1 (i)	$\begin{pmatrix}-1\\4\\3\end{pmatrix}\times\begin{pmatrix}-k\\4\\k+2\end{pmatrix}=\begin{pmatrix}4k-4\\2-2k\\4k-4\end{pmatrix}$ [$=2(k-1)\begin{pmatrix}2\\-1\\2\end{pmatrix}$]	B1 M1 A2 **4**	$\overrightarrow{\text{AB}}$ and $\overrightarrow{\text{CD}}$ *(Condone* $\overrightarrow{\text{BA}}$ *and* $\overrightarrow{\text{DC}}$ *)* Evaluating vector product Give A1 ft for one element correct
(ii)(*A*)	$k=1$	B1 **1**	
(*B*)	$\overrightarrow{\text{CA}}\times\overrightarrow{\text{AB}}=\begin{pmatrix}-3\\-8\\4\end{pmatrix}\times\begin{pmatrix}-1\\4\\3\end{pmatrix}=\begin{pmatrix}-40\\5\\-20\end{pmatrix}$ Distance is $\dfrac{\lvert\overrightarrow{\text{CA}}\times\overrightarrow{\text{AB}}\rvert}{\lvert\overrightarrow{\text{AB}}\rvert}=\dfrac{45}{\sqrt{26}}$ (≈ 8.825)	M1 M1 A1 M1 M1 A1 **6**	For appropriate vector product Evaluation *Dependent on previous M1* Method for finding shortest distance *Dependent on <u>first</u> M1* Calculating magnitudes *Dependent on previous M1* Accept 8.82 to 8.83
	OR $\overrightarrow{\text{CP}}\,.\,\overrightarrow{\text{AB}}=\begin{pmatrix}-2-\lambda-1\\-3+4\lambda-5\\2+3\lambda+2\end{pmatrix}\cdot\begin{pmatrix}-1\\4\\3\end{pmatrix}=0$ M2A1 $\overrightarrow{\text{CP}}=\dfrac{1}{26}\begin{pmatrix}-95\\-140\\155\end{pmatrix}$ Distance is $\dfrac{\sqrt{52650}}{26}$ M1 M1A1		Finding $\overrightarrow{\text{CP}}$ *Dependent on previous M1* *Dependent on previous M1*
(*C*)	Normal vector is $\overrightarrow{\text{CA}}\times\overrightarrow{\text{AB}}=\begin{pmatrix}-40\\5\\-20\end{pmatrix}=-5\begin{pmatrix}8\\-1\\4\end{pmatrix}$ Equation of plane is $8x-y+4z=-16+3+8$ $8x-y+4z+5=0$	M1 M1 A1 **3**	*Dependent on previous M1* Allow $-40x+5y-20z=25$ etc
(iii)	$\dfrac{\overrightarrow{\text{AC}}\,.\,(\overrightarrow{\text{AB}}\times\overrightarrow{\text{CD}})}{\lvert\overrightarrow{\text{AB}}\times\overrightarrow{\text{CD}}\rvert}=\dfrac{\begin{pmatrix}k+2\\8\\-4\end{pmatrix}\cdot\begin{pmatrix}2\\-1\\2\end{pmatrix}(2k-2)}{3(2k-2)}$ Shortest distance is $\left\lvert\dfrac{2k-12}{3}\right\rvert$	M1 M1 A1 ft A1 **4**	For $\overrightarrow{\text{AC}}\,.\,(\overrightarrow{\text{AB}}\times\overrightarrow{\text{CD}})$ Fully correct method *(evaluation not required) Dependent on previous M1* Correct evaluated expression for distance ft from (i) Simplified answer *Modulus not required*

(iv)	Intersect when $k = 6$ $-2 - \lambda = 6 - 6\mu$ $-3 + 4\lambda = 5 + 4\mu$ $2 + 3\lambda = -2 + 8\mu$ Solving, $\lambda = 4,\ \mu = 2$ Point of intersection is $(-6,\ 13,\ 14)$	B1 ft M1 A1 ft M1 A1 A1 **6**	 Forming at least two equations Two correct equations Solving to obtain λ or μ *Dependent on previous M1* One value correct
	$-2 - \lambda = k - k\mu$ OR $-3 + 4\lambda = 5 + 4\mu$ M1 $2 + 3\lambda = -2 + (k+2)\mu$ A1 Solving, $k = 6$ M1A1 $\lambda = 4,\ \mu = 2$ A1 Point of intersection is $(-6,\ 13,\ 14)$ A1		Forming three equations All equations correct *Dependent on previous M1* One value correct

2 (i)	Normal vector is $\begin{pmatrix}2x-4y\\-4x+6y\\-4z\end{pmatrix}$	M1 A1 A1 A1 **4**	Partial differentiation Condone $\mathbf{r}=\begin{pmatrix}x\\y\\z\end{pmatrix}+\lambda\begin{pmatrix}2x-4y\\-4x+6y\\-4z\end{pmatrix}$ *For 4 marks the normal must appear as a vector (isw)*
(ii)	At Q normal vector is $\begin{pmatrix}18\\-44\\-4\end{pmatrix}$ Tangent plane is $18x-44y-4z=306-176-4=126$ $9x-22y-2z=63$	M1 M1 M1 A1 **4**	For $18x-44y-4z$ *Dependent on previous M1* Using Q to find constant Accept any correct form
(iii)	$18\delta x-44\delta y-4\delta z\approx 0$ $18h-44p-4(-h)\approx 0$ $p\approx\frac{1}{2}h$	M1 A1 ft M1 A1 **4**	For $18\delta x-44\delta y-4\delta z$ *If left in terms of x, y, z: M1A0M1A0*
	OR $9(17+h)-22(4+p)-2(1-h)\approx 63$ M2A1 ft $p\approx\frac{1}{2}h$ A1		
	OR $(17+h)^2-4(17+h)(4+p)+...=0$ $-44p+22h\approx 0$ M2A1 $p\approx\frac{1}{2}h$ A1		Neglecting second order terms
	OR $p=\dfrac{4h+44\pm\sqrt{28h^2+88h+1936}}{6}$ M2A1 $p\approx\frac{1}{2}h$ A1		
(iv)	Normal parallel to *z*-axis requires $2x-4y=0$ and $-4x+6y=0$ $x=y=0$; then $-2z^2-63=0$ No solutions; hence no such points	M1A1 ft M1 A1 (ag) **4**	Correctly shown
	OR $2x-4y=-4x+6y$, so $y=\frac{3}{5}x$ $-\frac{8}{25}x^2-2z^2-63=0$, hence no points M2A2		Similarly if only $2x-4y=0$ used
(v)	$2x-4y=5\lambda$ $-4x+6y=-6\lambda$ $-4z=2\lambda$ $x=-\frac{3}{2}\lambda$, $y=-2\lambda$, $z=-\frac{1}{2}\lambda$ Substituting into equation of surface $\frac{9}{4}\lambda^2-12\lambda^2+12\lambda^2-\frac{1}{2}\lambda^2-63=0$ $\lambda=\pm 6$	M1A1 ft M1 M1 M1 M1	Obtaining *x, y, z* in terms of λ or $x=3z$, $y=4z$ Obtaining a value of λ (or equivalent)

	Point $(-9,\ -12,\ -3)$ gives $k = -45 + 72 - 6 = 21$ Point $(9,\ 12,\ 3)$ gives $k = 45 - 72 + 6 = -21$	A1 A1 **8**	Using a point to find *k* *If* $\lambda = 1$ *is assumed: M0M1M0M0M1*

3 (i)	$\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2=(6t^2-6)^2+(12t)^2$ $=36t^4+72t^2+36$ $=36(t^2+1)^2$ Arc length is $\int_0^1 6(t^2+1)\,dt$ $=\left[\ 2t^3+6t\ \right]_0^1$ $=8$	M1A1 A1 M1 A1 A1 **6**	Using $\int\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$ For $2t^3+6t$
(ii)	Curved surface area is $\int 2\pi\,y\,ds=\int_0^1 2\pi(6t^2)6(t^2+1)\,dt$ $=\pi\left[\ \frac{72}{5}t^5+24t^3\ \right]_0^1$ $=\frac{192\pi}{5}\quad(\approx 120.6)$	M1 A1 M1 A1 A1 **5**	Using $\int ...y\,ds$ (in terms of *t*) with 'ds' the same as in (i) Any correct integral form in terms of *t* *(limits required)* Integration For $\pi(\frac{72}{5}t^5+24t^3)$
(iii)	$\frac{dy}{dx}=\frac{12t}{6t^2-6}\quad\left(=\frac{2t}{t^2-1}\right)$ Equation of normal is $y-6t^2=\frac{1-t^2}{2t}(x-2t^3+6t)$ $y-6t^2=\frac{1}{2}\left(\frac{1}{t}-t\right)x-t^2(1-t^2)+3(1-t^2)$ $y=\frac{1}{2}\left(\frac{1}{t}-t\right)x+2t^2+t^4+3$	M1 A1 M1 A1 (ag) **4**	Method of differentiation *At least one intermediate step required* Correctly obtained
(iv)	Differentiating partially with respect to *t* $0=\frac{1}{2}\left(-\frac{1}{t^2}-1\right)x+4t+4t^3$ $\frac{1}{2t^2}(1+t^2)x=4t(1+t^2)$ $x=8t^3$ $t=\frac{1}{2}x^{\frac{1}{3}}$, so $y=\frac{1}{2}(2x^{-\frac{1}{3}}-\frac{1}{2}x^{\frac{1}{3}})x+\frac{1}{2}x^{\frac{2}{3}}+\frac{1}{16}x^{\frac{4}{3}}+3$ $y=\frac{3}{2}x^{\frac{2}{3}}-\frac{3}{16}x^{\frac{4}{3}}+3$	M1 A2 M1 M1 A1 **6**	Give A1 if just one error or omission For obtaining $a\,x=b\,t^3$ Eliminating *t*

(v)	P lies on the envelope of the normals Hence $a = \frac{3}{2} \times 64^{\frac{2}{3}} - \frac{3}{16} \times 64^{\frac{4}{3}} + 3$ $= -21$	M1 M1 A1 **3**	Or a fully correct method for finding the centre of curvature at a general pt $[\ (8t^3,\ 6t^2 - 3t^4 + 3)\ ]$ Or $t = 2$ and $a = 6 \times 2^2 - 3 \times 2^4 + 3$

<table>
<tr>
<td>**4 (i)**</td>
<td>
<table>
<tr><th></th><th>I</th><th>J</th><th>K</th><th>L</th><th>–I</th><th>–J</th><th>–K</th><th>–L</th></tr>
<tr><td>I</td><td>I</td><td>J</td><td>K</td><td>L</td><td>–I</td><td>–J</td><td>–K</td><td>–L</td></tr>
<tr><td>J</td><td>J</td><td>–I</td><td>L</td><td>–K</td><td>–J</td><td>I</td><td>–L</td><td>K</td></tr>
<tr><td>K</td><td>K</td><td>–L</td><td>–I</td><td>**J**</td><td>**–K**</td><td>**L**</td><td>**I**</td><td>**–J**</td></tr>
<tr><td>L</td><td>L</td><td>K</td><td>**–J**</td><td>**–I**</td><td>**–L**</td><td>**–K**</td><td>**J**</td><td>**I**</td></tr>
<tr><td>–I</td><td>–I</td><td>–J</td><td>**–K**</td><td>**–L**</td><td>**I**</td><td>**J**</td><td>**K**</td><td>**L**</td></tr>
<tr><td>–J</td><td>–J</td><td>I</td><td>**–L**</td><td>**K**</td><td>**J**</td><td>**–I**</td><td>**L**</td><td>**–K**</td></tr>
<tr><td>–K</td><td>–K</td><td>L</td><td>**I**</td><td>**–J**</td><td>**K**</td><td>**–L**</td><td>**–I**</td><td>**J**</td></tr>
<tr><td>–L</td><td>–L</td><td>–K</td><td>**J**</td><td>**I**</td><td>**L**</td><td>**K**</td><td>**–J**</td><td>**–I**</td></tr>
</table>
</td>
<td>B6
6</td>
<td>Give B5 for 30 (bold) entries correct
Give B4 for 24 (bold) entries correct
Give B3 for 18 (bold) entries correct
Give B2 for 12 (bold) entries correct
Give B1 for 6 (bold) entries correct</td>
</tr>
<tr>
<td>**(ii)**</td>
<td>
<table>
<tr><td>Element</td><td>**I**</td><td>**J**</td><td>**K**</td><td>**L**</td><td>**–I**</td><td>**–J**</td><td>**–K**</td><td>**–L**</td></tr>
<tr><td>Inverse</td><td>**I**</td><td>**–J**</td><td>**–K**</td><td>**–L**</td><td>**–I**</td><td>**J**</td><td>**K**</td><td>**L**</td></tr>
</table>
</td>
<td>B3
3</td>
<td>Give B2 for six correct
Give B1 for three correct</td>
</tr>
<tr>
<td>**(iii)**</td>
<td>
<table>
<tr><td>Element</td><td>**I**</td><td>**J**</td><td>**K**</td><td>**L**</td><td>**–I**</td><td>**–J**</td><td>**–K**</td><td>**–L**</td></tr>
<tr><td>Order</td><td>1</td><td>4</td><td>4</td><td>4</td><td>2</td><td>4</td><td>4</td><td>4</td></tr>
</table>
</td>
<td>B3
3</td>
<td>Give B2 for six correct
Give B1 for three correct</td>
</tr>
<tr>
<td>**(iv)**</td>
<td>Only two elements of *G* do not have order 4; so any subgroup of order 4 must contain an element of order 4
A subgroup of order 4 is cyclic if it contains an element of order 4
Hence any subgroup of order 4 is cyclic</td>
<td>M1A1
B1
A1
4</td>
<td>(may be implied)
For completion</td>
</tr>
<tr>
<td></td>
<td>OR If a group of order 4 is not cyclic, it contains three elements of order 2 B1
G has only one element of order 2; so this cannot occur M1A1
So any subgroup of order 4 is cyclic A1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>**(v)**</td>
<td>$\{\mathbf{I},\ -\mathbf{I}\}$
$\{\mathbf{I},\ \mathbf{J},\ -\mathbf{I},\ -\mathbf{J}\}$
$\{\mathbf{I},\ \mathbf{K},\ -\mathbf{I},\ -\mathbf{K}\}$
$\{\mathbf{I},\ \mathbf{L},\ -\mathbf{I},\ -\mathbf{L}\}$</td>
<td>B1
B1
B1
B1
B1
5</td>
<td>For $\{\mathbf{I},\ -\mathbf{I}\}$, at least one correct subgroup of order 4, and no wrong subgroups. This mark is lost if *G* or $\{\mathbf{I}\}$ is included</td>
</tr>
</table>

(vi)	The symmetry group has 5 elements of order 2 (4 reflections and rotation through $180°$) *G* has only one element of order 2, hence *G* is not isomorphic to the symmetry group	M1 A1 A1 **3**	Considering elements of order 2 (or self-inverse elements) Identification of at least two elements of order 2 in the symmetry group For completion

Pre-multiplication by transition matrix

5 (i)	$\mathbf{P}=\begin{pmatrix}0.8 & 0.1 & 0\\ 0.2 & 0.6 & 0.15\\ 0 & 0.3 & 0.85\end{pmatrix}$	B1B1B1 **3**	For the three columns
(ii)	$\mathbf{P}^7\begin{pmatrix}0.6\\0.4\\0\end{pmatrix}=\begin{pmatrix}0.3204 & 0.1545 & 0.0927\\ 0.3089 & 0.2895 & 0.2780\\ 0.3706 & 0.5560 & 0.6293\end{pmatrix}\begin{pmatrix}0.6\\0.4\\0\end{pmatrix}=\begin{pmatrix}0.254\\0.301\\0.445\end{pmatrix}$ Division 3 is the most likely	M1 M1 A1 M1 A1 A1 **6**	Considering $\mathbf{P}^7$ (or $\mathbf{P}^8$ or $\mathbf{P}^6$) Evaluating a power of **P** For $\mathbf{P}^7$ (Allow ± 0.001 throughout) Evaluation of probabilities One probability correct Correctly determined
(iii)	$\mathbf{P}^n \to \begin{pmatrix}0.1429 & 0.1429 & 0.1429\\ 0.2857 & 0.2857 & 0.2857\\ 0.5714 & 0.5714 & 0.5714\end{pmatrix}$ Equilibrium probabilities are 0.143, 0.286, 0.571	M1 M1 A1 **3**	Considering powers of **P** Obtaining limit *Must be accurate to 3 dp if given as decimals*
	OR $\mathbf{P}\begin{pmatrix}p\\q\\r\end{pmatrix}=\begin{pmatrix}p\\q\\r\end{pmatrix} \Rightarrow \begin{matrix}0.8p+0.1q=p\\ 0.2p+0.6q+0.15r=q\\ 0.3q+0.85r=r\end{matrix}$ M1 $q=2p,\ r=2q=4p$ and $p+q+r=1$ M1 $p=\frac{1}{7},\ q=\frac{2}{7},\ r=\frac{4}{7}$ A1		Obtaining at least two equations Solving (must use $p+q+r=1$)
(iv)	$\mathbf{Q}=\begin{pmatrix}0.8 & 0.1 & 0 & 0\\ 0.2 & 0.6 & 0.15 & 0\\ 0 & 0.3 & 0.75 & 0\\ 0 & 0 & 0.1 & 1\end{pmatrix}$	B1 B1 B1 **3**	Third column Fourth column Fully correct
(v)	$\mathbf{Q}^5\begin{pmatrix}0\\1\\0\\0\end{pmatrix}=\begin{pmatrix}0.4122 & 0.1566 & 0.0592 & 0\\ 0.3131 & 0.2767 & 0.2052 & 0\\ 0.2369 & 0.4105 & 0.4030 & 0\\ 0.0378 & 0.1563 & 0.3326 & 1\end{pmatrix}\begin{pmatrix}0\\1\\0\\0\end{pmatrix}$ $=\begin{pmatrix}0.1566\\0.2767\\0.4105\\0.1563\end{pmatrix}$ P(still in league) $=1-0.1563$ $=0.844$	M1 M1 A1 M1 A1 ft **5**	Considering $\mathbf{Q}^5$ (or $\mathbf{Q}^6$ or $\mathbf{Q}^4$) Evaluating a power of **Q** For 0.1563 (Allow 0.156 ± 0.001) For $1-a_{4,2}$ ft dependent on M1M1M1
(vi)	P(out of league) is element $a_{4,2}$ in $\mathbf{Q}^n$ When $n=15$, $a_{4,2}=0.4849$ When $n=16$, $a_{4,2}=0.5094$ First year is 2031	M1 M1 A1 A1 **4**	Considering $\mathbf{Q}^n$ for at least two more values of *n* Considering $a_{4,2}$ *Dep on previous M1* For $n=16$ *SR* With no working, $n=16$ stated B3 2031 stated B4

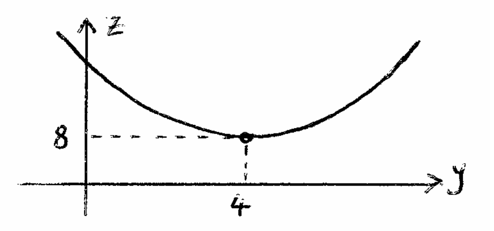
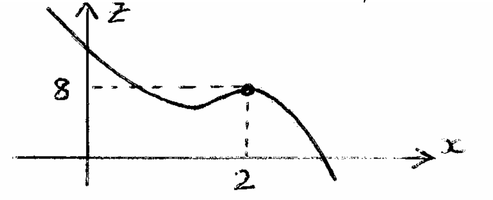
Post-multiplication by transition matrix

