

FP3 mark schemes from old P4, P5, P6 and FP1, FP2, FP3 papers (back to June 2002)

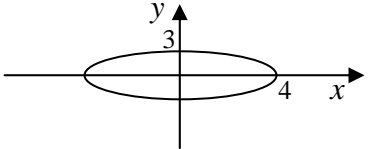
Please note that the following pages contain mark schemes for questions from past papers.

Where a question reference is marked with an asterisk (*), it is a partial version of the original.

The standard of the mark schemes is variable, depending on what we still have – many are scanned, some are handwritten and some are typed.

The questions are available on a separate document, originally sent with this one.

This document was circulated by e-mail in March 2009; the mark schemes for questions 2 and 7 have since been removed (18.3.09) since they are not on the specification.

1.	<p>(a) </p> <p>(b) $b^2 = a^2(1 - e^2) \Rightarrow 9 = 16(1 - e^2)$</p> <p>$e = \frac{\sqrt{7}}{4}$ oe awrt 0.661</p> <p>(c) Foci are at $(\pm ae, 0)$ use of ae</p> <p>$(\sqrt{7}, 0)$ <u>and</u> $(-\sqrt{7}, 0)$ awrt 2.65, 0 is required, ft their e</p>	<p>Closed shape 3, 4</p> <p>B1 (1)</p> <p>M1</p> <p>A1 (2)</p> <p>M1</p> <p>A1 ft (2)</p> <p>(5 marks)</p>
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[P5 June 2002 Qn 1]

3.	$10\left(\frac{e^x + e^{-x}}{2}\right) + 2\left(\frac{e^x - e^{-x}}{2}\right) = 11$ $6e^{2x} - 11e^x + 4 = 0$ $(2e^x - 1)(3e^x - 4) = 0$ $e^x = \frac{1}{2} \text{ and } \frac{4}{3}$ $x = \ln \frac{1}{2} \text{ and } \ln \frac{4}{3}$	<p>M1</p> <p>quadratic in e^x M1, A1</p> <p>M1</p> <p>A1</p> <p>M1, A1</p> <p>(7 marks)</p>
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Alt 3.	$10 \cosh x + 2 \sinh x \equiv R \cosh(x + \alpha)$ $R = \sqrt{96} \text{ and } \tan \alpha = \frac{1}{5}$ $\cosh(x + \alpha) = \frac{11}{\sqrt{96}}$ $x + \alpha = \ln \left[\frac{11}{\sqrt{96}} \pm \sqrt{\left(\frac{121}{96}\right) - 1} \right]$ $= \ln \frac{4}{\sqrt{6}} \text{ and } \ln \frac{\sqrt{6}}{4}$ $x = \ln \frac{4}{\sqrt{6}} - \frac{1}{2} \ln \frac{3}{2}, \ln \frac{\sqrt{6}}{4} - \frac{1}{2} \ln \frac{3}{2}$ $= \ln \left(\frac{4}{\sqrt{6}} - \frac{\sqrt{2}}{\sqrt{3}} \right), \ln \left(\frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{\sqrt{3}} \right)$ $x = \ln \frac{1}{2} \text{ and } \ln \frac{4}{3}$ <p>[One answer by alt. method gains max. M1A1M1M0M1A1A0]</p>	<p>M1, A1</p> <p>either A1</p> <p>both as single ln A1</p> <p>combine either into single ln. M1</p> <p>Dependent on first two Ms</p> <p>A1, A1</p> <p>(11 marks)</p>
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[P5 June 2002 Qn 3]

4.	<p>(a) $\int x^n \cos x \, dx = x^n \sin x - n \int x^{n-1} \sin x \, dx$</p> <p style="text-align: center;">$= \dots - [nx^{n-1} \cos x + \int n(n-1)x^{n-2} \cos x \, dx]$</p> <p>Using limits $I_n = \left(\frac{\pi}{2}\right)^n - n(n-1)I_{n-2}$ (*) cso</p> <p>(b) $I_0 = \int_0^{\frac{\pi}{2}} \cos x \, dx = [\sin x]_0^{\frac{\pi}{2}} = 1$ at any stage</p> <p>$I_6 = \left(\frac{\pi}{2}\right)^6 - 30I_4$</p> <p style="text-align: center;">$= \left(\frac{\pi}{2}\right)^6 - 30\left(\left(\frac{\pi}{2}\right)^6 - 12I_2\right)$</p> <p style="text-align: center;">$= \left(\frac{\pi}{2}\right)^6 - 30\left(\frac{\pi}{2}\right)^4 + 360\left(\frac{\pi}{2}\right)^2 - 720I_0$</p> <p>Hence $I_6 = \left(\frac{\pi}{2}\right)^6 - 30\left(\frac{\pi}{2}\right)^4 + 360\left(\frac{\pi}{2}\right)^2 - 720$ cao</p>	<p>M1, A1</p> <p>M1</p> <p>M1, A1 (5)</p> <p>B1</p> <p>M1</p> <p>M1</p> <p>A1 (4)</p> <p style="text-align: right;">(9 marks)</p>
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[P5 June 2002 Qn 4]

<p>5. (a)</p>	$y = \arctan 3x \Rightarrow \tan y = 3x$ $\sec^2 y \frac{dy}{dx} = 3$ $\frac{dy}{dx} = \frac{3}{1 + \tan^2 y} = \frac{3}{1 + 9x^2} \quad (*)$	<p>M1</p> <p>A1</p> <p>M1, A1 (4)</p>
<p>(b)</p>	$\int 6x \arctan 3x \, dx = 3x^2 \arctan 3x - \int \frac{9x^2}{1+9x^2} dx$ $= \dots \dots - \int \frac{1+9x^2-1}{1+9x^2} dx$ $= \dots \dots - x + \frac{1}{3} \arctan 3x$ $\left[\right]_0^{\frac{\sqrt{3}}{3}} = \frac{\pi}{3} - \frac{\sqrt{3}}{3} + \frac{\pi}{9}$ $= \frac{1}{9} (4\pi - 3\sqrt{3}) \quad (*) \quad \text{CSO}$	<p>M1, A1</p> <p>M1</p> <p>A1</p> <p>M1</p> <p>A1 (6)</p> <p>(10 marks)</p>

[P5 June 2002 Qn 6]

6.	(a)	M1, A1
	$y = 2x^{1/2}, \quad \frac{dy}{dx} = x^{-1/2}$ $\int 2\pi y \left[1 + \left(\frac{dy}{dx} \right)^2 \right] dx = 4\pi \int x^{1/2} \left[1 + \frac{1}{x} \right]^{1/2} dx$ $= 4\pi \int_0^1 \sqrt{1+x} dx \quad (*)$	M1 A1 (4)
	(b)	M1, A1 A1 (3)
	$S = 4\pi \int_0^1 \sqrt{1+x} dx = \left[4\pi \frac{2}{3} (1+x)^{3/2} \right]_{(0)}^{(1)}$ $= \frac{8\pi}{3} (2^{3/2} - 1)$ <p style="text-align: right;">or any exact equivalent</p>	
	(c)	M1 A1 A1 (3)
	$\int \left(1 + \left(\frac{dy}{dx} \right)^2 \right)^{1/2} dx = \int \left(1 + \frac{1}{x} \right)^{1/2} dx$ $\int \sqrt{\frac{x+1}{x}} dx$ <p>Using symmetry, $s = 2 \int_0^1 \sqrt{\frac{x+1}{x}} dx \quad (*)$</p>	
	(d)	oe B1 M1 M1 A1 M1 A1 (6)
	$x = \sinh^2 \theta, \quad \frac{dx}{d\theta} = 2 \sinh \theta \cosh \theta$ $I = 2 \int \sqrt{\frac{1 + \sinh^2 \theta}{\sinh^2 \theta}} \cdot 2 \sinh \theta \cosh \theta d\theta$ $= 4 \int \cosh^2 \theta d\theta$ $= 2 \int (1 + \cosh 2\theta) d\theta$ $= 2\theta + \sinh 2\theta$ <p>Limits are 0 and $\operatorname{arsinh} 1 (= \ln(1 + \sqrt{2}))$</p> $s = \left[2\theta + 2 \sinh \theta \sqrt{1 + \sinh^2 \theta} \right]_0^{\operatorname{arsinh} 1}$ $= 2 \operatorname{arsinh} 1 + 2 \sqrt{(1+1)^2} = 2 \left[\sqrt{2} + \ln(1 + \sqrt{2}) \right]$	

<p>6. (d) Alt</p>	<p>The last four marks can be gained:</p> $I = 4 \int \left(\frac{e^\theta + e^{-\theta}}{2} \right)^2 d\theta = \int (e^{2\theta} + 2 + e^{-2\theta}) d\theta$ $= \frac{e^{2\theta}}{2} + 2\theta - \frac{e^{-2\theta}}{2}$ $s = 2 \operatorname{arsinh} 1 + \frac{1}{2} \left[(1 + \sqrt{2})^2 - \frac{1}{(1 + \sqrt{2})^2} \right]$ $= \dots + \frac{1}{2} \left[1 + 2 + 2\sqrt{2} - \frac{1}{3 + 2\sqrt{2}} \cdot \frac{3 - 2\sqrt{2}}{3 - 2\sqrt{2}} \right]$ $= 2 \ln(1 + \sqrt{2}) + \frac{1}{2}(3 + 2\sqrt{2} - 3 + 2\sqrt{2}) = 2[\sqrt{2} + \ln(1 + \sqrt{2})] \quad (*)$	<p>M1</p> <p>A1</p> <p>M1, A1</p>
<p>6. (d) Alt</p>	<p>The last two marks may be gained by substituting back to the variable x</p> $s = [2\theta + \sinh 2\theta]_{\dots} = [2\theta + 2 \sinh \theta \cosh \theta]_{\dots}$ $= [2 \operatorname{arsinh} \sqrt{x} + 2\sqrt{x}\sqrt{1+x}]_0^1$ $= 2 \operatorname{arsinh} 1 + 2\sqrt{2} = 2 \ln(1 + \sqrt{2}) + 2\sqrt{2}$ $= 2[\sqrt{2} + \ln(1 + \sqrt{2})] \quad (*)$	<p>M1, A1</p>

[P5 June 2002 Qn 8]

<p>8.</p> <p>(a)</p>	$\begin{pmatrix} 1 & 0 & 4 \\ 0 & 5 & 4 \\ 4 & 4 & 3 \end{pmatrix} \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ -6 \\ 3 \end{pmatrix} = 3 \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}, \therefore \text{eigenvalue is } 3$	<p>M1A1, A1 (3)</p>
<p>(b)</p>	<p>Either $\begin{vmatrix} -8 & 0 & 4 \\ 0 & -4 & 4 \\ 4 & 4 & -6 \end{vmatrix} = -8(24 - 16) + 4(16) = -64 + 64 = 0$</p>	<p>M1 A1 (2)</p>
<p>Alt(b)</p>	<p>or $\begin{vmatrix} 1-\lambda & 0 & 4 \\ 0 & 5-\lambda & 4 \\ 4 & 4 & 3-\lambda \end{vmatrix} = 0$</p> <p>$\Rightarrow (1-\lambda)(5-\lambda)(3-\lambda) - 16(1-\lambda) - 16(5-\lambda) = 0$</p> <p>$\Rightarrow (3-\lambda)(\lambda-9)(\lambda+3) = 0 \Rightarrow \lambda$ is an eigenvalue</p>	<p>M1 A1</p>
<p>(c)</p>	<p>$\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ eigenvector $\Rightarrow x + 4z = 9x, 5y + 4z = 9y, 4x + 4y + 3z = 9z$</p> <p>At least two of these equations</p> <p>Attempt to solve $z = 2x, z = y, 2x + 2y = 3z$</p> <p>$\therefore \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$</p> <p>Make e.vectors unit to obtain $\mathbf{P} = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \end{pmatrix}$ columns in any order</p> <p>$\mathbf{D} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$, where $\lambda_3 = -3$, \mathbf{P} and \mathbf{D} consistent</p>	<p>M1</p> <p>A1</p> <p>A1 (3)</p> <p>M1, A1ft</p> <p>M1, A1, B1</p> <p>(5)</p>
<p>Alt</p>	<p>$\mathbf{P} = \begin{pmatrix} 2 & 1 & 2 \\ -2 & 2 & 1 \\ 1 & 2 & -2 \end{pmatrix}, \mathbf{D} = \begin{pmatrix} 27 & 0 & 0 \\ 0 & 81 & 0 \\ 0 & 0 & -27 \end{pmatrix}, \mathbf{P}$ and \mathbf{D} consistent</p>	<p>M1A1ft,</p> <p>M1A1,</p> <p>B1</p>

[P6 June 2002 Qn 5]

<p>9. (a)</p>	$\overrightarrow{AB} = 5\mathbf{i} + 3\mathbf{j} \quad \overrightarrow{AC} = 3\mathbf{i} + 2\mathbf{j} - \mathbf{k} \quad \text{or} \quad \overrightarrow{BC} = -2\mathbf{i} - \mathbf{j} - \mathbf{k}$ $\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 5 & 3 & 0 \\ 3 & 2 & -1 \end{vmatrix} = -3\mathbf{i} + 5\mathbf{j} + \mathbf{k}$ $\therefore \mathbf{r} = \mathbf{i} + 2\mathbf{j} + \mathbf{k} + \lambda(-3\mathbf{i} + 5\mathbf{j} + \mathbf{k})$	<p>M1, A1</p> <p>B1 ft (3)</p>
<p>(b)</p>	$\text{Volume} = \frac{1}{6} \overrightarrow{AD} \cdot (\overrightarrow{AB} \times \overrightarrow{AC}) \quad \overrightarrow{AD} = 2\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$ $= \frac{1}{6} (2\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}) \cdot (-3\mathbf{i} + 5\mathbf{j} + \mathbf{k})$ $= \frac{11}{6}$	<p>B1</p> <p>M1</p> <p>A1 (3)</p>
<p>(c)</p>	$\mathbf{r} \cdot (-3\mathbf{i} + 5\mathbf{j} + \mathbf{k}) = (2\mathbf{i} + \mathbf{j}) \cdot (-3\mathbf{i} + 5\mathbf{j} + \mathbf{k})$ $= -1$	<p>M1, A1 ft</p> <p>A1 (3)</p>
<p>(d)</p>	$[\mathbf{i}(1 - 3\lambda) + \mathbf{j}(2 + 5\lambda) + \mathbf{k}(1 + \lambda)] \cdot (-3\mathbf{i} + 5\mathbf{j} + \mathbf{k}) = -1$ $-3 + 9\lambda + 10 + 25\lambda + 3 + \lambda = -1$ $35\lambda + 10 = -1 \quad \Rightarrow \quad \lambda = -\frac{11}{35}$ $\therefore E \text{ is } \left(\frac{68}{35}, \frac{15}{35}, \frac{94}{35} \right)$	<p>M1, A1ft</p> <p>M1</p> <p>A1 (4)</p>
<p>(e)</p>	$\text{Distance} = -\frac{11}{35} -3i + 5j + k = \frac{11\sqrt{35}}{35} \quad (*)$	<p>M1 A1 (2)</p>
<p>(f)</p>	$\lambda = 2 \times \left(-\frac{11}{35} \right) = -\frac{22}{35}$ $\mathbf{r}_{D'} = \mathbf{i} + 2\mathbf{j} + \mathbf{k} + -\frac{22}{35}(-3\mathbf{i} + 5\mathbf{j} + \mathbf{k})$ $D' \text{ is } \left(\frac{101}{35}, -\frac{40}{35}, \frac{83}{35} \right)$	<p>B1</p> <p>M1</p> <p>A1 (3)</p> <p>(18 marks)</p>

[P6 June 2002 Qn 7]

10.	$4\left(\frac{e^x + e^{-x}}{2}\right) + \frac{e^x - e^{-x}}{2} = 8$ $5e^{2x} - 16e^x + 3 = 0$ $(5e^x - 1)(e^x - 3) = 0$ $e^x = \frac{1}{5}, 3$ $x = \ln\left(\frac{1}{5}\right), \ln 3$	M1 M1 A1 A1 accept – ln 5 M1 A1 (6) (6 marks)
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[P5 June 2003 Qn 1]

