# FP3 Specimen Paper Prime 2

1. $A=\begin{pmatrix}7 & 6\\ 6 & 2\end{pmatrix}$ , $A-\lambda I=\begin{pmatrix}7-\lambda & 6\\ 6 & 2-\lambda\end{pmatrix}$

$$\det(A-\lambda I) = (7-\lambda)(2-\lambda)-36$$
$$= -22-9\lambda+\lambda^2 = 0$$

$\det(A-\lambda I)=0 \Rightarrow \lambda^2-9\lambda-22=0$

$\therefore (\lambda-11)(\lambda+2)=0$

E. val :

$\Rightarrow$ $\lambda=-2$, $\lambda=11$

2. $\frac{9}{2}(e^x+e^{-x})-\frac{6}{2}(e^x-e^{-x})=7$

$\therefore \frac{9}{2}e^x+\frac{9}{2}e^{-x}-3e^x+3e^{-x}=7$

$\therefore \frac{3}{2}e^x+\frac{15}{2}e^{-x}=7$

$\Rightarrow 3e^x+15e^{-x}=14$

$\Rightarrow 3e^{2x}+15=14e^x$

$\Rightarrow 3e^{2x}-14e^x+15=0$

$(e^x-3)(3e^x-5)=0$ $\quad e^x=\frac{5}{3}$

$\Downarrow$

$\therefore e^x=3 \Rightarrow x=\ln 3$ $\quad x=\ln\frac{5}{3}$

3. Let length $= s$

$$s = \int_0^{2\pi} \sqrt{\left(\frac{\partial x}{\partial t}\right)^2 + \left(\frac{\partial y}{\partial t}\right)^2}\ \partial t$$

$$x = a(t - \sin t) \Rightarrow \frac{\partial x}{\partial t} = a(1 - \cos t)$$

$$y = a(1 - \cos t) \Rightarrow \frac{\partial y}{\partial t} = a(\sin t)$$

$$\therefore \quad s = \int_0^{2\pi} \sqrt{a^2(1-\cos t)^2 + a^2 \sin^2 t}\ dt$$

$$= \int_0^{2\pi} \sqrt{a^2(1 - 2\cos t + \cos^2 t + \sin^2 t)}\ dt$$

$$= \int_0^{2\pi} \sqrt{a^2(2 - 2\cos t)}\ \partial t$$

$$= \int_0^{2\pi} \sqrt{2a^2(1 - \cos t)}\ dt$$

$\cos 2t = \cos^2 t - \sin^2 t$

$\cos t = \cos^2 \frac{t}{2} - \sin^2 \frac{t}{2}$

$\cos^2 \frac{t}{2} - (1 - \cos^2 \frac{t}{2}$

$\underline{\cos t = 2\cos^2 \frac{t}{2} - 1}$

$\cancel{u = \sqrt{1-\cos t} \quad u' = \frac{1}{2}(1-\cos t) \sin t \quad v' = 1 \quad v = u}$

$$= \sqrt{2}\, a \int_0^{2\pi} \sqrt{1 - \cos t}\ dt$$

$\cancel{a = \cos} \quad \cancel{u = \cos t \quad \frac{du}{dt} = -\sin t \quad dt = \frac{du}{-\sin t}}$

$$= a\sqrt{2} \int_0^{2\pi} \sqrt{2\cos^2 \frac{t}{2}}\ \partial t = a\sqrt{2}\sqrt{2} \int \sin \frac{t}{2}\ dt$$

$$= 2a\left[-2\cos \frac{t}{2}\right]^{2\pi} = 8a$$

4. $\cancel{\int (x^2+4)^{1/2} \, \partial x}$

$= 2a(2 - -2)$

$= 8a$

$s = 8a$

$\cancel{u = (x^2+4)^{1/2}}$ $\quad$ $\cancel{u' = \frac{1}{2}(x^2+4)^{-1/2} \times 2x = x\sqrt{x^2+4}}$

