

June 2009
6669 Further Pure Mathematics FP3 (new)
Mark Scheme

Question Number	Scheme	Marks
Q1	$\frac{7}{\cosh x} - \frac{\sinh x}{\cosh x} = 5 \Rightarrow \frac{14}{e^x + e^{-x}} - \frac{(e^x - e^{-x})}{e^x + e^{-x}} = 5$ $\therefore 14 - (e^x - e^{-x}) = 5(e^x + e^{-x}) \Rightarrow 6e^x - 14 + 4e^{-x} = 0$ $\therefore 3e^{2x} - 7e^x + 2 = 0 \Rightarrow (3e^x - 1)(e^x - 2) = 0$ $\therefore e^x = \frac{1}{3} \text{ or } 2$ $x = \ln\left(\frac{1}{3}\right) \text{ or } \ln 2$	M1 A1 M1 A1 B1ft [5]
Alternative (i)	Write $7 - \sinh x = 5 \cosh x$, then use exponential substitution $7 - \frac{1}{2}(e^x - e^{-x}) = \frac{5}{2}(e^x + e^{-x})$ Then proceed as method above.	M1
Alternative (ii)	$(7 \operatorname{sech} x - 5)^2 = \tanh^2 x = 1 - \operatorname{sech}^2 x$ $50 \operatorname{sech}^2 x - 70 \operatorname{sech} x + 24 = 0$ $2(5 \operatorname{sech} x - 3)(5 \operatorname{sech} x - 4) = 0$ $\operatorname{sech} x = \frac{3}{5} \text{ or } \operatorname{sech} x = \frac{4}{5}$ $x = \ln\left(\frac{1}{3}\right) \text{ or } \ln 2$	M1 A1 M1 A1 B1ft
Q2 (a)	$\mathbf{b} \times \mathbf{c} = 0\mathbf{i} + 5\mathbf{j} + 5\mathbf{k}$	M1 A1 A1 (3)
(b)	$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = 0 + 5 = 5$	M1 A1 ft (2)
(c)	Area of triangle $OBC = \frac{1}{2} 5\mathbf{j} + 5\mathbf{k} = \frac{5}{2} \sqrt{2}$	M1 A1 (2)
(d)	Volume of tetrahedron = $\frac{1}{6} \times 5 = \frac{5}{6}$	B1 ft (1) [8]

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<p>Q3 (a)</p> $\begin{vmatrix} 6-\lambda & 1 & -1 \\ 0 & 7-\lambda & 0 \\ 3 & -1 & 2-\lambda \end{vmatrix} = 0 \quad \therefore (6-\lambda)(7-\lambda)(2-\lambda) + 3(7-\lambda) = 0$ <p>$(7-\lambda) = 0$ verifies $\lambda = 7$ is an eigenvalue (can be seen anywhere)</p> <p>$\therefore (7-\lambda)\{12-8\lambda+\lambda^2+3\} = 0 \quad \therefore (7-\lambda)\{\lambda^2-8\lambda+15\} = 0$</p> <p>$\therefore (7-\lambda)(\lambda-5)(\lambda-3) = 0$ and 3 and 5 are the other two eigenvalues</p> <p>(b)</p> $\text{Set } \begin{pmatrix} 6 & 1 & -1 \\ 0 & 7 & 0 \\ 3 & -1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 7 \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} -1 & 1 & -1 \\ 0 & 0 & 0 \\ 3 & -1 & -5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ <p>Solve $-x + y - z = 0$ and $3x - y - 5z = 0$ to obtain $x = 3z$ or $y = 4z$ and a second equation which can contain 3 variables</p> <p>Obtain eigenvector as $3\mathbf{i} + 4\mathbf{j} + \mathbf{k}$ (or multiple)</p>		<p>M1</p> <p>M1</p> <p>A1</p> <p>M1 A1 (5)</p> <p>M1</p> <p>M1 A1</p> <p>A1 (4)</p> <p>[9]</p>

