

1. The line $x=8$ is a directrix of the ellipse with equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad a > 0, b > 0,$$

and the point $(2, 0)$ is the corresponding focus.

Find the value of a and the value of b .

(5)

$$\text{Foci } (2, 0) \Rightarrow ae = 2$$

$$\text{Directrix } x=8 \Rightarrow \frac{a}{e} = 8$$

$$ae \times \frac{a}{e} = 2 \times 8$$

$$\Rightarrow a^2 = 16 \Rightarrow a = 4$$

$$e = \frac{1}{2}$$

$$b^2 = a^2(1 - e^2) = 16\left(1 - \frac{1}{4}\right)$$

$$= 16 \times \frac{3}{4} = 12$$

$$\Rightarrow b = \sqrt{12}$$

$$\therefore a = 4, b = \sqrt{12}$$

2. Use calculus to find the exact value of $\int_{-2}^1 \frac{1}{x^2 + 4x + 13} dx$.

(5)

$$\int_{-2}^1 \frac{1}{x^2 + 4x + 13} dx = \int_{-2}^1 \frac{1}{(x+2)^2 + 9} dx$$

$$= \left[\frac{1}{3} \arctan\left(\frac{x+2}{3}\right) \right]_{-2}^1$$

$$= \frac{\pi}{12} - 0 = \frac{\pi}{12}$$

3. (a) Starting from the definitions of $\sinh x$ and $\cosh x$ in terms of exponentials, prove that

$$\cosh 2x = 1 + 2\sinh^2 x \quad (3)$$

- (b) Solve the equation

$$\cosh 2x - 3\sinh x = 15,$$

giving your answers as exact logarithms.

(5)

(a)

$$\text{RHS} = 1 + 2\sinh^2 x = 1 + 2 \left(\frac{e^x - e^{-x}}{2} \right)^2$$

$$= 1 + \frac{(e^x - e^{-x})^2}{2}$$

$$= 1 + \frac{e^{2x} - 1 - 1 + e^{-2x}}{2}$$

$$= 1 + \frac{e^{2x} + e^{-2x} - 2}{2}$$

$$= 1 + \frac{e^{2x} + e^{-2x}}{2} - \frac{2}{2} = 1 + \frac{e^{2x} + e^{-2x}}{2} - 1$$

$$= \frac{e^{2x} + e^{-2x}}{2} = \cosh(2x) = \text{LHS}$$

as required.

Question 3 continued

(b)

$$1 + 2\sinh^2 x - 3\sinh x = 15$$

$$\therefore 2\sinh^2 x - 3\sinh x - 14 = 0$$

$$(2\sinh x - 7)(\sinh x + 2) = 0$$

$$\therefore \sinh x = \frac{7}{2}$$

$$\Rightarrow \cancel{\operatorname{arsh}} x = \ln \left(\frac{7 + \sqrt{53}}{2} \right)$$

$$\cancel{\sinh(x) = -2} \quad \sinh x = -2$$

$$\Rightarrow x = \ln(\sqrt{5} - 2)$$

4. $I_n = \int_0^a (a-x)^n \cos x \, dx, \quad a > 0, \quad n \geq 0$

(a) Show that, for $n \geq 2$,

$$I_n = na^{n-1} - n(n-1)I_{n-2} \quad (5)$$

(b) Hence evaluate $\int_0^{\frac{\pi}{2}} \left(\frac{\pi}{2} - x\right)^2 \cos x \, dx.$ (3)

4. (a). $I_n = \int_0^a (a-x)^n \cos x \, dx$

Let $u = (a-x)^n \quad u' = -n(a-x)^{n-1}$

$v' = \cos x \quad v = \sin x$

$$\therefore I_n = \left[\sin x (a-x)^n \right]_0^a + n \int_0^a (a-x)^{n-1} \sin x \, dx$$

