FP3 questions from old P4, P5, P6 and FP1, FP2, FP3 papers (back to June 2002)

The following pages contain questions from past papers which could conceivably appear on Edexcel's new FP3 papers from June 2009 onwards.

Where a question reference is marked with an asterisk (*), it is a partial version of the original.

Mark schemes are available on a separate document, originally sent with this one.

This document was circulated by e-mail in March 2009; questions 2 and 7 have since been removed (18.3.09) since they are not on the specification.

1. An ellipse has equation $\dfrac{x^2}{16} + \dfrac{y^2}{9} = 1$.

(*a*) Sketch the ellipse. **(1)**

(*b*) Find the value of the eccentricity *e*. **(2)**

(*c*) State the coordinates of the foci of the ellipse. **(2)**

[P5 June 2002 Qn 1]

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3. Solve the equation

$$10 \cosh x + 2 \sinh x = 11.$$

Give each answer in the form ln *a* where *a* is a rational number. **(7)**

[P5 June 2002 Qn 3]

4.
$$I_n = \int_0^{\frac{\pi}{2}} x^n \cos x \, \mathrm{d}x, \quad n \geq 0.$$

(*a*) Prove that $I_n = \left(\dfrac{\pi}{2}\right)^n - n(n-1)I_{n-2}$, $n \geq 2$. **(5)**

(*b*) Find an exact expression for I_6. **(4)**

[P5 June 2002 Qn 4]

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5. (*a*) Given that y = arctan $3x$, and assuming the derivative of tan x, prove that

$$\frac{dy}{dx} = \frac{3}{1+9x^2}.$$

(4)

(*b*) Show that

$$\int_0^{\frac{\sqrt{3}}{3}} 6x \arctan 3x \ dx = \tfrac{1}{9}(4\pi - 3\sqrt{3}).$$

(6)

[P5 June 2002 Qn 6]

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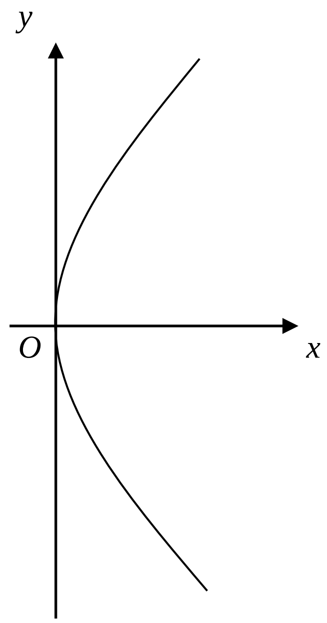
6. **Figure 1**



The curve C shown in Fig. 1 has equation $y^2 = 4x$, $0 \le x \le 1$.

The part of the curve in the first quadrant is rotated through 2π radians about the x-axis.

(a) Show that the surface area of the solid generated is given by

$$4\pi\int_0^1 \sqrt{(1+x)}\ \mathrm{d}x.$$

(4)

(b) Find the exact value of this surface area.

(3)

(c) Show also that the length of the curve C, between the points (1, –2) and (1, 2), is given by

$$2\int_0^1 \sqrt{\left(\frac{x+1}{x}\right)}\ \mathrm{d}x.$$

(3)

(d) Use the substitution $x = \sinh^2\theta$ to show that the exact value of this length is

$$2[\sqrt{2} + \ln(1 + \sqrt{2})].$$

(6)

[P5 June 2002 Qn 8]

7. Prove that $\sinh(\mathrm{i}\pi - \theta) = \sinh\theta$.

(4)

[P6 June 2002 Qn 1]

8.
$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 4 \\ 0 & 5 & 4 \\ 4 & 4 & 3 \end{pmatrix}.$$

(*a*) Verify that $\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$ is an eigenvector of **A** and find the corresponding eigenvalue.

(3)

(*b*) Show that 9 is another eigenvalue of **A** and find the corresponding eigenvector.

(5)

(*c*) Given that the third eigenvector of **A** is $\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$, write down a matrix **P** and a diagonal matrix **D** such that

$$\mathbf{P}^{\mathrm{T}}\mathbf{A}\mathbf{P} = \mathbf{D}.$$

(5)

[P6 June 2002 Qn 5]

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9. The plane Π passes through the points

$$A(-1, -1, 1),\ B(4, 2, 1) \text{ and } C\ (2, 1, 0).$$

(*a*) Find a vector equation of the line perpendicular to Π which passes through the point D (1, 2, 3).

(3)

(*b*) Find the volume of the tetrahedron *ABCD*.

(3)

(*c*) Obtain the equation of Π in the form $\mathbf{r.n} = p$.

(3)

The perpendicular from D to the plane Π meets Π at the point E.

(*d*) Find the coordinates of E.

(4)

(*e*) Show that $DE = \dfrac{11\sqrt{35}}{35}$.

(2)

The point D' is the reflection of D in Π.

(*f*) Find the coordinates of D'.

(3)

[P6 June 2002 Qn 7]

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10. Find the values of x for which

$$4 \cosh x + \sinh x = 8,$$

giving your answer as natural logarithms.

(6)

[P5 June 2003 Qn 1]

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11. (*a*) Prove that the derivative of artanh x, $-1 < x < 1$, is $\frac{1}{1-x^2}$.

(3)

(*b*) Find $\int \text{artanh}\, x \, dx$.

(4)

[P5 June 2003 Qn 2]

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12. **Figure 1**

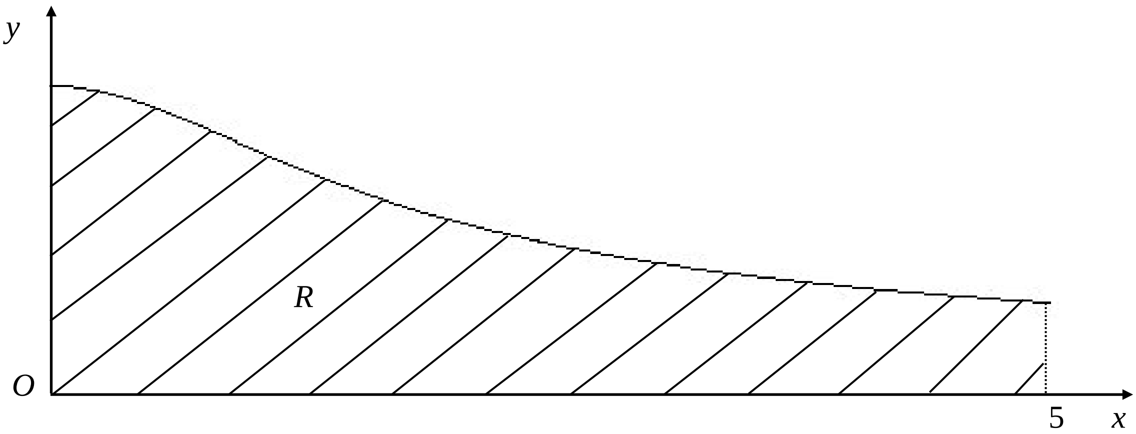


Figure 1 shows the cross-section R of an artificial ski slope. The slope is modelled by the curve with equation