11 (a)	$y = \operatorname{artanh} x$ $\tanh y = x$ $\operatorname{sech}^2 y \frac{dy}{dx} = 1$ $\frac{dy}{dx} = \frac{1}{1 - \tanh^2 y} = \frac{1}{1 - x^2} \quad (*)$ <p>cso</p>	M1 A1 A1 (3)
(b)	$\int 1 \cdot \operatorname{artanh} x \, dx = x \operatorname{artanh} x - \int \frac{x}{1 - x^2} \, dx$ $= x \operatorname{artanh} x + \frac{1}{2} \ln(1 - x^2) + c$	M1 A1 M1 A1 (4) (7 marks)

Alt.	$\frac{x}{1 - x^2} \equiv \frac{1}{2} \left[\frac{1}{1 - x} - \frac{1}{1 + x} \right]$ $\int \frac{x}{1 - x^2} \, dx = -\frac{1}{2} \ln(1 - x) - \frac{1}{2} \ln(1 + x)$ <p>This is acceptable (with the rest correct) for final M1 A1</p>	
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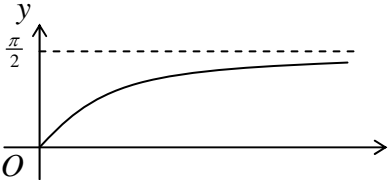
[P5 June 2003 Qn 2]

<p>12.</p>	$\int \frac{10}{\sqrt{4x^2+9}} dx = \frac{1}{2} \int \frac{10}{\sqrt{x^2+\frac{9}{4}}} dx$ $= \frac{10}{2} \operatorname{arsinh}\left(\frac{2x}{3}\right) \left(= 5 \ln \left[\frac{2x}{3} + \sqrt{\frac{4x^2}{9} + 1} \right] \right)$ $\left[\right]_0^5 = 5 \operatorname{arsinh} \frac{10}{3} \left(= 5 \ln \left(\frac{10}{3} + \sqrt{\frac{109}{9}} \right) \approx 9.594 \right)$ <p>ft on 5</p> <p>Area = $9.594 \times 100 = 960 \text{ (m}^2\text{)}$</p>	<p>M1</p> <p>M1 A1</p> <p>M1 A1 ft</p> <p>M1 A1</p> <p>(7 marks)</p>
	<p>Using a substitution</p> <p>(i) $2x = 3 \sinh \theta, \quad 2 dx = 3 \cosh \theta d\theta$</p> $\int \frac{10}{\sqrt{4x^2+9}} dx = \int \frac{10}{3 \cosh \theta} \times \frac{3}{2} \cosh \theta d\theta \quad \text{complete subs.}$ $= 5 \int d\theta = 5 \operatorname{arsinh} \frac{2x}{3}$ <p>then as before,</p> <p>or changing limits to 0 and $\operatorname{arsinh} \frac{10}{3}$ (or $\ln \left(\frac{10}{3} + \sqrt{\frac{109}{9}} \right)$) can gain</p> <p>this A1</p>	<p>M1</p> <p>M1 A1</p>
	<p>(ii) $2x = 3 \tan \theta, \quad 2 dx = 3 \sec^2 \theta d\theta$</p> $\int \frac{10}{\sqrt{4x^2+9}} dx = \int \frac{10}{\sqrt{9 \tan^2 \theta + 9}} \times \frac{3}{2} \sec^2 \theta d\theta$ $= 5 \int \sec \theta d\theta = 5 \ln (\sec \theta + \tan \theta)$ <p>Limits are 0 and $\arctan \frac{10}{3}$</p> $\left[\right]_0^{\arctan \frac{10}{3}} = 5 \ln \left(\sqrt{\frac{100}{9} + 1} + \frac{10}{3} \right) \text{ etc}$	<p>M1</p> <p>M1</p> <p>A1</p> <p>M1 A1ft</p>

[P5 June 2003 Qn 3]

<p>13. (a)</p>	$\frac{dy}{dx} = \sinh \frac{x}{a}$ $\int \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \int \sqrt{1 + \sinh^2\left(\frac{x}{a}\right)} dx = \int \cosh \frac{x}{a} dx = \sinh \frac{x}{a}$ $\text{Length} = 2 \left[a \sinh \frac{x}{a} \right]_0^{ka} = 2a \sinh k \quad (*)$	<p>B1</p> <p>M1, A1</p> <p>M1 A1 (5)</p>
<p>(b)</p>	$2a \sinh k = 8a$ $\sinh k = 4$ $x = ka = a \operatorname{arsinh} 4 = a \ln(4 + \sqrt{17})$ $y = a \cosh \frac{ka}{a} = a \sqrt{1 + \sinh^2 k} = a \sqrt{17}$	<p>B1</p> <p>B1</p> <p>M1 A1 (4)</p> <p>(9 marks)</p>

[P5 June 2003 Qn 5]

<p>14. (a)</p> <p>(b)</p> <p>(c)</p>	<p>$\sec y = e^x$</p> <p>$\sec y \tan y \frac{dy}{dx} = e^x \quad (= \sec y)$</p> <p>$\frac{dy}{dx} = \frac{e^x}{\sec y \tan y} = \frac{1}{\sqrt{\sec^2 y - 1}} = \frac{1}{\sqrt{e^{2x} - 1}} \quad (*)$</p>  <p>Shape, curve $\rightarrow (0, 0)$</p> <p>Asymptote, $(y =) \frac{\pi}{2}$</p> <p>$(x = \ln 2) \quad y = \operatorname{arcsec} 2 = \frac{\pi}{3}$</p> <p>$\frac{dy}{dx} = \frac{1}{\sqrt{4-1}} = \frac{1}{\sqrt{3}}$</p> <p>tangent is $y - \frac{\pi}{3} = \frac{1}{\sqrt{3}}(x - \ln 2)$</p> <p>$x = 0, \quad y = \frac{\pi}{3} - \frac{1}{\sqrt{3}} \ln 2$</p>	<p>B1</p> <p>M1 A1</p> <p>M1 A1 (5)</p> <p>cs0</p> <p>B1</p> <p>B1 (2)</p> <p>B1</p> <p>M1</p> <p>A1 (4)</p> <p>(11 marks)</p>
<p>Alt to (a)</p>	<p>$\cos y = e^{-x}$</p> <p>$-\sin y \frac{dy}{dx} = \frac{e^{-x}}{\sqrt{1 - \cos^2 y}}$</p> <p>$= \frac{e^{-x}}{\sqrt{1 - e^{-2x}}} = \frac{1}{\sqrt{e^{2x} - 1}} \quad (*)$</p>	<p>B1</p> <p>M1 A1</p> <p>M1 A1 (5)</p> <p>cs0</p>

[P5 June 2003 Qn 6]

15.	<p>(a) $I_n = \left[x^n e^x \right]_0^1 - n \int_0^1 x^{n-1} e^x dx = e - nI_{n-1} \quad (*)$</p>	cso	M1 A1 (2)
	<p>(b) $J_n = \left[-x^n e^{-x} \right]_0^1 + n \int_0^1 x^{n-1} e^{-x} dx$ $= -e^{-1} + nJ_{n-1}$</p>		M1 A1 A1 (3)
	<p>(c) $J_2 = -e^{-1} + 2J_1$ $J_1 = -e^{-1} + J_0$ $= -e^{-1} + \int_0^1 e^{-x} dx$ $= -e^{-1} + (1 - e^{-1}) \quad (= 1 - 2e^{-1})$</p>	J_2 and J_1	M1 A1
	<p>$J_2 = -e^{-1} + 2(1 - 2e^{-1}) = 2 - \frac{5}{e} \quad (*)$</p>		A1 (3)
	<p>(d) $\int_0^1 x^n \cosh x dx = \int_0^1 x^n \left(\frac{e^x + e^{-x}}{2} \right) dx = \frac{1}{2} (I_n + J_n) \quad (*)$</p>		B1 (1)
	<p>(e) $I_2 = e - 2I_1 = e - 2(e - I_0) = 2I_0 - e$ $= 2 \int_0^1 e^x dx - e = 2[e - 1] - e \quad (= e - 2)$</p>		M1 A1
	<p>$\frac{1}{2} (I_2 + J_2) = \frac{1}{2} (e - 2 + 2 - \frac{5}{e}) = \frac{1}{2} (e - \frac{5}{e})$</p>		M1 A1 (4) (13 marks)

[P5 June 2003 Qn 7]

<p>16. (a)</p> $\frac{dx}{dt} = a \sec t \tan t, \quad \frac{dy}{dt} = b \sec^2 t$ $\frac{dy}{dx} = \frac{b \sec^2 t}{a \sec t \tan t} \left(= \frac{b}{a \sin t} \right)$ <p>gradient of normal is $-\frac{a \sin t}{b}$</p> $y - b \tan t = -\frac{a \sin t}{b} (x - a \sec t)$ $ax \sin t + by = (a^2 + b^2) \tan t \quad (*)$ <p>(b)</p> $y = 0 \Rightarrow x = \frac{(a^2 + b^2) \tan t}{a \sin t} \left(= \frac{a^2 + b^2}{a \cos t} \right)$ $b^2 = a^2(e^2 - 1) \Rightarrow b^2 = \frac{5a^2}{4}$ <p>$OS = ae$ and $OA = 3AS$</p> $a^2 + \frac{5a^2}{4} = 3a^2 \times \frac{3}{2} \times \cos t$ $\cos t = \frac{1}{2}$ $t = \frac{\pi}{3}, \frac{5\pi}{3}$ <p>By symmetry or (as $OA = \left \frac{a^2 + b^2}{a \cos t} \right - \frac{a^2 + b^2}{a \cos t} = 3ae$)</p> $t = \frac{2\pi}{3}, \frac{4\pi}{3}$	<p>M1 A1</p> <p>M1 A1</p> <p>M1</p> <p>cso A1 (6)</p> <p>B1</p> <p>M1</p> <p>M1</p> <p>M1 A1</p> <p>A1</p> <p>M1 A1 (8)</p> <p>(14 marks)</p>	
<p>Alt. to (a)</p>	$\frac{2x}{a^2} - \frac{2y}{b^2} \frac{dy}{dx} = 0$ $\frac{dy}{dx} = \frac{2b^2 x}{2a^2 y} = \frac{b^2 a \sec t}{a^2 b \tan t} \dots$ <p>then as before</p>	<p>M1 A1</p> <p>M1 A1</p>

[P5 June 2003 Qn 8]

17.	$\overrightarrow{AB} \times \overrightarrow{AC} = (\mathbf{b} - \mathbf{a}) \times (\mathbf{c} - \mathbf{a}) = (\mathbf{b} \times \mathbf{c}) - (\mathbf{a} \times \mathbf{c}) - (\mathbf{b} \times \mathbf{a}) + (\mathbf{a} \times \mathbf{a})$ <p>Using $\mathbf{a} \times \mathbf{c} = -\mathbf{c} \times \mathbf{a}$ or $\mathbf{b} \times \mathbf{a} = -\mathbf{a} \times \mathbf{b}$ or $\mathbf{a} \times \mathbf{a} = \mathbf{0}$</p> $ \overrightarrow{AB} \times \overrightarrow{AC} = AB \cdot AC \sin \theta = 2 \times \text{area of triangle, or equivalent}$ <p>Final result : $\frac{1}{2} \mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{c} + \mathbf{c} \times \mathbf{a}$ (*)</p>	M1 A1 B1 M1 A1 cso [5]
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[P6 June 2003 Qn 1]

18.	<p>(a) Deriving characteristic equation $(4 - \lambda)(-9 - \lambda) + 30 = 0$</p> $\Rightarrow \lambda^2 + 5\lambda - 6 = 0 \Rightarrow (\lambda + 6)(\lambda - 1) = 0 \Rightarrow \lambda = -6, \lambda = 1$ <p>(b) Stating, implying or showing $\lambda = 1$ associated with point invariant line.</p> $\Rightarrow \begin{pmatrix} 4 & -5 \\ 6 & -9 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$ <p>Equation is $4x - 5y = x \Rightarrow 3x - 5y = 0$ any equivalent form</p>	M1 A1 M1 A1 (4) B1 M1 A1 (3) [7]
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[P6 June 2003 Qn 3]

<p>19.</p>	<p>(a) $\text{Det } \mathbf{A} = 3(u-3) - (u-5) - (3-5) = 2u-2 \quad [= 2(u-1)] \quad (*)$</p> <p>(b) Cofactors $\begin{pmatrix} u-3 & 5-u & -2 \\ -(u+3) & 3u+5 & -4 \\ 2 & -4 & 2 \end{pmatrix}$ $(-1 \text{ A mark for each term wrong})$</p> $\mathbf{A}^{-1} = \frac{1}{2(u-1)} \begin{pmatrix} u-3 & -(u+3) & 2 \\ 5-u & 3u+5 & -4 \\ -2 & -4 & 2 \end{pmatrix}$ <p>(c) $\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \mathbf{A}^{-1} \begin{pmatrix} 3 \\ 1 \\ 6 \end{pmatrix}$, Using $u = 6$: $\frac{1}{10} \begin{pmatrix} 3 & -9 & 2 \\ -1 & 23 & -4 \\ -2 & -4 & 2 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \\ 6 \end{pmatrix} = \frac{1}{10} \begin{pmatrix} 12 \\ -4 \\ 2 \end{pmatrix}$</p> <p>$a = \frac{6}{5}, \quad b = -\frac{2}{5}, \quad c = \frac{1}{5}$ $(\text{One correct A1, other 2 correct A1})$</p> <p>[Algebraic approach: Finding one value M1 A1, other two A1]</p>	<p>M1 A1 (2)</p> <p>M1 A3</p> <p>M1 A1 ft (6)</p> <p>M1</p> <p>A1, A1 (3)</p> <p>[11]</p>
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[P6 June 2003 Qn 6]

<p>20.</p>	<p>(a) Normal to plane is $(-\mathbf{i} + 5\mathbf{j} + 3\mathbf{k})$</p> <p>Equation of plane: $\mathbf{r} \cdot (-\mathbf{i} + 5\mathbf{j} + 3\mathbf{k}) = (\mathbf{i} + 2\mathbf{j} - \mathbf{k}) \cdot (-\mathbf{i} + 5\mathbf{j} + 3\mathbf{k})$</p> $\Rightarrow -x + 5y + 3z = -1 + 10 - 3 = 6 \text{ or equivalent} \quad (*)$ <p>[If vector equation of plane is by-passed, then B1 M2 A1]</p> <p>(b) $\frac{1}{\sqrt{35}}$</p> $ 6 - (\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}) \cdot (-\mathbf{i} + 5\mathbf{j} + 3\mathbf{k}) $ <p>or $\overrightarrow{PQ} \cdot (-\mathbf{i} + 5\mathbf{j} + 3\mathbf{k}) = 3\mathbf{k} \cdot (-\mathbf{i} + 5\mathbf{j} + 3\mathbf{k})$</p> $\text{Distance} = \frac{9}{\sqrt{35}} \text{ or a.w.r.t 1.52}$ <p>(c) Direction of one line in plane = $(-\mathbf{i} + 5\mathbf{j} + 3\mathbf{k})$</p> <p>Direction of another line in plane = $(3\mathbf{i} - 2\mathbf{k}) - (\mathbf{i} + 2\mathbf{j} + 2\mathbf{k})$</p> $\Rightarrow \mathbf{r} = (\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}) + s(-\mathbf{i} + 5\mathbf{j} + 3\mathbf{k}) + t(2\mathbf{i} - 2\mathbf{j} - 4\mathbf{k})$ <p>or $(3\mathbf{i} - 2\mathbf{k}) + s(-\mathbf{i} + 5\mathbf{j} + 3\mathbf{k}) + t(2\mathbf{i} - 2\mathbf{j} - 4\mathbf{k})$</p>	<p>B1</p> <p>M1</p> <p>M1 A1 (4)</p> <p>B1</p> <p>M1 A1</p> <p>A1 (4)</p> <p>M1</p> <p>M1</p> <p>M1 A1 (4)</p> <p>[12]</p>
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[P6 June 2003 Qn 7]