5 (i)	$\mathbf{P} = \begin{pmatrix} 0.8 & 0.2 & 0 \\ 0.1 & 0.6 & 0.3 \\ 0 & 0.15 & 0.85 \end{pmatrix}$	B1B1B1 **3**	For the three rows
(ii)	$\begin{pmatrix} 0.6 & 0.4 & 0 \end{pmatrix}\mathbf{P}^7$ $= \begin{pmatrix} 0.6 & 0.4 & 0 \end{pmatrix} \begin{pmatrix} 0.3204 & 0.3089 & 0.3706 \\ 0.1545 & 0.2895 & 0.5560 \\ 0.0927 & 0.2780 & 0.6293 \end{pmatrix}$ $= \begin{pmatrix} 0.254 & 0.301 & 0.445 \end{pmatrix}$ Division 3 is the most likely	M1 M1 A1 M1 A1 A1 **6**	Considering $\mathbf{P}^7$ (or $\mathbf{P}^8$ or $\mathbf{P}^6$) Evaluating a power of **P** For $\mathbf{P}^7$ (Allow ± 0.001 throughout) Evaluation of probabilities One probability correct Correctly determined
(iii)	$\mathbf{P}^n \to \begin{pmatrix} 0.1429 & 0.2857 & 0.5714 \\ 0.1429 & 0.2857 & 0.5714 \\ 0.1429 & 0.2857 & 0.5714 \end{pmatrix}$ Equilibrium probabilities are 0.143, 0.286, 0.571	M1 M1 A1 **3**	Considering powers of **P** Obtaining limit *Must be accurate to 3 dp if given as decimals*
	OR $\begin{pmatrix} p & q & r \end{pmatrix}\mathbf{P} = \begin{pmatrix} p & q & r \end{pmatrix}$ $0.8p + 0.1q = p$ $0.2p + 0.6q + 0.15r = q$ M1 $0.3q + 0.85r = r$ $q = 2p,\ r = 2q = 4p$ and $p + q + r = 1$ M1 $p = \frac{1}{7},\ q = \frac{2}{7},\ r = \frac{4}{7}$ A1		Obtaining at least two equations Solving (must use $p + q + r = 1$)
(iv)	$\mathbf{Q} = \begin{pmatrix} 0.8 & 0.2 & 0 & 0 \\ 0.1 & 0.6 & 0.3 & 0 \\ 0 & 0.15 & 0.75 & 0.1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$	B1 B1 B1 **3**	Third row Fourth row Fully correct
(v)	$\begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix}\mathbf{Q}^5$ $= \begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0.4122 & 0.3131 & 0.2369 & 0.0378 \\ 0.1566 & 0.2767 & 0.4105 & 0.1563 \\ 0.0592 & 0.2052 & 0.4030 & 0.3326 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ $= \begin{pmatrix} 0.1566 & 0.2767 & 0.4105 & 0.1563 \end{pmatrix}$ P(still in league) $= 1 - 0.1563$ $= 0.844$	M1 M1 A1 M1 A1 ft **5**	Considering $\mathbf{Q}^5$ (or $\mathbf{Q}^6$ or $\mathbf{Q}^4$) Evaluating a power of **Q** For 0.1563 (Allow 0.156 ± 0.001) For $1 - a_{2,4}$ ft dependent on M1M1M1

(vi)	P(out of league) is element $a_{2,4}$ in $\mathbf{Q}^n$ When $n = 15$, $a_{2,4} = 0.4849$ When $n = 16$, $a_{2,4} = 0.5094$ First year is 2031	M1 M1 A1 A1 **4**	Considering $\mathbf{Q}^n$ for at least two more values of *n* Considering $a_{2,4}$ *Dep on previous M1* For $n = 16$ *SR* With no working, $n = 16$ stated B3 2031 stated B4

1 (i)	$\mathbf{d}_K = \begin{pmatrix}8\\-1\\-14\end{pmatrix}\times\begin{pmatrix}6\\2\\-5\end{pmatrix}=\begin{pmatrix}33\\-44\\22\end{pmatrix}\quad [\ =11\begin{pmatrix}3\\-4\\2\end{pmatrix}\ ]$ $\mathbf{d}_L = \begin{pmatrix}8\\-1\\-14\end{pmatrix}\times\begin{pmatrix}2\\1\\-1\end{pmatrix}=\begin{pmatrix}15\\-20\\10\end{pmatrix}\quad [\ =5\begin{pmatrix}3\\-4\\2\end{pmatrix}\ ]$ Hence *K* and *L* are parallel For a point on *K*, $z=0,\ x=3,\ y=4$ i.e. $(3,\ 4,\ 0)$ For a point on *L*, $z=0,\ x=6,\ y=28$ i.e. $(6,\ 28,\ 0)$	M1* A1* A1 M1*A1* A1*	Finding direction of *K* or *L* One direction correct ** These marks can be earned anywhere in the question* Correctly shown Finding one point on *K* or *L* *or* $(6,\ 0,\ 2)$ *or* $(0,\ 8,\ -2)$ *etc* *Or* $(27,\ 0,\ 14)$ *or* $(0,\ 36,\ -4)$ *etc*
	$\left[\begin{pmatrix}6\\28\\0\end{pmatrix}-\begin{pmatrix}3\\4\\0\end{pmatrix}\right]\times\begin{pmatrix}3\\-4\\2\end{pmatrix}=\begin{pmatrix}3\\24\\0\end{pmatrix}\times\begin{pmatrix}3\\-4\\2\end{pmatrix}=\begin{pmatrix}48\\-6\\-84\end{pmatrix}$ Distance is $\dfrac{\sqrt{48^2+6^2+84^2}}{\sqrt{3^2+4^2+2^2}}=\dfrac{\sqrt{9396}}{\sqrt{29}}=18$	M1 M1 A1 **9**	For $(\mathbf{b}-\mathbf{a})\times\mathbf{d}$ Correct method for finding distance
	OR $\begin{pmatrix}6+3\lambda-3\\28-4\lambda-4\\2\lambda\end{pmatrix}\cdot\begin{pmatrix}3\\-4\\2\end{pmatrix}=0$ M1 $-87+29\lambda=0,\quad \lambda=3$ M1 Distance is $\sqrt{12^2+12^2+6^2}=18$ A1		For $(\mathbf{b}+\lambda\mathbf{d}-\mathbf{a}).\mathbf{d}=0$ Finding λ, and the magnitude
(ii)	Distance from $(3, 4, 0)$ to *R* is $\left\lvert\dfrac{2\times3+4-0-40}{\sqrt{2^2+1^2+1^2}}\right\rvert$ $=\dfrac{30}{\sqrt6}=\dfrac{30\sqrt6}{6}=5\sqrt6$	M1A1 ft A1 ag **3**	
(iii)	*K*, *M* intersect if $1+5\lambda=3+3\mu$ (1) $-4-4\lambda=4-4\mu$ (2) $3\lambda=2\mu$ (3) Solving (2) and (3): $\lambda=4,\ \mu=6$ Check in (1): LHS $=1+20=21$, RHS $=3+18=21$ Hence *K*, *M* intersect, at $(21,\ -20,\ 12)$	M1 A1 ft M1M1 M1A1 A1 **7**	At least 2 eqns, different parameters Two equations correct Intersection correctly shown *Can be awarded after M1A1M1M0M0*
	OR *M* meets *P* when M1 $8(1+5\lambda)-(-4-4\lambda)-14(3\lambda)=20$ A1 *M* meets *Q* when $6(1+5\lambda)+2(-4-4\lambda)-5(3\lambda)=26$ A1 Both equations have solution $\lambda=4$ A1 Point is on *P*, *Q* and *M*; hence on *K* and *M* M2 Point of intersection is $(21,\ -20,\ 12)$ A1		Intersection of *M* with **both** *P* and *Q*

(iv)	$\left[\begin{pmatrix}6\\28\\0\end{pmatrix}-\begin{pmatrix}1\\-4\\0\end{pmatrix}\right]\cdot\left[\begin{pmatrix}3\\-4\\2\end{pmatrix}\times\begin{pmatrix}5\\-4\\3\end{pmatrix}\right]=\begin{pmatrix}5\\32\\0\end{pmatrix}\cdot\begin{pmatrix}-4\\1\\8\end{pmatrix}=12$	M1A1 ft M1	For evaluating $\mathbf{d}_L\times\mathbf{d}_M$ For $(\mathbf{b}-\mathbf{c}).(\mathbf{d}_L\times\mathbf{d}_M)$
	Distance is $\dfrac{12}{\sqrt{4^2+1^2+8^2}}=\dfrac{12}{9}=\dfrac{4}{3}$	A1 ft A1 **5**	Numerical expression for distance

2 (i)	$\frac{\partial z}{\partial x} = y^2 - 8xy - 6x^2 + 54x - 36$ $\frac{\partial z}{\partial y} = 2xy - 4x^2$	B2 B1 **3**	Give B1 for 3 terms correct
(ii)	At stationary points, $\frac{\partial z}{\partial x} = 0$ and $\frac{\partial z}{\partial y} = 0$ When $x = 0$, $y^2 - 36 = 0$ $y = \pm 6$; points $(0,\ 6,\ 20)$ and $(0,\ -6,\ 20)$ When $y = 2x$, $4x^2 - 16x^2 - 6x^2 + 54x - 36 = 0$ $-18x^2 + 54x - 36 = 0$ $x = 1,\ 2$ Points $(1,\ 2,\ 5)$ and $(2,\ 4,\ 8)$	M1 M1 A1A1 M1 M1A1 A1 **8**	If A0, give A1 for $y = \pm 6$ or $y = 2,\ 4$ *A0 if any extra points given*
(iii)	When $x = 2$, $z = 2y^2 - 16y + 40$  When $y = 4$, $z = -2x^3 + 11x^2 - 20x + 20$ $\left(\frac{\mathrm{d}^2 z}{\mathrm{d}x^2} = -12x + 22 = -2 \text{ when } x = 2 \right)$  The point is a minimum on one section and a maximum on the other; so it is neither a maximum nor a minimum	B1 B1 B1 B1 B1 B1 **6**	'Upright' parabola $(2,\ 4,\ 8)$ identified as a minimum (in the first quadrant) 'Negative cubic' curve $(2,\ 4,\ 8)$ identified as a stationary point Fully correct (unambiguous minimum and maximum)
(iv)	Require $\frac{\partial z}{\partial x} = -36$ and $\frac{\partial z}{\partial y} = 0$ When $x = 0$, $y^2 - 36 = -36$ $y = 0$; point $(0,\ 0,\ 20)$ When $y = 2x$, $4x^2 - 16x^2 - 6x^2 + 54x - 36 = -36$ $-18x^2 + 54x = 0$ $x = 0,\ 3$ $x = 0$ gives $(0,\ 0,\ 20)$ same as above $x = 3$ gives $(3,\ 6,\ -7)$	M1 M1 A1 M1 M1 A1 A1 **7**	$\frac{\partial z}{\partial x} = 36$ can earn all M marks Solving to obtain *x* (or *y*) *or stating 'no roots' if appropriate (e.g. when +36 has been used)*

3 (i)	$1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2=1+\left(x-\frac{1}{4x}\right)^2$ $=1+x^2-\frac{1}{2}+\frac{1}{16x^2}=x^2+\frac{1}{2}+\frac{1}{16x^2}$ $=\left(x+\frac{1}{4x}\right)^2$ Arc length is $\int_1^a\left(x+\frac{1}{4x}\right)\mathrm{d}x$ $=\left[\frac{1}{2}x^2+\frac{1}{4}\ln x\right]_1^a$ $=\frac{1}{2}a^2+\frac{1}{4}\ln a-\frac{1}{2}$	M1 A1 M1 M1 A1 ag **5**	For $\int\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x$
(ii)	Curved surface area is $\int 2\pi x\,\mathrm{d}s$ $=\int_1^4 2\pi x\left(x+\frac{1}{4x}\right)\mathrm{d}x$ $=2\pi\left[\frac{1}{3}x^3+\frac{1}{4}x\right]_1^4$ $=\frac{87\pi}{2}\quad(\approx 137)$	M1 A1 ft M1 A1 A1 **5**	Any correct integral form (including limits) for $\frac{1}{3}x^3+\frac{1}{4}x$
(iii)	$\rho=\dfrac{\left(1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right)^{\frac{3}{2}}}{\frac{\mathrm{d}^2y}{\mathrm{d}x^2}}=\dfrac{\left(a+\frac{1}{4a}\right)^3}{1+\frac{1}{4a^2}}$ $=\dfrac{a\left(a+\frac{1}{4a}\right)^3}{a+\frac{1}{4a}}=a\left(a+\frac{1}{4a}\right)^2$	B1 B1 M1 A1 A1 ag **5**	any form, in terms of x or a any form, in terms of x or a Formula for ρ or κ ρ or κ correct, in any form, in terms of x or a
(iv)	At $(1,\ \frac{1}{2})$, $\rho=(\frac{5}{4})^2=\frac{25}{16}$ $\frac{\mathrm{d}y}{\mathrm{d}x}=1-\frac{1}{4}=\frac{3}{4}$, so $\hat{\mathbf{n}}=\begin{pmatrix}-\frac{3}{5}\\ \frac{4}{5}\end{pmatrix}$ $\mathbf{c}=\begin{pmatrix}1\\ \frac{1}{2}\end{pmatrix}+\frac{25}{16}\begin{pmatrix}-\frac{3}{5}\\ \frac{4}{5}\end{pmatrix}$ Centre of curvature is $\left(\frac{1}{16},\ \frac{7}{4}\right)$	M1 A1 M1 A1A1 **5**	Finding gradient Correct normal vector (not necessarily unit vector); may be in terms of x OR *M2A1 for obtaining equation of normal line at a general point and differentiating partially*

(v)	Differentiating partially w.r.t. *p* $0 = x^2 - 2p\ln x$ $p = \dfrac{x^2}{2\ln x}$ and $y = \dfrac{x^4}{2\ln x} - \dfrac{x^4}{4\ln x}$ $y = \dfrac{x^4}{4\ln x}$	M1 A1 M1 A1 **4**	

<table>
<tr><td>4 (i)</td><td>By Lagrange’s theorem,
a proper subgroup has order 2 or 5
A group of prime order is cyclic
Hence every proper subgroup is cyclic</td><td>M1
A1
M1
A1
4</td><td>Using Lagrange <i>(need not be mentioned explicitly)</i>
or equivalent
For completion</td></tr>
<tr><td>(ii)</td><td>e.g. $2^2=4,\ 2^3=8,\ 2^4=5,\ 2^5=10,$
$2^6=9,\ 2^7=7,\ 2^8=3,\ 2^9=6,\ 2^{10}=1$
2 has order 10, hence <i>M</i> is cyclic</td><td>M1
A1
A1
A1
4</td><td>Considering order of an element
Identifying an element of order 10 (2, 6, 7 or 8)
Fully justified
For conclusion (can be awarded after M1A1A0)</td></tr>
<tr><td>(iii)</td><td>$\{1,\ 10\}$
$\{1,\ 3,\ 4,\ 5,\ 9\}$</td><td>B1
B2
3</td><td>Ignore $\{1\}$ and <i>M</i>
Deduct 1 mark (from B1B2) for each (proper) subgroup given in excess of 2</td></tr>
<tr><td>(iv)</td><td>E is the identity
A, C, G, I are rotations
B, D, F, H, J are reflections</td><td>B1
M1
A1
A1
4</td><td>Considering elements of order 2 <i>(or equivalent) Implied by four of B, D, F, H, J in the same set</i>
Give A1 if one element is in the wrong set; or if two elements are interchanged</td></tr>
<tr><td>(v)</td><td><i>P</i> and <i>M</i> are not isomorphic
<i>M</i> is abelian, <i>P</i> is non-abelian</td><td>B1
B1
2</td><td>Valid reason
e.g. <i>M</i> has one element of order 2
<i>P</i> has more than one</td></tr>
<tr><td>(vi)</td><td><table><tr><td></td><td>A</td><td>B</td><td>C</td><td>D</td><td>E</td><td>F</td><td>G</td><td>H</td><td>I</td><td>J</td></tr><tr><td>Order</td><td>5</td><td>2</td><td>5</td><td>2</td><td>1</td><td>2</td><td>5</td><td>2</td><td>5</td><td>2</td></tr></table></td><td>B3
3</td><td>Give B2 for 7 correct
B1 for 4 correct</td></tr>
<tr><td>(vii)</td><td>{ E , B }, { E , D }, { E , F }, { E , H }, { E , J }
{ E , A , C , G , I }</td><td>M1
A1 ft
B2 cao
4</td><td>Ignore { E } and <i>P</i>
<i>Subgroups of order 2</i>
Using elements of order 2 (allow two errors/omissions)
Correct or ft. A0 if any others given
<i>Subgroups of order greater than 2</i>
Deduct 1 mark (from B2) for each extra subgroup given</td></tr>
</table>

Pre-multiplication by transition matrix

5 (i)	$\mathbf{P}=\begin{pmatrix}0&0&0.4&0.3\\0&0&0.6&0.7\\0.8&0.1&0&0\\0.2&0.9&0&0\end{pmatrix}$	B2 **2**	Give B1 for two columns correct
(ii)	$\mathbf{P}^4=\begin{pmatrix}0.3366&0.3317&0&0\\0.6634&0.6683&0&0\\0&0&0.3366&0.3317\\0&0&0.6634&0.6683\end{pmatrix}$ $\mathbf{P}^7=\begin{pmatrix}0&0&0.3334&0.3333\\0&0&0.6666&0.6667\\0.3335&0.3333&0&0\\0.6665&0.6667&0&0\end{pmatrix}$	B2 B2 **4**	Give B1 for two non-zero elements correct to at least 2dp Give B1 for two non-zero elements correct to at least 2dp
(iii)	$\mathbf{P}^7\begin{pmatrix}0.4\\0.3\\0.2\\0.1\end{pmatrix}=\begin{pmatrix}0.1000\\0.2000\\0.2334\\0.4666\end{pmatrix}$ P(8th letter is C) $=0.233$	M1 A1 **2**	Using $\mathbf{P}^7$ (or $\mathbf{P}^8$) and initial probs
(iv)	$0.1000\times0.3366\ +\ 0.2000\times0.6683$ $+\ 0.2334\times0.3366\ +\ 0.4666\times0.6683$ $=0.558$	M1 M1 A1 ft A1 **4**	Using probabilities for 8th letter Using diagonal elements from $\mathbf{P}^4$
(v)(*A*)	$\mathbf{P}^n\begin{pmatrix}0.4\\0.3\\0.2\\0.1\end{pmatrix}\approx\begin{pmatrix}\frac{1}{3}&\frac{1}{3}&0&0\\\frac{2}{3}&\frac{2}{3}&0&0\\0&0&\frac{1}{3}&\frac{1}{3}\\0&0&\frac{2}{3}&\frac{2}{3}\end{pmatrix}\begin{pmatrix}0.4\\0.3\\0.2\\0.1\end{pmatrix}=\begin{pmatrix}0.2333\\0.4667\\0.1\\0.2\end{pmatrix}$ P($(n+1)$ th letter is $A)=0.233$	M1 A1	Approximating $\mathbf{P}^n$ when n is large and even
(*B*)	$\mathbf{P}^n\begin{pmatrix}0.4\\0.3\\0.2\\0.1\end{pmatrix}\approx\begin{pmatrix}0&0&\frac{1}{3}&\frac{1}{3}\\0&0&\frac{2}{3}&\frac{2}{3}\\\frac{1}{3}&\frac{1}{3}&0&0\\\frac{2}{3}&\frac{2}{3}&0&0\end{pmatrix}\begin{pmatrix}0.4\\0.3\\0.2\\0.1\end{pmatrix}=\begin{pmatrix}0.1\\0.2\\0.2333\\0.4667\end{pmatrix}$ P($(n+1)$ th letter is $A)=0.1$	M1 A1 **4**	Approximating $\mathbf{P}^n$ when n is large and odd
(vi)	$\mathbf{Q}=\begin{pmatrix}0&0&0.4&0.3\\0&0&0.6&0.6\\0.8&0.1&0&0\\0.2&0.9&0&0.1\end{pmatrix}$	B1 **1**	