~~$\therefore I_n = 0$~~ $\therefore I_n = n \int_0^a (a-x)^{n-1} \sin x \, dx$

Let $u = (a-x)^{n-1} \quad u' = (1-n)(a-x)^{n-2}$

$v' = \sin x \quad v = -\cos x$

$$\therefore I_n = n \left(\left[-\cos x (a-x)^{n-1} \right]_0^a + \int_0^a (1-n) \cos x (a-x)^{n-2} \, dx \right)$$

$$\therefore I_n = n \left(0 + a^{n-1} + (1-n) I_{n-2} \right)$$

$$\Rightarrow I_n = n a^{n-1} + n(1-n) I_{n-2}$$

$$\therefore I_n = n a^{n-1} - n(n-1) I_{n-2}$$

as
required.

$$(b) I_2 = 2 \left(\frac{\pi}{2} \right) - 2 I_0$$

$$= \pi - 2 \int_0^{\pi/2} \left(\frac{\pi}{2} - x \right) \cos x \, dx$$

~~$$= \pi - 2 \int_0^{\pi/2} \left(\frac{\pi}{2} - x \right) \cos x \, dx$$~~

$$u = \left(\frac{\pi}{2} - x \right) \quad v' = -1$$

$$v' = \cos x \quad v = \sin x$$

~~$$= \pi - 2 \int_0^{\pi/2} \left(\frac{\pi}{2} - x \right) \cos x \, dx$$~~

$$= \pi - 2 \left(\left[\sin x \left(\frac{\pi}{2} - x \right) \right]_0^{\pi/2} + \int_0^{\pi/2} \sin x \, dx \right)$$

$$= \pi - 2 \int_0^{\pi/2} \sin x \, dx$$

$$= \pi - 2 \left[-\cos x \right]_0^{\pi/2}$$

$$= \pi - 2$$

5. Given that $y = (\operatorname{arcosh} 3x)^2$, where $3x > 1$, show that

$$(a) \quad (9x^2 - 1) \left(\frac{dy}{dx} \right)^2 = 36y, \quad (5)$$

$$(b) \quad (9x^2 - 1) \frac{d^2y}{dx^2} + 9x \frac{dy}{dx} = 18. \quad (4)$$

$$5(a). \quad y = (\operatorname{arcosh} 3x)^2$$

$$\Rightarrow \frac{\partial y}{\partial x} = (2 \operatorname{arcosh} 3x) \times \frac{d}{dx} (\operatorname{arcosh} 3x)$$

consider $\frac{d}{dx} (\operatorname{arcosh} 3x) :$

$$\text{Let } u = \operatorname{arcosh} 3x$$

$$c^2 - s^2 = 1$$

$$\therefore s^2 = c^2 - 1$$

$$\Rightarrow \cosh u = 3x$$

$$\frac{\partial u}{\partial x} \sinh u = 3$$

$$\therefore \frac{\partial u}{\partial x} = \frac{3}{\sinh u} = \frac{3}{\sqrt{\cosh^2 u - 1}}$$

$$\therefore \frac{\partial u}{\partial x} = \frac{3}{\sqrt{9x^2 - 1}}$$

$$\therefore \frac{d}{dx} (\operatorname{arcosh} 3x) = \frac{3}{\sqrt{9x^2 - 1}}$$

Question 5 continued

$$\therefore \frac{\partial y}{\partial x} = (2 \operatorname{arcosh} 3x) \times \frac{1}{2x} (\operatorname{arcosh} 3x)$$

$$= 2 \operatorname{arcosh} 3x \times \frac{3}{\sqrt{9x^2 - 1}}$$

$$\therefore \frac{\partial y}{\partial x} = \frac{6 \operatorname{arcosh} 3x}{\sqrt{9x^2 - 1}}$$

$$\Rightarrow \left(\frac{\partial y}{\partial x} \right)^2 = \frac{36 (\operatorname{arcosh} 3x)^2}{9x^2 - 1}$$