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Q4 (a)	$\frac{dy}{dx} = \frac{1}{2} x^{-\frac{1}{2}} \times \frac{1}{\sqrt{1+(\sqrt{x})^2}}$ $\therefore \frac{dy}{dx} = \frac{\frac{1}{2} x^{-\frac{1}{2}}}{\sqrt{1+x}} \quad \left(= \frac{1}{2\sqrt{x(1+x)}} \right)$	B1, M1 A1 (3)
(b)	$\therefore \int_{\frac{1}{4}}^4 \frac{1}{\sqrt{x(x+1)}} dx = [2\operatorname{ar sinh} \sqrt{x}]_{\frac{1}{4}}^4$ $= [2\operatorname{ar sinh} 2 - 2\operatorname{ar sinh}(\frac{1}{2})]$ $= [2\ln(2+\sqrt{5})] - [2\ln(\frac{1}{2} + \sqrt{\frac{5}{4}})]$ $2\ln \frac{(2+\sqrt{5})}{(\frac{1}{2} + \sqrt{\frac{5}{4}})} = 2\ln \frac{2(2+\sqrt{5})}{(1+\sqrt{5})} = 2\ln \frac{2(\sqrt{5}+2)(\sqrt{5}-1)}{(\sqrt{5}+1)(\sqrt{5}-1)} = 2\ln \frac{(3+\sqrt{5})}{2}$ $= \ln \frac{(3+\sqrt{5})(3+\sqrt{5})}{4} = \ln \frac{14+6\sqrt{5}}{4} = \ln \left(\frac{7}{2} + \frac{3\sqrt{5}}{2} \right)$	M1 M1 M1 M1 A1 A1 (6) [9]
Alternative (i) for part (a)	<p>Use $\sinh y = \sqrt{x}$ and state $\cosh y \frac{dy}{dx} = \frac{1}{2} x^{-\frac{1}{2}}$</p> $\therefore \frac{dy}{dx} = \frac{\frac{1}{2} x^{-\frac{1}{2}}}{\sqrt{1+\sinh^2 y}} = \frac{\frac{1}{2} x^{-\frac{1}{2}}}{\sqrt{1+(\sqrt{x})^2}}$ $\therefore \frac{dy}{dx} = \frac{\frac{1}{2} x^{-\frac{1}{2}}}{\sqrt{1+x}} \quad \left(= \frac{1}{2\sqrt{x(1+x)}} \right)$	B1 M1 A1 (3)
Alternative (i) for part (b) Alternative (ii) for part (b)	<p>Use $x = \tan^2 \theta$, $\frac{dx}{d\theta} = 2 \tan \theta \sec^2 \theta$ to give $2 \int \sec \theta d\theta = [2 \ln(\sec \theta + \tan \theta)]$</p> $= [2 \ln(\sec \theta + \tan \theta)]_{\tan \theta = \frac{1}{2}}^{\tan \theta = 2}$ <p>i.e. use of limits then proceed as before from line 3 of scheme</p> <p>Use $\int \frac{1}{\sqrt{[(x+\frac{1}{2})^2 - \frac{1}{4}]}} dx = \operatorname{arcosh} \frac{x+\frac{1}{2}}{\frac{1}{2}}$</p> $= [\operatorname{arcosh} 9 - \operatorname{arcosh}(\frac{3}{2})]$ $= [\ln(9+\sqrt{80})] - [\ln(\frac{3}{2} + \frac{1}{2}\sqrt{5})]$ $= \ln \frac{(9+\sqrt{80})}{(\frac{3}{2} + \frac{1}{2}\sqrt{5})} = \ln \frac{2(9+\sqrt{80})(3-\sqrt{5})}{(3+\sqrt{5})(3-\sqrt{5})}$ $= \ln \frac{2(9+4\sqrt{5})(3-\sqrt{5})}{(3+\sqrt{5})(3-\sqrt{5})} = \ln \left(\frac{7}{2} + \frac{3\sqrt{5}}{2} \right)$	M1 M1 M1 M1 M1 A1 A1 (6) [9]