$$y = \frac{10}{\sqrt{(4x^2+9)}}, \qquad 0 \le x \le 5.$$

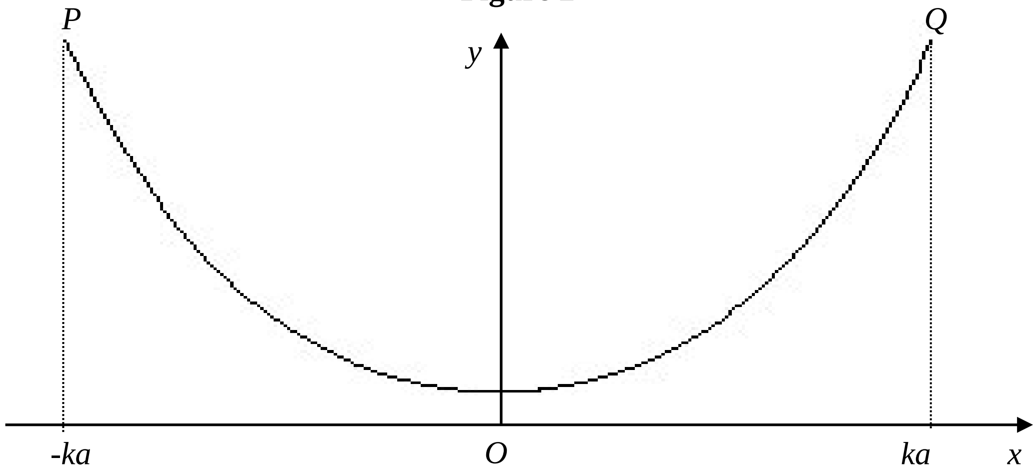
Given that 1 unit on each axis represents 10 metres, use integration to calculate the area R. Show your method clearly and give your answer to 2 significant figures.

(7)

[P5 June 2003 Qn 3]

13.

Figure 2



A rope is hung from points P and Q on the same horizontal level, as shown in Fig. 2. The curve formed by the rope is modelled by the equation

$$y = a\cosh\left(\frac{x}{a}\right), \qquad -ka \le x \le ka,$$

where a and k are positive constants.

(a) Prove that the length of the rope is $2a \sinh k$.

(5)

Given that the length of the rope is $8a$,

(b) find the coordinates of Q, leaving your answer in terms of natural logarithms and surds, where appropriate.

(4)

[P5 June 2003 Qn 5]

14. The curve C has equation

$$y = \text{arcsec } e^x, \qquad x > 0, \qquad 0 < y < \tfrac{1}{2}\pi .$$

(a) Prove that $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{(e^{2x} - 1)}}$.

(5)

(b) Sketch the graph of C.

(2)

The point A on C has x-coordinate ln 2. The tangent to C at A intersects the y-axis at the point B.

(c) Find the exact value of the y-coordinate of B.

(4)

[P5 June 2003 Qn 6]

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15. $$I_n = \int_0^1 x^n e^x \, dx \quad \text{and} \quad J_n = \int_0^1 x^n e^{-x} \, dx, \quad n \geq 0.$$

(a) Show that, for $n \geq 1$,

$$I_n = e - nI_{n-1}.$$

(2)

(b) Find a similar reduction formula for J_n .

(3)

(c) Show that $J_2 = 2 - \dfrac{5}{e}$.

(3)

(d) Show that $\displaystyle\int_0^1 x^n \cosh x \, dx \;=\; \tfrac{1}{2}(I_n + J_n)$.

(1)

(e) Hence, or otherwise, evaluate $\displaystyle\int_0^1 x^2 \cosh x \, dx$, giving your answer in terms of e.

(4)

[P5 June 2003 Qn 7]

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16. The hyperbola C has equation $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$.

(a) Show that an equation of the normal to C at the point P ($a \sec t$, $b \tan t$) is

$$ax \sin t + by = (a^2 + b^2) \tan t.$$

(6)

The normal to C at P cuts the x-axis at the point A and S is a focus of C. Given that the eccentricity of C is $\frac{3}{2}$, and that $OA = 3OS$, where O is the origin,

(b) determine the possible values of t, for $0 \leq t < 2\pi$.

(8)

[P5 June 2003 Qn 1]

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17. Referred to a fixed origin O, the position vectors of three non-collinear points A, B and C are $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ respectively. By considering $\overrightarrow{AB} \times \overrightarrow{AC}$, prove that the area of ΔABC can be expressed in the form $\frac{1}{2}\left|\, \mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{c} + \mathbf{c} \times \mathbf{a} \right|$.

(5)

[P6 June 2003 Qn 1]

18.

$$\mathbf{M} = \begin{pmatrix} 4 & -5 \\ 6 & -9 \end{pmatrix}$$

(a) Find the eigenvalues of $\mathbf{M}$.

(4)

A transformation T: $\mathbb{R}^2 \rightarrow \mathbb{R}^2$ is represented by the matrix $\mathbf{M}$. There is a line through the origin for which every point on the line is mapped onto itself under T.

(b) Find a cartesian equation of this line.

(3)

[P6 June 2003 Qn 3]

19.

$$\mathbf{A} = \begin{pmatrix} 3 & 1 & -1 \\ 1 & 1 & 1 \\ 5 & 3 & u \end{pmatrix}, \quad u \neq 1.$$

(*a*) Show that det $\mathbf{A}$ =2(u – 1).

(2)

(*b*) Find the inverse of $\mathbf{A}$.

(6)

The image of the vector $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ when transformed by the matrix $\begin{pmatrix} 3 & 1 & -1 \\ 1 & 1 & 1 \\ 5 & 3 & 6 \end{pmatrix}$ is $\begin{pmatrix} 3 \\ 1 \\ 6 \end{pmatrix}$.