(vii)	$\mathbf{Q}^n \to \begin{pmatrix} 0.1721 & 0.1721 & 0.1721 & 0.1721 \\ 0.3105 & 0.3105 & 0.3105 & 0.3105 \\ 0.1687 & 0.1687 & 0.1687 & 0.1687 \\ 0.3487 & 0.3487 & 0.3487 & 0.3487 \end{pmatrix}$ Probabilities are 0.172, 0.310, 0.169, 0.349	M1 M1 A2 **4**	Considering $\mathbf{Q}^n$ for large n OR at least two eqns for equilib probs Probabilities from equal columns OR solving to obtain equilib probs Give A1 for two correct
(viii)	$0.3487 \times 0.1 \times 0.1$ $= 0.0035$	M1M1 A1 **3**	Using 0.3487 and 0.1

Post-multiplication by transition matrix

5 (i)	$\mathbf{P} = \begin{pmatrix} 0 & 0 & 0.8 & 0.2 \\ 0 & 0 & 0.1 & 0.9 \\ 0.4 & 0.6 & 0 & 0 \\ 0.3 & 0.7 & 0 & 0 \end{pmatrix}$	B2 **2**	Give B1 for two rows correct
(ii)	$\mathbf{P}^4 = \begin{pmatrix} 0.3366 & 0.6634 & 0 & 0 \\ 0.3317 & 0.6683 & 0 & 0 \\ 0 & 0 & 0.3366 & 0.6634 \\ 0 & 0 & 0.3317 & 0.6683 \end{pmatrix}$ $\mathbf{P}^7 = \begin{pmatrix} 0 & 0 & 0.3335 & 0.6665 \\ 0 & 0 & 0.3333 & 0.6667 \\ 0.3334 & 0.6666 & 0 & 0 \\ 0.3333 & 0.6667 & 0 & 0 \end{pmatrix}$	B2 B2 **4**	Give B1 for two non-zero elements correct to at least 2dp Give B1 for two non-zero elements correct to at least 2dp
(iii)	$(0.4 \quad 0.3 \quad 0.2 \quad 0.1)\mathbf{P}^7$ $= (0.1000 \quad 0.2000 \quad 0.2334 \quad 0.4666)$ P(8th letter is C) $= 0.233$	M1 A1 **2**	Using $\mathbf{P}^7$ (or $\mathbf{P}^8$) and initial probs
(iv)	$0.1000 \times 0.3366 \ + \ 0.2000 \times 0.6683$ $+ \ 0.2334 \times 0.3366 \ + \ 0.4666 \times 0.6683$ $= 0.558$	M1 M1A1 ft A1 **4**	Using probabilities for 8th letter Using diagonal elements from $\mathbf{P}^4$
(v)(*A*)	$\mathbf{u}\,\mathbf{P}^n \approx (0.4 \quad 0.3 \quad 0.2 \quad 0.1)\begin{pmatrix} \frac{1}{3} & \frac{2}{3} & 0 & 0 \\ \frac{1}{3} & \frac{2}{3} & 0 & 0 \\ 0 & 0 & \frac{1}{3} & \frac{2}{3} \\ 0 & 0 & \frac{1}{3} & \frac{2}{3} \end{pmatrix}$ $= (0.2333 \quad 0.4667 \quad 0.1 \quad 0.2)$ P($(n+1)$ th letter is $A) = 0.233$	M1 A1	Approximating $\mathbf{P}^n$ when n is large and even
(*B*)	$\mathbf{u}\,\mathbf{P}^n \approx (0.4 \quad 0.3 \quad 0.2 \quad 0.1)\begin{pmatrix} 0 & 0 & \frac{1}{3} & \frac{2}{3} \\ 0 & 0 & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} & 0 & 0 \\ \frac{1}{3} & \frac{2}{3} & 0 & 0 \end{pmatrix}$ $= (0.1 \quad 0.2 \quad 0.2333 \quad 0.4667)$ P($(n+1)$ th letter is $A) = 0.1$	M1 A1 **4**	Approximating $\mathbf{P}^n$ when n is large and odd
(vi)	$\mathbf{Q} = \begin{pmatrix} 0 & 0 & 0.8 & 0.2 \\ 0 & 0 & 0.1 & 0.9 \\ 0.4 & 0.6 & 0 & 0 \\ 0.3 & 0.6 & 0 & 0.1 \end{pmatrix}$	B1 **1**	

(vii)	$\mathbf{Q}^n \to \begin{pmatrix} 0.1721 & 0.3105 & 0.1687 & 0.3487 \\ 0.1721 & 0.3105 & 0.1687 & 0.3487 \\ 0.1721 & 0.3105 & 0.1687 & 0.3487 \\ 0.1721 & 0.3105 & 0.1687 & 0.3487 \end{pmatrix}$ Probabilities are 0.172, 0.310, 0.169, 0.349	M1 M1 A2 **4**	Considering $\mathbf{Q}^n$ for large n OR at least two eqns for equilib probs Probabilities from equal rows OR solving to obtain equilib probs Give A1 for two correct
(viii)	$0.3487 \times 0.1 \times 0.1$ $= 0.0035$	M1M1 A1 **3**	Using 0.3487 and 0.1

4757 (FP3) Further Applications of Advanced Mathematics

1 (i)	$\overrightarrow{\mathrm{AB}}\times\overrightarrow{\mathrm{AC}}=\begin{pmatrix}6\\8\\5\end{pmatrix}\times\begin{pmatrix}10\\-5\\1\end{pmatrix}=\begin{pmatrix}33\\44\\-110\end{pmatrix}$ ABC is $3x+4y-10z=-9+20-20$ $3x+4y-10z+9=0$	B2 M1 A1 **4**	*Ignore subsequent working* Give B1 for one element correct SC1 for minus the correct vector For $3x+4y-10z$ Accept $33x+44y-110z=-99$ etc
(ii)	Distance is $\dfrac{3\times5+4\times4-10\times8+9}{\sqrt{3^2+4^2+10^2}}$ $=(-)\ \dfrac{40}{\sqrt{125}}\quad\left(=\dfrac{8}{\sqrt{5}}\right)$	M1 A1 ft A1 **3**	Using distance formula (or other complete method) *Condone negative answer* *Accept a.r.t. 3.58*
(iii)	$\overrightarrow{\mathrm{AB}}\times\overrightarrow{\mathrm{CD}}=\begin{pmatrix}6\\8\\5\end{pmatrix}\times\begin{pmatrix}-2\\4\\5\end{pmatrix}=\begin{pmatrix}20\\-40\\40\end{pmatrix}\quad\left[\ =20\begin{pmatrix}1\\-2\\2\end{pmatrix}\ \right]$ Distance is $\overrightarrow{\mathrm{AC}}\,.\,\hat{\mathbf{n}}=\dfrac{\begin{pmatrix}10\\-5\\1\end{pmatrix}.\begin{pmatrix}1\\-2\\2\end{pmatrix}}{\sqrt{1^2+2^2+2^2}}$ $=\dfrac{22}{3}$	M1 A1 M1 A1 **4**	Evaluating $\overrightarrow{\mathrm{AB}}\times\overrightarrow{\mathrm{CD}}$ *or* method for finding end-points of common perp PQ *or* P $(3/2,11,23/4)$ & Q $(71/18,\ 55/9,\ 383/36)$ *or* $\overrightarrow{\mathrm{PQ}}=(22/9,\ -44/9,\ 44/9)$
(iv)	Volume is $\frac{1}{6}(\overrightarrow{\mathrm{AB}}\times\overrightarrow{\mathrm{AC}})\,.\,\overrightarrow{\mathrm{AD}}$ $=\dfrac{1}{6}\begin{pmatrix}33\\44\\-110\end{pmatrix}.\begin{pmatrix}8\\-1\\6\end{pmatrix}$ $=(-)\ \dfrac{220}{3}$	M1 A1 M1 A1 **4**	Scalar triple product *Accept a.r.t. 73.3*
(v)	E is $(-3+10\lambda,\ 5-5\lambda,\ 2+\lambda)$ $3(-3+10\lambda)-2(2+\lambda)+5=0$ $\lambda=\dfrac{2}{7}$ F is $(-3+8\mu,\ 5-\mu,\ 2+6\mu)$ $3(-3+8\mu)-2(2+6\mu)+5=0$ $\mu=\dfrac{2}{3}$ Since $0<\lambda<1$, E is between A and C Since $0<\mu<1$, F is between A and D	M1 A1 M1 A1 B1 **5**	

(vi)	$V_{\text{ABEF}} = \frac{1}{6}(\overrightarrow{\text{AB}}\times\overrightarrow{\text{AE}})\,.\,\overrightarrow{\text{AF}}$ $= \frac{1}{6}\lambda\mu(\overrightarrow{\text{AB}}\times\overrightarrow{\text{AC}})\,.\,\overrightarrow{\text{AD}}$ $= \lambda\mu V_{\text{ABCD}}$ $= \frac{4}{21}V_{\text{ABCD}}$ Ratio of volumes is $\frac{4}{21} : \frac{17}{21}$ $= 4 : 17$	M1 A1 M1 A1 ag **4**	($13\frac{61}{63}$) *ft if numerical* Finding ratio of volumes of two parts SC1 for 4 : 17 deduced from $^{4}/_{21}$ without working
2 (i)	$\frac{\partial g}{\partial x} = 6z - 2(x+2y+3z) = -2x - 4y$ $\frac{\partial g}{\partial y} = -4(x+2y+3z)$ $\frac{\partial g}{\partial z} = 6x - 6(x+2y+3z) = -12y - 18z$	M1 A1 A1 A1 **4**	Partial differentiation *Any correct form, ISW*
(ii)	At P, $\frac{\partial g}{\partial x} = 16,\ \frac{\partial g}{\partial y} = -4,\ \frac{\partial g}{\partial z} = 36$ Normal line is $\mathbf{r} = \begin{pmatrix} 7 \\ -7.5 \\ 3 \end{pmatrix} + \lambda\begin{pmatrix} 4 \\ -1 \\ 9 \end{pmatrix}$	M1 A1 A1 ft **3**	Evaluating partial derivatives at P All correct *Condone omission of* ' **r =** '
(iii)	$\delta g \approx 16\,\delta x - 4\,\delta y + 36\,\delta z$ If $\overrightarrow{\text{PQ}} = \lambda\begin{pmatrix} 4 \\ -1 \\ 9 \end{pmatrix}$, $\delta g \approx 16(4\lambda) - 4(-\lambda) + 36(9\lambda) \quad (= 392\lambda)$ $h = \delta g,$ so $h \approx 392\lambda$ $\overrightarrow{\text{PQ}} \approx \frac{h}{392}\begin{pmatrix} 4 \\ -1 \\ 9 \end{pmatrix},$ so $\mathbf{n} = \frac{1}{392}\begin{pmatrix} 4 \\ -1 \\ 9 \end{pmatrix}$	M1 M1 A1 ft M1 A1 **5**	*Alternative:* *M3 for substituting* $x = 7 + 4\lambda$ *,* … *into* $g = 125 + h$ *and neglecting* λ^2 *A1 ft for linear equation in* λ *and* h *A1 for* **n** *correct*
(iv)	Require $\frac{\partial g}{\partial x} = \frac{\partial g}{\partial y} = 0$ $-2x - 4y = 0$ and $x + 2y + 3z = 0$ $x + 2y = 0$ and $z = 0$ $g(x, y, z) = 0 - 0^2 = 0 \neq 125$ Hence there is no such point on *S*	M1 M1 M1 A1 **4**	Useful manipulation using both eqns Showing there is no such point on *S* Fully correct proof
(v)	Require $\frac{\partial g}{\partial z} = 0$ and $\frac{\partial g}{\partial y} = 5\frac{\partial g}{\partial x}$ $-4x - 8y - 12z = 5(-2x - 4y)$	M1 M1 M1	Implied by $\frac{\partial g}{\partial x} = \lambda,\ \frac{\partial g}{\partial y} = 5\lambda$ *This M1 can be awarded for* $-2x - 4y = 1$ and $-4x - 8y - 12z = 5$

	$y=-\frac{3}{2}z$ and $x=5z$ $6(5z)z-(5z)^2=125$ $z=\pm 5$ Points are $(25,\ -7.5,\ 5)$ and $(-25,\ 7.5,\ -5)$	A1 M1 M1 A1 A1 ft **8**	*or* $z=-\frac{2}{3}y$ and $x=-\frac{10}{3}y$ *or* $y=-\frac{3}{10}x$ and $z=\frac{1}{5}x$ *or* $x=-\frac{5}{4}\lambda,\ y=\frac{3}{8}\lambda,\ z=-\frac{1}{4}\lambda$ *or* $x:y:z=10:-3:2$ Substituting into $\mathrm{g}(x,\ y,\ z)=125$ Obtaining one value of *x, y, z or* λ *Dependent on previous M1* *ft is minus the other point, provided all M marks have been earned*
3 (i)	$\dot{x}^2+\dot{y}^2=(24t^2)^2+(18t-8t^3)^2$ $=576t^4+324t^2-288t^4+64t^6$ $=324t^2+288t^4+64t^6$ $=(18t+8t^3)^2$ Arc length is $\int_0^2(18t+8t^3)\,\mathrm{d}t$ $=\left[\ 9t^2+2t^4\ \right]_0^2$ $=68$	B1 M1 A1 ag M1 A1 A1 **6**	*Note* $\int_0^2(18+8t^3)\,\mathrm{d}t\ =\left[\ 18t+2t^4\ \right]_0^2=68$ *earns M1A0A0*
(ii)	Curved surface area is $\int 2\pi\, y\,\mathrm{d}s$ $=\int_0^2 2\pi(9t^2-2t^4)(18t+8t^3)\,\mathrm{d}t$ $=\int_0^2 \pi(324t^3+72t^5-32t^7)\,\mathrm{d}t$ $=\pi\left[\ 81t^4+12t^6-4t^8\ \right]_0^2$ $=1040\pi\quad(\approx 3267)$	M1 M1 A1 M1 M1 A1 **6**	Using $\mathrm{d}s=(18t+8t^3)\,\mathrm{d}t$ Correct integral expression including limits *(may be implied by later work)*
(iii)	$\kappa=\frac{\dot{x}\ddot{y}-\ddot{x}\dot{y}}{(\dot{x}^2+\dot{y}^2)^{\frac{3}{2}}}=\frac{(24t^2)(18-24t^2)-(48t)(18t-8t^3)}{(18t+8t^3)^3}$ $=\frac{48t^2(9-12t^2-18+8t^2)}{8t^3(9+4t^2)^3}=\frac{-48t^2(9+4t^2)}{8t^3(9+4t^2)^3}$ $=\frac{-6}{t(4t^2+9)^2}$	M1 A1A1 M1 A1 ag **5**	Using formula for κ (or ρ) For numerator and denominator Simplifying the numerator

(iv)	When $t=1,\ x=8,\ y=7,\ \kappa=-\frac{6}{169}$ $\rho=(-)\,\dfrac{169}{6}$ $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{\dot{y}}{\dot{x}}=\dfrac{18t-8t^3}{24t^2}=\dfrac{10}{24}$ $\hat{\mathbf{n}}=\begin{pmatrix}5/13\\-12/13\end{pmatrix}$ $\mathbf{c}=\begin{pmatrix}8\\7\end{pmatrix}+\frac{169}{6}\begin{pmatrix}5/13\\-12/13\end{pmatrix}$ Centre of curvature is $(18\frac{5}{6},\ -19)$	B1 M1 M1 A1 M1 A1A1 **7**	Finding gradient (or tangent vector) Finding direction of the normal Correct unit normal (either direction)
4 (i)	*Commutative:* $x*y=y*x$ (for all *x*, *y*) *Associative:* $(x*y)*z=x*(y*z)$ (for all *x*, *y*, *z*)	B1 B2 **3**	Accept e.g. ‘Order does not matter’ Give B1 for a partial explanation, e.g. ‘Position of brackets does not matter’
(ii)	$2(x+\frac{1}{2})(y+\frac{1}{2})-\frac{1}{2}=2xy+x+y+\frac{1}{2}-\frac{1}{2}$ $=2xy+x+y=x*y$	B1 ag **1**	*Intermediate step required*
(iii)(*A*)	If $x,y\in S$ then $x>-\frac{1}{2}$ and $y>-\frac{1}{2}$ $x+\frac{1}{2}>0$ and $y+\frac{1}{2}>0$, so $2(x+\frac{1}{2})(y+\frac{1}{2})>0$ $2(x+\frac{1}{2})(y+\frac{1}{2})-\frac{1}{2}>-\frac{1}{2}$, so $x*y\in S$	M1 A1 A1 **3**	
(*B*)	0 is the identity since $0*x=0+x+0=x$ If $x\in S$ and $x*y=0$ then $2xy+x+y=0$ $y=\dfrac{-x}{2x+1}$ $y+\frac{1}{2}=\dfrac{1}{2(2x+1)}>0$ (since $x>-\frac{1}{2}$) so $y\in S$ *S* is closed and associative; there is an identity; and every element of *S* has an inverse in *S*	B1 B1 M1 A1 M1 A1 **6**	or $2(x+\frac{1}{2})(y+\frac{1}{2})-\frac{1}{2}=0$ or $y+\frac{1}{2}=\dfrac{1}{4(x+\frac{1}{2})}$ *Dependent on M1A1M1*
(iv)	If $x*x=0,\ \ 2x^2+x+x=0$ $x=0$ or -1 0 is the identity (and has order 1) -1 is not in *S*	M1 A1 A1 **3**	

(v)	$4*6=48+4+6=58$ $=56+2=7\times 8+2$ So $4\circ 6=2$	B1 B1 ag **2**	
(vi)	<table><tr><td>Element</td><td>0</td><td>1</td><td>2</td><td>4</td><td>5</td><td>6</td></tr><tr><td>Order</td><td>1</td><td>6</td><td>6</td><td>3</td><td>3</td><td>2</td></tr></table>	B3 **3**	Give B2 for 4 correct B1 for 2 correct
(vii)	$\{0\}$, G $\{0,\ 6\}$ $\{0,\ 4,\ 5\}$	B1 B1 B1 **3**	*Condone omission of G* If more than 2 non-trivial subgroups are given, deduct 1 mark (from final B1B1) for each non-trivial subgroup in excess of 2