$\cancel{v' = 1}$ $\quad$ $\cancel{v = x}$

$\cancel{x\sqrt{(x^2+4)} - \int x^2\sqrt{x^2+4}}$

$\cancel{s^2+c^2=}$

$c^2 - s^2 = 1$

$c^2 = 1 + s^2 = s^2 + 1$

4. $\int (x^2+4)^{1/2} \, \partial x$

$x = 2\sinh\theta$ $\quad$ $\frac{x}{2} = \sinh\theta$

$x^2 = 4\sinh^2\theta$ $\quad$ $\theta = \operatorname{arsinh}\frac{x}{2}$

$= \ln\left(\frac{x}{2} + \sqrt{\frac{x^2}{4}+1}\right)$

$= \int (4\sinh^2\theta + 4)^{1/2} \cdot 2\cosh\theta \, \partial\theta$

$\frac{\partial x}{\partial\theta} = 2\cosh\theta$

$dx = 2\cosh\theta \, \partial\theta$

$= 2\int \sqrt{4(\sinh^2\theta + 1)} \cdot \cosh\theta \, \partial\theta$

$\cos 2x = \cos^2 - \sin^2$

$= 2\int 2\cosh^2\theta \, \partial\theta$

$\cosh 2x = \cosh^2 x + \sinh^2 x$

$\cosh 2x = 2\cosh^2 x - 1$

$= 4\int \cosh^2\theta \, \partial\theta$

$\frac{\cosh(2x)+1}{2} = \cosh^2 x$

$= \frac{4}{2}\int \cosh(2\theta) + 1 \, \partial\theta$

$\sinh 2\theta = 2\sinh\theta\cosh\theta$

$= x\sqrt{1+\sinh^2\theta}$

$= x\sqrt{1+\frac{x^2}{4}}$

$= 2\left[\frac{1}{2}\sinh(2\theta) + \theta\right] + C$

$= \sinh 2\theta + 2\theta + C = x\sqrt{1+\frac{x^2}{4}} + 2\ln\left(\frac{x}{2} + \sqrt{\frac{x^2}{4}+1}\right) + C$

5 (a) $y = \arcsin x$

$\therefore \quad \sin y = x \qquad\qquad \cos y \equiv \sqrt{1-\sin^2 y} = \sqrt{1-x^2}$

$\therefore \quad \cos y \frac{\partial y}{\partial x} = 1$

$\therefore \quad \frac{\partial y}{\partial x} = \frac{1}{\cos(y)}$

$\therefore \quad \frac{\partial y}{\partial x} = \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}}$ //

$(1-x^2)^{-1/2}$

(b) $\frac{\partial^2 y}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{1}{\sqrt{1-x^2}}\right) = -\frac{1}{2}(1-x^2)^{-\frac{3}{2}}(-2x)$

$= x(1-x^2)^{-3/2}$

$\frac{\partial^2 y}{\partial x^2} = \frac{x}{(\sqrt{1-x^2})^3}$

$\therefore \quad (1-x^2)\frac{\partial^2 y}{\partial x^2} = x(1-x^2)^{-3/2}(1-x^2)$

$= x(1-x^2)^{-1/2}$

$= x \cdot \frac{1}{\sqrt{1-x^2}}$

$= x \cdot \frac{\partial y}{\partial x}$

$\Rightarrow \quad (1-x^2)\frac{\partial^2 y}{\partial x^2} - x\frac{\partial y}{\partial x} = 0$ //

6.(a).

$$I_n = \int_0^{\pi/2} x^n \sin x \, \partial x$$

Let $u = x^n$ $\quad u' = nx^{n-1}$

$v' = \sin x$ $\quad v = -\cos x$ .

$$\therefore \quad I_n = \left[-x^n \cos x\right]_0^{\frac{\pi}{2}} + n\int_0^{\pi/2} x^{n-1} \cos x \, \partial x$$

$$\cancel{= 0} \quad \cancel{= 0 +} \quad = n\int_0^{\pi/2} x^{n-1} \cos x \, dx$$