$$\therefore (9x^2 - 1) \left(\frac{\partial y}{\partial x} \right)^2 = (9x^2 - 1) \frac{36 (\operatorname{arcosh} 3x)^2}{9x^2 - 1}$$

$$= 36 (\operatorname{arcosh} 3x)^2$$

$$= 36 y^2 \quad \text{as required.}$$

Question 5 continued

$$(b) \quad \frac{\partial y}{\partial x} = \frac{6 \operatorname{arccosh} 3x}{\sqrt{9x^2 - 1}}$$

$$\therefore \frac{\partial^2 y}{\partial x^2} = \frac{(\sqrt{9x^2 - 1})(18)}{(\sqrt{9x^2 - 1})^2} - 6 \operatorname{arccosh} 3x$$

Via quotient rule...

$$\therefore \frac{\partial^2 y}{\partial x^2} = \frac{(\sqrt{9x^2 - 1}) \left(\frac{18}{\sqrt{9x^2 - 1}} \right) - 6 \operatorname{arccosh} 3x \left(\frac{1}{2} (9x^2 - 1)^{-1/2} \cdot 18x \right)}{9x^2 - 1}$$

$$\therefore \frac{\partial^2 y}{\partial x^2} = \frac{18 - 54x \operatorname{arccosh} 3x (9x^2 - 1)^{-1/2}}{9x^2 - 1}$$

$$\Rightarrow (9x^2 - 1) \frac{\partial^2 y}{\partial x^2} = (9x^2 - 1) \frac{18 - 54x \operatorname{arccosh} 3x (9x^2 - 1)^{-1/2}}{9x^2 - 1}$$

$$(9x^2 - 1) \frac{\partial^2 y}{\partial x^2} = 18 - 54x \operatorname{arccosh} 3x \cdot (9x^2 - 1)^{-1/2}$$

$$= \frac{18 - 54x \operatorname{arccosh} 3x}{\sqrt{9x^2 - 1}}$$

$$\& \quad 9x \frac{dy}{dx} = 9x \cdot \frac{6 \operatorname{arccosh} 3x}{\sqrt{9x^2 - 1}}$$

$$\therefore 9x \frac{dy}{dx} = \frac{54x \operatorname{arccosh} 3x}{\sqrt{9x^2 - 1}}$$

$$\therefore LHS = (9x^2 - 1) \frac{\partial^2 y}{\partial x^2} + 9x \frac{\partial y}{\partial x}$$

$$= \frac{18 - 54x \operatorname{arccosh} 3x}{\sqrt{9x^2 - 1}} + \frac{54x \operatorname{arccosh} 3x}{\sqrt{9x^2 - 1}}$$

$$= 18 + \frac{0}{\sqrt{9x^2 - 1}}$$

$$= 18 = RHS$$

as required.

6.

$$M = \begin{pmatrix} 1 & 0 & 3 \\ 0 & -2 & 1 \\ k & 0 & 1 \end{pmatrix}, \text{ where } k \text{ is a constant.}$$

Given that $\begin{pmatrix} 6 \\ 1 \\ 6 \end{pmatrix}$ is an eigenvector of M ,

(a) find the eigenvalue of M corresponding to $\begin{pmatrix} 6 \\ 1 \\ 6 \end{pmatrix}$, (2)

(b) show that $k = 3$, (2)

(c) show that M has exactly two eigenvalues. (4)

A transformation $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is represented by M .

The transformation T maps the line l_1 , with cartesian equations $\frac{x-2}{1} = \frac{y}{-3} = \frac{z+1}{4}$, onto the line l_2 .