Pre-multiplication by transition matrix

5 (i)	$\mathbf{P}=\begin{pmatrix}0.1 & 0.7 & 0.1\\ 0.4 & 0.2 & 0.6\\ 0.5 & 0.1 & 0.3\end{pmatrix}$	B2 **2**	Give B1 for two columns correct
(ii)	$\mathbf{P}^6\begin{pmatrix}\frac{1}{3}\\ \frac{1}{3}\\ \frac{1}{3}\end{pmatrix}=\begin{pmatrix}0.328864\\ 0.381536\\ 0.2896\end{pmatrix}$ P(*B* used on 7th day) $=0.3815$	M1 M1 M1 A1 **4**	Using $\mathbf{P}^6$ (or $\mathbf{P}^7$) For matrix of initial probabilities For evaluating matrix product *Accept 0.381 to 0.382*
(iii)	$0.328864\times0.1+0.381536\times0.2+0.2896\times0.3$ $=0.1961$	M1 M1 A1 **3**	Using diagonal elements from **P** Correct method *Accept a.r.t. 0.196*
(iv)	$\mathbf{P}^3=\begin{pmatrix}0.352 & 0.328 & 0.304\\ 0.364 & 0.404 & 0.372\\ 0.284 & 0.268 & 0.324\end{pmatrix}$ $0.328864\times0.352+0.381536\times0.404+0.2896\times0.324$ $=0.3637$	M1 M1 M1 A1 **4**	For evaluating $\mathbf{P}^3$ Using diagonal elements from $\mathbf{P}^3$ Correct method *Accept a.r.t. 0.364*
(v)	$\mathbf{Q}=\begin{pmatrix}0.3289 & 0.3289 & 0.3289\\ 0.3816 & 0.3816 & 0.3816\\ 0.2895 & 0.2895 & 0.2895\end{pmatrix}$ 0.3289, 0.3816, 0.2895 are the long-run probabilities for the routes *A, B, C*	B1 B1 B1 B1 **4**	*Deduct 1 if not given as a (3×3) matrix* *Deduct 1 if not 4 dp* *Accept 'equilibrium probabilities'*
(vi)	$\begin{pmatrix}0.1 & 0.7 & a\\ 0.4 & 0.2 & b\\ 0.5 & 0.1 & c\end{pmatrix}\begin{pmatrix}0.4\\ 0.2\\ 0.4\end{pmatrix}=\begin{pmatrix}0.4\\ 0.2\\ 0.4\end{pmatrix}$ $0.04+0.14+0.4a=0.4,\ \text{so}\ \ a=0.55$ $0.16+0.04+0.4b=0.2,\ \text{so}\ \ b=0$ $0.2+0.02+0.4c=0.4,\ \text{so}\ \ c=0.45$ After *C*, routes *A, B, C* will be used with probabilities 0.55, 0, 0.45	M1 M1 A2 **4**	Obtaining a value for *a, b* or *c* Give A1 for one correct
(vii)	$0.4\times0.1+0.2\times0.2+0.4\times0.45$ $=0.26$	M1 M1 A1 **3**	Using long-run probs 0.4, 0.2, 0.4 Using diag elements from new matrix

Post-multiplication by transition matrix

5 (i)	$\mathbf{P} = \begin{pmatrix} 0.1 & 0.4 & 0.5 \\ 0.7 & 0.2 & 0.1 \\ 0.1 & 0.6 & 0.3 \end{pmatrix}$	B2 **2**	Give B1 for two rows correct
(ii)	$\left(\tfrac{1}{3} \quad \tfrac{1}{3} \quad \tfrac{1}{3}\right)\mathbf{P}^6 = (0.328864 \quad 0.381536 \quad 0.2896)$ P(B used on 7th day) $= 0.3815$	M1 M1 M1 A1 **4**	Using $\mathbf{P}^6$ (or $\mathbf{P}^7$) For matrix of initial probabilities For evaluating matrix product *Accept 0.381 to 0.382*
(iii)	$0.328864 \times 0.1 + 0.381536 \times 0.2 + 0.2896 \times 0.3$ $= 0.1961$	M1 M1 A1 **3**	Using diagonal elements from **P** Correct method *Accept a.r.t. 0.196*
(iv)	$\mathbf{P}^3 = \begin{pmatrix} 0.352 & 0.364 & 0.284 \\ 0.328 & 0.404 & 0.268 \\ 0.304 & 0.372 & 0.324 \end{pmatrix}$ $0.328864 \times 0.352 + 0.381536 \times 0.404 + 0.2896 \times 0.324$ $= 0.3637$	M1 M1 M1 A1 **4**	For evaluating $\mathbf{P}^3$ Using diagonal elements from $\mathbf{P}^3$ Correct method *Accept a.r.t. 0.364*
(v)	$\mathbf{Q} = \begin{pmatrix} 0.3289 & 0.3816 & 0.2895 \\ 0.3289 & 0.3816 & 0.2895 \\ 0.3289 & 0.3816 & 0.2895 \end{pmatrix}$ 0.3289, 0.3816, 0.2895 are the long-run probabilities for the routes *A, B, C*	B1B1B1 B1 **4**	*Deduct 1 if not given as a (3×3) matrix* *Deduct 1 if not 4 dp* *Accept 'equilibrium probabilities'*
(vi)	$(0.4 \quad 0.2 \quad 0.4)\begin{pmatrix} 0.1 & 0.4 & 0.5 \\ 0.7 & 0.2 & 0.1 \\ a & b & c \end{pmatrix} = (0.4 \quad 0.2 \quad 0.4)$ $0.04 + 0.14 + 0.4a = 0.4, \text{ so } a = 0.55$ $0.16 + 0.04 + 0.4b = 0.2, \text{ so } b = 0$ $0.2 + 0.02 + 0.4c = 0.4, \text{ so } c = 0.45$ After *C*, routes *A, B, C* will be used with probabilities 0.55, 0, 0.45	M1 M1 A2 **4**	Obtaining a value for *a, b* or *c* Give A1 for one correct
(vii)	$0.4 \times 0.1 + 0.2 \times 0.2 + 0.4 \times 0.45$ $= 0.26$	M1 M1 A1 **3**	Using long-run probs 0.4, 0.2, 0.4 Using diag elements from new matrix

4757 Further Pure 3

1 (i)	Putting $x=0$, $-3y+10z=6,\ -4y-2z=8$ $y=-2,\ z=0$ Direction is given by $\begin{pmatrix}8\\-3\\10\end{pmatrix}\times\begin{pmatrix}3\\-4\\-2\end{pmatrix}$ $=\begin{pmatrix}46\\46\\-23\end{pmatrix}$ Equation of *L* is $\mathbf{r}=\begin{pmatrix}0\\-2\\0\end{pmatrix}+\lambda\begin{pmatrix}2\\2\\-1\end{pmatrix}$	M1 A1 M1 A1 A1 ft **5**	Finding coords of a point on the line or $(2,\ 0,\ -1)$, $(1,\ -1,\ -\frac{1}{2})$ etc or finding a second point *Dependent on M1M1* Accept any form Condone omission of '**r** ='
(ii)	$\overrightarrow{\mathrm{AB}}\times\mathbf{d}=\begin{pmatrix}7\\-14\\4\end{pmatrix}\times\begin{pmatrix}2\\2\\-1\end{pmatrix}=\begin{pmatrix}6\\15\\42\end{pmatrix}\quad[\ =3\begin{pmatrix}2\\5\\14\end{pmatrix}\ ]$ Distance is $\left[\begin{pmatrix}-1\\12\\5\end{pmatrix}-\begin{pmatrix}0\\-2\\0\end{pmatrix}\right].\hat{\mathbf{n}}=\dfrac{\begin{pmatrix}-1\\14\\5\end{pmatrix}.\begin{pmatrix}2\\5\\14\end{pmatrix}}{\sqrt{2^2+5^2+14^2}}$ $=\dfrac{138}{15}=\dfrac{46}{5}=9.2$	M1 A2 ft M1 A1 ft A1 **6**	Evaluating $\overrightarrow{\mathrm{AB}}\times\mathbf{d}$ Give A1 ft if just one error Appropriate scalar product Fully correct expression
(iii)	$\lvert\overrightarrow{\mathrm{AB}}\times\mathbf{d}\rvert=\left\lvert\begin{pmatrix}6\\15\\42\end{pmatrix}\right\rvert=\sqrt{6^2+15^2+42^2}$ Distance is $\dfrac{\lvert\overrightarrow{\mathrm{AB}}\times\mathbf{d}\rvert}{\lvert\mathbf{d}\rvert}=\dfrac{\sqrt{6^2+15^2+42^2}}{\sqrt{2^2+2^2+1^2}}$ $=\dfrac{45}{3}=15$	M1 M1 M1A1 ft A1 **5**	For $\lvert\overrightarrow{\mathrm{AB}}\times\mathbf{d}\rvert$ Evaluating magnitude *In this part, M marks are dependent on previous M marks*

(iv)	At D, $\begin{pmatrix}-1\\12\\5\end{pmatrix}+\lambda\begin{pmatrix}k+1\\-12\\-3\end{pmatrix}=\begin{pmatrix}6\\-2\\9\end{pmatrix}+\mu\begin{pmatrix}2\\2\\-1\end{pmatrix}$	M1	*Condone use of same parameter on both sides*
	$12-12\lambda=-2+2\mu$ $5-3\lambda=9-\mu$	A1 ft	Two equations for λ and μ
	$\lambda=\frac{1}{3},\quad\mu=5$	M1M1	Obtaining λ and μ (numerically) Give M1 for λ and μ in terms of *k*
	$-1+\frac{1}{3}(k+1)=6+10$ $k=50$	M1 A1	Equation for *k*
	D is $(6+2\mu,\ -2+2\mu,\ 9-\mu)$ i.e. $(16,\ 8,\ 4)$	M1 A1 **8**	Obtaining coordinates of D

Alternative solutions for Q1

1 (i)	e.g. $23x-23y=46$ $x=t,\ \ y=t-2$ $3t-4(t-2)-2z=8$ $x=t,\ \ y=t-2,\ \ z=-\frac{1}{2}t$	M1A1 M1A1 ft A1 **5**	Eliminating one of *x*, *y*, *z*
(ii)	$\overrightarrow{PQ}=\begin{pmatrix}-1+7\mu\\12-14\mu\\5+4\mu\end{pmatrix}-\begin{pmatrix}2\lambda\\-2+2\lambda\\-\lambda\end{pmatrix}\quad \overrightarrow{PQ}\,.\,\mathbf{d}=\overrightarrow{PQ}\,.\,\overrightarrow{AB}=0$ $2(-1+7\mu-2\lambda)+2(14-14\mu-2\lambda)-(5+4\mu+\lambda)=0$ $7(-1+7\mu-2\lambda)-14(14-14\mu-2\lambda)+4(5+4\mu+\lambda)=0\ \ \lambda=27/25,\ \ \mu=47/75$ $\lvert\overrightarrow{PQ}\rvert=\sqrt{(92/75)^2+(230/75)^2+(644/75)^2}\ =9.2$	M1 A1 ft A1 ft M1A1 ft A1 **6**	Two equations for λ and μ Expression for shortest distance
(iii)	$\overrightarrow{AX}\cdot\mathbf{d}=\begin{pmatrix}6+2\lambda+1\\-2+2\lambda-12\\9-\lambda-5\end{pmatrix}\cdot\begin{pmatrix}2\\2\\-1\end{pmatrix}=0$ $2(7+2\lambda)+2(2\lambda-14)-(4-\lambda)=0$ $\lambda=2$ $\overrightarrow{AX}=\begin{pmatrix}11\\-10\\2\end{pmatrix}$ $AX=\sqrt{11^2+10^2+2^2}$ $=15$	M1 A1 ft M1 M1 A1 **5**	

(iv)	$\begin{pmatrix}7\\-14\\4\end{pmatrix}\cdot\left[\begin{pmatrix}k+1\\-12\\-3\end{pmatrix}\times\begin{pmatrix}2\\2\\-1\end{pmatrix}\right]=0$ $\begin{pmatrix}7\\-14\\4\end{pmatrix}\cdot\begin{pmatrix}18\\k-5\\2k+26\end{pmatrix}=0$	M1	Appropriate scalar triple product equated to zero
	$126-14k+70+8k+104=0$	M1	Equation for *k*
	$k=50$	A1	
	At D, $\begin{pmatrix}-1\\12\\5\end{pmatrix}+\lambda\begin{pmatrix}51\\-12\\-3\end{pmatrix}=\begin{pmatrix}6\\-2\\9\end{pmatrix}+\mu\begin{pmatrix}2\\2\\-1\end{pmatrix}$	M1	*Condone use of same parameter on both sides*
	$-1+51\lambda=6+2\mu$ $12-12\lambda=-2+2\mu$ $5-3\lambda=9-\mu$	A1 ft	Two equations for λ and μ
	$\lambda=\frac{1}{3},\quad\mu=5$	M1	Obtaining λ or μ
	D is $(6+2\mu,\ -2+2\mu,\ 9-\mu)$ i.e. $(16,\ 8,\ 4)$	M1 A1	Obtaining coordinates of D
		8	

2 (i)	$\frac{\partial z}{\partial x} = 3(x+y)^3 + 9x(x+y)^2 - 6x^2 + 24$ $\frac{\partial z}{\partial y} = 9x(x+y)^2$	M1 A2 A1 **4**	Partial differentiation Give A1 if just one minor error
(ii)	At stationary points, $\frac{\partial z}{\partial x} = 0$ and $\frac{\partial z}{\partial y} = 0$ $9x(x+y)^2 = 0 \Rightarrow x = 0$ or $y = -x$ If $x = 0$ then $3y^3 + 24 = 0$ $y = -2$; one stationary point is $(0,\ -2,\ 0)$ If $y = -x$ then $-6x^2 + 24 = 0$ $x = \pm 2$; stationary points are $(2,\ -2,\ 32)$ and $(-2,\ 2,\ -32)$	M1 M1 A1A1 M1 A1 A1 **7**	If A0A0, give A1 for $x = \pm 2$
(iii)	At P $(1,\ -2,\ 19)$, $\frac{\partial z}{\partial x} = 24$, $\frac{\partial z}{\partial y} = 9$ Normal line is $\mathbf{r} = \begin{pmatrix}1\\-2\\19\end{pmatrix} + \lambda\begin{pmatrix}24\\9\\-1\end{pmatrix}$	B1 M1 A1 ft **3**	For normal vector (allow sign error) Condone omission of '**r** ='
(iv)	$\delta z \approx \frac{\partial z}{\partial x}\delta x + \frac{\partial z}{\partial y}\delta y$ $= 24\delta x + 9\delta y$ $3h \approx 24k + 9h$ $k \approx -\frac{1}{4}h$	M1 A1 ft M1 A1 **4**	
	OR Tangent plane is $24x + 9y - z = -13$ $24(1+k) + 9(-2+h) - (19+3h) \approx -13$ M2A1 ft $k \approx -\frac{1}{4}h$ A1		
(v)	$\frac{\partial z}{\partial x} = 27$ and $\frac{\partial z}{\partial y} = 0$ $9x(x+y)^2 = 0 \Rightarrow x = 0$ or $y = -x$ If $x = 0$ then $3y^3 + 24 = 27$ $y = 1,\ z = 0$; point is $(0,\ 1,\ 0)$ $d = 0$ If $y = -x$ then $-6x^2 + 24 = 27$ $x^2 = -\frac{1}{2}$; there are no other points	M1 M1 A1 A1 M1 A1 **6**	(Allow M1 for $\frac{\partial z}{\partial x} = -27$)

3 (i)	$\left(\frac{\mathrm{d}x}{\mathrm{d}\theta}\right)^2+\left(\frac{\mathrm{d}y}{\mathrm{d}\theta}\right)^2=[a(1+\cos\theta)]^2+(a\sin\theta)^2$ $=a^2(2+2\cos\theta)$ $=4a^2\cos^2\frac{1}{2}\theta$ $s=\int 2a\cos\frac{1}{2}\theta\,\mathrm{d}\theta$ $=4a\sin\frac{1}{2}\theta+C$ $s=0$ when $\theta=0 \Rightarrow C=0$	M1 A1 M1 M1 A1 A1 ag **6**	Forming $\left(\frac{\mathrm{d}x}{\mathrm{d}\theta}\right)^2+\left(\frac{\mathrm{d}y}{\mathrm{d}\theta}\right)^2$ Using half-angle formula Integrating to obtain $k\sin\frac{1}{2}\theta$ Correctly obtained (+*C* not needed) *Dependent on all previous marks*
(ii)	$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{a\sin\theta}{a(1+\cos\theta)}$ $=\frac{2\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta}{2a\cos^2\frac{1}{2}\theta}=\tan\frac{1}{2}\theta$ $\psi=\frac{1}{2}\theta$, and so $s=4a\sin\psi$	M1 M1 A1 A1 **4**	Using half-angle formulae
(iii)	$\rho=\frac{\mathrm{d}s}{\mathrm{d}\psi}=4a\cos\psi$ $=4a\cos\frac{1}{2}\theta$	M1 A1 ft A1 ag **3**	Differentiating intrinsic equation
	OR $\rho=\frac{\left(4a^2\cos^2\frac{1}{2}\theta\right)^{3/2}}{a(1+\cos\theta)(a\cos\theta)-(-a\sin\theta)(a\sin\theta)}$ M1A1 ft $=\frac{8a^3\cos^3\frac{1}{2}\theta}{a^2(1+\cos\theta)}=\frac{8a^3\cos^3\frac{1}{2}\theta}{2a^2\cos^2\frac{1}{2}\theta}=4a\cos\frac{1}{2}\theta$ A1 ag		Correct expression for ρ or κ
(iv)	When $\theta=\frac{2}{3}\pi,\ \psi=\frac{1}{3}\pi,\ x=a(\frac{2}{3}\pi+\frac{1}{2}\sqrt{3}),\ y=\frac{3}{2}a$ $\rho=2a$ $\hat{\mathbf{n}}=\begin{pmatrix}-\sin\psi\\ \cos\psi\end{pmatrix}=\begin{pmatrix}-\frac{1}{2}\sqrt{3}\\ \frac{1}{2}\end{pmatrix}$ $\mathbf{c}=\begin{pmatrix}a(\frac{2}{3}\pi+\frac{1}{2}\sqrt{3})\\ \frac{3}{2}a\end{pmatrix}+2a\begin{pmatrix}-\frac{1}{2}\sqrt{3}\\ \frac{1}{2}\end{pmatrix}$ Centre of curvature is $(\ a(\frac{2}{3}\pi-\frac{1}{2}\sqrt{3}),\ \frac{5}{2}a\ )$	B1 M1 A1 M1 A1A1 **6**	Obtaining a normal vector Correct unit normal *(possibly in terms of θ)* Accept $(1.23a,\ 2.5a)$

(v)	Curved surface area is $\int 2\pi\, y\, \mathrm{d}s$ $= \int_0^{\pi} 2\pi\, a(1-\cos\theta)\, 2a\cos\tfrac{1}{2}\theta\, \mathrm{d}\theta$ $= \int_0^{\pi} 8\pi a^2 \sin^2 \tfrac{1}{2}\theta \cos\tfrac{1}{2}\theta\, \mathrm{d}\theta$ $= \left[\ \tfrac{16}{3}\pi a^2 \sin^3 \tfrac{1}{2}\theta\ \right]_0^{\pi}$ $= \tfrac{16}{3}\pi a^2$	M1 A1 ft M1 M1 A1 **5**	Correct integral expression in any form *(including limits; may be implied by later working)* Obtaining an integrable form Obtaining $k\sin^3\tfrac{1}{2}\theta$ *or equivalent*