(*c*) Find the values of a, b and c.

(3)

[P6 June 2003 Qn 6]

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20. The plane Π_1 passes through the P, with position vector $\mathbf{i} + 2\mathbf{j} - \mathbf{k}$, and is perpendicular to the line L with equation

$$\mathbf{r} = 3\mathbf{i} - 2\mathbf{k} + \lambda(-\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}).$$

(*a*) Show that the Cartesian equation of Π_1 is $x - 5y - 3z = -6$.

(4)

The plane Π_2 contains the line L and passes through the point Q, with position vector $\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$.

(*b*) Find the perpendicular distance of Q from Π_1.

(4)

(*c*) Find the equation of Π_2 in the form $\mathbf{r} = \mathbf{a} + s\mathbf{b} + t\mathbf{c}$.

(4)

[P6 June 2003 Qn 7]

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21. Using the definitions of $\cosh x$ and $\sinh x$ in terms of exponentials,

(*a*) prove that $\cosh^2 x - \sinh^2 x = 1$,

(3)

(*b*) solve $\operatorname{cosech} x - 2 \coth x = 2$,

giving your answer in the form $k \ln a$, where k and a are integers.

(4)

[P5 June 2004 Qn 1]

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22. $$4x^2 + 4x + 17 \equiv (ax + b)^2 + c, \qquad a > 0.$$

(*a*) Find the values of a, b and c.

(3)

(*b*) Find the exact value of

$$\int_{-0.5}^{1.5} \frac{1}{4x^2 + 4x + 17} \, \mathrm{d}x.$$

(4)

[P5 June 2004 Qn 2]

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23. An ellipse, with equation $\frac{x^2}{9} + \frac{y^2}{4} = 1$, has foci S and S'.

(*a*) Find the coordinates of the foci of the ellipse.

(4)

(*b*) Using the focus-directrix property of the ellipse, show that, for any point P on the ellipse,

$$SP + S'P = 6.$$

(3)

[P5 June 2004 Qn 3]

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24. Given that $y = \sinh^{n-1} x \cosh x$,

(a) show that $\dfrac{\mathrm{d}y}{\mathrm{d}x} = (n - 1)\sinh^{n-2} x + n \sinh^n x$.

(3)

The integral I_n is defined by $I_n = \displaystyle\int_0^{\operatorname{arsinh} 1} \sinh^n x \,\mathrm{d}x, \quad n \geq 0.$

(b) Using the result in part (a), or otherwise, show that

$$nI_n = \sqrt{2} - (n - 1)I_{n-2}, \qquad n \geq 2$$

(2)

(c) Hence find the value of I_4.

(4)

[P5 June 2004 Qn 5]

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25. **Figure 1**

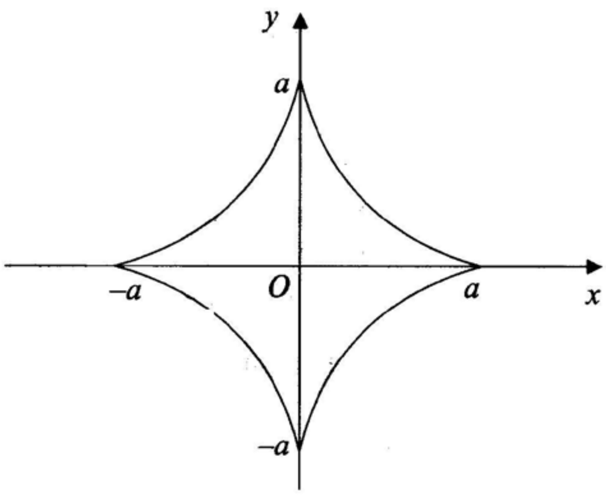

Figure 1 shows the curve with parametric equations

$$x = a\cos^3\theta, \quad y = a\sin^3\theta, \quad 0 \leq \theta < 2\pi.$$

(a) Find the total length of this curve.

(7)

The curve is rotated through π radians about the x-axis.

(b) Find the area of the surface generated.

(5)

[P5 June 2004 Qn 7]

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26. The points A, B and C lie on the plane Π and, relative to a fixed origin O, they have position vectors

$$\mathbf{a} = 3\mathbf{i} - \mathbf{j} + 4\mathbf{k}, \quad \mathbf{b} = -\mathbf{i} + 2\mathbf{j}, \quad \mathbf{c} = 5\mathbf{i} - 3\mathbf{j} + 7\mathbf{k}$$

respectively.

(*a*) Find $\overrightarrow{AB} \times \overrightarrow{AC}$.

(4)

(*b*) Find an equation of Π in the form $\mathbf{r.n} = p$.

(2)

The point D has position vector $5\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$.

(*c*) Calculate the volume of the tetrahedron $ABCD$.

(4)

[P6 June 2004 Qn 3]

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27. The matrix $\mathbf{M}$ is given by

$$\mathbf{M} = \begin{pmatrix} 1 & 4 & -1 \\ 3 & 0 & p \\ a & b & c \end{pmatrix},$$

where p, a, b and c are constants and $a > 0$.

Given that $\mathbf{MM}^{\mathrm{T}} = k\mathbf{I}$ for some constant k, find

(*a*) the value of p,

(2)

(*b*) the value of k,

(2)

(*c*) the values of a, b and c,

(6)

(*d*) $|\det \mathbf{M}|$.

(2)

[P6 June 2004 Qn 5]

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28. The transformation R is represented by the matrix $\mathbf{A}$, where

$$\mathbf{A} = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}.$$

(a) Find the eigenvectors of $\mathbf{A}$.

(5)

(b) Find an orthogonal matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that

$$\mathbf{A} = \mathbf{PDP}^{-1}.$$

(5)

(c) Hence describe the transformation R as a combination of geometrical transformations, stating clearly their order.

(4)

[P6 June 2004 Qn 6]

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29. (a) Find $\displaystyle\int \frac{1+x}{\sqrt{(1-4x^2)}}\,dx$.

(5)

(b) Find, to 3 decimal places, the value of

$$\int_0^{0.3} \frac{1+x}{\sqrt{(1-4x^2)}}\,dx.$$

(2)

(Total 7 marks)

[FP2/P5 June 2005 Qn 1]

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30. (*a*) Show that, for $x = \ln k$, where k is a positive constant,

$$\cosh 2x = \frac{k^4 + 1}{2k^2}.$$

(3)

Given that $f(x) = px - \tanh 2x$, where p is a constant,

(*b*) find the value of p for which $f(x)$ has a stationary value at $x = \ln 2$, giving your answer as an exact fraction.