<p>21.</p>	<p>(a)</p> $\begin{aligned} \cosh^2 x - \sinh^2 x &= \frac{1}{4}(e^x + e^{-x})^2 - \frac{1}{4}(e^x - e^{-x})^2 \\ &= \frac{1}{4}(e^{2x} + 2 + e^{-2x} - e^{2x} + 2 - e^{-2x}) \\ &= 1 \quad * \end{aligned}$ <p>(b)</p> $\frac{1}{\sinh x} - 2 \frac{\cosh x}{\sinh x} = 2 \quad \therefore 1 - (e^x + e^{-x}) = e^x - e^{-x}$ $\therefore 2e^x = 1$ <p>Make x the subject of the formula, $x = \ln\left(\frac{1}{2}\right) = -\ln 2$</p>	<p>M1</p> <p>A1</p> <p>A1 (3)</p> <p>M1</p> <p>A1</p> <p>M1, A1 (4)</p>
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[P5 June 2004 Qn 1]

<p>22. (a)</p> <p>(b)</p>	<p>$a = 2, \quad b = 1, \quad c = 16$</p> $\int_{-0.5}^{1.5} \frac{1}{(2x+1)^2 + 16} dx$ $= \left[\frac{1}{8} \arctan \left(\frac{2x+1}{4} \right) \right]_{-0.5}^{1.5}$ $= \frac{\pi}{32}$	<p>B1, B1, B1 (3)</p> <p>M1</p> <p>M1 A1</p> <p>B1 (4)</p>
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[P5 June 2004 Qn 2]

<p>23. (a)</p> <p>(b)</p>	<p>As $4 = 9(1 - e^2)$, $\therefore e^2 = \frac{5}{9}$</p> <p>Uses ae to obtain that the foci are at $(\pm\sqrt{5}, 0)$</p> $PS + PS' = e(PM + PM')$ $= e \times \frac{2a}{e}$ $= 2a = 6$	<p>M1, A1</p> <p>M1 A1 (4)</p> <p>M1 for single statement e.g. $PS = ePM$</p> <p>M1 needs complete method</p> <p>M1</p> <p>M1</p> <p>A1 (3)</p>
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[P5 June 2004 Qn 3]

<p>24.</p>	<p>(a) Using product rule $\frac{dy}{dx} = (n-1)\sinh^{n-2} x \cosh^2 x + \sinh^n x$</p> <p>Using $\cosh^2 x = 1 + \sinh^2 x$ in derived expression</p> <p>to obtain $\frac{dy}{dx} = (n-1)\sinh^{n-2} x(1 + \sinh^2 x) + \sinh^n x$</p> <p>and $\frac{dy}{dx} = (n-1)\sinh^{n-2} x + n\sinh^n x$ *</p> <p>(b) $[\sinh^{n-1} x \cosh x]_0^{\operatorname{arsinh} 1} = \int_0^{\operatorname{arsinh} 1} (n-1)\sinh^{n-2} x dx + \int_0^{\operatorname{arsinh} 1} n\sinh^n x dx$</p> <p>So $\cosh(\operatorname{arsinh} 1) = (n-1)I_{n-2} + nI_n$</p> <p>If $\sinh \alpha = 1$ then $\cosh \alpha = \sqrt{1 + \sinh^2 \alpha} = \sqrt{2}$</p> <p>$\therefore nI_n = \sqrt{2} - (n-1)I_{n-2}$ *</p> <p>OR</p> <p>$\int_0^{\operatorname{arsinh} 1} \sinh^{n-1} x \sinh x dx = [\sinh^{n-1} x \cosh x]_0^{\operatorname{arsinh} 1} - (n-1) \int_0^{\operatorname{arsinh} 1} \cosh^2 x \sinh^{n-2} x dx$</p> <p>and use $\cosh^2 x = 1 + \sinh^2 x$</p> <p>collect $I_n + (n-1)I_n$ to obtain $nI_n = \sqrt{2} - (n-1)I_{n-2}$ *</p> <p>(c) $I_0 = \operatorname{arsinh} 1$</p> <p>$2I_2 = \sqrt{2} - I_0$</p> <p>$4I_4 = \sqrt{2} - 3I_2$ and use with previous results to obtain...</p> <p>$= \frac{1}{8}(3\operatorname{arsinh} 1 - \sqrt{2}) = 0.154$ (either answer acceptable)</p>	<p>M1</p> <p>M1</p> <p>A1 (3)</p> <p>M1</p> <p>A1 (2)</p> <p>M1</p> <p>A1 (2)</p> <p>B1</p> <p>M1</p> <p>M1</p> <p>A1 (4)</p>
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[P5 June 2004 Qn 5]

<p>25. (a)</p>	$\frac{dx}{d\theta} = -3a \cos^2 \theta \sin \theta \quad \frac{dy}{d\theta} = 3a \sin^2 \theta \cos \theta$ $s = \int \sqrt{(9a^2(c^4 s^2 + s^4 c^2))} d\theta$ $= 3a \int \sqrt{c^2 s^2} d\theta$ $= 3a \int \cos \theta \sin \theta d\theta$ <p>Total length = $4 \times \frac{3a}{2} [\sin^2 \theta]_0^{\frac{\pi}{2}}$</p> $= 6a$	<p>B1</p> <p>M1</p> <p>M1</p> <p>A1</p> <p>M1</p> <p>M1 A1 (7)</p>
<p>(b)</p>	$A = 2\pi \int a \sin^3 \theta \times 3a \cos \theta \sin \theta d\theta$ $= 6\pi a^2 \int \sin^4 \theta \cos \theta d\theta$ $= \frac{6\pi a^2}{5} [\sin^5 \theta]_0^{\frac{\pi}{2}} \times 2$ $= \frac{12\pi a^2}{5}$	<p>M1 A1</p> <p>M1</p> <p>M1 A1 (5)</p>

[P5 June 2004 Qn 7]

26. (a) $\vec{AB} = \begin{pmatrix} -4 \\ 3 \\ -4 \end{pmatrix}$ $\vec{AC} = \begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix}$ M1
- $\vec{AB} \times \vec{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -4 & 3 & -4 \\ 2 & -2 & 3 \end{vmatrix} = \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix}$ A1: One value correct, A1: All correct M1 A1 A1 (4)
- (b) $\mathbf{r} \cdot \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix} = 3 - 4 + 8$ $\mathbf{r} \cdot \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix} = 7$ M1 A1ft (2)
- (c) $\vec{AD} \cdot \vec{AB} \times \vec{AC}$ (Attempt suitable triple scalar product) M1
- $\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$ or $\begin{pmatrix} -2 \\ -3 \\ 1 \end{pmatrix}$ (if using AD) B1
- Volume = $\frac{1}{6} \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix} = \frac{1}{6}(2 + 12 - 2) = 2$ M1 A1(cso)(4)

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[P6 June 2004 Qn 3]

27.
$$\mathbf{MM}^T = \begin{pmatrix} 1 & 4 & -1 \\ 3 & 0 & p \\ a & b & c \end{pmatrix} \begin{pmatrix} 1 & 3 & a \\ 4 & 0 & b \\ -1 & p & c \end{pmatrix} = \begin{pmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{pmatrix}$$

(a)
$$\begin{pmatrix} 1 & 4 & -1 \end{pmatrix} \begin{pmatrix} 3 \\ 0 \\ p \end{pmatrix} = 0 \Rightarrow p = 3$$
 M1 A1 (2)

(b)
$$\begin{pmatrix} 1 & 4 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 4 \\ -1 \end{pmatrix} = k \Rightarrow k = 18$$
 (ft on their p , if used) M1 A1ft (2)

(c) 2 equations: $a + 4b - c = 0$ $3a + 3c = 0$ M1

a and b in terms of c (or equiv.): $a = -c$ $b = \frac{1}{2}c$ (ft on their p) M1 A1ft

Using $\begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 18$ ($a^2 + b^2 + c^2 = 18$): $a = 2\sqrt{2}$, $b = -\sqrt{2}$, $c = -2\sqrt{2}$ M1 A2(1,0) (6)

(d) $|\det \mathbf{M}| = |(3\sqrt{2}) - 4(-12\sqrt{2}) - 1(-3\sqrt{2})| = 54\sqrt{2}$ M1 A1(cso) (2)

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Alternatives:

(c) Require $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ parallel to $\begin{pmatrix} 1 \\ 4 \\ -1 \end{pmatrix} \times \begin{pmatrix} 3 \\ 0 \\ 3 \end{pmatrix}$, $= \begin{pmatrix} 12 \\ -6 \\ -12 \end{pmatrix}$ M1, M1 A1

(Then as in main scheme, scaling to give a , b and c .) M1 A2(1,0) (6)

(d) $\det(\mathbf{MM}^T) = 18^3$, $\det \mathbf{M} = \det \mathbf{M}^T$, $|\det \mathbf{M}| = 18\sqrt{18} (=54\sqrt{2})$ M1 A1 (2)

[P6 June 2004 Qn 5]

28. (a) $\det \mathbf{A} = 0 \quad (3 - \lambda)^2 - 1 = 0$ M1
- $\lambda^2 - 6\lambda + 8 = 0 \quad (\lambda - 2)(\lambda - 4) = 0 \quad \lambda = 2, \lambda = 4$ A1
- $\lambda = 2: \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0, \quad x + y = 0, \quad \text{Eigenvector} \begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad (\text{or equiv.})$ M1 A1
- $\lambda = 4: \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0, \quad -x + y = 0, \quad \text{Eigenvector} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (\text{or equiv.})$ A1 (5)
- (b) $\mathbf{P} = k \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \quad \mathbf{M}: \text{eigenvectors as columns, } k = \frac{1}{\sqrt{2}}$ M1, A1
- $\left\{ \mathbf{P}^{-1} = \mathbf{P}^T = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \right\}$
- $\mathbf{D} = \mathbf{P}^{-1} \mathbf{A} \mathbf{P} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix}$ M1, M1 A1 (5)
- (c) 1. Rotation of $\frac{\pi}{4}$ clockwise (about (0, 0)).
2. Stretch, $\times 4$ parallel to x -axis, $\times 2$ parallel to y -axis.
3. Rotation of $\frac{\pi}{4}$ anticlockwise (about (0, 0)).
1. and 3. both rotation, or both reflection. M1
- Correct angles, opposite sense or correct lines (reflection). A1
- Stretch. B1
- All correct, including order. A1 (4)

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[P6 June 2004 Qn 6]

29.	<p>(a) $\int \frac{1+x}{\sqrt{1-4x^2}} dx = \int \frac{1}{\sqrt{1-4x^2}} dx + \int \frac{x}{\sqrt{1-4x^2}} dx$</p> $\frac{1}{2} \arcsin 2x + 2 \times \frac{-1}{8} \sqrt{1-4x^2}$ $= \frac{1}{2} \arcsin 2x - \frac{1}{4} \sqrt{1-4x^2} (+C)$	<p>M1</p> <p>M1 A1, M1</p> <p>A1 (5)</p>
	<p><u>Alternative</u> Let $x = \frac{1}{2} \sin \theta$, $\int \frac{1+0.5 \sin \theta}{\cos \theta} \times \frac{1}{2} \cos \theta d\theta$ M1</p> $= \frac{1}{2} \theta - \frac{1}{4} \cos \theta (+C), = \frac{1}{2} \arcsin 2x - \frac{1}{4} \sqrt{1-4x^2} (+C)$ M1 A1, M1 A1	
	<p>(b) $\int_0^{0.3} \frac{1+x}{\sqrt{1-4x^2}} dx = 0.5 \arcsin 0.6 - 0.25 \sqrt{0.64} + 0.25 = 0.372$</p>	<p>M1 A1 (2)</p> <p>(7)</p>

[FP2/P5 June 2005 Qn 1]

30.	<p>(a) $\cosh 2x = \frac{e^{2x} + e^{-2x}}{2} = \frac{e^{2 \ln k} + e^{-2 \ln k}}{2}$ (or use $e^x = k$)</p> $= \frac{k^2 + k^{-2}}{2} = \frac{k^4 + 1}{2k^2}$ (*)	<p>M1</p> <p>M1 A1 (3)</p>
	<p>(b) $f^{-1}(x) = p - 2 \operatorname{sech}^2 2x$</p>	<p>M1 A1</p>
	<p>For $x = \ln 2$, $\cosh 2x = \frac{2^4 + 1}{2 \times 2^2} = \frac{17}{8}$</p>	<p>B1</p>
	<p>$p - \frac{2}{\cosh^2 2x} = 0$, $p = 2 \times \frac{64}{289} = \frac{128}{289}$</p>	<p>A1 (4)</p> <p>(7)</p>

[FP2/P5 June 2005 Qn 2]

31.	$\frac{dx}{dt} = -3a \cos^2 t \sin t \quad \frac{dy}{dt} = 3a \sin^2 t \cos t$ $\text{Area} = 2\pi \int a \sin^3 t \sqrt{9a^2 (\cos^4 t \sin^2 t + \sin^4 t \cos^2 t)} dt$ $= 6\pi a^2 \int \sin^3 t \sin t \cos t dt = 6\pi a^2 \left[\frac{\sin^5 t}{5} \right]_0^{\pi/2} = \frac{6\pi a^2}{5}$	M1 A1 M1 A1 M1 A1 A1 (7) (7)
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[FP2/P5 June 2005 Qn 3]

32.	<p>(a) $I_n = \frac{1}{2} x^n e^{2x} - \frac{n}{2} \int x^{n-1} e^{2x} dx, \quad I_n = \frac{1}{2} (x^n e^{2x} - n I_{n-1})$ (*)</p> <p>(b) $\int_0^1 x^2 e^{2x} dx = I_2 = \left[\frac{1}{2} x^2 e^{2x} \right]_0^1 - I_1 = \frac{1}{2} e^2 - I_1$</p> <p>$I_1 = \left[\frac{1}{2} x e^{2x} \right]_0^1 - \frac{I_0}{2} = \frac{1}{2} e^2 - \frac{1}{2} I_0$</p> <p>$I_0 = \int_0^1 e^{2x} dx = \left[\frac{e^{2x}}{2} \right]_0^1 = \frac{e^2}{2} - \frac{1}{2}$</p> <p>$I_2 = \frac{e^2}{2} - \left(\frac{e^2}{2} - \left(\frac{e^2}{4} - \frac{1}{4} \right) \right) = \frac{1}{4} (e^2 - 1)$</p>	M1 A1 A1 (3) <i>one correct stat</i> M1 <i>linking all three</i> M1 A1 <i>use limit</i> M1 A1 (5) (8)
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[FP2/P5 June 2005 Qn 4]

33.	$\int x \operatorname{arcosh} x dx = \frac{x^2}{2} \operatorname{arcosh} x - \int \frac{x^2}{2\sqrt{x^2-1}} dx$ $\left[\frac{x^2}{2} \operatorname{arcosh} x \right]_1^2 = 2 \operatorname{arcosh} 2$ <p>Let $x = \cosh \theta$ $\int \frac{\cosh^2 \theta}{2 \sinh \theta} \sinh \theta d\theta$</p> $= \int \frac{\cosh^2 \theta}{2} d\theta = \int \frac{1 + \cosh 2\theta}{4} d\theta = \frac{\theta}{4} + \frac{\sinh 2\theta}{8}$ $= \left[\frac{\theta}{4} + \frac{\sinh 2\theta}{8} \right]_0^{\operatorname{arcosh} 2} = \frac{1}{4} \operatorname{arcosh} 2 + \frac{2 \times \sqrt{3} \times 2}{8}$ $\text{Area} = \frac{7}{4} \operatorname{arcosh} 2 - \frac{\sqrt{3}}{2} = \frac{7}{4} \ln(2 + \sqrt{3}) - \frac{\sqrt{3}}{2}$	M1 A1 M1 A1 M1, M1 A1 limits M1 A1 (*) A1 (10) (10)
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[FP2/P5 June 2005 Qn 6]