<table>
<tr><td>4 (i)</td><td>In <i>G</i>, $3^2=2,\ 3^3=6,\ 3^4=4,\ 3^5=5,\ 3^6=1$
[or $5^2=4,\ 5^3=6,\ 5^4=2,\ 5^5=3,\ 5^6=1$]
In <i>H</i>, $5^2=7,\ 5^3=17,\ 5^4=13,\ 5^5=11,\ 5^6=1$
[or $11^2=13,\ 11^3=17,\ 11^4=7,\ 11^5=5,\ 11^6=1$]
<i>G</i> has an element 3 (or 5) of order 6
<i>H</i> has an element 5 (or 11) of order 6</td><td>M1
A1
B1
B1
4</td><td>All powers of an element of order 6
All powers correct in both groups</td></tr>
<tr><td>(ii)</td><td>$\{1,\ 6\}$
$\{1,\ 2,\ 4\}$</td><td>B1
B2
3</td><td><i>Ignore</i> $\{1\}$ <i>and G</i>
<i>Deduct 1 mark (from B1B2) for each proper subgroup in excess of two</i></td></tr>
<tr><td>(iii)</td><td>$\begin{array}{cc} G & H \\ 1 \leftrightarrow 1 \\ 2 \leftrightarrow 7 \\ 3 \leftrightarrow 5 \\ 4 \leftrightarrow 13 \\ 5 \leftrightarrow 11 \\ 6 \leftrightarrow 17 \end{array}$ OR $\begin{array}{cc} G & H \\ 1 \leftrightarrow 1 \\ 2 \leftrightarrow 13 \\ 3 \leftrightarrow 11 \\ 4 \leftrightarrow 7 \\ 5 \leftrightarrow 5 \\ 6 \leftrightarrow 17 \end{array}$</td><td>B4
4</td><td>Give B3 for 4 correct, B2 for 3 correct,
B1 for 2 correct</td></tr>
<tr><td>(iv)</td><td>$\mathrm{ad}(1)=\mathrm{a}(3)=1$
$\mathrm{ad}(2)=\mathrm{a}(2)=3$
$\mathrm{ad}(3)=\mathrm{a}(1)=2$, so $\mathrm{ad}=\mathrm{c}$
$\mathrm{da}(1)=\mathrm{d}(2)=2$
$\mathrm{da}(2)=\mathrm{d}(3)=1$
$\mathrm{da}(3)=\mathrm{d}(1)=3$, so $\mathrm{da}=\mathrm{f}$</td><td>M1
A1
M1
A1
4</td><td>Evaluating e.g. $\mathrm{ad}(1)$ <i>(one case sufficient; intermediate value must be shown)</i>
For $\mathrm{ad}=\mathrm{c}$ correctly shown
Evaluating e.g. $\mathrm{da}(1)$ <i>(one case sufficient; no need for any working)</i></td></tr>
<tr><td>(v)</td><td><i>S</i> is not abelian; <i>G</i> is abelian</td><td>B1
1</td><td><i>or S</i> has 3 elements of order 2;
<i>G</i> has 1 element of order 2
<i>or S</i> is not cyclic <i>etc</i></td></tr>
<tr><td>(vi)</td><td><table><tr><td>Element</td><td>a</td><td>b</td><td>c</td><td>d</td><td>e</td><td>f</td></tr><tr><td>Order</td><td>3</td><td>3</td><td>2</td><td>2</td><td>1</td><td>2</td></tr></table></td><td>B4
4</td><td>Give B3 for 5 correct, B2 for 3 correct,
B1 for 1 correct</td></tr>
<tr><td>(vii)</td><td>$\{\mathrm{e},\ \mathrm{c}\},\ \{\mathrm{e},\ \mathrm{d}\},\ \{\mathrm{e},\ \mathrm{f}\}$
$\{\mathrm{e},\ \mathrm{a},\ \mathrm{b}\}$</td><td>B1B1B1
B1
4</td><td>Ignore { e } and <i>S</i>
If more than 4 proper subgroups are given, deduct 1 mark for each proper subgroup in excess of 4</td></tr>
</table>

Pre-multiplication by transition matrix

5 (i)	$\mathbf{P}=\begin{pmatrix}0&0.1&0&0.3\\0.7&0.8&0&0.6\\0.1&0&1&0.1\\0.2&0.1&0&0\end{pmatrix}$	B2 **2**	Give B1 for two columns correct
(ii)	$\mathbf{P}^{13}\begin{pmatrix}0.6\\0.4\\0\\0\end{pmatrix}=\begin{pmatrix}0.0810\\0.5684\\0.2760\\0.0746\end{pmatrix}$	M1 A2 **3**	Using $\mathbf{P}^{13}$ (or $\mathbf{P}^{14}$) Give A1 for 2 probabilities correct *(Max A1 if not at least 3dp)* *Tolerance* ± 0.0001
(iii)	$0.5684\times 0.8+0.2760$ $=0.731$	M1M1 A1 ft **3**	For 0.5684×0.8 and 0.2760 *Accept 0.73 to 0.7312*
(iv)	$\mathbf{P}^{30}\begin{pmatrix}0.6\\0.4\\0\\0\end{pmatrix}=\begin{pmatrix}.\\.\\0.4996\\.\end{pmatrix},\quad \mathbf{P}^{31}\begin{pmatrix}0.6\\0.4\\0\\0\end{pmatrix}=\begin{pmatrix}.\\.\\0.5103\\.\end{pmatrix}$ Level 32	M1 A1 A1 **3**	Finding P(C) for some powers of **P** For identifying $\mathbf{P}^{31}$
(v)	Expected number of levels including the next change of location is $\frac{1}{0.2}=5$ Expected number of further levels in B is 4	M1 A1 A1 **3**	For $1/(1-0.8)$ or $0.8/(1-0.8)$ For 5 or 4 For 4 as final answer
(vi)	$\mathbf{Q}=\begin{pmatrix}0&0.1&0&0.3\\0.7&0.8&0&0.6\\0.1&0&0.9&0.1\\0.2&0.1&0.1&0\end{pmatrix}$ $\mathbf{Q}^n\to\begin{pmatrix}0.0916&0.0916&0.0916&0.0916\\0.6183&0.6183&0.6183&0.6183\\0.1908&0.1908&0.1908&0.1908\\0.0992&0.0992&0.0992&0.0992\end{pmatrix}$ A: 0.0916 B: 0.6183 C: 0.1908 D: 0.0992	B1 M1 M1 A2 **5**	*Can be implied* Evaluating powers of **Q** *or Obtaining (at least) 3 equations from* $\mathbf{Qp}=\mathbf{p}$ Limiting matrix with equal columns *or Solving to obtain one equilib prob* *or M2 for other complete method* Give A1 for two correct *(Max A1 if not at least 3dp)* *Tolerance* ± 0.0001

(vii)	$\begin{pmatrix} 0 & 0.1 & a & 0.3 \\ 0.7 & 0.8 & b & 0.6 \\ 0.1 & 0 & c & 0.1 \\ 0.2 & 0.1 & d & 0 \end{pmatrix}\begin{pmatrix} 0.11 \\ 0.75 \\ 0.04 \\ 0.1 \end{pmatrix} = \begin{pmatrix} 0.11 \\ 0.75 \\ 0.04 \\ 0.1 \end{pmatrix}$	M1 A1	Transition matrix and $\begin{pmatrix} 0.11 \\ 0.75 \\ 0.04 \\ 0.1 \end{pmatrix}$
	$0.075 + 0.04a + 0.03 = 0.11$ $0.077 + 0.6 + 0.04b + 0.06 = 0.75$ $0.011 + 0.04c + 0.01 = 0.04$ $0.022 + 0.075 + 0.04d = 0.1$	M1	Forming at least one equation *or* $a + b + c + d = 1$
	$a = 0.125, \quad b = 0.325, \quad c = 0.475, \quad d = 0.075$	A2 **5**	Give A1 for two correct

Post-multiplication by transition matrix

5 (i)	$\mathbf{P}=\begin{pmatrix}0&0.7&0.1&0.2\\0.1&0.8&0&0.1\\0&0&1&0\\0.3&0.6&0.1&0\end{pmatrix}$	B2 **2**	Give B1 for two rows correct
(ii)	$(0.6\quad 0.4\quad 0\quad 0)\,\mathbf{P}^{13}$ $=(0.0810\quad 0.5684\quad 0.2760\quad 0.0746)$	M1 A2 **3**	Using $\mathbf{P}^{13}$ (or $\mathbf{P}^{14}$) Give A1 for 2 probabilities correct *(Max A1 if not at least 3dp)* *Tolerance* ± 0.0001
(iii)	$0.5684\times 0.8+0.2760$ $=0.731$	M1M1 A1 ft **3**	For 0.5684×0.8 and 0.2760 *Accept 0.73 to 0.7312*
(iv)	$(0.6\quad 0.4\quad 0\quad 0)\,\mathbf{P}^{30}=(.\quad .\quad 0.4996\quad .)$ $(0.6\quad 0.4\quad 0\quad 0)\,\mathbf{P}^{31}=(.\quad .\quad 0.5103\quad .)$ Level 32	M1 A1 A1 **3**	Finding P(C) for some powers of **P** For identifying $\mathbf{P}^{31}$
(v)	Expected number of levels including the next change of location is $\frac{1}{0.2}=5$ Expected number of further levels in B is 4	M1 A1 A1 **3**	For $1/(1-0.8)$ or $0.8/(1-0.8)$ For 5 or 4 For 4 as final answer
(vi)	$\mathbf{Q}=\begin{pmatrix}0&0.7&0.1&0.2\\0.1&0.8&0&0.1\\0&0&0.9&0.1\\0.3&0.6&0.1&0\end{pmatrix}$ $\mathbf{Q}^n\to\begin{pmatrix}0.0916&0.6183&0.1908&0.0992\\0.0916&0.6183&0.1908&0.0992\\0.0916&0.6183&0.1908&0.0992\\0.0916&0.6183&0.1908&0.0992\end{pmatrix}$ A: 0.0916 B: 0.6183 C: 0.1908 D: 0.0992	B1 M1 M1 A2 **5**	*Can be implied* Evaluating powers of **Q** *or Obtaining (at least) 3 equations from* $\mathbf{pQ}=\mathbf{p}$ Limiting matrix with equal rows *or Solving to obtain one equilib prob* *or M2 for other complete method* Give A1 for two correct *(Max A1 if not at least 3dp)* *Tolerance* ± 0.0001

(vii)	$(0.11 \quad 0.75 \quad 0.04 \quad 0.1)\begin{pmatrix} 0 & 0.7 & 0.1 & 0.2 \\ 0.1 & 0.8 & 0 & 0.1 \\ a & b & c & d \\ 0.3 & 0.6 & 0.1 & 0 \end{pmatrix}$	M1	Transition matrix and $(0.11 \quad 0.75 \quad 0.04 \quad 0.1)$
	$= (0.11 \quad 0.75 \quad 0.04 \quad 0.1)$	A1	
	$0.075 + 0.04a + 0.03 = 0.11$ $0.077 + 0.6 + 0.04b + 0.06 = 0.75$ $0.011 + 0.04c + 0.01 = 0.04$ $0.022 + 0.075 + 0.04d = 0.1$	M1	Forming at least one equation *or* $a + b + c + d = 1$
	$a = 0.125, \quad b = 0.325, \quad c = 0.475, \quad d = 0.075$	A2 **5**	Give A1 for two correct

1 (i)	$\overrightarrow{AC}\times\overrightarrow{AB}=\begin{pmatrix}5\\-8\\-26\end{pmatrix}\times\begin{pmatrix}2\\1\\-2\end{pmatrix}=\begin{pmatrix}42\\-42\\21\end{pmatrix}$	B2	Give B1 for one component correct
	Perpendicular distance is $\dfrac{\lvert\overrightarrow{AC}\times\overrightarrow{AB}\rvert}{\lvert\overrightarrow{AB}\rvert}$	M1	
	$=\dfrac{\sqrt{42^2+42^2+21^2}}{\sqrt{2^2+1^2+2^2}}=\dfrac{63}{3}$	M1	Calculating magnitude of a vector product
	$=21$	A1 **5**	www
	OR $\left[\begin{pmatrix}3+2\lambda\\8+\lambda\\27-2\lambda\end{pmatrix}-\begin{pmatrix}8\\0\\1\end{pmatrix}\right].\begin{pmatrix}2\\1\\-2\end{pmatrix}=0$ M1 A1		Appropriate scalar product
	$2(2\lambda-5)+(\lambda+8)-2(-2\lambda+26)=0$ A1 ft		
	$\lambda=6$ [F is $(15,\ 14,\ 15)$]		
	$CF=\sqrt{7^2+14^2+14^2}=21$ M1A1		
(ii)	$\overrightarrow{AB}\times\overrightarrow{CD}=\begin{pmatrix}2\\1\\-2\end{pmatrix}\times\begin{pmatrix}3\\p\\p-1\end{pmatrix}=\begin{pmatrix}3p-1\\-2p-4\\2p-3\end{pmatrix}$	B1 B1 B1	
	$\overrightarrow{AC}\,.\,(\overrightarrow{AB}\times\overrightarrow{CD})=\begin{pmatrix}5\\-8\\-26\end{pmatrix}.\begin{pmatrix}3p-1\\-2p-4\\2p-3\end{pmatrix}$	M1	
	$=5(3p-1)-8(-2p-4)-26(2p-3)$ $[=-21p+105]$	A1 ft	
	$\lvert\overrightarrow{AB}\times\overrightarrow{CD}\rvert=\sqrt{(3p-1)^2+(-2p-4)^2+(2p-3)^2}$	B1 ft	
	$=\sqrt{17p^2-2p+26}$		
	Distance is $\dfrac{\lvert\overrightarrow{AC}\,.\,(\overrightarrow{AB}\times\overrightarrow{CD})\rvert}{\lvert\overrightarrow{AB}\times\overrightarrow{CD}\rvert}=\dfrac{21\lvert p-5\rvert}{\sqrt{17p^2-2p+26}}$	M1A1 (ag) **8**	Correctly obtained
(iii)	$V=(\pm)\ \tfrac{1}{6}(\overrightarrow{AC}\times\overrightarrow{AB}).\overrightarrow{AD}=(\pm)\ \tfrac{1}{6}\begin{pmatrix}42\\-42\\21\end{pmatrix}.\begin{pmatrix}8\\p-8\\p-27\end{pmatrix}$	M1 A1 ft	Appropriate scalar triple product In any form
	$=(\pm)\ 56-7(p-8)+\tfrac{7}{2}(p-27)$	M1	Evaluation of scalar triple product *Dependent on previous M1*
	$=(\pm)\ \tfrac{35}{2}-\tfrac{7}{2}p$	A1 **4**	$\tfrac{1}{6}(105-21p)$ or better
	$=\tfrac{7}{2}\lvert p-5\rvert$		
(iv)	Intersect when $p=5$	B1	
	$\begin{pmatrix}3\\8\\27\end{pmatrix}+\lambda\begin{pmatrix}2\\1\\-2\end{pmatrix}=\begin{pmatrix}8\\0\\1\end{pmatrix}+\mu\begin{pmatrix}3\\5\\4\end{pmatrix}$	B1 ft M1	Equations of both lines *(may involve p)* Equation for intersection *(must have different parameters)*
	$3+2\lambda=8+3\mu$	A1 ft	Equation involving λ and μ
	$8+\lambda=5\mu$ $[\,8+\lambda=p\mu\,]$	A1 ft	Second equation involving λ and μ
	$27-2\lambda=1+4\mu$ $[\,27-2\lambda=1+(p-1)\mu\,]$		*or* Two equations in $\lambda,\ \mu,\ p$
	$\lambda=7,\quad\mu=3$	M1	Obtaining λ or μ
	Point of intersection is $(17,\ 15,\ 13)$	A1 **7**	

2 (i)	$\frac{\partial \mathrm{g}}{\partial x} = (y + xy + z^2)\,\mathrm{e}^{x-2y}$ $\frac{\partial \mathrm{g}}{\partial y} = (x - 2xy - 2z^2)\,\mathrm{e}^{x-2y}$ $\frac{\partial \mathrm{g}}{\partial z} = 2z\,\mathrm{e}^{x-2y}$	M1 A1 A1 A1 **4**	Partial differentiation
(ii)	At $(2, 1, -1)$, $\frac{\partial \mathrm{g}}{\partial x} = 4$, $\frac{\partial \mathrm{g}}{\partial y} = -4$, $\frac{\partial \mathrm{g}}{\partial z} = -2$ Normal has direction $\begin{pmatrix} 4 \\ -4 \\ -2 \end{pmatrix}$ L passes through $(2, 1, -1)$ and has this direction	M1 A1 M1 A1 (ag) **4**	
(iii)	When $\mathrm{g} = 0$, $xy + z^2 = 0$ $(2-2\lambda)(1+2\lambda) + (-1+\lambda)^2 = 0$ $3 - 3\lambda^2 = 0$ $\lambda = \pm 1$ $\lambda = 1$ gives $\mathrm{P}(0, 3, 0)$ $\lambda = -1$ gives $\mathrm{Q}(4, -1, -2)$	M1 M1 A1 (ag) A1 **4**	Obtaining a value of λ Or B1 for verifying $\mathrm{g}(0, 3, 0) = 0$ *and* showing that P is on L
(iv)	At P, $\frac{\partial \mathrm{g}}{\partial x} = 3\mathrm{e}^{-6}$, $\frac{\partial \mathrm{g}}{\partial y} = 0$, $\frac{\partial \mathrm{g}}{\partial z} = 0$ $\delta \mathrm{g} \approx \frac{\partial \mathrm{g}}{\partial x}\delta x + \frac{\partial \mathrm{g}}{\partial y}\delta y + \frac{\partial \mathrm{g}}{\partial z}\delta z$ $= 3\mathrm{e}^{-6}(-2\mu) + 0 + 0 = -6\mu\mathrm{e}^{-6}$	M1 M1 A1 (ag) **3**	OR give M2 A1 www for $\mathrm{g}(-2\mu,\ 3+2\mu,\ \mu)$ $= (-3\mu^2 - 6\mu)\,\mathrm{e}^{-6\mu-6} \approx -6\mu\,\mathrm{e}^{-6}$
(v)	When $-6\mu\,\mathrm{e}^{-6} \approx h$, $\mu \approx -\frac{1}{6}\mathrm{e}^{6}h$ Point $(-2\mu,\ 3+2\mu,\ \mu)$ is approximately $(\frac{1}{3}\mathrm{e}^{6}h,\ 3-\frac{1}{3}\mathrm{e}^{6}h,\ -\frac{1}{6}\mathrm{e}^{6}h)$	M1 A1 (ag) **2**	
(vi)	At Q, $\frac{\partial \mathrm{g}}{\partial x} = -\mathrm{e}^{6}$, $\frac{\partial \mathrm{g}}{\partial y} = 4\mathrm{e}^{6}$, $\frac{\partial \mathrm{g}}{\partial z} = -4\mathrm{e}^{6}$ When $x = 4 - 2\mu$, $y = -1 + 2\mu$, $z = -2 + \mu$ $\delta \mathrm{g} \approx (-\mathrm{e}^{6})(-2\mu) + (4\mathrm{e}^{6})(2\mu) + (-4\mathrm{e}^{6})(\mu)$ $= 6\mu\,\mathrm{e}^{6}$ If $6\mu\,\mathrm{e}^{6} \approx h$, then $\mu \approx \frac{1}{6}\mathrm{e}^{-6}h$ Point is approximately $(4 - \frac{1}{3}\mathrm{e}^{-6}h,\ -1 + \frac{1}{3}\mathrm{e}^{-6}h,\ -2 + \frac{1}{6}\mathrm{e}^{-6}h)$	M1 M1 M1A1 M1 A2 **7**	OR give M1 M2 A1 www for $\mathrm{g}(4-2\mu,\ -1+2\mu,\ -2+\mu)$ $= (-3\mu^2 + 6\mu)\,\mathrm{e}^{-6\mu+6} \approx 6\mu\,\mathrm{e}^{6}$ Give A1 for one coordinate correct *If partial derivatives are not evaluated at Q, max mark is M0M1M0M0*

