FP3 June 2017 (MA)

Q1) let $y = \operatorname{arsinh}(t)$

$t = \tanh x$

$\frac{dt}{dx} = \operatorname{sech}^2 x$

then $\frac{dy}{dt} = \frac{1}{\sqrt{1+t^2}}$

$\therefore$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{\operatorname{sech}^2 x}{\sqrt{1+t^2}} = \frac{\operatorname{sech}^2 x}{\sqrt{1+\tanh^2 x}}$$

Q2a) $x = 6\cos\theta$ $\quad y = 5\sin\theta$

$\frac{dx}{dt} = -6\sin\theta$ $\quad \frac{dy}{dt} = 5\cos\theta$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = -\frac{5\cos\theta}{6\sin\theta} \quad \therefore \; m_\perp = \frac{6\sin\theta}{5\cos\theta}$$

(perpendicular)

$$y - 5\sin\theta = \frac{6\sin\theta}{5\cos\theta}(x - 6\cos\theta)$$

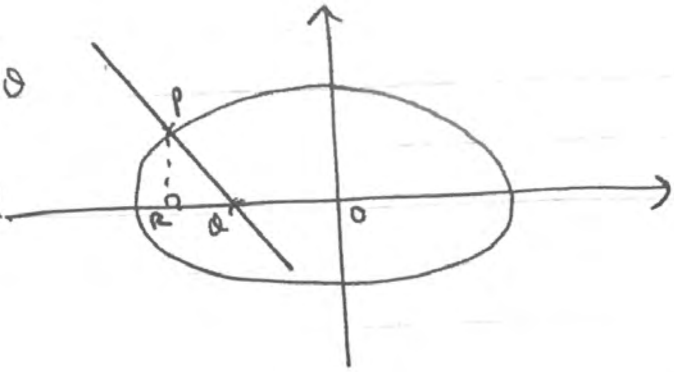
$$y - 5\sin\theta = \left(\frac{6\sin\theta}{5\cos\theta}\right)x - \frac{36\sin\theta}{5}$$

$\times 5\cos\theta$: $\quad 5y\cos\theta - 25\sin\theta\cos\theta = 6x\sin\theta - 36\sin\theta\cos\theta$

$\Rightarrow \quad 6x\sin\theta - 5y\cos\theta = (36-25)\sin\theta\cos\theta$

$\Rightarrow \quad 6x\sin\theta - 5y\cos\theta = 11\sin\theta\cos\theta$

b) $\underline{y=0}$: $6x\cancel{\sin\theta} = 11\cancel{\sin\theta}\cos\theta$



$(x>0)$ $\quad x = \frac{11}{6}\cos\theta = OQ$ //

so $OQ = \frac{11}{6}\cos\theta$

and $OR = 6\cos\theta$

$$\frac{OQ}{OR} = \frac{\frac{11}{6}\cancel{\cos\theta}}{6\cancel{\cos\theta}} = \frac{11}{36} \text{ //}$$

$\underline{\text{finding } e^2}$: $\quad b^2 = a^2(1-e^2)$

$$25 = 36(1-e^2)$$

$$\frac{25}{36} = 1-e^2$$

$$\therefore \; e^2 = 1 - \frac{25}{36} = \frac{11}{36} = \frac{OQ}{OR}$$

Q3a) $LHS = \cosh 2x = \frac{e^{2x}+e^{-2x}}{2}$

$RHS = 2\cosh^2 x - 1 = 2\left(\frac{e^{x}+e^{-x}}{2}\right)\left(\frac{e^{x}+e^{-x}}{2}\right) - 1$

$= \left(\frac{e^{2x}+2+e^{-2x}}{2}\right) - 1$

$= \frac{e^{2x}+2+e^{-2x}-2}{2} = \frac{e^{2x}+e^{-2x}}{2}$

$= \cosh 2x = LHS$

b) $29\cosh x - 3(2\cosh^2 x - 1) = 38$

$29\cosh x - 6\cosh^2 x + 3 = 38$

$6\cosh^2 x - 29\cosh x + 35 = 0$

By Quadratic Formula, $\cosh x = \frac{5}{2}$, $\cosh x = \frac{7}{3}$

$\Rightarrow \frac{e^{x}+e^{-x}}{2} = \frac{5}{2}$

$\times 2e^{x}$

$\Rightarrow e^{2x}+1 = 5e^{x}$

$\Rightarrow e^{2x} - 5e^{x} + 1 = 0$

$e^{x} = \frac{5 \pm \sqrt{21}}{2}$

$\boxed{x = \ln\left(\frac{5+\sqrt{21}}{2}\right)}$, $\boxed{x = \ln\left(\frac{5-\sqrt{21}}{2}\right)}$