$u = x^{n-1}$ $\quad u' = (n-1)x^{n-2}$

$v' = \cos x$ $\quad v = \sin x$

$$\therefore \quad I_n = \cancel{n[} \; n\left( \left[x^{n-1} \sin x\right]_0^{\pi/2} - (n-1)\int_0^{\pi/2} x^{n-2} \sin x \right)$$

$$= n\left( \left(\frac{\pi}{2}\right)^{n-1} - (n-1) I_{n-2} \right)$$

$$= n\left(\frac{\pi}{2}\right)^{n-1} - n(n-1) I_{n-2}$$

as required.

(b) $I_3 = 3\left(\frac{\pi}{2}\right)^2 - 6\, I_1$

$u = x \quad u' = 1$

$v' = \sin x \quad v = -\cos x$

$$= \frac{3}{4}\pi^2 - 6\int_0^{\pi/2} x \sin x \, dx$$

$$= \frac{3}{4}\pi^2 - 6\left(\left[-x\cos x\right]_0^{\pi/2} + \int_0^{\pi/2} \cos x \, dx\right)$$

$$= \frac{3\pi^2}{4} - 6\left(\left[\sin x\right]_0^{\pi/2}\right)$$

$$= \frac{3\pi^2}{4} - 6(1-0)$$

$$= \frac{3\pi^2}{4} - 6$$

7(a). $\begin{pmatrix} 1 & x & -1 \\ 3 & 0 & 2 \\ 1 & 1 & 0 \end{pmatrix}$

$$\det[A(x)] = \cancel{2(} \; -2 - x(-2) - 3 = \underline{\underline{2x-5}}$$

$\cancel{C}$ $\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$

$$M = \begin{pmatrix} -2 & -2 & 3 \\ 1 & 1 & 1-x \\ 2x & 5 & -3x \end{pmatrix}$$

$$C = \begin{pmatrix} -2 & 2 & 3 \\ -1 & 1 & x-1 \\ 2x & -5 & -3x \end{pmatrix}$$

$$C^{T} = \begin{pmatrix} -2 & -1 & 2x \\ 2 & 1 & -5 \\ 3 & x-1 & -3x \end{pmatrix}$$

$$\therefore \left[A(x)\right]^{-1} = \frac{1}{2x-5}\begin{pmatrix} -2 & -1 & 2x \\ 2 & 1 & -5 \\ 3 & x-1 & -3x \end{pmatrix}$$

(b) $\begin{pmatrix} 1 & 3 & -1 \\ 3 & 0 & 2 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} p \\ q \\ r \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$

$\therefore \quad \begin{pmatrix} p + 3q - r \\ 3p + 2r \\ p + q \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$

$\therefore \quad p + 3q - r = 2$

$\Rightarrow r = p + 3q - 2$

$\therefore \quad 3p + 2r = 3p + 2p + 6q - 4 = 5p + 6q - 4$

$5p + 6q - 4 = 3 \Rightarrow \quad 5p + 6q = 7$

$p + q = 4$

$\therefore \quad 5p + 5q = 20$

$5p + 6q = 7 \Rightarrow 5p + 5q + q = 7$

$\Rightarrow 20 + q = 7$

$\Rightarrow q = -13$

$\therefore 5p = 20 - 5(-13) = 85$

$\Rightarrow p = 17$

$\cancel{2r =} \quad r = \dfrac{3 - 3p}{2} = -24$

$\therefore \quad p = 17$

$q = -13$

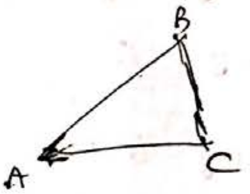
$r = -24$

8.
(a) $\vec{AB} = \begin{pmatrix}1\\3\\0\end{pmatrix} - \begin{pmatrix}2\\0\\1\end{pmatrix} = \begin{pmatrix}-1\\3\\-1\end{pmatrix}$