(d) Taking $k = 3$, find cartesian equations of l_2 . (5)

6(a). $Mx = \lambda x$

$$\therefore \begin{pmatrix} 1 & 0 & 3 \\ 0 & -2 & 1 \\ k & 0 & 1 \end{pmatrix} \begin{pmatrix} 6 \\ 1 \\ 6 \end{pmatrix} = \begin{pmatrix} 6\lambda \\ \lambda \\ 6\lambda \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 24 \\ 4 \\ 6k+6 \end{pmatrix} = \begin{pmatrix} 6\lambda \\ \lambda \\ 6\lambda \end{pmatrix} \Rightarrow \lambda = 4$$

E. Value = 4

$$(b) \quad Mx = \begin{pmatrix} 1 & 0 & 3 \\ 0 & -2 & 1 \\ k & 0 & 1 \end{pmatrix} \begin{pmatrix} 6 \\ 1 \\ 6 \end{pmatrix} = \begin{pmatrix} 24 \\ 4 \\ 6k+6 \end{pmatrix} = \begin{pmatrix} 6\lambda \\ \lambda \\ 6\lambda \end{pmatrix} = \begin{pmatrix} 24 \\ 4 \\ 24 \end{pmatrix}$$

Question 6 continued

$$\therefore 6k + 6 = 24 \Rightarrow k = \frac{24-6}{6}$$

$$\Rightarrow k = 3 \quad \text{as required}$$

$$(c) \quad A - \lambda I = \begin{pmatrix} 1-\lambda & 0 & 3 \\ 0 & -2-\lambda & 1 \\ 3 & 0 & 1-\lambda \end{pmatrix}$$

$$\det(A - \lambda I) = 0$$

$$\Rightarrow \det(A - \lambda I) = (1-\lambda)(-2-\lambda)(1-\lambda) + 3(-3(-2-\lambda))$$

$$= (-2-\lambda)[(1-\lambda)^2 + 9] = 0$$

$$= (-2-\lambda)(\lambda^2 - 2\lambda + 10) \Rightarrow -2-\lambda = 0 \Rightarrow \lambda = -2$$

For an E. Value.

$$= (\lambda+2)(\lambda+4)(\lambda-4)$$

$$= -(\lambda+2)^2(\lambda-4) = -(\lambda+2)(\lambda-4) \quad (1-\lambda)^2 + 9 = 0$$

$$\Rightarrow (1-\lambda)^2 = -9$$

$$\Rightarrow \lambda = -2$$

$$\lambda = 4$$

are

only E. vals.

$$= -(\lambda+2)$$

no solution since
discriminant < 0 .

$\therefore$ only eigen values are

$$\lambda = -2 \quad \text{and} \quad \lambda = 4$$

$$(a) \quad x - 2 = -\frac{1}{3}y \quad , \quad -\frac{1}{3}y = \frac{z}{4} + \frac{1}{4}$$

$$x = 2 - \frac{y}{3} \quad \leftarrow \text{and } x - 2 = \frac{z}{4} + \frac{1}{4}$$

$$\Rightarrow \frac{2 - \frac{y}{3} - 2}{3} = \frac{z}{4} + \frac{1}{4}$$

$$z + 1 = -\frac{4}{3}y$$

$$\therefore z = -1 - \frac{4}{3}y$$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 - \frac{y}{3} \\ y \\ -1 - \frac{4}{3}y \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} + y \begin{pmatrix} -1/3 \\ 1 \\ -4/3 \end{pmatrix}$$

$$\text{Let } y = \lambda \Rightarrow \underline{r} = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} -1/3 \\ 1 \\ -4/3 \end{pmatrix}$$

$$\therefore \underline{r} = \begin{pmatrix} 2 - \lambda/3 \\ \lambda \\ -1 - 4\lambda/3 \end{pmatrix} \quad \text{is eqn of } l_1$$

$$x = -1 - \frac{13}{3}\lambda \Rightarrow -\frac{3(x+1)}{13} = \lambda$$

$$y = -\frac{10}{3}\lambda - 1 \Rightarrow \frac{3(y+1)}{-10} = \lambda$$

$$z = 5 - \frac{7\lambda}{3} \Rightarrow \frac{3(5-z)}{7} = \lambda$$

$$\therefore -\frac{3x+3}{13} = \frac{3y+3}{-10} = \frac{15-3z}{7}$$

~~3~~

$$-\frac{(x+1)}{13} = \frac{y+1}{-10} = \frac{5-z}{7}$$

7. The plane Π has vector equation

$$\mathbf{r} = 3\mathbf{i} + \mathbf{k} + \lambda(-4\mathbf{i} + \mathbf{j}) + \mu(6\mathbf{i} - 2\mathbf{j} + \mathbf{k})$$