34.	<p>(a) $\ln\left(\frac{1-\sqrt{1-x^2}}{x}\right) = \ln\left(\frac{1-\sqrt{1-x^2}}{x} \times \frac{1+\sqrt{1-x^2}}{1+\sqrt{1-x^2}}\right)$</p> $= \ln\left(\frac{1-(1-x^2)}{x(1+\sqrt{1-x^2})}\right) = -\ln\left(\frac{1+\sqrt{1-x^2}}{x}\right)$	M1	(*)	M1 A1	(3)
	<p>(b) Let $y = \operatorname{arsech} x$ $\operatorname{sech} y = \frac{2}{e^y + e^{-y}}$</p>	B1			
	$xe^y + xe^{-y} = 2 \quad xe^{2y} - 2e^y + x = 0$	M1			
	$e^y = \frac{2 \pm \sqrt{4-4x^2}}{2x} = \frac{1 \pm \sqrt{1-x^2}}{x}$	M1 A1			
	$y = \ln\left(\frac{1 \pm \sqrt{1-x^2}}{x}\right) = (\pm) \ln\left(\frac{1 + \sqrt{1-x^2}}{x}\right)$		(*)	A1	(5)
	<p>(c) $3(1 - \operatorname{sech}^2 x) - 4\operatorname{sech} x + 1 = 0$</p>	M1			
	$(3\operatorname{sech} x - 2)(\operatorname{sech} x + 2) = 0 \quad \operatorname{sech} x = \frac{2}{3}$	M1 A1			
	$x = \pm \ln\left(\frac{3}{2}\left(1 + \sqrt{\frac{5}{9}}\right)\right) = \pm \ln\left(\frac{3 + \sqrt{5}}{2}\right)$	M1 A1			(5)
					(13)

[FP2/P5 June 2005 Qn 8]

35.	<p>(a) (i) $\mathbf{b} \times \mathbf{a}$ is perpendicular to \mathbf{a} (and \mathbf{b})</p> <p>$\mathbf{a} \cdot \mathbf{b} \times \mathbf{a} = \mathbf{a} \mathbf{b} \times \mathbf{a} \cos 90^\circ = 0$ or equivalent</p> <p>(ii) $\mathbf{a} \times \mathbf{b} = \mathbf{a} \times \mathbf{c} \Rightarrow \mathbf{a} \times (\mathbf{b} - \mathbf{c}) = \mathbf{0}$</p> <p>As $\mathbf{a} \neq \mathbf{0}$ and $\mathbf{b} \neq \mathbf{c}$,</p> <p>$\mathbf{a}$ is parallel to $(\mathbf{b} - \mathbf{c})$, so $\mathbf{b} - \mathbf{c} = \lambda \mathbf{a}$</p> <p>(b) (i) If \mathbf{A} non-singular, then $\mathbf{A}^{-1} \mathbf{A} \mathbf{B} = \mathbf{A}^{-1} \mathbf{A} \mathbf{C} \Rightarrow \mathbf{B} = \mathbf{C}$ (*)AG</p> <p>(ii) $\begin{pmatrix} 3 & 6 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 5 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 21 \\ 1 & 7 \end{pmatrix}$</p> <p>Set $\begin{pmatrix} 3 & 6 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 3 & 21 \\ 1 & 7 \end{pmatrix}$ and finding two equations</p> <p>Any non-zero values of a, b, c and d such that $a + 2c = 1$ and $b + 2d = 7$.</p>	<p>B1</p> <p>B1 (2)</p> <p>M1</p> <p>A1 (2)</p> <p>M1A1 (2)</p> <p>B1</p> <p>M1</p> <p>A1 (3)</p> <p>[9]</p>
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[FP3/P6 June 2005 Qn 2]

36.	<p>(a) Normal to plane is $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 0 & 3 \\ 1 & -2 & 1 \end{vmatrix} = 6\mathbf{i} + \mathbf{j} - 4\mathbf{k}$ (or any multiple)</p> <p>(b) Equation of plane is $6x + y - 4z = d$</p> <p>Substituting appropriate point in equation to give $6x + y - 4z = 16$ [e.g. (1, 6, -1), (3, -2, 0), (3, 6, 2) etc.]</p> <p>(c) $p = -2$</p> <p>(d) Direction of line is perpendicular to both normals</p> $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 1 \\ 6 & 1 & -4 \end{vmatrix} = -9\mathbf{i} + 10\mathbf{j} - 11\mathbf{k}$ <p>[Planes are: $6x + y - 4z = 16$, $x + 2y + z = 2$]</p> <p>Finding a point on line</p> <p>a and b identified</p> <p>Any correct equation of correct form e.g. $\left[r - \begin{pmatrix} -3 \\ 6 \\ -7 \end{pmatrix} \right] \times \begin{pmatrix} 9 \\ -10 \\ 11 \end{pmatrix} = \mathbf{0}$.</p> <p><i>Alternative: Using equations of planes to find general point on line</i></p> <p>Using equations of planes to form any two of $10x + 9y = 24$, $11x - 9z = 30$, $11y + 10z = -4$ M1 Putting in parametric form M1</p> <p>e.g. $\left(\lambda, \frac{24 - 10\lambda}{9}, \frac{-30 + 11\lambda}{9} \right)$ A1</p> <p>a and b identified M1 Writing in required form; a correct equation A1</p>	<p>M1A1 (2)</p> <p>M1</p> <p>A1 (2)</p> <p>B1 (1)</p> <p>M1</p> <p>M1A1</p> <p>M1</p> <p>A1 (5) [10]</p>
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[FP3/P6 June 2005 Qn 3]

37.

(a) $\text{Det} = -12 - 2(2k - 8) + 16 = 20 - 4k$ (*) AG

M1A1 (2)

(b) Cofactors $\begin{pmatrix} -4 & 8 - 2k & 4 \\ 8 - 2k & 3k - 16 & 2 \\ 4 & 2 & -4 \end{pmatrix}$ [A1 each error]

M1A3

$$\mathbf{A}^{-1} = \frac{1}{20 - 4k} \begin{pmatrix} -4 & 8 - 2k & 4 \\ 8 - 2k & 3k - 16 & 2 \\ 4 & 2 & -4 \end{pmatrix}$$

M1A1√(6)

(c) Setting $\begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}$

M1

$$\lambda = -1$$

A1 (2)

(d) Forming equations in x , y and z : $\begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 8 \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

M1

$$-5x + 2y + 4z = 0, \quad 2x + 2z = 8y, \quad 4x + 2y - 5z = 0$$

A1

Establishing ratio $x : y : z$: $[x = 2y, x = z]$

Eigenvector $(k) \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$

M1

A1 (4)

[14]

[FP3/P6 June 2005 Qn 7]

38.	$x^2 - 2x + 17 = (x - 1)^2 + 16$ $I = \int_1^4 \frac{1}{\sqrt{(x-1)^2 + 16}} dx = \left[\operatorname{arsinh} \frac{(x-1)}{4} \right] \text{ or equiv. } = \operatorname{arsinh} \frac{3}{4}$ <p>[M1 does not require limits; A1 f.t. on completing square, providing arsinh]</p> <p>Into ln form $\left[\ln \left[\frac{3}{4} + \sqrt{\frac{9}{16} + 1} \right] \right]; = \ln 2$</p> <p>[If straight to ln form : B1, $\ln \left[(x-1) + \sqrt{(x-1)^2 + 16} \right]$ M1</p> <p>Using limits correctly M1A1√, ln2 A1]</p>	B1 M1 A1√ M1A1 [5]
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[FP2/P5 January 2006 Qn 1]

39.	<p>(a) Using $b^2 = a^2 (e^2 - 1)$; $[4 = 16 (e^2 - 1)]$ $e = \frac{\sqrt{5}}{2}$ or equiv. (1.12)</p> <p>(b) Distance between foci = $2ae$ $[2 \times 4 \times \frac{\sqrt{5}}{2}]$; $= 4\sqrt{5}$</p> <p>[A1√ dependent on both Ms]</p> <p>(c)</p> <p style="padding-left: 100px;">Ellipse, centred on origin</p> <p style="padding-left: 100px;">Hyperbola, both branches</p> <p style="padding-left: 100px;">Totally correct, touching, with correct intercepts</p>	M1A1 (2) M1A1√ (2) B1 B1 B1 (3) [7]
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[FP2/P5 January 2006 Qn 2]

40.	$\frac{dx}{dt} = 1 + \cos t, \quad \frac{dy}{dt} = \sin t, \quad (\text{both})$ $s = \int \sqrt{(1 + \cos t)^2 + (\sin t)^2} dt ; = \int \sqrt{2 + 2 \cos t} dt$ Use of "half-angle formula" $[\int \sqrt{4 \cos^2 t} dt]$; $s = \left[4 \sin \frac{t}{2} \right]_{(0)}^{\left(\frac{\pi}{2}\right)}$ Using limits correctly and surd form; $= 2\sqrt{2}$ (allow $\frac{4}{\sqrt{2}}$)	B1 M1A1 M1A1√ M1A1 [7]
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[FP2/P5 January 2006 Qn 3]

41.	Using $\cosh x = \frac{e^x + e^{-x}}{2}$ and attempt to progress	M1
	Correct intermediate step as far as $4 \left(\frac{e^{3x} + 3e^x + 3e^{-x} + e^{-3x}}{8} \right) - \left[3 \left(\frac{e^x + e^{-x}}{2} \right) \right]$	A1
	$= \frac{e^{3x} + e^{-3x}}{2} = \cosh 3x$	A1 (3)
	(b) Using part (a) to reduce to $\cosh^2 x = [2]$	M1
	Correct method to form $\ln x$ or find e^x or e^{2x}	M1
	$x = \ln(\sqrt{2} + 1), \quad \ln(\sqrt{2} - 1)$ or equivalent	A1 A1√
	or $\frac{1}{2} \ln(3 + 2\sqrt{2}), \quad \frac{1}{2} \ln(3 - 2\sqrt{2}),$ (after finding $e^{2x} = \dots$)	(4)
		[7]

[FP2/P5 January 2006 Qn 4]

<p>42.</p>	<p>(a) $I_n = -\frac{2}{3} \left[x^n (4-x)^{\frac{3}{2}} \right]_0^4 + \frac{2}{3} n \int_0^4 x^{n-1} (4-x)^{\frac{3}{2}} dx$</p> $= \frac{2}{3} n \int_0^4 x^{n-1} (4-x)^{\frac{3}{2}} dx$ $= \frac{2}{3} n \int_0^4 4x^{n-1} (4-x)^{\frac{1}{2}} dx - \frac{2}{3} n \int_0^4 x^n (4-x)^{\frac{1}{2}} dx$ $\Rightarrow I_n = \frac{8}{3} n I_{n-1} - \frac{2}{3} n I_n$ <p>[$(2n+3)I_n = 8nI_{n-1}$] $\Rightarrow I_n = \frac{8n}{2n+3} I_{n-1}$ AG</p> <p>(b) Relating I_2 to I_0 using result from (a)</p> $I_2 = \frac{16}{7} \cdot \frac{8}{5} I_0 = \frac{2048}{105} \left(19 \frac{53}{105} \right)$	<p>M1A1</p> <p>A1√</p> <p>M1A1</p> <p>A1* (6)</p> <p>M1</p> <p>A1A1 (3)</p> <p>[9]</p>
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[FP2/P5 January 2006 Qn 7]

<p>43.</p>	<p>(a) $\operatorname{ar\,tanh} \frac{1}{\sqrt{2}} = \frac{1}{2} \ln \left(\frac{1 + \frac{1}{\sqrt{2}}}{1 - \frac{1}{\sqrt{2}}} \right)$ or equivalent</p> <p>$= \frac{1}{2} \ln \left[\frac{\sqrt{2} + 1}{\sqrt{2} - 1} \times \frac{\sqrt{2} + 1}{\sqrt{2} + 1} \right]$ or equivalent</p> <p>$= \frac{1}{2} \ln(\sqrt{2} + 1)^2 = \ln(\sqrt{2} + 1)$ AG</p> <p><i>Alternative Approach</i></p> <p>If using $y = \operatorname{ar\,tanh}(\sin x) \Rightarrow \tanh y = \left(\frac{1}{\sqrt{2}} \right)$ [or $\cosh^2 y = 2$] and then use exponentials:</p> <p>Progression as far as $e^y = \dots$ or $e^{2y} = \dots$ M1 Converting to ln form M1 Answer as given A1*</p> <p>Note: $\frac{1}{2} \ln(3 + 2\sqrt{2})$ can earn M1M1 but for A1* there must be a convincing further step.</p> <p>(b) $\frac{dy}{dx} = \frac{1}{1 - \sin^2 x} \cdot \cos x; = \frac{\cos x}{\cos^2 x} = \sec x$</p> <p>Note: If $\tanh y = \sin x$ is differentiated M1 requires $\frac{dy}{dx} = f(x)$</p> <p>(c) Attempt at by parts and use of result in (b)</p> <p>$= -\cos x \operatorname{ar\,tanh}(\sin x) + \int \cos x \sec x dx$</p> <p>$= -\cos x \operatorname{ar\,tanh}(\sin x) + x$</p> <p>Using limits correctly : $= -\frac{1}{\sqrt{2}} \ln(1 + \sqrt{2}) + \frac{\pi}{4}$ or exact equivalent</p>	<p>M1</p> <p>M1</p> <p>A1* (3)</p> <p>M1A1 (2)</p> <p>M1</p> <p>A1</p> <p>M1</p> <p>M1A1 (5) [10]</p>
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[FP2/P5 January 2006 Qn 8]

<p>44.</p>	<p>(a) $\begin{vmatrix} k - \lambda & 2 \\ 2 & -1 - \lambda \end{vmatrix} = 0$</p> <p>Characteristic equation: $\lambda^2 - \lambda - 6 = 0$</p> <p>Solving : $(\lambda - 3)(\lambda + 2) = 0 \Rightarrow \lambda = \dots$</p> <p style="text-align: center;">$\lambda = -2, \lambda = 3$ (both)</p> <p>(b) Method for finding an eigenvector</p> <p>$\begin{pmatrix} 2 & 2 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 3 \begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and</p>	<p>M1</p> <p>A1</p> <p>M1</p> <p>A1</p> <p style="text-align: right;">(4)</p> <p>M1</p>
	<p>$\begin{pmatrix} 2 & 2 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = -2 \begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$</p> <p>Equations are: $y = \frac{1}{2}x$ and $y = -2x$.</p> <p>Alt: $\begin{pmatrix} 2 & 2 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} x \\ mx \end{pmatrix} = \begin{pmatrix} y \\ my \end{pmatrix} \Rightarrow 2m^2 + 3m - 2 = 0$ M1A1</p> <p>$[m = \frac{1}{2}, -2]$ Correct equations A1</p>	<p>A1✓</p> <p>A1</p> <p style="text-align: right;">(3)</p> <p>Total 9 marks</p>

[*FP3/P6 January 2006 Qn 3]

45.

$$\mathbf{A} = \begin{pmatrix} k & 1 & -2 \\ 0 & -1 & k \\ 9 & 1 & 0 \end{pmatrix}$$

(a) Det. $\mathbf{A} = -k^2 + 9k - 18$

Setting to zero and solving for k $[(k-6)(k-3) = 0]$
 $\Rightarrow k = 3, k = 6$

(b) Cofactors $\begin{pmatrix} -k & 9k & 9 \\ -2 & 18 & 9-k \\ k-2 & -k^2 & -k \end{pmatrix}$

[B1 for each row (or column)]

$$\mathbf{A}^{-1} = \frac{1}{\det} \begin{pmatrix} -k & -2 & k-2 \\ 9k & 18 & -k^2 \\ 9 & 9-k & -k \end{pmatrix}$$

[A1 f.t. is on determinant or cofactors]

M1A1

M1
A1

(4)

B3

M1A1√

(5)

Total 9 marks

[FP3/P6 January 2006 Qn 4]

<p>46.</p>	<p>(a) $R\vec{Q} = \begin{pmatrix} 1 \\ -1 \\ -1-c \end{pmatrix}$, $RP = \begin{pmatrix} -4 \\ 3 \\ -2-c \end{pmatrix}$ (both)</p> $R\vec{P} \times R\vec{Q} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -4 & 3 & -2-c \\ 1 & -1 & -1-c \end{vmatrix}$ $= (-5-4c)\mathbf{i} - (6+5c)\mathbf{j} + \mathbf{k}$ <p>(b) $c = -2$ $d = -6 - 5c = 4$ AG</p> <p>(c) $\mathbf{r} \cdot \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix} = p$</p> <p>Substituting point in plane to give p, $\mathbf{r} \cdot \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix} = 7$.</p> <p>(d) Equation of normal to plane through S: $\mathbf{r} = \begin{pmatrix} 1 \\ 5 \\ 10 \end{pmatrix} + t \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix}$</p> <p>Meets plane where $\begin{pmatrix} 1+3t \\ 5+4t \\ 10+t \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix} = 7 \Rightarrow t = -1$</p>	<p>B1</p> <p>M1A1√ (3)</p> <p>A1√ A1*(cso) (2)</p> <p>M1</p> <p>M1A1 (3)</p> <p>B1</p> <p>M1A1√</p>
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[FP3/P6 January 2006 Qn 7]