(4)

(Total 7 marks)

[FP2/P5 June 2005 Qn 2]

31.

Figure 1

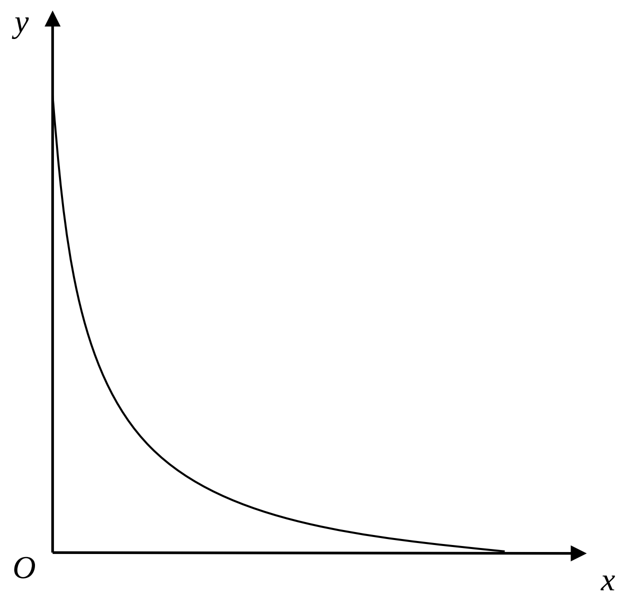

Figure 1 shows a sketch of the curve with parametric equations

$$x = a \cos^3 t, \qquad y = a \sin^3 t, \qquad 0 \leq t \leq \frac{\pi}{2},$$

where a is a positive constant.

The curve is rotated through 2π radians about the x-axis. Find the exact value of the area of the curved surface generated.

[FP2/P5 June 2005 Qn 3]

32.

$$I_n = \int x^n e^{2x} \, dx, \quad n \geq 0.$$

(*a*) Prove that, for $n \geq 1$,

$$I_n = \frac{1}{2}(x^n e^{2x} - nI_{n-1}).$$

(3)

(*b*) Find, in terms of e, the exact value of

$$\int_0^1 x^2 e^{2x} \, dx.$$

(5)

[FP2/P5 June 2005 Qn 4]

33.

Figure 2

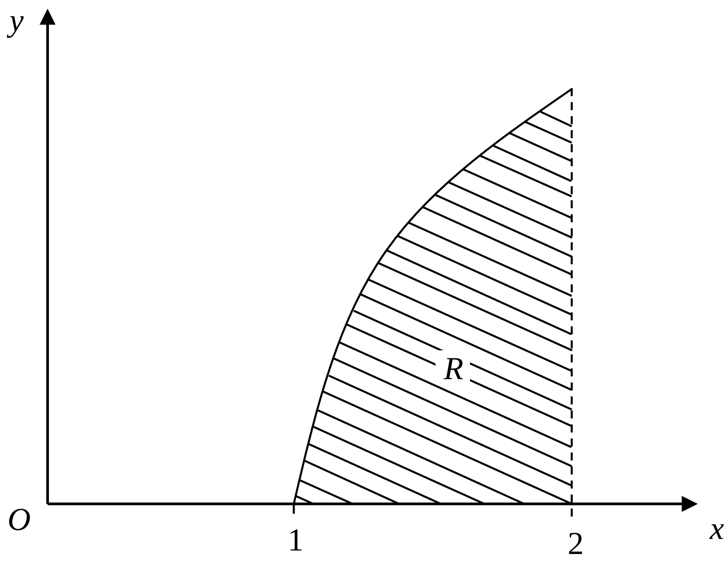

Figure 2 shows a sketch of the curve with equation

$$y = x \operatorname{arcosh} x, \qquad 1 \leq x \leq 2.$$

The region R, as shown shaded in Figure 2, is bounded by the curve, the x-axis and the line $x = 2$.

Show that the area of R is

$$\frac{7}{4} \ln (2 + \sqrt{3}) - \frac{\sqrt{3}}{2}.$$

(Total 10 marks)

[FP2/P5 June 2005 Qn 6]

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34. (*a*) Show that, for $0 < x \leq 1$,

$$\ln\left(\frac{1-\sqrt{(1-x^2)}}{x}\right) = -\ln\left(\frac{1+\sqrt{(1-x^2)}}{x}\right).$$

(3)

(*b*) Using the definition of cosh x or sech x in terms of exponentials, show that, for $0 < x \leq 1$,

$$\text{arsech } x = \ln\left(\frac{1+\sqrt{(1-x^2)}}{x}\right).$$

(5)

(*c*) Solve the equation

$$3\tanh^2 x - 4\,\text{sech } x + 1 = 0,$$

giving exact answers in terms of natural logarithms.

(5)

(Total 13 marks)

[FP2/P5 June 2005 Qn 8]

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35. (*a*) (i) Explain why, for any two vectors **a** and **b**, $\mathbf{a.b} \times \mathbf{a} = 0$.

(2)

(ii) Given vectors **a**, **b** and **c** such that $\mathbf{a} \times \mathbf{b} = \mathbf{a} \times \mathbf{c}$, where $\mathbf{a} \neq \mathbf{0}$ and $\mathbf{b} \neq \mathbf{c}$, show that

$$\mathbf{b} - \mathbf{c} = \lambda\mathbf{a}, \quad \text{where } \lambda \text{ is a scalar.}$$

(2)

(*b*) **A**, **B** and **C** are 2×2 matrices.

(i) Given that **AB** = **AC**, and that **A** is not singular, prove that **B** = **C**.

(2)

(ii) Given that **AB** = **AC**, where $\mathbf{A} = \begin{pmatrix} 3 & 6 \\ 1 & 2 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & 5 \\ 0 & 1 \end{pmatrix}$, find a matrix **C** whose elements are all non-zero.

(3)

[FP3/P6 June 2005 Qn 2]

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36. The line l_1 has equation

$$\mathbf{r} = \mathbf{i} + 6\mathbf{j} - \mathbf{k} + \lambda(2\mathbf{i} + 3\mathbf{k})$$

and the line l_2 has equation

$$\mathbf{r} = 3\mathbf{i} + p\mathbf{j} + \mu(\mathbf{i} - 2\mathbf{j} + \mathbf{k}), \text{ where } p \text{ is a constant.}$$

The plane Π_1 contains l_1 and l_2.

