering $\mathbf{Q}^5$ (or $\mathbf{Q}^6$ or $\mathbf{Q}^4$) Evaluating a power of **Q** For 0.1563 (Allow 0.156 ± 0.001) For $1-a_{2,4}$ ft dependent on M1M1M1

(vi)	P(out of league) is element $a_{2,4}$ in $\mathbf{Q}^n$ When $n=15$, $a_{2,4}=0.4849$ When $n=16$, $a_{2,4}=0.5094$ First year is 2031	M1 M1 A1 A1 **4**	Considering $\mathbf{Q}^n$ for at least two more values of *n* Considering $a_{2,4}$ *Dep on previous M1* For $n=16$ *SR* With no working, $n=16$ stated B3 2031 stated B4