<p>47.</p>	$5\left(\frac{e^x + e^{-x}}{2}\right) - 2\left(\frac{e^x - e^{-x}}{2}\right) = 11$ $3e^{2x} - 22e^x + 7 = 0$ <p>M: Simplify to form <u>quadratic</u> in e^x</p> $(3e^x - 1)(e^x - 7) = 0 \quad e^x = \frac{1}{3}, \quad e^x = 7 \quad \text{M: Solve 3 term quadratic.}$ $x = \ln \frac{1}{3} \text{ (or } -\ln 3) \quad x = \ln 7$	<p>B1</p> <p>M1 A1</p> <p>M1 A1</p> <p>A1 (6)</p> <p>6 Marks</p>
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[FP2 June 2006 Qn 1]

<p>48.</p> <p>(a)</p> <p>(b)</p>	<p>Using $b^2 = a^2(1 - e^2)$ or equiv. to find e or ae: ($a = 2$ and $b = 1$)</p> $e = \frac{\sqrt{3}}{2}$ <p>Using $y^2 = 4(ae)x$ $y^2 = 4\sqrt{3}x$ (M requires <u>values</u> for a and e)</p> $x = -\sqrt{3}$	<p>M1 A1</p> <p>M1 A1 (4)</p> <p>B1ft (1)</p> <p>5 Marks</p>
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[FP2 June 2006 Qn 2]

<p>49. (a)</p>	$\frac{dy}{dx} = 4 \operatorname{sech}^2 4x - 1$ <p>Put $\frac{dy}{dx} = 0$ ($\cosh^2 4x = 4$ $\cosh 4x = 2$)</p> $4x = \ln(2 \pm \sqrt{3}) \text{ or } 8x = \ln(7 \pm 4\sqrt{3}) \text{ or } e^{4x} = 2 \pm \sqrt{3} \text{ or}$ $e^{4x} = 7 \pm 4\sqrt{3} \text{ (}\pm \text{ or } +\text{)}$ $x = \frac{1}{4} \ln(2 + \sqrt{3}) \text{ or } x = \frac{1}{8} \ln(7 + 4\sqrt{3}) \quad (\text{or equiv.})$	<p>B1</p> <p>M1</p> <p>A1</p> <p>A1 (4)</p>
<p>(b)</p>	$y = -\frac{1}{4} \ln(2 + \sqrt{3}) + \tanh(\dots) \quad (\text{Substitute for } x)$ $\operatorname{sech} 4x = \frac{1}{2} = \sqrt{1 - \tanh^2 4x}, \quad \tanh 4x = \frac{\sqrt{3}}{2}$ $y = \frac{\sqrt{3}}{2} - \frac{1}{4} \ln(2 + \sqrt{3}) = \frac{1}{4} \{2\sqrt{3} - \ln(2 + \sqrt{3})\} \quad (*)$	<p>M1</p> <p>M1</p> <p>A1 (3)</p> <p>7 Marks</p>
	<p>(a) ‘Second solution’, if seen, must be rejected to score the final mark.</p> <p>(b) 2nd M requires an expression in terms of $\sqrt{3}$ without hyperbolics, exponentials and logarithms.</p>	

[FP2 June 2006 Qn 5]

<p>50. (a)</p> $\frac{dx}{dt} = 1 - \frac{1}{t} \quad \frac{dy}{dt} = 2t^{-\frac{1}{2}}$ $\sqrt{\left(1 - \frac{1}{t}\right)^2 + \left(2t^{-\frac{1}{2}}\right)^2}, \quad = \sqrt{1 + \frac{2}{t} + \frac{1}{t^2}} = 1 + \frac{1}{t} \quad \text{or} \quad \frac{t+1}{t}$ $\text{Length} = \int_1^4 \left(1 + \frac{1}{t}\right) dt = [t + \ln t]_1^4 = (4 + \ln 4) - 1 = 3 + \ln 4 \quad (*)$ <p>(b) Surface area =</p> $2\pi \int_1^4 4\sqrt{t} \sqrt{\left(1 - \frac{1}{t}\right)^2 + \left(2t^{-\frac{1}{2}}\right)^2} dt \quad \left(= 8\pi \int_1^4 \left(t^{\frac{1}{2}} + t^{-\frac{1}{2}}\right) dt \right)$ $= (8\pi) \left[\frac{2t^{\frac{3}{2}}}{3} + 2t^{\frac{1}{2}} \right]_1^4 = (8\pi) \left\{ \left(\frac{16}{3} + 4 \right) - \left(\frac{2}{3} + 2 \right) \right\} = \frac{160\pi}{3} \quad \left(53\frac{1}{3}\pi \right)$	<p>B1 B1</p> <p>M1, A1</p> <p>M1 M1 A1 (7)</p> <p>M1</p> <p>M1 M1 A1 (4)</p> <p>(11 marks)</p>
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[FP2 June 2006 Qn 6]

<p>51.</p> $\int x^2 \operatorname{arsinh} x dx = \frac{x^3}{3} \operatorname{arsinh} x - \int \frac{x^3}{3\sqrt{x^2+1}} dx$ $\left[\frac{x^3}{3} \operatorname{arsinh} x \right]_0^3 = 9 \operatorname{arsinh} 3 \quad \left(\text{or } 9 \ln(3 + \sqrt{10}) \right)$ <p>Let $u = x^2 + 1 \quad \frac{du}{dx} = 2x \quad \left[u^2 = x^2 + 1 \quad 2u \frac{du}{dx} = 2x \right]$</p> $\frac{1}{3} \int \frac{x^3}{u^{\frac{1}{2}} \cdot 2x} du = \frac{1}{6} \int \frac{u-1}{u^{\frac{1}{2}}} du = \frac{1}{6} \int \left(u^{\frac{1}{2}} - u^{-\frac{1}{2}} \right) du \quad \left[\frac{1}{3} \int (u^2 - 1) du \right]$ $= \frac{1}{6} \left[\frac{2u^{\frac{3}{2}}}{3} - 2u^{\frac{1}{2}} \right] \quad \left[= \frac{1}{3} \left[\frac{u^3}{3} - u \right] \right]$ <p>When $x = 0, u = 1$ and when $x = 3, u = 10 \quad [\dots u = \sqrt{10}]$</p> $\frac{1}{6} \left[\frac{2u^{\frac{3}{2}}}{3} - 2u^{\frac{1}{2}} \right]_1^{10} = \frac{1}{6} \left\{ \left(\frac{20\sqrt{10}}{3} - 2\sqrt{10} \right) - \left(\frac{2}{3} - 2 \right) \right\}$ $\text{Area} = 9 \operatorname{arsinh} 3 - \frac{1}{6} \left(\frac{14\sqrt{10}}{3} + \frac{4}{3} \right) = 9 \ln(3 + \sqrt{10}) - \frac{1}{9} (7\sqrt{10} + 2) \quad (*)$	<p>M1 A1 A1</p> <p>B1</p> <p>M1</p> <p>M1</p> <p>M1</p> <p>M1 A1</p> <p>A1cso (10)</p>
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	<p><u>Dependent M marks:</u></p> <p>M: Choose an appropriate substitution & find $\frac{du}{dx}$ <u>or</u> 'Set up' integration by parts.</p> <p>M: Get <u>all</u> in terms of 'u' <u>or</u> Use integration by parts.</p> <p>M: Sound integration.</p> <p>M: Substitute both limits (for the correct variable) and subtract.</p>	
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51.	<p><u>Alternative solution:</u></p> <p>Let $x = \sinh \theta$ $\frac{dx}{d\theta} = \cosh \theta$</p> $\int x^2 \operatorname{ar} \sinh x \, dx = \int \theta \sinh^2 \theta \cosh \theta \, d\theta$ $= \left[\frac{\theta \sinh^3 \theta}{3} \right] - \int \frac{1}{3} \sinh \theta (\cosh^2 \theta - 1) \, d\theta$ $\left[\frac{\theta \sinh^3 \theta}{3} \right]_0^{\operatorname{arsinh} 3} = 9 \operatorname{arsinh} 3$ $\int \frac{1}{3} \sinh \theta (\cosh^2 \theta - 1) \, d\theta = \frac{1}{3} \left[\frac{\cosh^3 \theta}{3} - \cosh \theta \right]$ $\frac{1}{3} \left[\frac{\cosh^3 \theta}{3} - \cosh \theta \right]_0^{\operatorname{arsinh} 3} = \frac{1}{3} \left\{ \left(\frac{10\sqrt{10}}{3} - \sqrt{10} \right) - \left(\frac{1}{3} - 1 \right) \right\}$ $\text{Area} = 9 \operatorname{arsinh} 3 - \frac{1}{3} \left(\frac{7\sqrt{10}}{3} + \frac{2}{3} \right) = 9 \ln(3 + \sqrt{10}) - \frac{1}{9} (7\sqrt{10} + 2) \quad (*)$	<p>M1</p> <p>M1</p> <p>M1 A1 A1</p> <p>B1</p> <p>M1</p> <p>M1 A1</p> <p>A1cso (10)</p> <p>10 Marks</p>
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<p>51.</p> <p>(i)</p>	<p><u>A few alternatives for:</u> $\int \frac{x^3}{\sqrt{x^2+1}} dx$.</p> <p>Let $u = x^2$ $\frac{du}{dx} = 2x$</p> $\int \frac{u^{\frac{3}{2}}}{\sqrt{1+u}} \cdot \frac{1}{2u^{\frac{1}{2}}} du = \frac{1}{2} \int \frac{u}{\sqrt{1+u}} du$ <p>No marks yet... needs another substitution, or parts, or perhaps...</p> $\frac{u}{\sqrt{1+u}} = \sqrt{1+u} - \frac{1}{\sqrt{1+u}}$ $\frac{1}{2} \int \sqrt{1+u} du - \frac{1}{2} \int \frac{1}{\sqrt{1+u}} du$ $\frac{1}{3} (1+u)^{\frac{3}{2}} - (1+u)^{\frac{1}{2}}$ <p>Limits (0 to 9)</p>	<p>M1</p> <p>M1</p> <p>M1</p> <p>M1</p>
<p>(ii)</p>	<p>Let $x = \sinh \theta$ $\frac{dx}{d\theta} = \cosh \theta$</p> $\int \frac{\sinh^3 \theta}{\cosh \theta} \cdot \cosh \theta d\theta = \int \sinh \theta (\cosh^2 \theta - 1) d\theta$ <p>Then, as in the alternative solution,</p> $\int \frac{1}{3} \sinh \theta (\cosh^2 \theta - 1) d\theta = \frac{1}{3} \left[\frac{\cosh^3 \theta}{3} - \cosh \theta \right]$ <p>Limits (0 to arsinh3)</p>	<p>M1</p> <p>M1</p> <p>M1</p> <p>M1</p>

<p>51. (iii)</p> <p>(iv)</p>	<p>Let $u = \tan \theta \quad \frac{du}{d\theta} = \sec^2 \theta$</p> $\int \frac{\tan^3 \theta}{\sec \theta} \cdot \sec^2 \theta d\theta = \int \tan \theta \sec \theta (\sec^2 \theta - 1) d\theta$ $= \int \sec^2 \theta (\sec \theta \tan \theta) d\theta - \int (\sec \theta \tan \theta) d\theta = \frac{\sec^3 \theta}{3} - \sec \theta$ <p>Limits ($\sec \theta = 1$ to $\sec \theta = \sqrt{10}$)</p> <p>(By parts... must be the 'right way round', not integrating x^2)</p> $u = x^2, \quad \frac{du}{dx} = 2x \quad \frac{dv}{dx} = \frac{x}{\sqrt{1+x^2}}, \quad v = \sqrt{1+x^2}$ $x^2 \sqrt{1+x^2} - \int 2x \sqrt{1+x^2} dx$ $x^2 \sqrt{1+x^2} - \frac{2}{3} (x^2 + 1)^{\frac{3}{2}}$ <p>Limits</p>	<p>M1</p> <p>M1</p> <p>M1</p> <p>M1</p> <p>M1</p> <p>M1</p> <p>M1</p> <p>M1</p>
<p>(v)</p> <p>(vi)</p>	<p>(By parts)</p> $u = x^3, \quad \frac{du}{dx} = 3x^2 \quad \frac{dv}{dx} = \frac{1}{\sqrt{1+x^2}}, \quad v = \operatorname{arsinh} x$ <p>No progress</p> $\frac{x^3}{\sqrt{1+x^2}} = \frac{x(x^2+1) - x}{\sqrt{1+x^2}} = \frac{x(x^2+1)}{\sqrt{1+x^2}} - \frac{x}{\sqrt{1+x^2}}$ $\int x \sqrt{1+x^2} dx - \int \frac{x}{\sqrt{1+x^2}} dx$ $= \frac{1}{3} (1+x^2)^{\frac{3}{2}} - (1+x^2)^{\frac{1}{2}}$ <p>Limits</p>	<p>M0</p> <p>M1</p> <p>M1</p> <p>M1</p> <p>M1</p>

[FP2 June 2006 Qn7]

<p>52. (a)</p> <p>(b)</p> <p>(c)</p>	$\int x^n \cosh x \, dx = x^n \sinh x - \int nx^{n-1} \sinh x \, dx$ $= x^n \sinh x - nx^{n-1} \cosh x + n(n-1) \int x^{n-2} \cosh x \, dx$ $I_n = x^n \sinh x - nx^{n-1} \cosh x + n(n-1)I_{n-2} \quad (*)$ $I_4 = x^4 \sinh x - 4x^3 \cosh x + 12I_2$ $I_4 = x^4 \sinh x - 4x^3 \cosh x + 12(x^2 \sinh x - 2x \cosh x + 2I_0)$ <p>(This M may also be scored by finding I_2 by integration.)</p> $I_0 = \int \cosh x \, dx = \sinh x + k$ $I_4 = (x^4 + 12x^2 + 24)\sinh x, + (-4x^3 - 24x)\cosh x \quad (+ C)$ $\left[(x^4 + 12x^2 + 24)\sinh x + (-4x^3 - 24x)\cosh x \right]_0^1$ $= 37\sinh 1 - 28\cosh 1 \quad \text{M: } x = 1 \text{ substituted throughout (at some stage)}$ $= 37\left(\frac{e - e^{-1}}{2}\right) - 28\left(\frac{e + e^{-1}}{2}\right)$ <p>M: Use of exp. Definitions (can be in terms of x)</p> $= \frac{1}{2}(9e - 65e^{-1})$ <p>(b) Integration constant missing <u>throughout</u> loses the B mark</p>	<p>M1 A1</p> <p>M1</p> <p>A1 (4)</p> <p>M1</p> <p>M1</p> <p>B1</p> <p>A1, A1(5)</p> <p>M1</p> <p>M1</p> <p>A1 (3)</p> <p>12 Marks</p>
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[FP2 June 2006 Qn 8]