(*a*) Find a vector which is normal to Π_1.

(2)

(*b*) Show that an equation for Π_1 is $6x + y - 4z = 16$.

(2)

(*c*) Find the value of p.

(1)

The plane Π_2 has equation $\mathbf{r}.(\mathbf{i} + 2\mathbf{j} + \mathbf{k}) = 2$.

(*d*) Find an equation for the line of intersection of Π_1 and Π_2, giving your answer in the form

$$(\mathbf{r} - \mathbf{a}) \times \mathbf{b} = \mathbf{0}.$$

(5)

[FP3/P6 June 2005 Qn 3]

37.
$$\mathbf{A} = \begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & k \end{pmatrix}.$$

(*a*) Show that det $\mathbf{A} = 20 - 4k$.

(2)

(*b*) Find $\mathbf{A}^{-1}$.

(6)

Given that $k = 3$ and that $\begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}$ is an eigenvector of $\mathbf{A}$,

(*c*) find the corresponding eigenvalue.

(2)

Given that the only other distinct eigenvalue of $\mathbf{A}$ is 8,

(*d*) find a corresponding eigenvector.

(4)

[FP3/P6 June 2005 Qn 7]

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38. Evaluate $\displaystyle\int_1^4 \frac{1}{\sqrt{(x^2 - 2x + 17)}}\,\mathrm{d}x$, giving your answer as an exact logarithm.

(5)

[FP2/P5 January 2006 Qn 1]

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39. The hyperbola H has equation $\frac{x^2}{16} - \frac{y^2}{4} = 1$.

Find

(a) the value of the eccentricity of H,

(2)

(b) the distance between the foci of H.

(2)

The ellipse E has equation $\frac{x^2}{16} + \frac{y^2}{4} = 1$.

(c) Sketch H and E on the same diagram, showing the coordinates of the points where each curve crosses the axes.

(3)

[FP2/P5 January 2006 Qn 2]

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40. A curve is defined by

$$x = t + \sin t, \quad y = 1 - \cos t,$$

where t is a parameter.

Find the length of the curve from $t = 0$ to $t = \frac{\pi}{2}$, giving your answer in surd form.

(7)

[FP2/P5 January 2006 Qn 3]

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41. (a) Using the definition of cosh x in terms of exponentials, prove that

$$4 \cosh^3 x - 3 \cosh x = \cosh 3x.$$

(3)

(b) Hence, or otherwise, solve the equation

$$\cosh 3x = 5 \cosh x,$$

giving your answer as natural logarithms.

(4)

[FP2/P5 January 2006 Qn 4]

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42. Given that

$$I_n = \int_0^4 x^n \sqrt{(4-x)}\ \mathrm{d}x, \qquad n \geq 0,$$

(*a*) show that $I_n = \dfrac{8n}{2n+3} I_{n-1}$, $\quad n \geq 1$.

(6)

Given that $\displaystyle\int_0^4 \sqrt{(4-x)}\ \ \mathrm{d}x = \frac{16}{3}$,

(*b*) use the result in part (*a*) to find the exact value of $\displaystyle\int_0^4 x^2 \sqrt{(4-x)}\ \ \mathrm{d}x$.

(3)

[FP2/P5 January 2006 Qn 7]

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43. (*a*) Show that artanh $\left(\sin \dfrac{\pi}{4}\right)$ = ln (1 + √2).

(3)

(*b*) Given that y = artanh (sin x), show that $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ = sec x.

(2)

(*c*) Find the exact value of $\displaystyle\int_0^{\frac{\pi}{4}} \sin x \ \operatorname{artanh}(\sin x)\ \ \mathrm{d}x$.

(5)

[FP2/P5 January 2006 Qn 8]

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44. A transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ is represented by the matrix

$$\mathbf{A} = \begin{pmatrix} 2 & 2 \\ 2 & -1 \end{pmatrix}, \text{ where } k \text{ is a constant.}$$

Find

(*a*) the two eigenvalues of **A**,

(4)

(*b*) a cartesian equation for each of the two lines passing through the origin which are invariant under T.

(3)

[*FP3/P6 January 2006 Qn 3]

45.
$$\mathbf{A} = \begin{pmatrix} k & 1 & -2 \\ 0 & -1 & k \\ 9 & 1 & 0 \end{pmatrix}, \text{ where } k \text{ is a real constant.}$$

(*a*) Find values of k for which **A** is singular.

(4)

Given that **A** is non-singular,

(*b*) find, in terms of k, $\mathbf{A}^{-1}$.

(5)

[FP3/P6 January 2006 Qn 4]

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46. The plane Π passes through the points

$$P(-1, 3, -2),\ \ Q(4, -1, -1) \text{ and } R(3, 0, c), \text{ where } c \text{ is a constant.}$$

(a) Find, in terms of c, $\overrightarrow{RP} \times \overrightarrow{RQ}$.

(3)

Given that $\overrightarrow{RP} \times \overrightarrow{RQ} = 3\mathbf{i} + d\mathbf{j} + \mathbf{k}$, where d is a constant,

(b) find the value of c and show that $d = 4$,

(2)

(c) find an equation of Π in the form $\mathbf{r.n} = p$, where p is a constant.

(3)

The point S has position vector $\mathbf{i} + 5\mathbf{j} + 10\mathbf{k}$. The point S' is the image of S under reflection in Π.

(d) Find the position vector of S'.

(5)

[FP3/P6 January 2006 Qn 7]

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47. Find the values of x for which

$$5 \cosh x - 2 \sinh x = 11,$$

giving your answers as natural logarithms.

(6)

[FP2 June 2006 Qn 1]

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48. The point S, which lies on the positive x-axis, is a focus of the ellipse with equation $\dfrac{x^2}{4} + y^2 = 1$.

Given that S is also the focus of a parabola P, with vertex at the origin, find

(a) a cartesian equation for P,

(4)

(b) an equation for the directrix of P.

(1)

[FP2 June 2006 Qn 2]

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49. The curve with equation

$$y = -x + \tanh 4x, \quad x \geq 0,$$

has a maximum turning point A.

(a) Find, in exact logarithmic form, the x-coordinate of A.