$\vec{AC} = \begin{pmatrix}1\\3\\2\end{pmatrix} - \begin{pmatrix}2\\0\\1\end{pmatrix} = \begin{pmatrix}-1\\3\\1\end{pmatrix}$

$\therefore \vec{AB} \times \vec{AC} = \begin{matrix} \times & -1 & 3 & \times & -1 & \times & -1 & \times & 3 & -1 \\ & -1 & 3 & & 1 & & -1 & & 3 & 1 \end{matrix}$

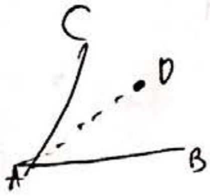
$= \begin{pmatrix}6\\2\\0\end{pmatrix}$



$\therefore \text{Area} = \left|\frac{1}{2}\begin{pmatrix}6\\2\\0\end{pmatrix}\right| = \left|\begin{pmatrix}3\\1\\0\end{pmatrix}\right| = \sqrt{3^2+1}$

$= \sqrt{10}$

(b)



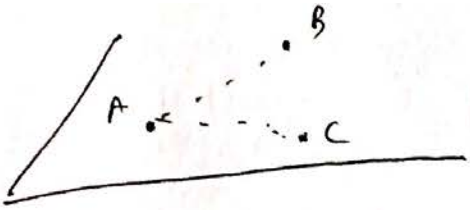
$AD = \begin{pmatrix}0\\4\\1\end{pmatrix} - \begin{pmatrix}2\\0\\1\end{pmatrix} = \begin{pmatrix}-2\\4\\0\end{pmatrix}$

$\text{Area} = \frac{1}{6}\left|\vec{AD}\cdot(\vec{AB}\times\vec{AC})\right|$

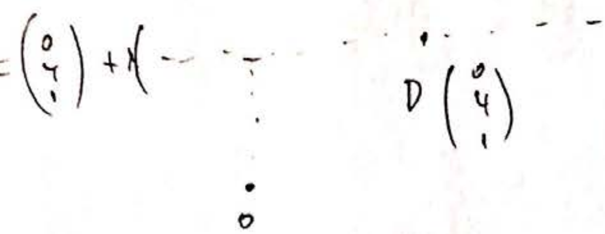
$= \frac{1}{6}\left|\begin{pmatrix}-2\\4\\0\end{pmatrix}\cdot\begin{pmatrix}6\\2\\0\end{pmatrix}\right|$

$= \frac{1}{6}\left|-12+8\right| = \frac{2}{3}$

(c)




$\underset{\sim}{n} = AB \times AC = \begin{pmatrix}6\\2\\0\end{pmatrix}$




$\Rightarrow \underset{\sim}{r} \cdot \begin{pmatrix}6\\2\\0\end{pmatrix} = \begin{pmatrix}2\\0\\1\end{pmatrix} \cdot \begin{pmatrix}6\\2\\0\end{pmatrix}$

$\underset{\sim}{r} \cdot \begin{pmatrix}6\\2\\0\end{pmatrix} = 12$

$\frac{1}{\sqrt{36+4}}$

$\hat{\underset{\sim}{n}} = \frac{1}{\sqrt{40}} \begin{pmatrix}6\\2\\0\end{pmatrix}$

$\cancel{\begin{pmatrix}1\\3\\0\end{pmatrix} \begin{pmatrix}6\\2\\0\end{pmatrix}}$

$\underset{\sim}{r} \cdot \underset{\sim}{n} = 12$

$\Rightarrow$ eqn of plane:

$6x + 2y = 12$

$3x + y = 6$

$3x + y - 6 = 0$

$D = \begin{pmatrix}0\\4\\1\end{pmatrix}$

$\therefore$ distance $= \left| \frac{3(0) + 4 - 6}{\sqrt{3^2 + 1^2}} \right|$

$= \left| \frac{-2}{\sqrt{10}} \right| = \frac{\sqrt{10}}{5}$

9(a). $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

$\therefore \frac{2}{a^2}x - \frac{2}{b^2}y\frac{\partial y}{\partial x} = 0$

$\therefore \frac{2}{a^2}x = \frac{2}{b^2}y\frac{\partial y}{\partial x}$

$\frac{1}{a^2}x = \frac{1}{b^2}y\frac{\partial y}{\partial x}$

$\frac{b^2}{a^2}\frac{x}{y} = \frac{\partial y}{\partial x}$ $\qquad \frac{\frac{1}{\cos}}{\frac{\sin}{\cos}}$