<p>53.(a)</p> <p>(b)</p> <p>(c)</p> <p>(d)</p> <p>(e)</p>	$\frac{x^2}{a^2} + \frac{(mx+c)^2}{b^2} = 1 \quad b^2x^2 + a^2(mx+c)^2 = a^2b^2$ $(b^2 + a^2m^2)x^2 + 2a^2mcx + a^2(c^2 - b^2) = 0 \quad (*)$ $(2a^2mc)^2 = 4(b^2 + a^2m^2)a^2(c^2 - b^2)$ $4a^4m^2c^2 = 4a^2(b^2c^2 - b^4 + a^2m^2c^2 - a^2m^2b^2) \quad c^2 = b^2 + a^2m^2 \quad (*)$ <p>Find height and base of triangle (perhaps in terms of c).</p> $OB = c \left(= \sqrt{b^2 + a^2m^2} \right) \quad \text{and} \quad AO = \frac{c}{m} \left(= \frac{\sqrt{b^2 + a^2m^2}}{m} \right)$ <p>Area of triangle $OAB = \frac{c^2}{2m} = \frac{b^2 + a^2m^2}{2m}$ M: Find area and subs. for c.</p> $\Delta = \frac{b^2 + a^2m^2}{2m} = \frac{b^2}{2}m^{-1} + \frac{a^2}{2}m$ $\frac{d\Delta}{dm} = -\frac{b^2}{2}m^{-2} + \frac{a^2}{2} = 0 \quad \frac{b^2}{m^2} = a^2 \quad m = \frac{b}{a}$ $\Delta = \left(\frac{b^2}{2}\right)\left(\frac{a}{b}\right) + \left(\frac{a^2}{2}\right)\left(\frac{b}{a}\right) = ab \quad (*)$ <p>Root of quadratic: $x = \frac{-a^2mc}{b^2 + a^2m^2}$ (Should be <u>correct</u> if quoted directly)</p> <p>Using $m = \frac{b}{a}$ and $c = \sqrt{b^2 + a^2m^2}$: $x = -\frac{a}{\sqrt{2}}$</p> <p>(The 2nd M is dependent on using the quadratic equation).</p>	<p>M1</p> <p>A1 (2)</p> <p>M1</p> <p>A1 (2)</p> <p>M1</p> <p>A1</p> <p>M1 A1 (4)</p> <p>M1 A1</p> <p>A1 (3)</p> <p>M1</p> <p>M1 A1 (3)</p> <p>14 Marks</p>
<p>(d)</p> <p>(e)</p>	<p><u>Alternative:</u> $b^2 + a^2m^2 \geq 2bam$ (since $(b - am)^2 \geq 0$)</p> $\frac{b^2 + a^2m^2}{2m} \geq ab \quad [A1]$ <p style="text-align: right;">Conclusion [A1]</p> <p><u>Alternative:</u></p> <p>Begin with full eqn. $(b^2 + a^2m^2)x^2 + 2a^2mcx + a^2(c^2 - b^2) = 0$.</p> <p>In the eqn., use conditions $m = \frac{b}{a}$ and $c = \sqrt{b^2 + a^2m^2}$ ($= b\sqrt{2}$)</p> <p>Simplify and solve eqn., e.g. $2x^2 + 2a\sqrt{2}x + a^2 = 0 \quad x = -\frac{a}{\sqrt{2}}$</p>	<p>[M1]</p> <p>[M1]</p>

54.	$\mathbf{A}^1 = \begin{pmatrix} 1 & 1 & \frac{1}{2}(1+3) \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \mathbf{A}$ <p>(Hence true for $n=1$)</p> $\mathbf{A}^{k+1} = \mathbf{A}^k \cdot \mathbf{A} = \begin{pmatrix} 1 & k & \frac{1}{2}(k^2+3k) \\ 0 & 1 & k \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ $= \begin{pmatrix} 1 & k+1 & 2+k+\frac{1}{2}(k^2+3k) \\ 0 & 1 & k+1 \\ 0 & 0 & 1 \end{pmatrix}$ $2+k+\frac{1}{2}(k^2+3k) = \frac{1}{2}(k^2+5k+4) = \frac{1}{2}(k^2+2k+1+3k+3)$ $= \frac{1}{2}((k+1)^2+3(k+1))$ <p>(Hence, if result is true for $n=k$, then it is true for $n=k+1$).</p> <p>By Mathematical Induction, above implies true for all positive integers. cso</p>	<p>B1</p> <p>M1</p> <p>M1 Dep</p> <p>A1</p> <p>A1 cso (5)</p> <p>[5 marks]</p>
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[FP3 June 2006 Qn 1]

55.	<p>(a) $(4-\lambda)(1-\lambda)+2=0$</p> $\lambda^2 - 5\lambda + 6 = (\lambda-3)(\lambda-2) = 0$ $\lambda_1 = 2, \lambda_2 = 3 \quad \text{both}$ <p>(b) $M^{-1} = \frac{1}{\underline{6}} \begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix}$</p> <p>(c) $\begin{vmatrix} \frac{1}{6} - \frac{1}{2} & \frac{1}{3} \\ -\frac{1}{6} & \frac{2}{3} - \frac{1}{2} \end{vmatrix} = -\frac{1}{3} \times \frac{1}{6} + \frac{1}{3} \times \frac{1}{6} = 0$ M1 for either value</p> <p>(hence $\frac{1}{2}$ is an eigenvalue of \mathbf{M}^{-1})</p>	<p>M1</p> <p>M1</p> <p>A1 (3)</p> <p><u>B1</u> B1 (2)</p> <p>M1 A1</p>
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	$\begin{vmatrix} \frac{1}{6} - \frac{1}{3} & \frac{1}{3} \\ -\frac{1}{6} & \frac{2}{3} - \frac{1}{3} \end{vmatrix} = -\frac{1}{6} \times \frac{1}{3} + \frac{1}{3} \times \frac{1}{6} = 0$ <p>(hence $\frac{1}{3}$ is an eigenvalue of \mathbf{M}^{-1})</p> <p>(d) Using eigenvalues</p> $\begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = 2 \begin{pmatrix} x \\ y \end{pmatrix}$ $4x - 2y = 2x \Rightarrow y = x$ $\begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = 3 \begin{pmatrix} x \\ y \end{pmatrix}$ $4x - 2y = 3x \Rightarrow y = \frac{1}{2}x$	<p>A1</p> <p>(3)</p> <p>M1 A1</p> <p>M1 A1 (4)</p> <p>[12]</p>
	<p><i>Alternative to (c), using characteristic polynomial of \mathbf{M}^{-1}</i></p> $\left(\frac{1}{6} - \lambda\right)\left(\frac{2}{3} - \lambda\right) + \frac{1}{3} \times \frac{1}{6} = 0$ <p>Leading to $6\lambda^2 - 5\lambda + 1 = (3\lambda - 1)(2\lambda - 1) = 0 \Rightarrow \lambda = \frac{1}{2}, \frac{1}{3}$</p> <p><i>Alternative to (d)</i> $\begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ mx \end{pmatrix} = \begin{pmatrix} x' \\ mx' \end{pmatrix}$</p> $4x - 2mx = x', \quad x + mx = mx' \quad \text{both}$ $\frac{1+m}{4-2m} = m$ <p>Leading to $2m^2 - 3m + 1 = (2m - 1)(m - 1) = 0 \Rightarrow m = \frac{1}{2}, 1$</p> $y = \frac{1}{2}x, \quad y = x \quad \text{both}$	<p>M1</p> <p>A1, A1 (3)</p> <p>M1</p> <p>A1</p> <p>M1</p> <p>A1 (4)</p>

[FP3 June 2006 Qn 5]

<p>56.</p>	<p>(a) $(\mathbf{b}-\mathbf{a})\times(\mathbf{c}-\mathbf{a}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 0 & -3 \\ 4 & -5 & -1 \end{vmatrix}$</p> <p style="text-align: center;">$= -15\mathbf{i} - 10\mathbf{j} - 10\mathbf{k}$</p> <p style="text-align: center;">Allow M1 A1 for negative of above</p> <p>(b) $\mathbf{r}\cdot(3\mathbf{i}+2\mathbf{j}+2\mathbf{k}) = (3\mathbf{i}+2\mathbf{j}+2\mathbf{k})\cdot(\mathbf{i}+3\mathbf{j}-\mathbf{k})$ or equivalent</p> <p style="text-align: center;">$\mathbf{r}\cdot(3\mathbf{i}+2\mathbf{j}+2\mathbf{k}) = 7$ or multiple</p> <p>(c) Let $x = \lambda$, $z = 3 - \lambda$,</p> <p style="text-align: center;">then $2y = 7 - 3\lambda - 2(3 - \lambda) \Rightarrow y = \frac{1}{2} - \frac{1}{2}\lambda$</p> <p style="text-align: center;">x, y and z in terms of a single parameter</p> <p style="text-align: center;">The direction of l is any multiple of $(2\mathbf{i} - \mathbf{j} - 2\mathbf{k})$</p> <p style="text-align: center;">$(\mathbf{r} - (\frac{1}{2}\mathbf{j} + 3\mathbf{k})) \times (2\mathbf{i} - \mathbf{j} - 2\mathbf{k}) = \mathbf{0}$ or equivalent</p> <p>Possible equivalents are $(\mathbf{r} - (\mathbf{i} + 2\mathbf{k})) \times (-2\mathbf{i} + \mathbf{j} + 2\mathbf{k}) = \mathbf{0}$</p> <p style="text-align: center;">and $(\mathbf{r} - (3\mathbf{i} - \mathbf{j})) \times (-\mathbf{i} + \frac{1}{2}\mathbf{j} + \mathbf{k}) = \mathbf{0}$</p> <p>The general form is</p> <p style="text-align: center;">$\{\mathbf{r} - [\mathbf{i} + 2\mathbf{k} + c_1(2\mathbf{i} - \mathbf{j} - 2\mathbf{k})]\} \times c_2(2\mathbf{i} - \mathbf{j} - 2\mathbf{k}) = \mathbf{0}$</p> <p>(d) $(\lambda\mathbf{i} + (\frac{1}{2} - \frac{1}{2}\lambda)\mathbf{j} + (3 - \lambda)\mathbf{k}) \cdot (2\mathbf{i} - \mathbf{j} - 2\mathbf{k}) = 0$</p> <p style="text-align: center;">$2\lambda - \frac{1}{2} + \frac{1}{2}\lambda - 6 + 2\lambda = 0$</p> <p style="text-align: center;">Leading to $\lambda = \frac{13}{9}$</p> <p style="text-align: center;">$P: (\frac{13}{9}, -\frac{2}{9}, \frac{14}{9})$</p> <p><i>Alternative to (d)</i></p>	<p>M1</p> <p>A1+A1+A1 (4)</p> <p>M1</p> <p>A1 (2)</p> <p>M1</p> <p>M1</p> <p>M1 A1 (4)</p> <p>M1</p> <p>M1 A1</p> <p>A1 (4)</p> <p>[14]</p>
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	$OP^2 = \lambda^2 + \left(\frac{1}{2} - \frac{1}{2}\lambda\right)^2 + (3 - \lambda)^2 \quad \left(= \frac{1}{4}(9\lambda^2 - 26\lambda + 37)\right)$	M1
	$\frac{d}{d\lambda}(OP^2) = 0 \Rightarrow \lambda = \frac{13}{9}$	M1 A1
	$P: \left(\frac{13}{9}, -\frac{2}{9}, \frac{14}{9}\right)$	A1 (4)

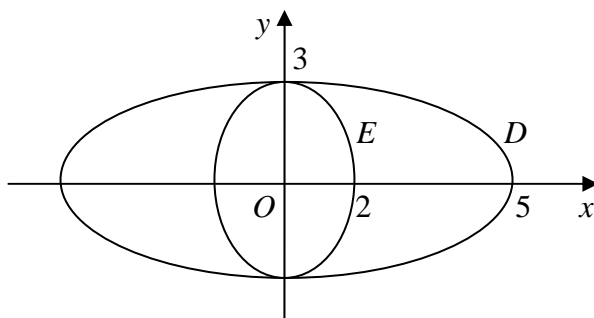
[FP3 June 2006 Qn 7]

57.	$x^2 + 4x - 5 = (x + 2)^2 - 9$ $\int \frac{1}{\sqrt{((x+2)^2 - 9)}} dx = \operatorname{arcosh} \frac{x+2}{3}$ <p>ft their completing the square, requires arcosh</p> $\left[\operatorname{arcosh} \frac{x+2}{3} \right]_1^3 = \operatorname{arcosh} \frac{5}{3} - \operatorname{arcosh} 1$ $= \ln \left(\frac{5}{3} + \sqrt{\frac{25}{9} - 1} \right) = \ln \left(\frac{5}{3} + \frac{4}{3} \right) = \ln 3$	B1 M1 A1ft M1 A1 (5) [5]
	<p><i>Alternative</i></p> $x^2 + 4x - 5 = (x + 2)^2 - 9$ <p>Let $x + 2 = 3 \sec \theta$, $\frac{dx}{d\theta} = 3 \sec \theta \tan \theta$</p> $\int \frac{1}{\sqrt{((x+2)^2 - 9)}} dx = \int \frac{3 \sec \theta \tan \theta}{\sqrt{9 \sec^2 \theta - 9}} d\theta$ $= \int \sec \theta d\theta$ $\left[\ln(\sec \theta + \tan \theta) \right]_{\operatorname{arcsec} 1}^{\operatorname{arcsec} \frac{5}{3}} = \ln \left(\frac{5}{3} + \frac{4}{3} \right) = \ln 3$	B1 M1 A1ft M1 A1 (5)

[FP2 June 2007 Qn 1]

58.

(a)



One ellipse centred at O
 Another ellipse, centred at O , touching on y -axis
 Intersections: At least 2, 5, and 3 shown correctly

B1

B1

B1

(3)

(b) Using $b^2 = a^2(1 - e^2)$, or equivalent, to find e or ae for D or E .

M1

For S : $a = 5$ and $b = 3$, $e = \frac{4}{5}$, $ae = 4$ ignore sign with ae

A1

For T : $a' = 3$ and $b' = 2$, $e' = \frac{\sqrt{5}}{3}$, $a'e' = \sqrt{5}$ ignore sign with $a'e'$

A1

$$ST = \sqrt{(16+5)} = \sqrt{21}$$

M1 A1

(5)

[8]

[FP2 June 2007 Qn 2]

59.	$\frac{dy}{dx} = \frac{1}{4} \left(4x - \frac{1}{x} \right)$ $\int \left(1 + \left(\frac{dy}{dx} \right)^2 \right)^{\frac{1}{2}} dx = \int \left(1 + \left(x - \frac{1}{4x} \right)^2 \right)^{\frac{1}{2}} dx$ $= \int \left(1 + x^2 + \frac{1}{16x^2} - \frac{1}{2} \right)^{\frac{1}{2}} dx = \int \left(\left(x + \frac{1}{4x} \right)^2 \right)^{\frac{1}{2}} dx = \int \left(x + \frac{1}{4x} \right) dx$ $= \frac{x^2}{2} + \frac{\ln x}{4}$ $\left[\frac{x^2}{2} + \frac{\ln x}{4} \right]_{0.5}^{-2} = 2 + \frac{\ln 2}{4} - \frac{1}{8} - \frac{\ln 0.5}{4} = \frac{15}{8} + \frac{1}{2} \ln 2$ $\left(a = \frac{15}{8}, b = \frac{1}{2} \right)$	<p>B1</p> <p>M1</p> <p>M1 A1</p> <p>A1</p> <p>M1 A1</p> <p>(7)</p> <p>(7 marks)</p>
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[FP2 June 2007 Qn 3]

60.	<p>(a)</p> $\cosh A \cosh B - \sinh A \sinh B = \left(\frac{e^A + e^{-A}}{2} \right) \left(\frac{e^B + e^{-B}}{2} \right) - \left(\frac{e^A - e^{-A}}{2} \right) \left(\frac{e^B - e^{-B}}{2} \right)$ $= \frac{1}{4} (e^{A+B} + e^{-A+B} + e^{A-B} + e^{-A-B} - e^{A+B} + e^{-A+B} + e^{A-B} - e^{-A-B})$ $= \frac{1}{4} (2e^{-A+B} + 2e^{A-B}) = \frac{e^{A-B} + e^{-(A-B)}}{2} = \cosh(A-B) \quad *$ <p style="text-align: right;">cso M1 A1 (3)</p> <p>(b)</p> $\cosh x \cosh 1 - \sinh x \sinh 1 = \sinh x$ $\cosh x \cosh 1 = \sinh x(1 + \sinh 1) \Rightarrow \tanh x = \frac{\cosh 1}{1 + \sinh 1}$ $\tanh x = \frac{\frac{e+e^{-1}}{2}}{1 + \frac{e-e^{-1}}{2}} = \frac{e+e^{-1}}{2+e-e^{-1}} = \frac{e^2+1}{e^2+2e-1} \quad *$ <p style="text-align: right;">cso M1 A1 (4)</p> <p><i>Alternative for (b)</i></p> $\frac{e^{x-1} + e^{-(x-1)}}{2} = \frac{e^x - e^{-x}}{2}$ <p style="text-align: right;">M1</p>	<p>M1</p> <p>M1</p> <p>M1 A1 (4)</p> <p>[7]</p>
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	Leading to $e^{2x} = \frac{e^2 + e}{e - 1}$ $\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{e^{2x} - 1}{e^{2x} + 1} = \frac{e^2 + e - (e - 1)}{e^2 + e + (e - 1)} = \frac{e^2 + 1}{e^2 + 2e - 1} \quad *$	M1 M1 A1 (4)
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[FP2 June 2007 Qn 4]