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<p>Q5 (a)</p> <p>(b)</p> <p>(c)</p>	<p>$-(25 - x^2)^{\frac{1}{2}} + c$</p> <p>$I_n = \int x^{n-1} \cdot \frac{x}{\sqrt{(25 - x^2)}} dx = -x^{n-1} \sqrt{25 - x^2} + \int (n-1)x^{n-2} \sqrt{(25 - x^2)} dx$</p> <p>$I_n = \left[-x^{n-1} \sqrt{25 - x^2} \right]_0^5 + \int_0^5 \frac{(n-1)x^{n-2} (25 - x^2)}{\sqrt{(25 - x^2)}} dx$</p> <p>$I_n = 0 + 25(n-1) I_{n-2} - (n-1) I_n$</p> <p>$\therefore nI_n = 25(n-1)I_{n-2} \text{ and so } I_n = \frac{25(n-1)}{n} I_{n-2} \quad *$</p> <p>$I_0 = \int_0^5 \frac{1}{\sqrt{(25 - x^2)}} dx = \left[\arcsin\left(\frac{x}{5}\right) \right]_0^5 = \frac{\pi}{2}$</p> <p>$I_4 = \frac{25 \times 3}{4} \times \frac{25 \times 1}{2} I_0 = \frac{1875}{16} \pi$</p>	<p>M1A1 (2)</p> <p>M1 A1ft</p> <p>M1</p> <p>M1</p> <p>A1 (5)</p> <p>M1 A1</p> <p>M1 A1 (4)</p> <p>[11]</p>
<p>Alternative for (b)</p>	<p>Using substitution $x = 5\sin\theta$</p> <p>$I_n = 5^n \int_0^{\frac{\pi}{2}} \sin^n \theta d\theta = \left[-5^n \sin^{n-1} \theta \cos \theta \right]_0^{\frac{\pi}{2}} + 5^n (n-1) \int_0^{\frac{\pi}{2}} \sin^{n-2} \theta \cos^2 \theta d\theta$</p> <p>$= \left[-5^n \sin^{n-1} \theta \cos \theta \right]_0^{\frac{\pi}{2}} + 5^n (n-1) \int_0^{\frac{\pi}{2}} \sin^{n-2} \theta (1 - \sin^2 \theta) d\theta$</p> <p>$I_n = 0 + 25(n-1) I_{n-2} - (n-1) I_n$</p> <p>$\therefore nI_n = 25(n-1)I_{n-2} \text{ and so } I_n = \frac{25(n-1)}{n} I_{n-2} \quad *$</p> <p>(need to see that $I_{n-2} = 5^{n-2} \int_0^{\frac{\pi}{2}} \sin^{n-2} \theta d\theta$ for final A1)</p>	<p>M1A1</p> <p>M1</p> <p>M1</p> <p>A1 (5)</p>

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Q6 (a)	$\frac{x^2}{a^2} - \frac{(mx+c)^2}{b^2} = 1 \quad \text{and so} \quad b^2x^2 - a^2(mx+c)^2 = a^2b^2$ $\therefore (b^2 - a^2m^2)x^2 - 2a^2mcx - a^2(c^2 + b^2) = 0$ <p style="text-align: center;">Or $(a^2m^2 - b^2)x^2 + 2a^2mcx + a^2(c^2 + b^2) = 0$ *</p>	M1
(b)	$(2a^2mc)^2 = 4(a^2m^2 - b^2) \times a^2(c^2 + b^2)$ $4a^4m^2c^2 = -4a^2(b^2c^2 + b^4 - a^2m^2c^2 - a^2m^2b^2)$ $c^2 = a^2m^2 - b^2 \quad \text{or} \quad a^2m^2 = b^2 + c^2 \quad *$	M1 A1 (2)
(c)	<p>Substitute (1, 4) into $y = mx + c$ to give $4 = m + c$ and Substitute $a = 5$ and $b = 4$ into $c^2 = a^2m^2 - b^2$ to give $c^2 = 25m^2 - 16$ Solve simultaneous equations to eliminate m or c : $(4 - m)^2 = 25m^2 - 16$ To obtain $24m^2 + 8m - 32 = 0$ Solve to obtain $8(3m + 4)(m - 1) = 0 \dots m = \dots$ or...</p> $m = 1 \quad \text{or} \quad -\frac{4}{3}$ <p>Substitute to get $c = 3$ or $\frac{16}{3}$</p> <p>Lines are $y = x + 3$ and $3y + 4x = 16$</p>	B1 M1 A1 M1 A1 M1 A1 (7) [11]