$\left(\frac{\partial y}{\partial x}\right)_{\substack{x = a\sec\theta \\ y = b\tan\theta}} = \frac{b^2}{a^2} \cdot \frac{a\sec\theta}{b\tan\theta} = \frac{b^{\cancel{2}}\cancel{a}}{a^{\cancel{2}}\cancel{b}}\operatorname{cosec}\theta$

$= \frac{b}{a}\operatorname{cosec}\theta$

$\therefore m_N = \frac{-1}{\frac{b}{a}\operatorname{cosec}\theta} = -\frac{1}{\frac{b}{a}}\sin\theta = -\frac{a}{b}\sin\theta$

$\therefore y - b\tan\theta = -\frac{a}{b}\sin\theta(x - a\sec\theta)$

$\therefore by - b^2\tan\theta = -a\sin\theta(x - a\sec\theta)$

$\therefore by - b^2\tan\theta = -ax\sin\theta + a^2\sin\theta\sec\theta$

$\therefore by\tan x\sin\theta = a^2\frac{\sin\theta}{\cos\theta} + b^2\tan\theta$

$by\tan x\sin\theta = a^2\tan\theta + b^2\tan\theta$

$\therefore by + ax\sin\theta = (a^2 + b^2)\tan\theta$ as required

(b) when $x=0$

~~At~~ $by = (a^2+b^2)\tan\theta$

$$\Rightarrow y = \frac{a^2+b^2}{b}\tan\theta$$

$$\left(0, \frac{a^2+b^2}{b}\tan\theta\right)$$

when $y=0$

$$ax\sin\theta = (a^2+b^2)\tan\theta$$

$\frac{\frac{1}{\cos}\sin}{\sin}$

$$\therefore x = \frac{(a^2+b^2)}{a}\sec\theta$$

$$\therefore \left(\frac{a^2+b^2}{a}\sec\theta, 0\right)$$

$$\therefore M = \left(\overset{x=}{\frac{a^2+b^2}{2a}\sec\theta}, \overset{y=}{\frac{a^2+b^2}{2b}\tan\theta}\right)$$

$\frac{s^2+c^2}{c^2} = \frac{1}{c^2}$

$t^2+1 = \sec^2$

$$X = \frac{a^2+b^2}{2a}\sec\theta \qquad Y = \frac{a^2+b^2}{2b}\tan\theta$$

$$\sec\theta = \sqrt{\tan^2\theta+1} \quad \& \quad \frac{2bY}{a^2+b^2} = \tan\theta$$

$$\therefore X = \frac{a^2+b^2}{2a}\sqrt{\tan^2\theta+1} = \frac{a^2+b^2}{2a}\sqrt{\frac{4b^2Y^2}{(a^2+b^2)^2}+1}$$

$$\therefore X^2 = \frac{(a^2+b^2)^2}{4a^2}\left(\frac{4b^2Y^2}{(a^2+b^2)^2}+1\right)$$

$$x^2 = \frac{4b^2y^2}{4a^2} + \frac{(a^2+b^2)^2}{4a^2}$$

$$x = \frac{b^2}{a^2}y^2 + \frac{(a^2+b^2)^2}{4a^2} \quad \times 4a^2$$

$$x = \frac{b^2}{a^2}y^2 + \frac{(a^2+b^2)^2}{4a^2}$$

$$4a^2x^2 = 4b^2y^2 + (a^2+b^2)^2$$

$$\therefore 4a^2x^2 - 4b^2y^2 = (a^2+b^2)^2$$