61.	<p>(a) $I_n = -\frac{3}{4} \left[x^n (8-x)^{4/3} \right]_0^8 + \frac{3}{4} \int nx^{n-1} (8-x)^{4/3} dx$</p> <p>$= \frac{3}{4} \int nx^{n-1} (8-x)^{4/3} dx \quad \text{ft numeric constants only}$</p> <p>$\int nx^{n-1} (8-x)(8-x)^{1/3} dx = \int nx^{n-1} 8(8-x)^{1/3} dx - \int nx^{n-1} x(8-x)^{1/3} dx$</p> <p>$I_n = 6nI_{n-1} - \frac{3}{4}nI_n \Rightarrow I_n = \frac{24n}{3n+4} I_{n-1} \quad *$</p> <p>(b) $I_0 = \int_0^8 (8-x)^{1/3} dx = \left[-\frac{3}{4} (8-x)^{4/3} \right]_0^8 = \frac{3}{4} \times 8^{4/3} = 12$</p> <p>$I = \int_0^8 x(x+5)(8-x)^{1/3} dx = I_2 + 5I_1$</p> <p>$I_1 = \frac{24}{7} I_0, \quad I_2 = \frac{48}{10} I_1 = \frac{48}{10} \times \frac{24}{7} I_0 \left(= \frac{576}{35} I_0 \right)$</p> <p>$\left(\text{The previous line can be implied by } I = I_2 + 5I_1 = \frac{168}{5} I_0 \right)$</p> <p>$I = \left(\frac{576}{35} + 5 \times \frac{24}{7} \right) \times 12 = \frac{2016}{5} (= 403.2)$</p>	M1 A1 A1ft M1 A1 A1 (6) * cso M1 A1 M1 M1 A1 A1 (6) (12 marks)
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[FP2 June 2007 Qn 6]

62.	<p>(a) $\frac{d}{dx}(\operatorname{arsinh} x^{1/2}) = \frac{1}{\sqrt{1+x}} \times \frac{1}{2} x^{-1/2} \left(= \frac{1}{2\sqrt{x}\sqrt{1+x}} \right)$</p> <p>At $x = 4$, $\frac{dy}{dx} = \frac{1}{4\sqrt{5}}$ accept equivalents</p>	<p>M1 A1</p> <p>A1 (3)</p>
	<p>(b) $x = \sinh^2 \theta, \quad \frac{dx}{d\theta} = 2 \sinh \theta \cosh \theta$</p> <p>$\int \operatorname{arsinh} \sqrt{x} dx = \int \theta \times 2 \sinh \theta \cosh \theta d\theta$</p> <p>$= \int \theta \sinh 2\theta d\theta = \frac{\theta \cosh 2\theta}{2} - \int \frac{\cosh 2\theta}{2} d\theta$</p> <p>$= \dots - \frac{\sinh 2\theta}{4}$</p> <p>$\left[\frac{\theta \cosh 2\theta}{2} - \frac{\sinh 2\theta}{4} \right]_0^{\operatorname{arsinh} 2} = \dots$ attempt at substitution</p> <p>$= \left[\frac{\theta(1+2\sinh^2 \theta)}{2} - \frac{2 \sinh \theta \cosh \theta}{4} \right] = \frac{1}{2} \operatorname{arsinh} 2 \times (1+8) - \frac{4\sqrt{5}}{4}$</p> <p>$= \frac{9}{2} \ln(2+\sqrt{5}) - \sqrt{5}$</p>	<p>M1 A1</p> <p>M1 A1 + A1</p> <p>M1</p> <p>M1</p> <p>M1 A1</p> <p>A1 (10)</p> <p>[13]</p>

Alternative for (a)

	<p>$x = \sinh^2 y, \quad 2 \sinh y \cosh y \frac{dy}{dx} = 1$</p> <p>$\frac{dy}{dx} = \frac{1}{2 \sinh y \cosh y} = \frac{1}{2 \sinh y \sqrt{(\sinh^2 y + 1)}} \left(= \frac{1}{2\sqrt{x}\sqrt{1+x}} \right)$</p> <p>At $x = 4$, $\frac{dy}{dx} = \frac{1}{4\sqrt{5}}$ accept equivalents</p>	<p>M1</p> <p>A1</p> <p>A1 (3)</p>
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An alternative for (b) is given on the next page

62.

Alternative for (b)

$$\int 1 \times \operatorname{arsinh} \sqrt{x} \, dx = x \operatorname{arsinh} \sqrt{x} - \int x \times \frac{1}{2\sqrt{x}\sqrt{(1+x)}} \, dx$$

$$= x \operatorname{arsinh} \sqrt{x} - \int \frac{\sqrt{x}}{2\sqrt{(1+x)}} \, dx$$

$$\text{Let } x = \sinh^2 \theta, \quad \frac{dx}{d\theta} = 2 \sinh \theta \cosh \theta$$

$$\int \frac{\sqrt{x}}{\sqrt{(1+x)}} \, dx = \int \frac{\sinh \theta}{\cosh \theta} \times 2 \sinh \theta \cosh \theta \, d\theta$$

$$= 2 \int \sinh^2 \theta \, d\theta = 2 \int \frac{\cosh 2\theta - 1}{2} \, d\theta = \frac{\sinh 2\theta}{2} - \theta$$

$$\left[\frac{\sinh 2\theta}{2} - \theta \right]_0^{\operatorname{arsinh} 2} = \left[\frac{2 \sinh \theta \cosh \theta}{2} - \theta \right]_0^{\operatorname{arsinh} 2} = \frac{2 \times 2 \times \sqrt{5}}{2} - \operatorname{arsinh} 2$$

$$\int_0^4 \operatorname{arsinh} \sqrt{x} \, dx = 4 \operatorname{arsinh} 2 - \frac{1}{2} \left(\frac{2 \times 2 \times \sqrt{5}}{2} - \operatorname{arsinh} 2 \right) = \frac{9}{2} \ln(2 + \sqrt{5}) - \sqrt{5} \quad \text{A1} \quad (10)$$

The last 7 marks of the alternative solution can be gained as follows

$$\text{Let } x = \tan^2 \theta, \quad \frac{dx}{d\theta} = 2 \tan \theta \sec^2 \theta$$

$$\int \frac{\sqrt{x}}{\sqrt{(1+x)}} \, dx = \int \frac{\tan \theta}{\sec \theta} \times 2 \tan \theta \sec^2 \theta \, d\theta \quad \text{dependent on first M1}$$

$$= \int 2 \sec \theta \tan^2 \theta \, d\theta$$

$$\int (\sec \theta \tan \theta) \tan \theta \, d\theta = \sec \theta \tan \theta - \int \sec^3 \theta \, d\theta$$

$$= \sec \theta \tan \theta - \int \sec \theta (1 + \tan^2 \theta) \, d\theta$$

$$\text{Hence } \int \sec \theta \tan^2 \theta \, d\theta = \frac{1}{2} \sec \theta \tan \theta - \frac{1}{2} \int \sec \theta \, d\theta$$

$$= \frac{1}{2} \sec \theta \tan \theta - \frac{1}{2} \ln(\sec \theta + \tan \theta)$$

$$\left[\dots \right]_0^{\arctan 2} = \frac{1}{2} \times \sqrt{5} \times 2 - \frac{1}{2} \ln(\sqrt{5} + 2)$$

$$\int_0^4 \operatorname{arsinh} \sqrt{x} \, dx = 4 \operatorname{arsinh} 2 - \sqrt{5} + \frac{1}{2} \ln(2 + \sqrt{5}) = \frac{9}{2} \ln(2 + \sqrt{5}) - \sqrt{5} \quad \text{A1}$$

[FP2 June 2007 Qn 7]

63.	(a)	$\begin{pmatrix} 3 & 4 & p \\ -1 & q & -4 \\ 1 & 1 & 3 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = \lambda \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$	
	Third row	$1-3 = -\lambda \Rightarrow \lambda = 2$	M1 A1 (2)
	(b)	$\begin{pmatrix} 3 & 4 & p \\ -1 & q & -4 \\ 1 & 1 & 3 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 4-p \\ q+4 \\ -2 \end{pmatrix} = 2 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$	M1 A1
	First row	$4-p = 0 \Rightarrow p = 4$	Method for either
	Second row	$q+4 = 2 \Rightarrow q = -2$	Both correct
	M1 A1 ft (4)		
	(c)	$\begin{pmatrix} 3 & 4 & 4 \\ -1 & -2 & -4 \\ 1 & 1 & 3 \end{pmatrix} \cdot \begin{pmatrix} l \\ m \\ n \end{pmatrix} = \begin{pmatrix} 10 \\ -4 \\ 3 \end{pmatrix}$	M1
		$3l + 4m + 4n = 10$	Obtaining 3 linear equations
		$-l - 2m - 4n = -4$	
		$l + m + 3n = 3$	
		$2l + 2m = 6$	Reducing to a pair of equations and solving for one variable
		$3l + 2m = 8$	
		$l = 2, m = 1, n = 0$	Solving for all three variables.
	M1 M1 A1 (4) [10]		
	Alternative to (c)	$\mathbf{A}^{-1} = \frac{1}{6} \begin{pmatrix} 2 & 8 & 8 \\ -1 & -5 & -8 \\ -1 & -1 & 2 \end{pmatrix}$	M1 M1
		$\frac{1}{6} \begin{pmatrix} 2 & 8 & 8 \\ -1 & -5 & -8 \\ -1 & -1 & 2 \end{pmatrix} \cdot \begin{pmatrix} 10 \\ -4 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$	M1 A1 (4)

[FP3 June 2007 Qn 3]

64.	<p>(a) $\overrightarrow{AB} = -\mathbf{i} + 3\mathbf{j} + 3\mathbf{k}$, $\overrightarrow{AC} = 4\mathbf{j} + 2\mathbf{k}$ any two</p> $\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 3 & 3 \\ 0 & 4 & 2 \end{vmatrix} = -6\mathbf{i} + 2\mathbf{j} - 4\mathbf{k}$ <p>Give A1 for any two components correct or the negative of the correct answer. (4)</p> <p>(b) Cartesian equation has form $3x - y + 2z = p$ $(2, -1, 0) \Rightarrow 6 + 1 = p$ or use of another point $3x - y + 2z = 7$ * or any multiple (2)</p> <p>(c) Parametric form of line is $\mathbf{r} = \begin{pmatrix} 5 \\ 5 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix}$ or equivalent form M1 A1</p> <p>Substituting into equation of plane $3(5 + 2\lambda) - (5 - \lambda) + 2(3 - 2\lambda) = 7$ M1 Leading to $\lambda = -3$ A1 $T : (-1, 8, 9)$ A1 (5)</p> <p>(d) $\overrightarrow{AT} = -3\mathbf{i} + 9\mathbf{j} + 9\mathbf{k}$, $\overrightarrow{BT} = -2\mathbf{i} + 6\mathbf{j} + 6\mathbf{k}$ both M1 These are parallel and hence A, B and T are collinear * (by the axiom of parallels) M1 A1 (3) [14]</p> <p>Alternative to (d) The equation of AB: $\mathbf{r} = 2\mathbf{i} - \mathbf{j} + \mu(-\mathbf{i} + 3\mathbf{j} + 3\mathbf{k})$ or equivalent i: $-1 = 2 - \mu \Rightarrow \mu = 3$ M1 $\mu = 3 \Rightarrow \overrightarrow{OT} = -\mathbf{i} + 8\mathbf{j} + 9\mathbf{k}$ M1 Hence A, B and T are collinear * cso A1 (3)</p> <p>Note: Column vectors or bold-faced vectors may be used at any stage.</p>	<p>B1</p> <p>M1 A1 A1</p> <p>M1</p> <p>A1 (2)</p> <p>M1 A1</p> <p>M1</p> <p>A1</p> <p>A1 (5)</p> <p>M1</p> <p>M1 A1 (3)</p> <p>[14]</p> <p>M1</p> <p>M1</p> <p>A1 (3)</p>
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[FP3 June 2007 Qn 7]

<p>65.</p>	$\frac{d}{dx}(\ln(\tanh x)) = \frac{\operatorname{sech}^2 x}{\tanh x}$ $= \frac{1}{\sinh x \cosh x} = \frac{2}{\sinh 2x} = 2 \operatorname{cosech} 2x$ <p>(*)</p> <p>Notes</p> <p>1M1 Any valid differentiation attempt including $\ln(e^x - e^{-x}) - \ln(e^x + e^{-x})$</p> <p>1A1 c.a.o. (o.e e.g. $\frac{\cosh x}{\sinh x} - \frac{\sinh x}{\cosh x}$)</p> <p>2M1 Proceeding to a hyperbolic expression in $2x$</p> <p>2A1 c.s.o.</p>	<p>M1 A1</p> <p>M1 A1 (4)</p> <p>4</p>
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[FP2 June 2008 Qn 1]

<p>66.</p>	$8\left(\frac{e^x + e^{-x}}{2}\right) - 4\left(\frac{e^x - e^{-x}}{2}\right) = 13$ $4e^x + 4e^{-x} - 2e^x + 2e^{-x} = 13$ $2e^{2x} - 13e^x + 6 = 0 \quad (\text{or equiv.})$ $(2e^x - 1)(e^x - 6) = 0$ $e^x = \frac{1}{2}, \quad e^x = 6$ $x = \ln \frac{1}{2} \text{ (or } -\ln 2), \quad x = \ln 6$ <p>Notes</p> <p>B1 Correctly substituting exponentials for all hyperbolics</p> <p>1M1 To a three term quadratic in e^x</p> <p>1A1 c.a.o. (o.e.)</p> <p>2M1 Solving their equation to $e^x =$</p> <p>2A1ft f.t. their equation.</p> <p>3A1 c.a.o.</p>	<p>B1</p> <p>M1 A1</p> <p>M1 A1ft</p> <p>A1 (6)</p> <p>6</p>
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[FP2 June 2008 Qn 2]

<p>67.</p>	$\int \frac{3}{\sqrt{x^2-9}} dx + \int \frac{x}{\sqrt{x^2-9}} dx$ $= \left[3 \operatorname{arcosh} \frac{x}{3} + \sqrt{x^2-9} \right]$ $= \left[3 \ln \left(\frac{x + \sqrt{x^2-9}}{3} \right) + \sqrt{x^2-9} \right]_5^6$ $= \left(3 \ln \left(\frac{6 + \sqrt{27}}{3} \right) + \sqrt{27} \right) - \left(3 \ln \left(\frac{5+4}{3} \right) + 4 \right)$ $= 3 \ln \frac{6 + \sqrt{27}}{9} + \sqrt{27} - 4 = 3 \ln \frac{2 + \sqrt{3}}{3} + 3\sqrt{3} - 4$ <p>Notes</p> <p>B1 Correctly changing to an integrable form. 1M1 Complete attempt to integrate at least one bit. 1A1 One term correct 2A1 All correct 2DM1 Substituting limits in all. Must have got first M1 3A1 Correctly (no follow through) 4A1 c.s.o.</p>	<p>B1</p> <p>M1 A1 A1</p> <p>M1 A1</p> <p>A1 (7)</p> <p>7</p>
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[FP2 June 2008 Qn 3]