- (a) Find an equation of Π in the form $\mathbf{r} \cdot \mathbf{n} = p$, where $\mathbf{n}$ is a vector perpendicular to Π and p is a constant. (5)

The point P has coordinates $(6, 13, 5)$. The line l passes through P and is perpendicular to Π . The line l intersects Π at the point N .

- (b) Show that the coordinates of N are $(3, 1, -1)$. (4)

The point R lies on Π and has coordinates $(1, 0, 2)$.

- (c) Find the perpendicular distance from N to the line PR . Give your answer to 3 significant figures. (5)

Let:

$$(a) \quad \vec{A} = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} \quad \vec{B} = \begin{pmatrix} -4 \\ 1 \\ 0 \end{pmatrix} \quad \vec{C} = \begin{pmatrix} 6 \\ -2 \\ 1 \end{pmatrix}$$

$$\vec{AB} = \begin{pmatrix} -4 \\ 1 \\ 0 \end{pmatrix} \quad \vec{AC} = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$$

$$\vec{AC} = \begin{pmatrix} 6 \\ -2 \\ 1 \end{pmatrix}$$

$$\therefore \vec{AB} \times \vec{AC} = \begin{vmatrix} -4 & 1 & 0 \\ 6 & -2 & 1 \end{vmatrix} = \begin{vmatrix} -4 & 1 & 0 \\ 6 & -2 & 1 \end{vmatrix}$$

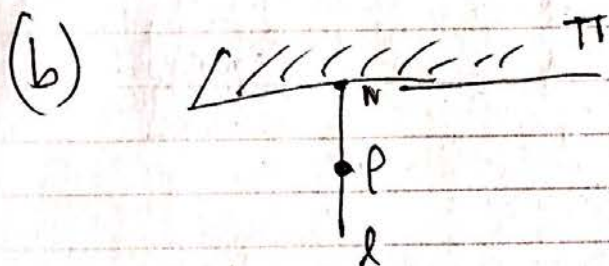
$$\vec{AB} \times \vec{AC} = \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix} \Rightarrow \vec{n} = \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix}$$

Question 7 continued

$$\therefore \underline{r} \cdot \underline{n} = A \cdot \underline{n}$$

$$\therefore \underline{r} \cdot \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix}$$

$$\underline{r} \cdot \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix} = 5$$



line l has direction vector $\begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix}$

eqn:

$$\therefore \underline{r} = \begin{pmatrix} 6 \\ 13 \\ 5 \end{pmatrix} + t \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix}$$

$$N \text{ lies on } l \Rightarrow N = \begin{pmatrix} 6+t \\ 13+4t \\ 5+2t \end{pmatrix} \text{ for some } t.$$

$$\underline{r} \cdot \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix} = 5 \Rightarrow \begin{pmatrix} 6+t \\ 13+4t \\ 5+2t \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix} = 5$$