(4)

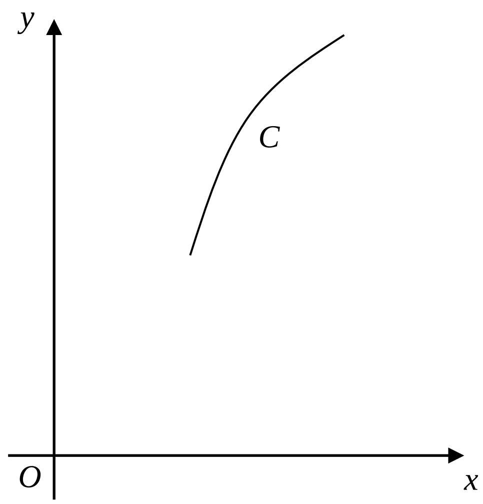
(b) Show that the y-coordinate of A is $\frac{1}{4}\{2\sqrt{3} - \ln(2 + \sqrt{3})\}$.

(3)

[FP2 June 2006 Qn 5]

50.

Figure 1



The curve C, shown in Figure 1, has parametric equations

$$x = t - \ln t,$$
$$y = 4\sqrt{t}, \qquad 1 \leq t \leq 4.$$

(a) Show that the length of C is $3 + \ln 4$.

(7)

The curve is rotated through 2π radians about the x-axis.

(b) Find the exact area of the curved surface generated.

(4)

[FP2 June 2006 Qn 6]

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51. **Figure 2**

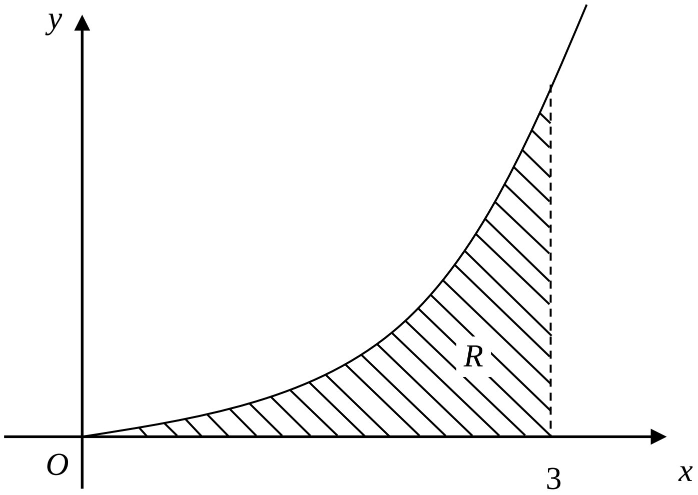

Figure 2 shows a sketch of part of the curve with equation

$$y = x^2 \text{ arsinh } x.$$

The region R, shown shaded in Figure 2, is bounded by the curve, the x-axis and the line $x = 3$.

Show that the area of R is

$$9 \ln (3 + \sqrt{10}) - \tfrac{1}{9}(2 + 7\sqrt{10}).$$

(10)

[FP2 June 2006 Qn 7]

52. $$I_n = \int x^n \cosh x \ \mathrm{d}x, \quad n \geq 0.$$

(*a*) Show that, for $n \geq 2$,

$$I_n = x^n \sinh x - nx^{n-1} \cosh x + n(n-1) I_{n-2}.$$

(4)

(*b*) Hence show that

$$I_4 = \mathrm{f}(x) \sinh x + \mathrm{g}(x) \cosh x + C,$$

where f(x) and g(x) are functions of x to be found, and C is an arbitrary constant.

(5)

(*c*) Find the exact value of $\int_0^1 x^4 \cosh x \ \mathrm{d}x$, giving your answer in terms of e.

(3)

[FP2 June 2006 Qn 8]

53. The ellipse E has equation $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ and the line L has equation $y = mx + c$, where $m > 0$ and $c > 0$.

(*a*) Show that, if L and E have any points of intersection, the x-coordinates of these points are the roots of the equation

$$(b^2 + a^2m^2)x^2 + 2a^2mcx + a^2(c^2 - b^2) = 0.$$

(2)

Hence, given that L is a tangent to E,

(*b*) show that $c^2 = b^2 + a^2m^2$.

(2)

The tangent L meets the negative x-axis at the point A and the positive y-axis at the point B, and O is the origin.

(*c*) Find, in terms of m, a and b, the area of triangle OAB.

(4)

(*d*) Prove that, as m varies, the minimum area of triangle OAB is ab.

(3)

(*e*) Find, in terms of a, the x-coordinate of the point of contact of L and E when the area of triangle OAB is a minimum.

(3)

[FP2 June 2006 Qn 9]

54.

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Prove by induction, that for all positive integers n,

$$\mathbf{A}^n = \begin{pmatrix} 1 & n & \frac{1}{2}(n^2 + 3n) \\ 0 & 1 & n \\ 0 & 0 & 1 \end{pmatrix}.$$

(5)

[FP3 June 2006 Qn 1]

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55. The eigenvalues of the matrix **M**, where

$$\mathbf{M} = \begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix},$$

are λ_1 and λ_2, where $\lambda_1 < \lambda_2$.

(*a*) Find the value of λ_1 and the value of λ_2.

(3)

(*b*) Find $\mathbf{M}^{-1}$.

(2)

(*c*) Verify that the eigenvalues of $\mathbf{M}^{-1}$ are λ_1^{-1} and λ_2^{-1}.

(3)

A transformation $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is represented by the matrix **M**. There are two lines, passing through the origin, each of which is mapped onto itself under the transformation T.

(*d*) Find cartesian equations for each of these lines.

(4)

[FP3 June 2006 Qn 5]

56. The points A, B and C lie on the plane Π_1 and, relative to a fixed origin O, they have position vectors

$$\mathbf{a} = \mathbf{i} + 3\mathbf{j} - \mathbf{k}, \quad \mathbf{b} = 3\mathbf{i} + 3\mathbf{j} - 4\mathbf{k} \quad \text{and} \quad \mathbf{c} = 5\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$$

respectively.

(*a*) Find $(\mathbf{b} - \mathbf{a}) \times (\mathbf{c} - \mathbf{a})$.

(4)

(*b*) Find an equation for Π_1, giving your answer in the form $\mathbf{r.n} = p$.

(2)

The plane Π_2 has cartesian equation $x + z = 3$ and Π_1 and Π_2 intersect in the line l.