<p>68.</p>	<p>(a) $\frac{dy}{dx} = \frac{3x^2}{\sqrt{1+x^6}}$, At $x = \sqrt{2}$ $\frac{dy}{dx} = \frac{6}{3} = 2$</p> <p>$y - \operatorname{arsinh}(2\sqrt{2}) = 2(x - \sqrt{2})$</p> <p>$y = 2x - 2\sqrt{2} + \ln(3 + 2\sqrt{2})$ (*)</p> <p>(b) $\frac{3a^2}{\sqrt{1+a^6}} = 2$ $9a^4 = 4(1+a^6)$</p> <p>$4a^6 - 9a^4 + 4 = 0$ $(a^2 - 2)(4a^4 - a^2 - 2) = 0$</p> <p>$a^2 = \frac{1 \pm \sqrt{1+32}}{8}$ $a = \sqrt{\frac{1+\sqrt{33}}{8}} \approx 0.92$</p> <p>Notes</p> <p>(a)1M1 Attempt to differentiate need $(1+x^6)^{-\frac{1}{2}}$ at least</p> <p>1A1 correct</p> <p>2A1 c.a.o.</p> <p>2M1 Substituting into straight line equation (linear). Must use $x = \sqrt{2}$</p> <p>3A1 c.s.o.</p> <p>(b)1M1 Their derivative = their gradient (condone x throughout)</p> <p>2M1= A mark cao, any form</p> <p>1A1 quartic cao</p> <p>3M1 Solving their quartic to 'a' =</p> <p>2A1 c.a.o. (a.w.r.t. 0.92 to 2dp)</p>	<p>M1 A1, A1</p> <p>M1</p> <p>A1 (5)</p> <p>M1 A1</p> <p>A1</p> <p>M1 A1 (5)</p> <p>10</p>
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[FP2 June 2008 Qn 4]

<p>69.</p>	<p>(a) $I_n = \int_0^\pi e^x \sin^n x \, dx = [e^x \sin^n x] - \int e^x n \sin^{n-1} x \cos x \, dx$</p> <p>$[e^x \sin^n x - n e^x \sin^{n-1} x \cos x] + n \int e^x (-\sin^n x + (n-1) \cos x \sin^{n-2} x \cos x) \, dx$</p> <p>$[e^x \sin^n x - n e^x \sin^{n-1} x \cos x]_0^\pi = 0$</p> <p>$I_n = -n \int e^x \sin^n x \, dx + n(n-1) \int \sin^{n-2} x (1 - \sin^2 x) \, dx$</p> <p>$I_n = -n I_n + n(n-1) I_{n-2} - n(n-1) I_n \quad I_n = \frac{n(n-1)}{n^2+1} I_{n-2} \quad (*)$</p> <p>(b) $I_4 = \frac{4 \times 3}{17} I_2, \quad = \frac{12}{17} \times \frac{2}{5} I_0$</p> <p>$I_0 = \int_0^\pi e^x \, dx = [e^x]_0^\pi = \dots, \quad I_4 = \frac{24}{85} (e^\pi - 1)$</p>	<p>M1 A1</p> <p>M1 A1</p> <p>B1</p> <p>M1</p> <p>M1 A1 (8)</p> <p>M1, A1</p> <p>M1, A1 (4)</p> <p>12</p>
	<p>(a) 1M1 Complete attempt to use parts once in the right direction need $\sin^{n-1} x$</p> <p>1A1 cao</p> <p>2M1 Attempt to use parts again with sensible choice of parts, not reversing. Need to be differentiating a product.</p> <p>2A1 cao</p> <p>1B1 both = 0 at some point. (doesn't need to be correct, must = 0)</p> <p>3DM1 $I_n =$ expressions in $\int e^x \sin^k x \, dx$ Depends on 2nd M</p> <p>4DM1 Expression in I_n and I_{n-2} to $I_n =$. Depends on 3rd M</p> <p>3A1 c.s.o.</p> <p>(b) 1M1 I_4 in terms of I_2</p> <p>1A1 I_4 correctly in terms of I_0 [o.e.]</p> <p>2M1 $\int e^x \, dx$</p> <p>2A1 c.a.o for I_4.</p>	

[FP2 June 2008 Qn 5]

<p>70.</p>	<p>(a) $\int \cosh x \arctan(\sinh x) dx = \sinh x \arctan(\sinh x) - \int \sinh x \frac{\cosh x}{1 + \sinh^2 x} dx$</p> <p>$= \sinh x \arctan(\sinh x) - \frac{1}{2} \ln(1 + \sinh^2 x) (+C)$</p> <p>Or: $\dots\dots\dots - \int \tanh x dx$</p> <p>$= \sinh x \arctan(\sinh x) - \ln(\cosh x) (+C)$ M1 A1</p> <p><u>Alternative:</u></p> <p>Let $t = \sinh x$, $\frac{dt}{dx} = \cosh x$, $\int \arctan t dt = t \arctan t - \int \frac{t}{1+t^2} dt$ M1 A1 A1</p> <p>$= \dots\dots - \frac{1}{2} \ln(1+t^2)$ M1</p> <p>$= \sinh x \arctan(\sinh x) - \frac{1}{2} \ln(1 + \sinh^2 x) (+C)$ (or equiv.) A1</p> <p>(b) $\frac{1}{10} [\sinh x \arctan(\sinh x) - \ln(\cosh x)]_0^2 = \dots\dots$, 0.34 (*) M1, A1 (2)</p>	<p>M1 A1 A1</p> <p>M1 A1 (5)</p> <p>M1 A1</p> <p>M1 A1 A1</p> <p>M1</p> <p>A1</p> <p>M1, A1 (2)</p> <p>7</p>
	<p>(a) <u>Alternative:</u></p> <p>Let $\tan t = \sinh x$, $\sec^2 t \frac{dt}{dx} = \cosh x$, $\int t \sec^2 t dt = t \tan t - \int \tan t dt$ M1 A1 A1</p> <p>$= \dots\dots - \ln(\sec t)$ M1</p> <p>$= \sinh x \arctan(\sinh x) - \ln \sqrt{1 + \sinh^2 x} (+C)$ (or equiv.) A1</p> <p>Notes</p> <p>(a)1M1 Complete attempt to use parts</p> <p>1A1 One term correct.</p> <p>2A1 All correct.</p> <p>2M1 All integration completed. Need a ln term.</p> <p>3A1 c.a.o. (in x) o.e, any correct form, simplified or not</p> <p>(b)1M1 Use of limits 0 and 2 and 1/10.</p> <p>1A1 c.s.o.</p>	

[FP2 June 2008 Qn 6]

<p>71.</p>	<p>(a) $\frac{2x}{16} - \frac{2y}{9} \frac{dy}{dx} = 0$ $\left[\frac{dx}{dt} = 4 \sec t \tan t, \frac{dy}{dt} = 3 \sec^2 t \right]$</p> <p>$\frac{dy}{dx} = \frac{9x}{16y} = \frac{36 \sec t}{48 \tan t} = \frac{3}{4 \sin t}$</p> <p>$y - 3 \tan t = \frac{-4 \sin t}{3} (x - 4 \sec t)$</p> <p>$4x \sin t + 3y = 25 \tan t$ (*)</p> <p>(b) Using $b^2 = a^2(e^2 - 1)$: $ae = \sqrt{a^2 + b^2} = 5$ or $e = \frac{5}{4}$</p> <p>P: $4 \sec t = 5$ $\cos t = \frac{4}{5}$</p> <p>Coordinates of P: $(4 \sec t, 3 \tan t) = \left(5, \frac{9}{4}\right)$</p> <p>(c) R: $x = \frac{25 \tan t}{4 \sin t} = \frac{125}{16}$</p> <p>Area of PRS: $\frac{1}{2}(SR \times SP) = \frac{1}{2} \times \left(\frac{125}{16} - 5\right) \times \frac{9}{4} = \frac{405}{128} \left(= 3 \frac{21}{128}\right)$</p> <p>Notes</p> <p>(a)1M1 Differentiating 1A1 c.a.o. 2M1 $\frac{dy}{dx}$ in terms of t. 2A1 c.a.o. 3M1 Substituting gradient of normal into straight line equation. 3A1 c.s.o.</p> <p>(b)1M1 Use of $b^2 = a^2(e^2 - 1)$ 1A1 c.a.o. for ae or for e 2M1 Using x coordinate of focus = x coordinate of P, to get single term $f(t) = \text{constant}$. (Allow recovery in (c)) 3M1 Substituting into P coordinates to a number for x and for y. 2A1 c.a.o.</p> <p>(c)1M1 Attempt to find x coordinate of R. 2M1 Substituting into correct template i.e. $\frac{1}{2} \times \text{their } R_x - \text{their } H_x \times \text{their } P_y$ 1A1 c.a.o. 3 s.f. or better.</p>	<p>M1 A1</p> <p>M1 A1</p> <p>M1</p> <p>A1 (6)</p> <p>M1 A1</p> <p>M1</p> <p>M1 A1 (5)</p> <p>M1</p> <p>M1 A1 (3)</p> <p>14</p>
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[FP2 June 2008 Qn 7]

72. (a)	$\begin{pmatrix} 1 & p & 2 \\ 0 & 3 & q \\ 2 & p & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \quad \begin{pmatrix} 1+2p+2 \\ 6+q \\ 2+2p+1 \end{pmatrix} = \begin{pmatrix} \lambda \\ 2\lambda \\ \lambda \end{pmatrix} \text{ is M1 A1 (2 eqns implied)}$ $\begin{pmatrix} 3+2p \\ 6+q \\ 3+2p \end{pmatrix} \Rightarrow 6+q = 2(3+2p) \text{ is M1 A1 (2 eqns, use of parameter implied)}$ $1+2p+2 = \lambda \quad 6+q = 2\lambda \quad \text{M: Two equations, one in } p, \text{ one in } q$ $\therefore 6+q = 6+4p \Rightarrow q = 4p \quad (*)$	M1 A1 A1 (3)
(b)	$\begin{vmatrix} -4 & p & 2 \\ 0 & -2 & 4p \\ 2 & p & -4 \end{vmatrix} = 0 \text{ or } \begin{vmatrix} 1-\lambda & p & 2 \\ 0 & 3-\lambda & 4p \\ 2 & p & 1-\lambda \end{vmatrix} = 0 \text{ (or with } q \text{ instead of } 4p)$ $[-4(8-4p^2) - p(0-8p) + 2(0+4)] = 0 \quad p^2 = 1 \text{ or } pq = 4$ $p < 0 \quad p = -1 \quad q = -4 \quad \text{M: Use } q = 4p \text{ to find value of } p \text{ and of } q$ <p style="text-align: center;">A1: Positive values must be rejected</p>	M1 A1 dM1 A1 (4)
(c)	$-4x - y + 2z = 0, \quad -2y - 4z = 0, \quad 2x - y - 4z = 0 \text{ Any 2 eqns, with value of } p$ $2x = -y = 2z \quad \text{(or 2 separate equations)}$ <p style="text-align: center;">E. vector is $k \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$ (Any non-zero value of k)</p>	M1 M1 A1 (3)
(10)		
	<p>(a) Assuming a value for λ, e.g. $\lambda = 1$, gives M1 A0 A0.</p> <p>(a) Assuming result and working 'backwards':</p> $\begin{pmatrix} 1 & p & 2 \\ 0 & 3 & 4p \\ 2 & p & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3+2p \\ 6+4p \\ 3+2p \end{pmatrix} = (3+2p) \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \text{ gives M1 A0 A0}$ <p>(b) <u>Alternative:</u></p> $\begin{pmatrix} 1 & p & 2 \\ 0 & 3 & 4p \\ 2 & p & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 5 \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ or } \begin{pmatrix} -4 & p & 2 \\ 0 & -2 & 4p \\ 2 & p & -4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \text{ (or } q \text{ instead of } 4p)$ $x + py + 2z = 5x, \quad 3y + 4pz = 5y, \quad 2x + py + z = 5z$ $py + 2z = 4x \text{ (i), } 2pz = y \text{ (ii), } 2x + py = 4z \text{ (iii)}$ <p>From (i) and (iii) $py = 2z$</p> <p>From (ii) $p^2 = 1$ (or equiv. in terms of p and/or q)</p> $p < 0, p = -1, q = -4 \quad \text{A1: Positive values must be rejected}$ <p>(b) Using the eigenvector $\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ scores <u>no marks</u> in this part.</p>	M1 A1 dM1 A1

73. (a)	$\overrightarrow{PQ} = \mathbf{i} - \mathbf{j} + 2\mathbf{k}, \quad \overrightarrow{PR} = 2\mathbf{i} - 3\mathbf{j} + 3\mathbf{k}$ $\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 2 \\ 2 & -3 & 3 \end{vmatrix} = 3\mathbf{i} + \mathbf{j} - \mathbf{k}$	B1 M1 A1 (3)
(b)	$\mathbf{r} \cdot (3\mathbf{i} + \mathbf{j} - \mathbf{k}) = (\mathbf{i} - \mathbf{k}) \cdot (3\mathbf{i} + \mathbf{j} - \mathbf{k})$ $\mathbf{r} \cdot (3\mathbf{i} + \mathbf{j} - \mathbf{k}) = 4 \quad \text{o.e.}$	[may use \overrightarrow{OQ} or \overrightarrow{OR}] M1 A1 ft (2)
(c)	$3x + y - z = 4 \text{ (i)}, x - 2y - 5z = 6 \text{ (ii)}$ $\text{(i)} \times 2 + \text{(ii)} \quad 7x - 7z = 14, \quad x = z + 2 \quad \text{(M: Eliminate one variable)}$ $\text{In (ii)} \quad z + 2 - 2y - 5z = 6, \quad y + 2 = -2z \quad \text{(M: Substitute back)}$ $\therefore x = z + 2 \text{ and } y + 2 = -2z \quad \text{o.e. (} y = 2 - 2z \text{)}$ <p style="text-align: center;">(Two correct '3-term' equations)</p> $\frac{x-2}{(1)} = \frac{y+2}{-2} = \frac{z}{(1)} \quad \text{o.e. (M: Form cartesian equations)}$	M1 M1 A1 M1 A1 (5)
(d)	<p>Writing down direction vector of \overrightarrow{PS} from part (c).</p> $\overrightarrow{QR} = \mathbf{i} - 2\mathbf{j} + \mathbf{k} = \overrightarrow{PS} \quad \therefore PS \parallel QR \quad \text{(or cross-product} = 0 \text{)}$	M1 A1 (2)
(e)	$\overrightarrow{PT} = 4\mathbf{i} + 2\mathbf{j} \quad \text{(or } \overrightarrow{QT} = 3\mathbf{i} + 3\mathbf{j} - 2\mathbf{k} \text{ or } \overrightarrow{RT} = 2\mathbf{i} + 5\mathbf{j} - 3\mathbf{k} \text{)}$ $\text{Volume} = \frac{1}{3} \overrightarrow{PQ} \times \overrightarrow{PR} \cdot \overrightarrow{PT} = \frac{1}{3} (3\mathbf{i} + \mathbf{j} - \mathbf{k}) \cdot (4\mathbf{i} + 2\mathbf{j}) \quad \text{ft from (a)}$ <p>(Instead of $\overrightarrow{PQ} \times \overrightarrow{PR}$, it could be $\overrightarrow{PQ} \times \overrightarrow{QR}$ or $\overrightarrow{PR} \times \overrightarrow{QR}$)</p> $= \frac{1}{3} (12 + 2)$ $= 4\frac{2}{3} \quad \text{o.e.}$	M1 A1 ft A1 (3) (15)
	<p>(a) If both vectors are 'reversed', B0 M1 A1 is possible</p> <p>(c) <u>Alternative:</u></p> $\text{Direction of line: } \begin{pmatrix} 1 \\ -2 \\ -5 \end{pmatrix} \times \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix} = 7 \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \quad \text{M2 A1}$ <p>Through $P(1, 0, -1)$: $\frac{x-1}{1} = \frac{y}{-2} = \frac{z+1}{1} \quad \text{M1 A1}$</p> <p>(e) <u>Alternative:</u></p> $\frac{1}{3} \begin{vmatrix} 4 & 2 & 0 \\ 1 & -1 & 2 \\ 2 & -3 & 3 \end{vmatrix} \text{ gives M1 A1 directly. Here ft from 1}^{\text{st}} \text{ line of part (a).}$ <p><u>Special case:</u></p> $\frac{1}{6} \text{ or } \frac{1}{2} \text{ instead of } \frac{1}{3}, \text{ but method otherwise correct: M1 A0 A0}$	

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