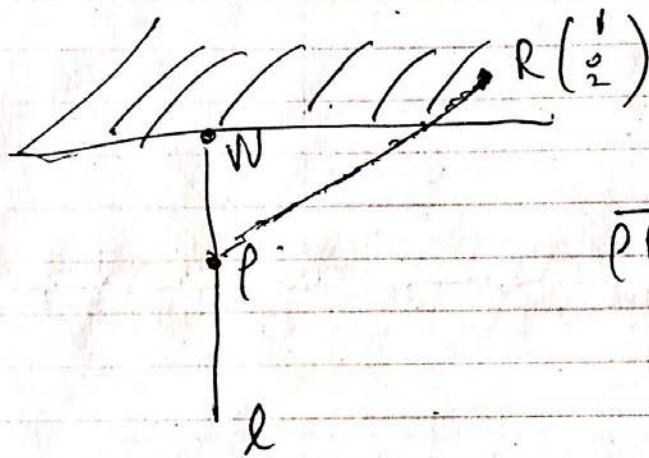
$$\therefore 6+t + 52 + 16t + 10 + 4t = 5$$

$$\Rightarrow 68 + 21t = 5 \Rightarrow t = -3$$

Question 7 continued

$$\therefore N = \begin{pmatrix} 6-3 \\ 13+4(-3) \\ 5+2(-3) \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ -1 \end{pmatrix} \quad \text{as required.}$$

(c)



$$\vec{PR} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} - \begin{pmatrix} 6 \\ 13 \\ 5 \end{pmatrix} = \begin{pmatrix} -5 \\ -13 \\ -3 \end{pmatrix}$$

Line \$\vec{PR}\$ has eqn \$\underline{r} = \begin{pmatrix} 6 \\ 13 \\ 5 \end{pmatrix} + s \begin{pmatrix} -5 \\ -13 \\ -3 \end{pmatrix}\$

$$\begin{aligned} x &= 6 - 5s & \Rightarrow & \quad 6 - 2s \\ y &= 13 - 13s & \Rightarrow & \quad 13 - 8 = 5 \end{aligned}$$

Diagram showing a line passing through point P and point R. Point N is the projection of point P onto the line. The distance from P to N is labeled 'd'. The line is defined by the vector equation $L = \begin{pmatrix} 6 \\ 13 \\ 5 \end{pmatrix} + s \begin{pmatrix} -5 \\ -13 \\ -3 \end{pmatrix}$. The point P is $\begin{pmatrix} 6 \\ 13 \\ 5 \end{pmatrix}$ and the point R is $\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$. The point N is $\begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix}$.

$$\vec{PN} = \begin{pmatrix} -3 \\ -12 \\ -6 \end{pmatrix} \quad \text{and} \quad \vec{PR} = \begin{pmatrix} -5 \\ -13 \\ -3 \end{pmatrix}$$

$$\therefore \cos \theta = \frac{\begin{pmatrix} -3 \\ -12 \\ -6 \end{pmatrix} \cdot \begin{pmatrix} -5 \\ -13 \\ -3 \end{pmatrix}}{\sqrt{3^2 + 12^2 + 6^2} \times \sqrt{5^2 + 13^2 + 3^2}}$$

$$\cos \theta = \frac{189}{21\sqrt{87}}$$

$$\Rightarrow \sin \theta = \frac{\sqrt{58}}{21} = \frac{d}{|PN|}$$

$$|PN| = 3\sqrt{21} \Rightarrow d = 3\sqrt{21} \sin \theta = 3.61$$

$$\therefore d = \underline{\underline{3.61}} \text{ units.}$$

8. The hyperbola H has equation $\frac{x^2}{16} - \frac{y^2}{4} = 1$.

The line l_1 is the tangent to H at the point $P(4 \sec t, 2 \tan t)$.

- (a) Use calculus to show that an equation of l_1 is

$$2y \sin t = x - 4 \cos t \quad (5)$$

The line l_2 passes through the origin and is perpendicular to l_1 .

The lines l_1 and l_2 intersect at the point Q .