3 (i)	$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{2}x^{-1/2}-\frac{1}{2}x^{1/2}$	B1	
	$1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2=1+(\frac{1}{2}x^{-1/2}-\frac{1}{2}x^{1/2})^2$	M1	
	$=1+\frac{1}{4}x^{-1}-\frac{1}{2}+\frac{1}{4}x=\frac{1}{4}x^{-1}+\frac{1}{2}+\frac{1}{4}x$		
	$=(\frac{1}{2}x^{-1/2}+\frac{1}{2}x^{1/2})^2$	A1	
	Arc length is $\int_0^a(\frac{1}{2}x^{-1/2}+\frac{1}{2}x^{1/2})\,\mathrm{d}x$		
	$=\left[\ x^{1/2}+\frac{1}{3}x^{3/2}\ \right]_0^a$	M1	
	$=a^{1/2}+\frac{1}{3}a^{3/2}$	A1 (ag) **5**	
(ii)	Curved surface area is $\int 2\pi y\,\mathrm{d}s$	M1	For $\int y\,\mathrm{d}s$
	$=\int_0^3 2\pi(\ x^{1/2}-\frac{1}{3}x^{3/2}\ )(\ \frac{1}{2}x^{-1/2}+\frac{1}{2}x^{1/2}\ )\,\mathrm{d}x$	A1	Correct integral form *including limits*
	$=2\pi\int_0^3(\ \frac{1}{2}+\frac{1}{3}x-\frac{1}{6}x^2\ )\,\mathrm{d}x$		
	$=2\pi\left[\ \frac{1}{2}x+\frac{1}{6}x^2-\frac{1}{18}x^3\ \right]_0^3$	M1A1	For $\frac{1}{2}x+\frac{1}{6}x^2-\frac{1}{18}x^3$
	$=3\pi$	A1 **5**	
(iii)	When $x=4$, $\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{3}{4}$	B1	
	Unit normal vector is $\begin{pmatrix}-\frac{3}{5}\\-\frac{4}{5}\end{pmatrix}$	M1 A1 ft	Finding a normal vector Correct unit normal (either direction)
	$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=-\frac{1}{4}x^{-3/2}-\frac{1}{4}x^{-1/2}\quad(=-\frac{5}{32})$	B1	
	$\rho=\frac{\left\{1+(-\frac{3}{4})^2\right\}^{3/2}}{(-)\frac{5}{32}}\quad(=\frac{125/64}{5/32}=\frac{25}{2})$	M1 A1 ft	Applying formula for ρ or κ
	$\mathbf{c}=\begin{pmatrix}4\\-\frac{2}{3}\end{pmatrix}+\frac{25}{2}\begin{pmatrix}-\frac{3}{5}\\-\frac{4}{5}\end{pmatrix}$	M1	
	$=\begin{pmatrix}-3\frac{1}{2}\\-10\frac{2}{3}\end{pmatrix}$	A1 A1 **9**	
(iv)	Differentiating partially w.r.t. p	M1	
	$0=2p\,x^{1/2}-p^2x^{3/2}$	A1	
	$p=\frac{2}{x}$		
	Envelope is $y=\frac{4}{x^2}x^{1/2}-\frac{1}{3}\frac{8}{x^3}x^{3/2}$	M1 A1	
	$y=\frac{4}{3}x^{-3/2}$	A1 **5**	

4 (i)	$\mathrm{s\,t}(x)=\mathrm{s}\left(\frac{x}{x-1}\right)=\frac{\frac{x}{x-1}-1}{\frac{x}{x-1}}$ $=\frac{x-(x-1)}{x}=\frac{1}{x}=\mathrm{r}(x)$ $\mathrm{t\,s}(x)=\mathrm{t}\left(\frac{x-1}{x}\right)=\frac{\frac{x-1}{x}}{\frac{x-1}{x}-1}$ $=\frac{x-1}{(x-1)-x}=1-x=\mathrm{q}(x)$	M1 A1 (ag) M1 A1 **4**	
(ii)	$\begin{array}{c\|cccccc} & \mathrm{p} & \mathrm{q} & \mathrm{r} & \mathrm{s} & \mathrm{t} & \mathrm{u} \\ \hline \mathrm{p} & \mathrm{p} & \mathrm{q} & \mathrm{r} & \mathrm{s} & \mathrm{t} & \mathrm{u} \\ \mathrm{q} & \mathrm{q} & \mathrm{p} & \mathrm{s} & \mathrm{r} & \mathrm{u} & \mathrm{t} \\ \mathrm{r} & \mathrm{r} & \mathrm{u} & \mathrm{p} & \mathrm{t} & \mathrm{s} & \mathrm{q} \\ \mathrm{s} & \mathrm{s} & \mathrm{t} & \mathrm{q} & \mathrm{u} & \mathrm{r} & \mathrm{p} \\ \mathrm{t} & \mathrm{t} & \mathrm{s} & \mathrm{u} & \mathbf{q} & \mathbf{p} & \mathbf{r} \\ \mathrm{u} & \mathrm{u} & \mathrm{r} & \mathrm{t} & \mathbf{p} & \mathbf{q} & \mathbf{s} \end{array}$	B3 **3**	Give B2 for 4 correct, B1 for 2 correct
(iii)	$\begin{array}{\|c\|c\|c\|c\|c\|c\|c\|} \text{Element} & \mathrm{p} & \mathrm{q} & \mathrm{r} & \mathrm{s} & \mathrm{t} & \mathrm{u} \\ \hline \text{Inverse} & \mathrm{p} & \mathrm{q} & \mathrm{r} & \mathrm{u} & \mathrm{t} & \mathrm{s} \end{array}$	B3 **3**	Give B2 for 4 correct, B1 for 2 correct
(iv)	$\{\,\mathrm{p}\,\},\ F$ $\{\,\mathrm{p},\ \mathrm{q}\,\},\ \{\,\mathrm{p},\ \mathrm{r}\,\},\ \{\,\mathrm{p},\ \mathrm{t}\,\}$ $\{\,\mathrm{p},\ \mathrm{s},\ \mathrm{u}\,\}$	B1B1B1 B1 **4**	*Ignore these in the marking* Deduct one mark for each non-trivial subgroup in excess of four
(v)	$\begin{array}{\|c\|c\|c\|c\|c\|c\|c\|} \text{Element} & 1 & -1 & \mathrm{e}^{\frac{\pi}{3}\mathrm{j}} & \mathrm{e}^{-\frac{\pi}{3}\mathrm{j}} & \mathrm{e}^{\frac{2\pi}{3}\mathrm{j}} & \mathrm{e}^{-\frac{2\pi}{3}\mathrm{j}} \\ \hline \text{Order} & 1 & 2 & 6 & 6 & 3 & 3 \end{array}$	B4 **4**	Give B3 for 4 correct, B2 for 3 correct B1 for 2 correct
(vi)	$2^1=2,\ 2^2=4,\ 2^3=8,\ 2^4=16,\ 2^5=13,\ 2^6=\mathbf{7}$ $2^7=14,\ 2^8=9,\ 2^9=\mathbf{18},\ 2^{10}=17,\ 2^{11}=15,\ 2^{12}=11$ $2^{13}=3,\ 2^{14}=6,\ 2^{15}=12,\ 2^{16}=5,\ 2^{17}=10,\ 2^{18}=1$ Hence 2 has order 18	M1 A1 A1 **3**	Finding (at least two) powers of 2 For $2^6=7$ and $2^9=18$ Correctly shown *All powers listed implies final A1*
(vii)	*G* is abelian (so all its subgroups are abelian) *F* is not abelian	B1 **1**	*Can have 'cyclic' instead of 'abelian'*
(viii)	Subgroup of order 6 is $\{1,\ 2^3,\ 2^6,\ 2^9,\ 2^{12},\ 2^{15}\}$ i.e. $\{1,\ 7,\ 8,\ 11,\ 12,\ 18\}$	M1 A1 **2**	or B2

Pre-multiplication by transition matrix

5 (i)	$\mathbf{P}=\begin{pmatrix}0.16 & 0.28 & 0.43 & 1\\ 0.84 & 0 & 0 & 0\\ 0 & 0.72 & 0 & 0\\ 0 & 0 & 0.57 & 0\end{pmatrix}$	B2 **2**	*Allow tolerance of* ± 0.0001 *in probabilities throughout this question* Give B1 for two columns correct
(ii)	$\mathbf{P}^9\begin{pmatrix}1\\0\\0\\0\end{pmatrix}=\begin{pmatrix}0.3349\\0.3243\\0.2231\\0.1177\end{pmatrix}$ Prob$(C)=0.2231$	M2 A1 **3**	Using $\mathbf{P}^9$ Give M1 for using $\mathbf{P}^{10}$
(iii)	Week 5 $\mathbf{P}^4\begin{pmatrix}1\\0\\0\\0\end{pmatrix}=\begin{pmatrix}0.5020\\0.2851\\0.1577\\0.0552\end{pmatrix}$	B1 M1 A1 **3**	First column of a power of **P** *SC* Give B0M1A1 for Week 9 and 0.3860 0.3098 0.2066 0.0976
(iv)	$\mathbf{P}^7=\begin{pmatrix}. & . & . & .\\ 0.2869 & . & . & .\\ . & . & . & .\\ . & . & . & .\end{pmatrix}$ $\mathbf{P}^8=\begin{pmatrix}. & . & . & .\\ . & . & . & .\\ . & 0.2262 & . & .\\ . & . & . & .\end{pmatrix}$ Probability is 0.2869×0.2262 $=0.0649$	M1M1 M1 A1 **4**	Elements from $\mathbf{P}^7$ and $\mathbf{P}^8$ Multiplying appropriate probabilities
(v)	Expected run length is $\dfrac{1}{1-0.16}=1.19$ (3 sf)	M1 A1 **2**	Allow 1.2
(vi)	$\mathbf{P}^n\ \rightarrow\begin{pmatrix}0.3585 & 0.3585 & 0.3585 & 0.3585\\ 0.3011 & 0.3011 & 0.3011 & 0.3011\\ 0.2168 & 0.2168 & 0.2168 & 0.2168\\ 0.1236 & 0.1236 & 0.1236 & 0.1236\end{pmatrix}$ A: 0.3585 B: 0.3011 C: 0.2168 D: 0.1236	M1 M1 A2 **4**	Evaluating $\mathbf{P}^n$ with $n\geq 10$ *or Obtaining (at least) 3 equations from* $\mathbf{P}\mathbf{p}=\mathbf{p}$ Limiting matrix with equal columns *or Solving to obtain one equilib prob* Give A1 for two correct
(vii)	Expected number is 145×0.3585 ≈ 52	M1 A1 ft **2**	
(viii)	$\begin{pmatrix}a & b & c & 1\\ 1-a & 0 & 0 & 0\\ 0 & 1-b & 0 & 0\\ 0 & 0 & 1-c & 0\end{pmatrix}\begin{pmatrix}0.4\\0.25\\0.2\\0.15\end{pmatrix}=\begin{pmatrix}0.4\\0.25\\0.2\\0.15\end{pmatrix}$ $0.4a+0.25b+0.2c+0.15=0.4$ $0.4(1-a)=0.25$ $0.25(1-b)=0.2$ $0.2(1-c)=0.15$ $a=0.375,\quad b=0.2,\quad c=0.25$	M1 A1 M1 A1 **4**	Transition matrix and $\begin{pmatrix}0.4\\0.25\\0.2\\0.15\end{pmatrix}$ Forming at least one equation *Dependent on previous M1*

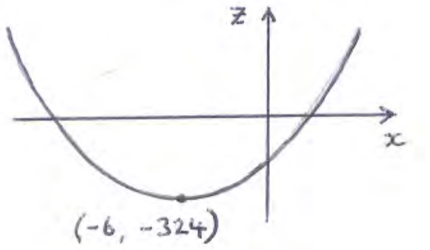
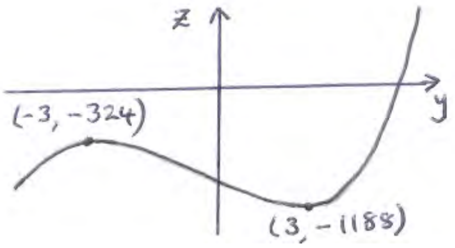
Post-multiplication by transition matrix

5 (i)	$$\mathbf{P}=\begin{pmatrix}0.16 & 0.84 & 0 & 0\\ 0.28 & 0 & 0.72 & 0\\ 0.43 & 0 & 0 & 0.57\\ 1 & 0 & 0 & 0\end{pmatrix}$$	B2 **2**	*Allow tolerance of* ± 0.0001 *in probabilities throughout this question* Give B1 for two rows correct
(ii)	$\begin{pmatrix}1 & 0 & 0 & 0\end{pmatrix}\mathbf{P}^9$ $=\begin{pmatrix}0.3349 & 0.3243 & 0.2231 & 0.1177\end{pmatrix}$ $\text{Prob}(C)=0.2231$	M2 A1 **3**	Using $\mathbf{P}^9$ Give M1 for using $\mathbf{P}^{10}$
(iii)	Week 5 $\begin{pmatrix}1 & 0 & 0 & 0\end{pmatrix}\mathbf{P}^4$ $=\begin{pmatrix}0.5020 & 0.2851 & 0.1577 & 0.0552\end{pmatrix}$	B1 M1 A1 **3**	First row of a power of **P** *SC* Give B0M1A1 for Week 9 and 0.3860 0.3098 0.2066 0.0976
(iv)	$\mathbf{P}^7=\begin{pmatrix}. & 0.2869 & . & .\\ . & . & . & .\\ . & . & . & .\\ . & . & . & .\end{pmatrix}$ $\mathbf{P}^8=\begin{pmatrix}. & . & . & .\\ . & . & 0.2262 & .\\ . & . & . & .\\ . & . & . & .\end{pmatrix}$ Probability is 0.2869×0.2262 $=0.0649$	M1M1 M1 A1 **4**	Elements from $\mathbf{P}^7$ and $\mathbf{P}^8$ Multiplying appropriate probabilities
(v)	Expected run length is $\dfrac{1}{1-0.16}=1.19$ (3 sf)	M1 A1 **2**	Allow 1.2
(vi)	$$\mathbf{P}^n\to\begin{pmatrix}0.3585 & 0.3011 & 0.2168 & 0.1236\\ 0.3585 & 0.3011 & 0.2168 & 0.1236\\ 0.3585 & 0.3011 & 0.2168 & 0.1236\\ 0.3585 & 0.3011 & 0.2168 & 0.1236\end{pmatrix}$$ A: 0.3585 B: 0.3011 C: 0.2168 D: 0.1236	M1 M1 A2 **4**	Evaluating $\mathbf{P}^n$ with $n\geq 10$ *or Obtaining (at least) 3 equations from* $\mathbf{p}\mathbf{P}=\mathbf{p}$ Limiting matrix with equal rows *or Solving to obtain one equilib prob* Give A1 for two correct
(vii)	Expected number is 145×0.3585 ≈ 52	M1 A1 ft **2**	
(viii)	$$\begin{pmatrix}0.4 & 0.25 & 0.2 & 0.15\end{pmatrix}\begin{pmatrix}a & 1-a & 0 & 0\\ b & 0 & 1-b & 0\\ c & 0 & 0 & 1-c\\ 1 & 0 & 0 & 0\end{pmatrix}$$ $=\begin{pmatrix}0.4 & 0.25 & 0.2 & 0.15\end{pmatrix}$ $0.4a+0.25b+0.2c+0.15=0.4$ $0.4(1-a)=0.25$ $0.25(1-b)=0.2$ $0.2(1-c)=0.15$ $a=0.375,\quad b=0.2,\quad c=0.25$	M1 A1 M1 A1 **4**	Transition matrix and $\begin{pmatrix}0.4 & 0.25 & 0.2 & 0.15\end{pmatrix}$ Forming at least one equation *Dependent on previous M1*

1 (i)	Distance is $\dfrac{2(-2)-(-7)+2(7)-11}{\sqrt{2^2+1^2+2^2}}$ $=2$	M1 A1 A1 **3**	Formula, or other complete method Numerical expression for distance
(ii)	$\begin{pmatrix}1\\4\\7\end{pmatrix}\times\begin{pmatrix}2\\-1\\2\end{pmatrix}=\begin{pmatrix}15\\12\\-9\end{pmatrix}$ Equation of AD is $\mathbf{r}=\begin{pmatrix}2\\-1\\3\end{pmatrix}+\lambda\begin{pmatrix}5\\4\\-3\end{pmatrix}$	M1 A1 A1 ft **3**	Vector product of normals, or finding a point on AD, e.g. $(0,-2.6,4.2)$, $(3.25,0,2.25)$, $(7,3,0)$ Correct direction Accept any form
(iii)	$\overrightarrow{\mathrm{CA}}\times\mathbf{d}=\begin{pmatrix}-5\\-6\\2\end{pmatrix}\times\begin{pmatrix}5\\4\\-3\end{pmatrix}=\begin{pmatrix}10\\-5\\10\end{pmatrix}$ Distance is $\dfrac{\lvert\overrightarrow{\mathrm{CA}}\times\mathbf{d}\rvert}{\lvert\mathbf{d}\rvert}=\dfrac{\sqrt{10^2+5^2+10^2}}{\sqrt{5^2+4^2+3^2}}=\dfrac{\sqrt{225}}{\sqrt{50}}$ $=\sqrt{4.5}=\dfrac{3}{\sqrt{2}}=\dfrac{3\sqrt{2}}{2}\approx 2.12$	M1 A2 ft M1 M1 A1 **6**	Appropriate vector product Give A1 if just one error Formula for distance Finding magnitude *Both dependent on first M1*
(iv)	$\mathbf{d}\times\overrightarrow{\mathrm{BC}}=\begin{pmatrix}5\\4\\-3\end{pmatrix}\times\begin{pmatrix}9\\12\\-6\end{pmatrix}=\begin{pmatrix}12\\3\\24\end{pmatrix}=3\begin{pmatrix}4\\1\\8\end{pmatrix}$ Distance is $\dfrac{\begin{pmatrix}-4\\-6\\4\end{pmatrix}.\begin{pmatrix}4\\1\\8\end{pmatrix}}{\sqrt{4^2+1^2+8^2}}=\dfrac{10}{9}$	M1 A1 ft M1 A1 ft M1 A1 **6**	Vector product of directions Appropriate scalar product Evaluation of scalar product For denominator
(v)	$V=\frac{1}{6}(\overrightarrow{\mathrm{AB}}\times\overrightarrow{\mathrm{AC}}).\overrightarrow{\mathrm{AD}}=\frac{1}{6}\left[\begin{pmatrix}-4\\-6\\4\end{pmatrix}\times\begin{pmatrix}5\\6\\-2\end{pmatrix}\right].\lambda\begin{pmatrix}5\\4\\-3\end{pmatrix}$ $=\frac{1}{6}\lambda\begin{pmatrix}-12\\12\\6\end{pmatrix}.\begin{pmatrix}5\\4\\-3\end{pmatrix}=-5\lambda$ $V=\pm 20\ \Rightarrow\ \lambda=\pm 4$ D is $(22,15,-9)$ or $(-18,-17,15)$	M1 M1 A1 ft M1 A1A1 **6**	Appropriate scalar triple product Evaluation of scalar triple product or $-2a+2b+c+3$ (simplified) for $\mathrm{D}(a,b,c)$ Obtain a value of λ, or one of a, b, c