(*c*) Find an equation for l, giving your answer in the form $(\mathbf{r} - \mathbf{p}) \times \mathbf{q} = \mathbf{0}$.

(4)

The point P is the point on l that is the nearest to the origin O.

(*d*) Find the coordinates of P.

(4)

[FP3 June 2006 Qn 7]

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57. Evaluate $\displaystyle\int_1^3 \frac{1}{\sqrt{(x^2 + 4x - 5)}}\,dx$, giving your answer as an exact logarithm.

(5)

[FP2 June 2007 Qn 1]

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58. The ellipse D has equation $\dfrac{x^2}{25} + \dfrac{y^2}{9} = 1$ and the ellipse E has equation $\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$.

(*a*) Sketch D and E on the same diagram, showing the coordinates of the points where each curve crosses the axes.

(3)

The point S is a focus of D and the point T is a focus of E.

(*b*) Find the length of ST.

(5)

[FP2 June 2007 Qn 2]

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59. The curve C has equation

$$y = \tfrac{1}{4}(2x^2 - \ln x), \quad x > 0.$$

Find the length of C from $x = 0.5$ to $x = 2$, giving your answer in the form $a + b \ln 2$, where a and b are rational numbers.

(7)

[FP2 June 2007 Qn 3]

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60. (*a*) Starting from the definitions of cosh and sinh in terms of exponentials, prove that

$$\cosh(A - B) = \cosh A \cosh B - \sinh A \sinh B.$$

(3)

(*b*) Hence, or otherwise, given that $\cosh(x - 1) = \sinh x$, show that

$$\tanh x = \frac{e^2 + 1}{e^2 + 2e - 1}.$$

(4)

[FP2 June 2007 Qn 4]

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61. Given that $I_n = \displaystyle\int_0^8 x^n (8 - x)^{\frac{1}{3}} \, dx, \quad n \geq 0,$

(*a*) show that $I_n = \dfrac{24n}{3n + 4} I_{n-1}, \quad n \geq 1.$

(6)

(*b*) Hence find the exact value of $\displaystyle\int_0^8 x(x + 5)(8 - x)^{\frac{1}{3}} \, dx$.

(6)

[FP2 June 2007 Qn 6]

62.

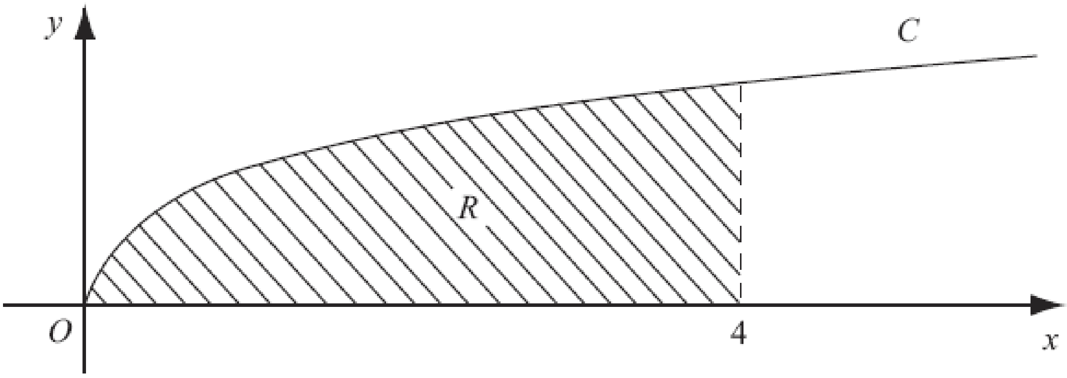

Figure 1

Figure 1 shows part of the curve C with equation $y = \text{arsinh}(\sqrt{x})$, $x \geq 0$.

(a) Find the gradient of C at the point where $x = 4$.

(3)

The region R, shown shaded in Figure 1, is bounded by C, the x-axis and the line $x = 4$.

(b) Using the substitution $x = \sinh^2 \theta$, or otherwise, show that the area of R is

$$k \ln (2 + \sqrt{5}) - \sqrt{5},$$

where k is a constant to be found.

(10)

[FP2 June 2007 Qn 7]

63. Given that $\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$ is an eigenvector of the matrix **A**, where

$$\mathbf{A} = \begin{pmatrix} 3 & 4 & p \\ -1 & q & -4 \\ 1 & 1 & 3 \end{pmatrix},$$

(*a*) find the eigenvalue of **A** corresponding to $\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$,

(2)

(*b*) find the value of *p* and the value of *q*.

(4)

The image of the vector $\begin{pmatrix} l \\ m \\ n \end{pmatrix}$ when transformed by **A** is $\begin{pmatrix} 10 \\ -4 \\ 3 \end{pmatrix}$.

(*c*) Using the values of *p* and *q* from part (*b*), find the values of the constants *l*, *m* and *n*.

(4)

[FP3 June 2007 Qn 3]

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64. The points A, B and C have position vectors, relative to a fixed origin O,

$$\mathbf{a} = 2\mathbf{i} - \mathbf{j},$$

$$\mathbf{b} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k},$$

$$\mathbf{c} = 2\mathbf{i} + 3\mathbf{j} + 2\mathbf{k},$$

respectively. The plane Π passes through A, B and C.

(*a*) Find $\overrightarrow{AB} \times \overrightarrow{AC}$.

(4)

(*b*) Show that a cartesian equation of Π is $3x - y + 2z = 7$.

(2)

The line l has equation $(\mathbf{r} - 5\mathbf{i} - 5\mathbf{j} - 3\mathbf{k}) \times (2\mathbf{i} - \mathbf{j} - 2\mathbf{k}) = \mathbf{0}$. The line l and the plane Π intersect at the point T.

(*c*) Find the coordinates of T.

(5)

(*d*) Show that A, B and T lie on the same straight line.

(3)

[FP3 June 2007 Qn 7]

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65. Show that

$$\frac{\mathrm{d}}{\mathrm{d}x}[\ln(\tanh x)] = 2 \operatorname{cosech} 2x, \quad x > 0.$$

(4)

[FP2 June 2008 Qn 1]

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66. Find the values of x for which

$$8 \cosh x - 4 \sinh x = 13,$$

giving your answers as natural logarithms.