- (b) Show that, as t varies, an equation of the locus of Q is

$$(x^2 + y^2)^2 = 16x^2 - 4y^2 \quad (8)$$

$$8(a). \quad \frac{x^2}{16} - \frac{y^2}{4} = 1$$

$$\frac{1}{8} x \Rightarrow \frac{1}{2} y \frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{2x}{8y} = + \frac{x}{4y}$$

$$\therefore \frac{dy}{dx} = \frac{2x}{8y} = \frac{x}{4y}$$

$$\therefore \left(\frac{dy}{dx} \right)_P = \frac{4 \sec t}{8 \tan t} = \frac{\sec t}{2 \tan t}$$

$$= \frac{1/\cos t}{2 \frac{\sin t}{\cos t}}$$

$$= \frac{1}{2 \sin t}$$

$$m_T = -\frac{1}{2} \operatorname{cosec} t$$

$$\therefore y - y_1 = m(x - x_1)$$

$$\begin{aligned}\therefore y - 2 \tan t &= \frac{1}{2} \operatorname{cosec} t (x - 4 \sec t) \\ &= \frac{x}{2} \operatorname{cosec} t - 2 \operatorname{cosec} t \sec t\end{aligned}$$

$$2y \sin t - \frac{4 \sin^2 t}{\cos t} = x - 4 \sec t$$

$$\therefore 2y \sin t = x - 4 \sec t + \frac{4 \sin^2 t}{\cos t}$$

$$\therefore 2y \sin t = x + \frac{4 \sin^2 t - 4}{\cos t}$$

$$\therefore 2y \sin t = x + \frac{4(\sin^2 t - 1)}{\cos t}$$

$$\therefore 2y \sin t = x - \frac{4(1 - \sin^2 t)}{\cos t}$$

$$\therefore 2y \sin t = x - 4 \cdot \frac{\cos^2 t}{\cos t}$$

$$\therefore 2y \sin t = x - 4 \sec t$$

as required.

Question 8 continued

$$\therefore 2y \sin t = \underline{x - 4 \cos t} \quad \text{as required.}$$

(b) Gradient of $l_1 = \frac{-1}{\frac{1}{2} \csc t} = -2 \sin t$

$\therefore l_1$ has eqn: $y = -2 \sin t x$

& l_2 has eqn: $2y \sin t = x - 4 \cos t$

$$\Rightarrow -4 \sin^2 t x = x - 4 \cos t$$

$$\therefore (-4 \sin^2 t - 1)x = -4 \cos t$$

$$\therefore x = \frac{4 \cos t}{1 + 4 \sin^2 t}$$

$$\therefore y = \frac{-8 \sin t \cos t}{1 + 4 \sin^2 t}$$

$$\therefore Q \left(\frac{4 \cos t}{1 + 4 \sin^2 t}, \frac{-8 \sin t \cos t}{1 + 4 \sin^2 t} \right)$$

$$x^2 = \frac{16 \cos^2 t}{(1 + 4 \sin^2 t)^2} \quad \& \quad y^2 = \frac{64 \sin^2 t \cos^2 t}{(1 + 4 \sin^2 t)^2}$$

(Total 13 marks)

$$\therefore \text{LHS} = x^2 + y^2 = \frac{16\cos^2 t + 64\sin^2 t \cos^2 t}{(1+4\sin^2 t)^2}$$

$$\therefore x^2 + y^2 = \frac{16\cos^2 t (1 + 4\sin^2 t)}{(1+4\sin^2 t)^2}$$

$$\therefore x^2 + y^2 = \frac{16\cos^2 t}{1+4\sin^2 t}$$

$$\therefore (x^2 + y^2)^2 = \frac{256\cos^4 t}{(1+4\sin^2 t)^2}$$

$$\text{RHS} = 16x^2 - 4y^2 = \frac{256\cos^2 t}{(1+4\sin^2 t)^2} - \frac{256\sin^2 t \cos^2 t}{(1+4\sin^2 t)^2}$$

$$= \frac{256\cos^2 t (1 - \sin^2 t)}{(1+4\sin^2 t)^2}$$

$$= \frac{256\cos^4 t}{(1+4\sin^2 t)^2} = \underline{\underline{\text{LHS}}} \quad \text{as required.}$$