Alternative methods for Question 1

1 (ii)	Eliminating x $3y+4z=9$ $x=7-\frac{5}{3}t,\ y=3-\frac{4}{3}t,\ z=t$	M1 A1A1	**3**	Eliminating one of x, y, z or $3x+5z=21$ or $4x-5y=13$
1 (iii)	$\left[\begin{pmatrix}2\\-1\\3\end{pmatrix}+\lambda\begin{pmatrix}5\\4\\-3\end{pmatrix}-\begin{pmatrix}7\\5\\1\end{pmatrix}\right].\begin{pmatrix}5\\4\\-3\end{pmatrix}=0$ $25\lambda-25+16\lambda-24+9\lambda-6=0$ $\lambda=1.1$, F is $(7.5, 3.4, -0.3)$ $\text{CF}=\sqrt{(0.5)^2+(-1.6)^2+(-1.3)^2}$ $=\sqrt{4.5}$	M1 A1 ft A1 ft M1 M1 A1	**6**	Appropriate scalar product Obtaining a value for λ Finding magnitude
1 (iv)	$\left[\begin{pmatrix}-2\\-7\\7\end{pmatrix}+\mu\begin{pmatrix}9\\12\\-6\end{pmatrix}-\begin{pmatrix}2\\-1\\3\end{pmatrix}-\lambda\begin{pmatrix}5\\4\\-3\end{pmatrix}\right].\begin{pmatrix}5\\4\\-3\end{pmatrix}=0$ and $\begin{pmatrix}-5\lambda+9\mu-4\\-4\lambda+12\mu-6\\3\lambda-6\mu+4\end{pmatrix}.\begin{pmatrix}9\\12\\-6\end{pmatrix}=0$ $\lambda=\frac{4}{81},\quad \mu=\frac{128}{243}$ Distance is $\sqrt{\left(\frac{40}{81}\right)^2+\left(\frac{10}{81}\right)^2+\left(\frac{80}{81}\right)^2}$ $=\frac{10}{9}$	M1 A1 ft A1 ft M1 M1 A1	**6**	Two appropriate scalar products Obtaining values for λ and μ Obtaining distance

2 (i)	When $y=-3,\quad z=3x^2+36x-216$	B1	
		B1 B1	Correct shape (parabola) and position For $(-6,-324)$
	When $x=-6,\quad z=8y^3-216y-756$	B1	
		B1 B1 B1 **7**	Correct shape and position For $(-3,-324)$ For $(3,-1188)$ *If B0B0 then give B1 for* $x=\pm3$
(ii)	$(-6,-3,-324)$ is a SP on both sections; hence it is a SP on *S*	B1	
	Saddle point	B1 **2**	
(iii)	$\frac{\partial z}{\partial x}=-12xy-30x+36,\quad \frac{\partial z}{\partial y}=24y^2-6x^2$	B1B1	
	At a SP, $\frac{\partial z}{\partial x}=\frac{\partial z}{\partial y}=0$	M1	
	$24y^2-6x^2=0\ \Rightarrow\ y=\pm\frac{1}{2}x$		
	$y=\frac{1}{2}x\ \Rightarrow\ -6x^2-30x+36=0$	M1	
	$\Rightarrow\ x=-6,\ 1;$ SPs are $(-6,-3,-324)$ $(1,\ 0.5,\ 19)$	A1	
	$y=-\frac{1}{2}x\ \Rightarrow\ 6x^2-30x+36=0$	M1	
	$\Rightarrow\ x=2,\ 3;$ SPs are $(2,-1,\ 28)$	A1	
	$(3,-1.5,\ 27)$	A1 **8**	
(iv)	$\frac{\partial z}{\partial x}=120$ and $\frac{\partial z}{\partial y}=0$	M1	(Allow M1 for $\frac{\partial z}{\partial x}=-120$)
	$y=\frac{1}{2}x\ \Rightarrow\ -6x^2-30x-84=0;\ D=30^2-4\times6\times84$	M1	
	$D\ (=-1116)<0$; so this has no roots	A1	
	$y=-\frac{1}{2}x\ \Rightarrow\ 6x^2-30x-84=0\ \Rightarrow\ x=7,-2$	M1 M1	Obtaining at least one value of x Obtaining a value of k
	When $x=7,\ y=-3.5,\ z=203;$ so $k=637$	A1	
	When $x=-2,\ y=1,\ z=-148;$ so $k=-92$	A1 **7**	

3(a)(i)	$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{2}\mathrm{e}^{\frac{1}{2}x}-\frac{1}{2}\mathrm{e}^{-\frac{1}{2}x}$ $\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2=\frac{1}{4}\mathrm{e}^{x}-\frac{1}{2}+\frac{1}{4}\mathrm{e}^{-x}$ $1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2=\frac{1}{4}\mathrm{e}^{x}+\frac{1}{2}+\frac{1}{4}\mathrm{e}^{-x}=(\frac{1}{2}\mathrm{e}^{\frac{1}{2}x}+\frac{1}{2}\mathrm{e}^{-\frac{1}{2}x})^2$	B1 M1 A1 (ag) **3**	 Correct completion
(ii)	Length is $\int_0^{\ln a}(\frac{1}{2}\mathrm{e}^{\frac{1}{2}x}+\frac{1}{2}\mathrm{e}^{-\frac{1}{2}x})\,\mathrm{d}x$ $=\left[\mathrm{e}^{\frac{1}{2}x}-\mathrm{e}^{-\frac{1}{2}x}\right]_0^{\ln a}$ $=\left(\sqrt{a}-\frac{1}{\sqrt{a}}\right)-(1-1)=\frac{a-1}{\sqrt{a}}$	M1 A1 A1 (ag) **3**	For $\int(\frac{1}{2}\mathrm{e}^{\frac{1}{2}x}+\frac{1}{2}\mathrm{e}^{-\frac{1}{2}x})\,\mathrm{d}x$ For $\mathrm{e}^{\frac{1}{2}x}-\mathrm{e}^{-\frac{1}{2}x}$ Correctly shown
(iii)	Curved surface area is $\int 2\pi y\,\mathrm{d}s$ $=\int_0^{\ln a}2\pi(\mathrm{e}^{\frac{1}{2}x}+\mathrm{e}^{-\frac{1}{2}x})(\frac{1}{2}\mathrm{e}^{\frac{1}{2}x}+\frac{1}{2}\mathrm{e}^{-\frac{1}{2}x})\,\mathrm{d}x$ $=\pi\int_0^{\ln a}(\mathrm{e}^{x}+2+\mathrm{e}^{-x})\,\mathrm{d}x$ $=\pi\left[\mathrm{e}^{x}+2x-\mathrm{e}^{-x}\right]_0^{\ln a}$ $=\pi\left(a+2\ln a-\frac{1}{a}\right)$	M1 A1 M1 A1 A1 **5**	For $\int y\,\mathrm{d}s$ Correct integral form *including limits* Obtaining integrable expression For $\mathrm{e}^{x}+2x-\mathrm{e}^{-x}$
(b)(i)	$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\cos\theta}{-2\sin\theta}$ Gradient of normal is $\frac{2\sin\theta}{\cos\theta}$ $(=2\tan\theta)$ Normal is $y-\sin\theta=2\tan\theta(x-2\cos\theta)$ $y=2x\tan\theta-3\sin\theta$	B1 M1 A1 (ag) **3**	 Correctly shown
(ii)	Differentiating partially w.r.t. θ $0=2x\sec^2\theta-3\cos\theta$ $x=\frac{3}{2}\cos^3\theta$ $y=3\cos^3\theta\tan\theta-3\sin\theta$ $=3\sin\theta(\cos^2\theta-1)=-3\sin^3\theta$ $(2x)^{\frac{2}{3}}+y^{\frac{2}{3}}=(3\cos^3\theta)^{\frac{2}{3}}+(-3\sin^3\theta)^{\frac{2}{3}}$ $=3^{\frac{2}{3}}(\cos^2\theta+\sin^2\theta)=3^{\frac{2}{3}}$	M1 A1 M1 A1 M1 A1 (ag) **6**	 Obtaining an expression for y Any correct form Using $1-\cos^2\theta=\sin^2\theta$ Correctly shown
(iii) (*A*) (*B*)	$(2,0)$ has $\theta=0$ Centre of curvature is $(\frac{3}{2},\ 0)$ $\rho=\frac{1}{2}$ $(0,1)$ has $\theta=\frac{1}{2}\pi$ Centre of curvature is $(0,\ -3)$ $\rho=4$	M1 A1 M1 A1 **4**	Using param eqn with $\theta=0$ *(or other method for ρ or cc)* Using param eqn with $\theta=\frac{1}{2}\pi$ *(or other method for ρ or cc)*

<table>
<tr><td>4 (i)</td><td>$$\begin{array}{c|ccccc} & 1 & 3 & 4 & 5 & 9 \\ \hline 1 & 1 & 3 & 4 & 5 & 9 \\ 3 & 3 & 9 & 1 & 4 & 5 \\ 4 & 4 & 1 & 5 & 9 & 3 \\ 5 & 5 & 4 & 9 & 3 & 1 \\ 9 & 9 & 5 & 3 & 1 & 4 \end{array}$$
Composition table shows closure
Identity is 1
$$\begin{array}{l|ccccc} \text{Element} & 1 & 3 & 4 & 5 & 9 \\ \hline \text{Inverse} & 1 & 4 & 3 & 9 & 5 \end{array}$$
So every element has an inverse</td><td>B2
B1
B1
B2
6</td><td>Give B1 if not more than 4 errors
<i>Dependent on B2 for table</i>
Give B1 for 3 correct</td></tr>
<tr><td>(ii)</td><td>Since 5 is prime,
a group of order 5 must be cyclic
Two cyclic groups of the same order must be isomorphic</td><td>B1
B1
B1
3</td><td></td></tr>
<tr><td>(iii)</td><td>$$\begin{array}{r|ccccc} H & 1 & e^{\frac{2}{5}\pi j} & e^{\frac{4}{5}\pi j} & e^{\frac{6}{5}\pi j} & e^{\frac{8}{5}\pi j} \\ \hline G & 1 & 3 & 9 & 5 & 4 \\ \textit{or} & 1 & 4 & 5 & 9 & 3 \\ \textit{or} & 1 & 5 & 3 & 4 & 9 \\ \textit{or} & 1 & 9 & 4 & 3 & 5 \end{array}$$</td><td>B1
B2
3</td><td>For $1 \leftrightarrow 1$
For non-identity elements</td></tr>
<tr><td>(iv)</td><td>Identity is $(1,\ 1)$
Inverse of $(9,\ 3)$ is $(5,\ 4)$</td><td>B1
B1
2</td><td></td></tr>
<tr><td>(v)</td><td>$(x,\ y)^5 = (x^5,\ y^5)$
Since G has order 5, $x^5 = 1$ and $y^5 = 1$
Hence $(x,\ y)^5 = (1,\ 1)$</td><td>M1
M1
A1 (ag)
3</td><td></td></tr>
<tr><td>(vi)</td><td>Order of $(x,\ y)$ is a factor of 5 (so must be 1 or 5)
Only identity $(1,\ 1)$ can have order 1
Hence all other elements have order 5</td><td>M1
B1
A1 (ag)
3</td><td></td></tr>
<tr><td>(vii)</td><td>$\{(1,\ 1),\ (9,\ 3),\ (4,\ 9),\ (3,\ 5),\ (5,\ 4)\}$</td><td>B2
2</td><td>Give B1 ft for 5 elements including $(1,\ 1),\ (9,\ 3),\ (5,\ 4)$</td></tr>
<tr><td>(viii)</td><td>An element of order 5 generates a subgroup, and so can be in only one subgroup of order 5
Number is $24 \div 4 = 6$</td><td>M1
A1
2</td><td><i>Or</i> for $24 \div 4$
<i>Or</i> listing at least 2 other subgroups
<i>Give B1 for unsupported answer 6</i></td></tr>
</table>

Pre-multiplication by transition matrix

5 (i)	$\mathbf{P}=\begin{pmatrix}1&0.07&0&0\\0&0.8&0.15&0\\0&0.13&0.75&0\\0&0&0.1&1\end{pmatrix}$	B2 **2**	*Allow tolerance of* ± 0.0001 *in probabilities throughout this question* Give B1 for two columns correct
(ii)	If system enters an absorbing state, it remains in that state A and D are absorbing states	B1 B1 **2**	
(iii)	$\mathbf{P}^9\begin{pmatrix}0\\0.4\\0.6\\0\end{pmatrix}=\begin{pmatrix}0.2236\\0.2505\\0.1998\\0.3261\end{pmatrix}$ Prob(owned by B) = 0.2505	M1M1 A1 **3**	For $\mathbf{P}^9$ (or $\mathbf{P}^{10}$) and $\begin{pmatrix}0\\0.4\\0.6\\0\end{pmatrix}$
(iv)	$\mathbf{P}^4=\begin{pmatrix}1&\ldots&\ldots&\ldots\\\ldots&0.4818&\ldots&\ldots\\\ldots&\ldots&0.3856&\ldots\\\ldots&\ldots&\ldots&1\end{pmatrix}$ $0.2236+0.2505\times0.4818+0.1998\times0.3856+0.3261$ $=0.7474$	M1 M1 A1 **3**	Using diagonal elements from $\mathbf{P}^4$ Using probabilities for 10th day
(v)	$\begin{pmatrix}1&0&0&1\end{pmatrix}\mathbf{P}^n\begin{pmatrix}0\\0.4\\0.6\\0\end{pmatrix}$ $=(\ 0.8971\ )$ when $n=26$ $=(\ 0.9057\ )$ when $n=27$ i.e. on the 28th day	M1 M1 A1 ft A1 **4**	Considering $\mathbf{P}^n$ for some $n>20$ Evaluating Prob(A or D) for some values of n Identifying $n=26$ or $n=27$ (*Implies M1M1*)
(vi)	$\mathbf{P}^n\ \rightarrow\begin{pmatrix}1&\mathbf{0.5738}&\mathbf{0.3443}&0\\0&0&0&0\\0&0&0&0\\0&\mathbf{0.4262}&\mathbf{0.6557}&1\end{pmatrix}=\mathbf{Q}$	B2 **2**	Give B1 for two bold elements correct (to 3 dp)
(vii)	$\mathbf{Q}\begin{pmatrix}0\\0.4\\0.6\\0\end{pmatrix}=\begin{pmatrix}0.4361\\0\\0\\0.5639\end{pmatrix}$ Prob(eventually owned by A) = 0.4361	M1M1 A1 **3**	Using $\mathbf{Q}$ and $\begin{pmatrix}0\\0.4\\0.6\\0\end{pmatrix}$
(viii)	$\mathbf{Q}\begin{pmatrix}0\\p\\q\\0\end{pmatrix}=\begin{pmatrix}0.5\\0\\0\\0.5\end{pmatrix}$ (where $q=1-p$) $0.5738p+0.3443q=0.5$ $p=0.6786,\quad q=0.3214$	M1M1 A1 ft M1 A1 **5**	For LHS and RHS Or $0.4262p+0.6557q=0.5$ Solving to obtain a value of p Allow 0.678 - 0.679, 0.321 – 0.322

Question			Answer	Marks	Guidance	
1	(i)		$\begin{pmatrix}-1\\4\\1\end{pmatrix}\times\overrightarrow{\mathrm{BC}}=\begin{pmatrix}-1\\4\\1\end{pmatrix}\times\begin{pmatrix}-6\\18\\3\end{pmatrix}=\begin{pmatrix}-6\\-3\\6\end{pmatrix}\quad[=3\begin{pmatrix}-2\\-1\\2\end{pmatrix}]$	M1* A1	Vector product of directions	*Intention sufficient*
			Shortest distance is $\dfrac{\overrightarrow{\mathrm{AB}}\,.\,\mathbf{d}}{\lvert\,\mathbf{d}\,\rvert}=\dfrac{\begin{pmatrix}8\\-2\\-13\end{pmatrix}.\begin{pmatrix}-2\\-1\\2\end{pmatrix}}{\sqrt{2^2+1^2+2^2}}$	M1* M1	Appropriate scalar product Evaluation of $\lvert\,\mathbf{d}\,\rvert$	*Dep* * *Dep* **
			Shortest distance is $\dfrac{40}{3}$	A1 **[5]**		
		OR	$\left[\begin{pmatrix}11-6\lambda\\18\lambda\\-3+3\lambda\end{pmatrix}-\begin{pmatrix}3-\mu\\2+4\mu\\10+\mu\end{pmatrix}\right].\begin{pmatrix}-1\\4\\1\end{pmatrix}=0$ and $\begin{pmatrix}8-6\lambda+\mu\\-2+18\lambda-4\mu\\-13+3\lambda-\mu\end{pmatrix}.\begin{pmatrix}-6\\18\\3\end{pmatrix}=0$		M1* Two appropriate scalar products	
			$81\lambda-18\mu=29,\quad 123\lambda-27\mu=41$		A1 Two correct equations	
			$\lambda=-\dfrac{5}{3},\ \mu=-\dfrac{82}{9},\quad \overrightarrow{\mathrm{XY}}=\begin{pmatrix}{}^{80}\!/_{9}\\{}^{40}\!/_{9}\\-{}^{80}\!/_{9}\end{pmatrix}$		M1* Obtaining $\overrightarrow{\mathrm{XY}}$	*Dep* *
			Shortest distance is $\sqrt{\left(\dfrac{80}{9}\right)^2+\left(\dfrac{40}{9}\right)^2+\left(\dfrac{80}{9}\right)^2}$		M1	*Dep* **
			Shortest distance is $\dfrac{40}{3}$		A1	

Question			Answer	Marks	Guidance	
1	(ii)		$\overrightarrow{AB}\times\overrightarrow{BC}=\begin{pmatrix}8\\-2\\-13\end{pmatrix}\times\begin{pmatrix}-6\\18\\3\end{pmatrix}=\begin{pmatrix}228\\54\\132\end{pmatrix}$	M1* A2	Appropriate vector product Give A1 if one error	
			$\left\lvert\overrightarrow{AB}\times\overrightarrow{BC}\right\rvert=\sqrt{228^2+54^2+132^2}$ $\left\lvert\overrightarrow{BC}\right\rvert=\sqrt{6^2+18^2+3^2}$	M1*		*Dep **
			Shortest distance is $\dfrac{\left\lvert\overrightarrow{AB}\times\overrightarrow{BC}\right\rvert}{\left\lvert\overrightarrow{BC}\right\rvert}=\sqrt{\dfrac{72324}{369}}$	M1		*Dep ***
			Shortest distance is 14	A1 **[6]**		*Sign error in vector product can earn M1A1M1M1A1*
		OR	$\left[\begin{pmatrix}11-6\lambda\\18\lambda\\-3+3\lambda\end{pmatrix}-\begin{pmatrix}3\\2\\10\end{pmatrix}\right].\begin{pmatrix}-6\\18\\3\end{pmatrix}=0$		M1* Allow one error A1	
					M1* Obtaining a value of λ	*Dep **
			$\lambda=\dfrac{1}{3}$		A1	
			Shortest distance is $\sqrt{(6)^2+(4)^2+(-12)^2}$		M1	*Dep ***
			Shortest distance is 14		A1	

Question			Answer	Marks	Guidance	
1	(iii)		$\begin{pmatrix}11\\0\\-3\end{pmatrix}+\lambda\begin{pmatrix}-6\\18\\k+3\end{pmatrix}=\begin{pmatrix}3\\2\\10\end{pmatrix}+\mu\begin{pmatrix}-1\\4\\1\end{pmatrix}$			
			$11-6\lambda=3-\mu$	M1	Allow one error	Must use different parameters
			$18\lambda=2+4\mu$	A1	Two correct equations	
			$\lambda=5,\quad \mu=22$	A1		
			$-3+\lambda(k+3)=10+\mu$	M1	Obtaining a value of *k*	*Other methods possible*
			$k=4$	A1		*(e.g. distance between lines is 0)*
			Point of intersection is $\begin{pmatrix}3\\2\\10\end{pmatrix}+22\begin{pmatrix}-1\\4\\1\end{pmatrix}$	M1		
			Point of intersection is $(-19,\ 90,\ 32)$	A1 **[7]**		
1	(iv)		$\left\|\begin{pmatrix}-1\\4\\1\end{pmatrix}\right\|=\sqrt{18}$, so $\overrightarrow{\text{AD}}=(\pm)\dfrac{12}{\sqrt{18}}\begin{pmatrix}-1\\4\\1\end{pmatrix}=2\sqrt{2}\begin{pmatrix}-1\\4\\1\end{pmatrix}$	M1* A1	Obtaining $\overrightarrow{\text{AD}}$ or D	
			Volume is $\frac{1}{6}(\overrightarrow{\text{AB}}\times\overrightarrow{\text{AC}})\cdot\overrightarrow{\text{AD}}$	M1*	Appropriate scalar triple product	
			$=\dfrac{1}{6}\left[\begin{pmatrix}8\\-2\\-13\end{pmatrix}\times\begin{pmatrix}2\\16\\-10\end{pmatrix}\right].(2\sqrt{2})\begin{pmatrix}-1\\4\\1\end{pmatrix}$	A1 ft	Correct expression	*Can be implied*
			$=\dfrac{\sqrt{2}}{3}\begin{pmatrix}228\\54\\132\end{pmatrix}\cdot\begin{pmatrix}-1\\4\\1\end{pmatrix}=\dfrac{\sqrt{2}}{3}(120)$	M1	Evaluating scalar triple product	*Dep* **
			$=40\sqrt{2}$	A1 **[6]**	Accept 56.6	