(6)

[FP2 June 2008 Qn 2]

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67. Show that

$$\int_5^6 \frac{3+x}{\sqrt{(x^2-9)}}\,\mathrm{d}x = 3\ln\left(\frac{2+\sqrt{3}}{3}\right) + 3\sqrt{3} - 4.$$

(7)

[FP2 June 2008 Qn 3]

---

68. The curve C has equation

$$y = \text{arsinh}\,(x^3), \qquad x \geq 0.$$

The point P on C has x-coordinate $\sqrt{2}$.

(a) Show that an equation of the tangent to C at P is

$$y = 2x - 2\sqrt{2} + \ln(3 + 2\sqrt{2}).$$

(5)

The tangent to C at the point Q is parallel to the tangent to C at P.

(b) Find the x-coordinate of Q, giving your answer to 2 decimal places.

(5)

[FP2 June 2008 Qn 4]

---

69. Given that

$$I_n = \int_0^{\pi} \mathrm{e}^x \sin^n x \;\mathrm{d}x, \qquad n \geq 0,$$

(a) show that, for $n \geq 2$,

$$I_n = \frac{n(n-1)}{n^2+1} I_{n-2}.$$

(8)

(b) Find the exact value of I_4.

(4)

[FP2 June 2008 Qn 5]

70.

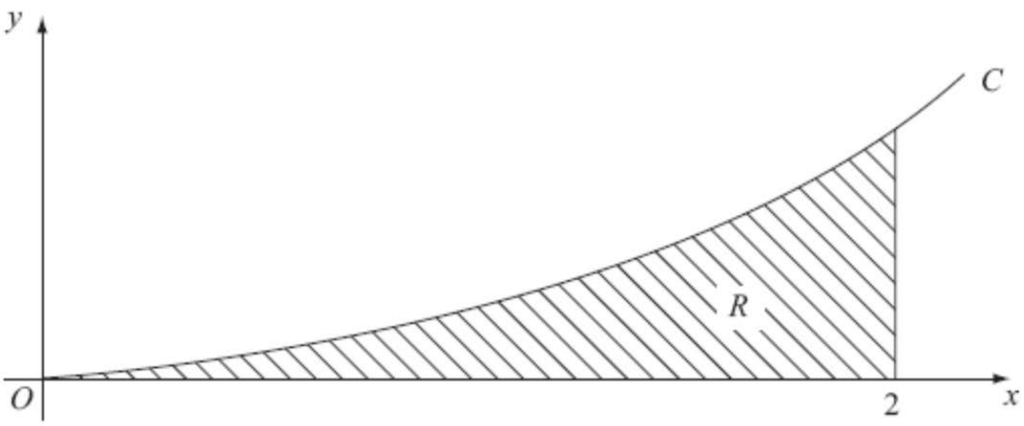

Figure 1

Figure 1 shows the curve C with equation

$$y = \frac{1}{10} \cosh x \arctan(\sinh x), \qquad x \geq 0.$$

The shaded region R is bounded by C, the x-axis and the line $x = 2$.

(a) Find $\int \cosh x \arctan(\sinh x) \, \mathrm{d}x$.

(5)

(b) Hence show that, to 2 significant figures, the area of R is 0.34.

(2)

[FP2 June 2008 Qn 6]

71. The hyperbola H has equation

$$\frac{x^2}{16} - \frac{y^2}{9} = 1.$$

(a) Show that an equation for the normal to H at a point P $(4 \sec t, 3 \tan t)$ is

$$4x \sin t + 3y = 25 \tan t.$$

(6)

The point S, which lies on the positive x-axis, is a focus of H. Given that PS is parallel to the y-axis and that the y-coordinate of P is positive,

(b) find the values of the coordinates of P.

(5)

Given that the normal to H at this point P intersects the x-axis at the point R,

(c) find the area of triangle PRS.

(3)

[FP2 June 2008 Qn 7]

72.

$$\mathbf{M} = \begin{pmatrix} 1 & p & 2 \\ 0 & 3 & q \\ 2 & p & 1 \end{pmatrix},$$

where p and q are constants.

Given that $\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ is an eigenvector of $\mathbf{M}$,

(a) show that $q = 4p$.

(3)

Given also that $\lambda = 5$ is an eigenvalue of $\mathbf{M}$, and $p < 0$ and $q < 0$, find

(b) the values of p and q,

(4)

(c) an eigenvector corresponding to the eigenvalue $\lambda = 5$.

(3)

[FP3 June 2008 Qn 2]

73.

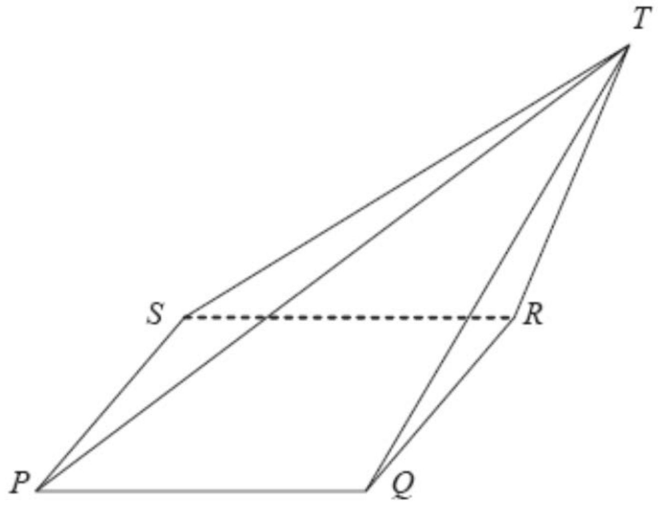

Figure 1

Figure 1 shows a pyramid *PQRST* with base *PQRS*.

The coordinates of *P*, *Q* and *R* are *P* (1, 0, –1), *Q* (2, –1, 1) and *R* (3, –3, 2).

Find

(*a*) $\overrightarrow{PQ} \times \overrightarrow{PR}$

(3)

(*b*) a vector equation for the plane containing the face *PQRS*, giving your answer in the form $\mathbf{r} \cdot \mathbf{n} = d$.

(2)

The plane Π contains the face *PST*. The vector equation of Π is $\mathbf{r} \cdot (\mathbf{i} - 2\mathbf{j} - 5\mathbf{k}) = 6$.

(*c*) Find cartesian equations of the line through *P* and *S*.

(5)

(*d*) Hence show that *PS* is parallel to *QR*.

(2)

Given that *PQRS* is a parallelogram and that *T* has coordinates (5, 2, –1),

(*e*) find the volume of the pyramid *PQRST*.

(3)

[FP3 June 2008 Qn 7]