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Q7 (a)	<p>If the lines meet, $-1+3\lambda = -4+3\mu$ and $2+4\lambda = 2\mu$</p> <p>Solve to give $\lambda = 0$ ($\mu = 1$ but this need not be seen).</p> <p>Also $1-\lambda = \alpha$ and so $\alpha = 1$.</p> <p>(b) $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 3 & 4 \\ 0 & 3 & 2 \end{vmatrix} = -6\mathbf{i}+2\mathbf{j}-3\mathbf{k}$ is perpendicular to both lines and hence to the plane</p> <p>The plane has equation $\mathbf{r}\cdot\mathbf{n}=\mathbf{a}\cdot\mathbf{n}$, which is $-6x + 2y - 3z = -14$, i.e. $-6x + 2y - 3z + 14 = 0$.</p>	<p>M1</p> <p>M1 A1</p> <p>B1</p> <p>(4)</p> <p>M1 A1</p> <p>M1</p> <p>A1 o.a.e.</p> <p>(4)</p>
OR (b)	<p>Alternative scheme</p> <p>Use $(1, -1, 2)$ and $(\alpha, -4, 0)$ in equation $ax+by+cz+d=0$</p> <p>And third point so three equations, and attempt to solve</p> <p>Obtain $6x - 2y + 3z =$ $(6x - 2y + 3z) - 14 = 0$</p>	<p>M1</p> <p>M1</p> <p>A1</p> <p>A1 o.a.e.</p> <p>(4)</p>
(c)	<p>$(\mathbf{a}_1 - \mathbf{a}_2) = \mathbf{i} - 3\mathbf{j} - 2\mathbf{k}$</p> <p>Use formula $\frac{(\mathbf{a}_1 - \mathbf{a}_2) \cdot \mathbf{n}}{ \mathbf{n} } = \frac{(\mathbf{i} - 3\mathbf{j} - 2\mathbf{k}) \cdot (-6\mathbf{i} + 2\mathbf{j} - 3\mathbf{k})}{\sqrt{(36+4+9)}} = \left(\frac{-6}{7}\right)$</p> <p>Distance is $\frac{6}{7}$</p>	<p>M1</p> <p>M1</p> <p>A1</p> <p>(3)</p> <p>[11]</p>

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Q8 (a)	$\frac{dx}{d\theta} = -3\sin\theta, \quad \frac{dy}{d\theta} = 5\cos\theta$ <p>so $S = 2\pi \int 5\sin\theta \sqrt{(-3\sin\theta)^2 + (5\cos\theta)^2} d\theta$</p> $\therefore S = 2\pi \int 5\sin\theta \sqrt{9 - 9\cos^2\theta + 25\cos^2\theta} d\theta$ <p>Let $c = \cos\theta$, $\frac{dc}{d\theta} = -\sin\theta$, limits 0 and $\frac{\pi}{2}$ become 1 and 0</p> <p>So $S = k\pi \int_0^{\alpha} \sqrt{16c^2 + 9} dc$, where $k = 10$, and α is 1</p>	<p>B1</p> <p>M1</p> <p>M1</p> <p>M1</p> <p>A1, A1 (6)</p>
(b)	<p>Let $c = \frac{3}{4}\sinh u$. Then $\frac{dc}{du} = \frac{3}{4}\cosh u$</p> <p>So $S = k\pi \int \sqrt{9\sinh^2 u + 9} \frac{3}{4}\cosh u du$</p> $= k\pi \int \frac{9}{4}\cosh^2 u du = k\pi \int \frac{9}{8}(\cosh 2u + 1) du$ $= k\pi \left[\frac{9}{16}\sinh 2u + \frac{9}{8}u \right]_0^d$ $= \frac{45\pi}{4} \left[\frac{20}{9} + \ln 3 \right] \quad \text{i.e. } \underline{117}$	<p>M1</p> <p>A1</p> <p>M1</p> <p>A1</p> <p>B1</p> <p>(5)</p> <p>[11]</p>