Question			Answer	Marks	Guidance	
2	(i)		$\frac{\partial z}{\partial x}=6x^2+6x+12y$	B1		
			$\frac{\partial z}{\partial y}=6y^2+6y+12x$	B1		
			If $\frac{\partial z}{\partial x}=\frac{\partial z}{\partial y}$, $6x^2+6x+12y=6y^2+6y+12x$ $x^2-y^2-x+y=0$			
			$(x-y)(x+y-1)=0$	M1	Identifying factor $(x-y)$	*SC* If M0, then give
			$y=x$ or $y=1-x$	E1E1		B1 for verifying $y=x$ B1 for verifying $y=1-x$
				[5]		
2	(ii)		$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial y}=0$	M1		Can be implied
			If $y=x$ then $6x^2+6x+12x=0$	M1	Obtaining quadratic in *x* (or *y*)	Or quartic, and factorising as x(linear)(quadratic)
			$x=0,\ -3$	M1	Obtaining a non-zero value of *x*	
			Stationary points $(0,\ 0,\ 0)$ and $(-3,\ -3,\ 54)$	B1A1	Condone $(0,\ 0)$ for B1	
			If $y=1-x$ then $6x^2+6x+12(1-x)=0$			
			$x^2-x+2=0$	M1	Obtaining quadratic with no real roots	
			Which has no real roots ($D=-7<0$)	A1	Correctly shown	Just stating 'No real roots' M1A0
				[7]		
2	(iii)		At P, $\frac{\partial z}{\partial x}=\frac{21}{2}$, $\frac{\partial z}{\partial y}=\frac{21}{2}$	M1 A1	Substituting into $\frac{\partial z}{\partial x}$ or $\frac{\partial z}{\partial y}$	Correct value, or substitution seen
			$\delta z\approx\frac{\partial z}{\partial x}\delta x+\frac{\partial z}{\partial y}\delta y$	M1		
			$w\approx\frac{21}{2}h+\frac{21}{2}h$	A1 ft		
			$h\approx\frac{w}{21}$	A1		

Question			Answer	Marks	Guidance	
				[5]		
2	**(iv)**		$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial y}=24$	M1	Allow sign error	24λ is M0 unless $\lambda=\pm1$ appears later
			If $y=x$ then $6x^2+6x+12x=24$ $x=1,\ -4$	M1	Obtaining quadratic in x (or y)	Or quartic, and one linear factor
			Points $(1,\ 1,\ 22)$ and $(-4,\ -4,\ 32)$	A1A1	If neither correct, give A1 for $x=1,\ -4$	
			If $y=1-x$ then $6x^2+6x+12(1-x)=24$ $x=2,\ -1$	M1	Obtaining quadratic in x (or y)	Or third linear factor of quartic
			Points $(2,\ -1,\ 5)$ and $(-1,\ 2,\ 5)$	A1A1 **[7]**	If neither correct, give A1 for $x=2,\ -1$	
3	**(a)**		$r^2+\left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2=a^2(1+\cos\theta)^2+(-a\sin\theta)^2$	B1	Condone ... $+(a\sin\theta)^2$	
			$=a^2(1+2\cos\theta+\cos^2\theta+\sin^2\theta)=2a^2(1+\cos\theta)$		or $4a^2\cos^4\tfrac{1}{2}\theta+4a^2\sin^2\tfrac{1}{2}\theta\cos^2\tfrac{1}{2}\theta$	
				M1	Using $1+\cos\theta=2\cos^2\tfrac{1}{2}\theta$	
			$=4a^2\cos^2\tfrac{1}{2}\theta$	A1		
			Arc $\int\sqrt{r^2+\left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2}\,\mathrm{d}\theta=\int_0^{\frac{1}{2}\pi}2a\cos\tfrac{1}{2}\theta\,\mathrm{d}\theta$	M1	For $\int\sqrt{r^2+\left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2}\,\mathrm{d}\theta$ in terms of θ	Limits not required
			$=\Big[\ 4a\sin\tfrac{1}{2}\theta\ \Big]_0^{\frac{1}{2}\pi}$	A1	For $4a\sin\tfrac{1}{2}\theta$	
			$=2\sqrt{2}\ a$	A1 **[6]**		

Question			Answer	Marks	Guidance	
3	(b)	(i)	$1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 = 1+\left(\frac{x^2}{2}-\frac{1}{2x^2}\right)^2$	B1		
			$= \frac{x^4}{4}+\frac{1}{2}+\frac{1}{4x^4}$	M1		
			$= \left(\frac{x^2}{2}+\frac{1}{2x^2}\right)^2$	A1		
			Area is $\int 2\pi y\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x$	M1*		
			$= \int_1^2 2\pi\left(\frac{x^3}{6}+\frac{1}{2x}\right)\left(\frac{x^2}{2}+\frac{1}{2x^2}\right)\mathrm{d}x$	A1 ft	Integral expression including limits	
			$= 2\pi\int_1^2\left(\frac{x^5}{12}+\frac{x}{3}+\frac{1}{4x^3}\right)\mathrm{d}x$	M1	Obtaining integrable form	*Dep **
			$= 2\pi\left[\frac{x^6}{72}+\frac{x^2}{6}-\frac{1}{8x^2}\right]_1^2$	A1	Allow one error	
			$= \frac{47\pi}{16}$	A1 **[8]**		

Question			Answer	Marks	Guidance	
3	(b)	(ii)	$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = x + \frac{1}{x^3} \quad (= \frac{17}{8})$	B1		
			$\rho = \frac{\left(\frac{x^2}{2} + \frac{1}{2x^2}\right)^3}{x + \frac{1}{x^3}}$	M1 A1 ft	Using formula for ρ or κ Correct expression for ρ or κ	
			$= \frac{\left(1+\left(\frac{15}{8}\right)^2\right)^{\frac{3}{2}}}{2+\frac{1}{8}} = \frac{\left(\frac{17}{8}\right)^3}{\frac{17}{8}}$	A1 ft	Correct numerical expression for ρ	
			$= \frac{289}{64}$	E1 **[5]**	Correctly shown	
3	(b)	(iii)	$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{15}{8}$, so unit normal is $\frac{1}{17}\begin{pmatrix} -15 \\ 8 \end{pmatrix}$	M1 A1	Obtaining a normal vector Correct unit normal	Allow M1 for $\begin{pmatrix} \pm 8 \\ \pm 15 \end{pmatrix}$ or $\begin{pmatrix} \pm 15 \\ \pm 8 \end{pmatrix}$
			$\mathbf{c} = \begin{pmatrix} 2 \\ 19/12 \end{pmatrix} + \frac{289}{64}\begin{pmatrix} -15/17 \\ 8/17 \end{pmatrix}$	M1	Allow sign errors	Must use a unit vector
			Centre of curvature is $\left(-\frac{127}{64}, \frac{89}{24}\right)$	A1A1 **[5]**		

Question			Answer	Marks	Guidance	
4	(a)	(i)	Identity is e Element: a b c d e f g h Inverse: b a c g e h d f	B1 B2 **[3]**	Give B1 for four correct	
4	(a)	(ii)	$d^2 = a,\ \ d^4 = c$ Hence d has order 8, and G is cyclic	M1 A1 A1 E1 **[4]**	Finding powers of an element Identifying d (or f or g or h) as a generator Or $f^2 = b,\ \ f^4 = c$ Or $g^2 = b,\ \ g^4 = c$ Or $h^2 = a,\ \ h^4 = c$ Correctly shown	At least fourth power *Implies previous M1*
4	(a)	(iii)	H: 0 2 4 6 8 10 12 14 G: e d a f c h b g or: e f b d c g a h or: e g b h c f a d or: e h a g c d b f	B1 B1 B1 **[3]**	For $e \leftrightarrow 0$ and $c \leftrightarrow 8$ For $\{d, f, g, h\} \leftrightarrow \{2, 6, 10, 14\}$ For a fully correct isomorphism	In any order
4	(a)	(iv)	Rotations have order 2 or 4 Reflections have order 2 There is no element of order 8 Hence not isomorphic	B1 E1 **[2]**	Correct statement about rotations and/or reflections which implies non-IM *Or* More than one element of order 2 *Or* Not commutative Fully correct explanation	*Or* (4) reflections (and 180° rotation) have order 2 *Or* composition of reflections (or 90° rotation and reflection) is not commutative Dependent on previous B1

Question			Answer	Marks	Guidance	
4	**(b)**	**(i)**	$\mathrm{f}_m\mathrm{f}_n(x) = \dfrac{\dfrac{x}{1+nx}}{1+m\left(\dfrac{x}{1+nx}\right)}$	M1	Composition of functions	In either order
			$= \dfrac{x}{1+nx+mx} = \dfrac{x}{1+(m+n)x} = \mathrm{f}_{m+n}(x)$	E1	Correctly shown	E0 if in wrong order
				[2]		
4	**(b)**	**(ii)**	$(\mathrm{f}_m\mathrm{f}_n)\mathrm{f}_p = \mathrm{f}_{m+n}\mathrm{f}_p = \mathrm{f}_{m+n+p}$ $\mathrm{f}_m(\mathrm{f}_n\mathrm{f}_p) = \mathrm{f}_m\mathrm{f}_{n+p} = \mathrm{f}_{m+n+p}$	M1	Combining three functions	
			Hence *S* is associative	E1	Correctly shown	M1E1 bod for $(\mathrm{f}_m\mathrm{f}_n)\mathrm{f}_p = \mathrm{f}_{m+n+p} = \mathrm{f}_m(\mathrm{f}_n\mathrm{f}_p)$
				[2]		
4	**(b)**	**(iii)**	For any f_m , f_n in *S*, $\mathrm{f}_m\mathrm{f}_n = \mathrm{f}_{m+n}$	M1	Referring to this in context	
			$\mathrm{f}_m\mathrm{f}_n$ is in *S* (so *S* is closed)	A1		
			Identity is f_0	B1	B0 for *x* B1 for $n = 0$	
			Inverse of f_n is f_{-n}	B1		
			since $\mathrm{f}_n\mathrm{f}_{-n} = \mathrm{f}_{n-n} = \mathrm{f}_0$	B1		
			S is also associative, and hence is a group	E1	Closure, associativity, identity and inverses must all be mentioned in (iii)	Dependent on previous 5 marks
				[6]		
4	**(b)**	**(iv)**	$\{\ \mathrm{f}_{2n}\ \}$ for all integers *n*	B2	Or $\{\ \mathrm{f}_{3n}\ \}$, etc Give B1 for multiples of 2 (or 3, etc) but not completely correctly described	e.g. $\{\ \mathrm{f}_0\ ,\ \mathrm{f}_2\ ,\ \mathrm{f}_4\ ,\ \mathrm{f}_6\ ,\ ...\}$
				[2]		

Question			Answer	Marks	Guidance	
5	(i)		***Pre-multiplication by transition matrix*** $\mathbf{P}=\begin{pmatrix}1 & 0.5 & 0 & 0 & 0\\ 0 & 0.05 & 0.5 & 0 & 0\\ 0 & 0.45 & 0.05 & 0.5 & 0\\ 0 & 0 & 0.45 & 0.05 & 0\\ 0 & 0 & 0 & 0.45 & 1\end{pmatrix}$	B3 **[3]**	Allow tolerance of ± 0.0001 in probabilities throughout this question Give B2 for four columns correct Give B1 for two columns correct	
5	(ii)		$\mathbf{P}^8\begin{pmatrix}0\\ 1/3\\ 1/3\\ 1/3\\ 0\end{pmatrix}=\begin{pmatrix}0.5042\\ 0.0230\\ 0.0278\\ \mathbf{0.02071}\\ 0.4242\end{pmatrix}$ P(3 lives) = 0.0207 (4 dp)	M1 E1 **[2]**	For $\mathbf{P}^8$ (allow $\mathbf{P}^7$ or $\mathbf{P}^9$) and initial column matrix Correctly shown	
5	(iii)		Let q(*n*) = P(not yet ended after *n* tasks) $=\begin{pmatrix}0 & 1 & 1 & 1 & 0\end{pmatrix}\mathbf{P}^n\begin{pmatrix}0\\ 1/3\\ 1/3\\ 1/3\\ 0\end{pmatrix}$ $q(10)=0.0371$	M1 M1 A1 **[3]**	Obtaining probabilities after 10 tasks Adding probabilities of 1, 2, 3 lives	Allow M1 for using $\mathbf{P}^9$ or $\mathbf{P}^{11}$

Question			Answer	Marks	Guidance	
5	(iv)		$\text{q}(9)-\text{q}(10)$ $=0.05072-0.03709$ $=0.0136$	M1 M1 A1 **[3]**	Using $\text{q}(9)$ and $\text{q}(10)$ Evaluating $\text{q}(9)$	
		OR	$\mathbf{P}^9\begin{pmatrix}0\\ \frac{1}{3}\\ \frac{1}{3}\\ \frac{1}{3}\\ 0\end{pmatrix}=\begin{pmatrix}.\\ 0.01506\\ .\\ 0.01355\\ .\end{pmatrix}$ $0.01506\times0.5+0.01355\times0.45$ $=0.0136$		M1 Probs of 1 and 3 lives after 9 tasks M1 A1	
5	(v)		$\text{q}(13)=0.01374$ $\text{q}(14)=0.00998$ Smallest N is 14	M1 M1 A1 **[3]**	Evaluating q(n) for some $n>10$ Consecutive values each side of 0.01 Must be clear that their answer is 14	Just $N=14$ www earns B3
5	(vi)		$\mathbf{P}^n\to\begin{pmatrix}1&0.7880&0.5525&0.2908&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0.2120&0.4475&0.7092&1\end{pmatrix}=\mathbf{L}$	B2 **[2]**	Give B1 for any element correct to 3 dp (other than 0 or 1)	
5	(vii)		$\mathbf{L}\begin{pmatrix}0\\ \frac{1}{3}\\ \frac{1}{3}\\ \frac{1}{3}\\ 0\end{pmatrix}=\begin{pmatrix}0.5438\\0\\0\\0\\0.4562\end{pmatrix}$ P(wins a prize) = 0.4562	M1M1 A1 **[3]**	Using **L** and the initial column matrix	

Question			Answer	Marks	Guidance	
5	(viii)		Maximum probability is 0.7092 Always start with 3 lives	B1 ft B1 **[2]**		
5	(ix)		$\mathbf{L}\begin{pmatrix}0\\0.1\\p\\q\\0\end{pmatrix}=\begin{pmatrix}0.4\\0\\0\\0\\0.6\end{pmatrix}$ $0.7880\times0.1+0.5525p+0.2908(0.9-p)=0.4$ P(2 lives) = 0.2273 , P(3 lives) = 0.6727	M1 M1 A1 **[3]**	Or $0.0212+0.4475p+0.7092(0.9-p)=0.6$ Obtaining a value for p or q Accept values rounding to 0.227, 0.673	Allow use of $p+q=1$
5			***Post-multiplication by transition matrix***		Allow tolerance of ±0.0001 in probabilities throughout this question	
5	(i)		$\mathbf{P}=\begin{pmatrix}1&0&0&0&0\\0.5&0.05&0.45&0&0\\0&0.5&0.05&0.45&0\\0&0&0.5&0.05&0.45\\0&0&0&0&1\end{pmatrix}$	B3 **[3]**	Give B2 for four rows correct Give B1 for two rows correct	
5	(ii)		$\begin{pmatrix}0&\frac{1}{3}&\frac{1}{3}&\frac{1}{3}&0\end{pmatrix}\mathbf{P}^8$ $=\begin{pmatrix}0.5042&0.0230&0.0278&\mathbf{0.02071}&0.4242\end{pmatrix}$ P(3 lives) = 0.0207 (4 dp)	M1 E1 **[2]**	For $\mathbf{P}^8$ (allow $\mathbf{P}^7$ or $\mathbf{P}^9$) and initial row matrix Correctly shown	

Question			Answer	Marks	Guidance	
5	**(iii)**		Let q(*n*) = P(not yet ended after *n* tasks) $=\begin{pmatrix}0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0\end{pmatrix}\mathbf{P}^n\begin{pmatrix}0\\1\\1\\1\\0\end{pmatrix}$ $\mathrm{q}(10)=0.0371$	M1 M1 A1 **[3]**	Obtaining probabilities after 10 tasks Adding probabilities of 1, 2, 3 lives	Allow M1 for using $\mathbf{P}^9$ or $\mathbf{P}^{11}$
5	**(iv)**		$\mathrm{q}(9)-\mathrm{q}(10)$ $=0.05072-0.03709$ $=0.0136$	M1 M1 A1 **[3]**	Using $\mathrm{q}(9)$ and $\mathrm{q}(10)$ Evaluating $\mathrm{q}(9)$	
		OR	$\begin{pmatrix}0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0\end{pmatrix}\mathbf{P}^9$ $=\begin{pmatrix}. & 0.01506 & . & 0.01355 & .\end{pmatrix}$ $0.01506\times0.5+0.01355\times0.45$ $=0.0136$		M1 Probs of 1 and 3 lives after 9 tasks M1 A1	
5	**(v)**		$\mathrm{q}(13)=0.01374$ $\mathrm{q}(14)=0.00998$ Smallest *N* is 14	M1 M1 A1 **[3]**	Evaluating q(*n*) for some $n>10$ Consecutive values each side of 0.01 Must be clear that their answer is 14	Just $N=14$ www earns B3
5	**(vi)**		$\mathbf{P}^n\to\begin{pmatrix}1 & 0 & 0 & 0 & 0\\0.7880 & 0 & 0 & 0 & 0.2120\\0.5525 & 0 & 0 & 0 & 0.4475\\0.2908 & 0 & 0 & 0 & 0.7092\\0 & 0 & 0 & 0 & 1\end{pmatrix}=\mathbf{L}$	B2 **[2]**	Give B1 for any element correct to 3 dp (other than 0 or 1)	

Question			Answer	Marks	Guidance	
5	(vii)		$\begin{pmatrix} 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \end{pmatrix}\mathbf{L}$ $= \begin{pmatrix} 0.5438 & 0 & 0 & 0 & 0.4562 \end{pmatrix}$ P(wins a prize) = 0.4562	M1M1 A1 **[3]**	Using **L** and the initial row matrix	
5	(viii)		Maximum probability is 0.7092 Always start with 3 lives	B1 ft B1 **[2]**		
5	(ix)		$\begin{pmatrix} 0 & 0.1 & p & q & 0 \end{pmatrix}\mathbf{L}$ $= \begin{pmatrix} 0.4 & 0 & 0 & 0 & 0.6 \end{pmatrix}$ $0.7880 \times 0.1 + 0.5525p + 0.2908(0.9 - p) = 0.4$ P(2 lives) = 0.2273 , P(3 lives) = 0.6727	M1 M1 A1 **[3]**	Or $0.0212 + 0.4475p + 0.7092(0.9 - p) = 0.6$ Accept values rounding to 0.227, 0.673	Allow use of $p + q = 1$