$\Rightarrow \frac{e^{x}+e^{-x}}{2} = \frac{7}{3}$

$\Rightarrow e^{2x}+1 = \frac{14}{3}e^{x}$

$\Rightarrow e^{2x} - \frac{14}{3}e^{x} + 1 = 0$

$e^{x} = \frac{7 \pm 2\sqrt{10}}{3}$

$$\boxed{x = \ln\left(\frac{7+2\sqrt{10}}{3}\right)}, \quad \boxed{x = \ln\left(\frac{7-2\sqrt{10}}{3}\right)}$$

Q4) $\displaystyle\int_{-1}^{7}\left[\frac{(x+2)^{\frac{1}{2}}}{x+5}\right]dx$

$$\left[\begin{array}{l} u^2 = x+2 \\ u^2+3 = x+5 \end{array}\right]$$

$$\left[\begin{array}{l} 2u\dfrac{du}{dx} = 1 \\ \therefore\ dx = 2u\,du \end{array}\right]$$

$$\Big\Downarrow$$

$$\int_{1}^{3}\left[\left(\frac{u}{u^2+3}\right)\times 2u\right]du$$

x	u
-1	1
7	3

$$\Rightarrow\ 2\int_{1}^{3}\left[\frac{u^2}{u^2+3}\right]du$$

$$\Rightarrow\ 2\int_{1}^{3}\left[\frac{u^2+3-3}{u^2+3}\right]du = 2\int_{1}^{3}\left[1-\frac{3}{u^2+3}\right]du$$

From formula book.

$$\Rightarrow\ 2\left[u - \frac{3}{\sqrt{3}}\arctan\left(\frac{u}{\sqrt{3}}\right)\right]_{1}^{3}$$

$$\Rightarrow\ 2\left[3-\frac{3}{\sqrt{3}}\arctan\left(\frac{3}{\sqrt{3}}\right)\right]-2\left[1-\frac{3}{\sqrt{3}}\arctan\left(\frac{1}{\sqrt{3}}\right.\right.$$

$$\Rightarrow\ 2\left[3-\frac{\pi}{\sqrt{3}}\right]-2\left[1-\frac{\pi}{2\sqrt{3}}\right]$$

$$= 6 - 2 + 2\left(\frac{\pi}{2\sqrt{3}} - \frac{\pi}{\sqrt{3}}\right)$$

$$= 4 + 2\left(\frac{-\pi}{2\sqrt{3}}\right)$$

$$= 4 - \frac{\pi}{\sqrt{3}} = \boxed{4 - \frac{\sqrt{3}}{3}\pi}$$

Q5a) $\Pi_1 : \quad r \cdot \begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix} = 5$

$\Pi_2 : \quad r \cdot \begin{pmatrix} 6 \\ 1 \\ -4 \end{pmatrix} = 7$

<u>using normals</u> : $\quad \begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix} \cdot \begin{pmatrix} 6 \\ 1 \\ -4 \end{pmatrix} = 6 - 2 + 12 = 16$ //

$\left| \begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix} \right| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$

$\left| \begin{pmatrix} 6 \\ 1 \\ -4 \end{pmatrix} \right| = \sqrt{6^2 + 1^2 + 4^2} = \sqrt{53}$

$\therefore \cos\theta = \dfrac{16}{\sqrt{14 \times 53}}$

$\cos\theta = \dfrac{16}{\sqrt{742}}$ //

so $\quad \theta = \cos^{-1}\left(\dfrac{16}{\sqrt{742}}\right) = \boxed{54°}$

b) l is perp. to Π_1 $\therefore$ direction vector : $\begin{pmatrix}1\\-2\\-3\end{pmatrix}$

and passes through $\begin{pmatrix}2\\3\\-1\end{pmatrix}$

$$\therefore \quad r_1 = \begin{pmatrix}2\\3\\-1\end{pmatrix} + \lambda\begin{pmatrix}1\\-2\\-3\end{pmatrix}$$

sub l into Π_2 : $\begin{pmatrix}2+\lambda\\3-2\lambda\\-1-3\lambda\end{pmatrix}\cdot\begin{pmatrix}6\\1\\-4\end{pmatrix} = 7$

$$\Rightarrow \quad 12 + \underline{6\lambda} + 3 - \underline{2\lambda} + 4 + \underline{12\lambda} = 7$$

$$\Rightarrow \quad 16\lambda = -12$$

$$\therefore \quad \lambda = -\frac{3}{4} \longrightarrow Q = \begin{pmatrix}2-3/4\\3-2(-3/4)\\-1-3(-3/4)\end{pmatrix}$$

$$Q = \boxed{\begin{pmatrix}5/4\\9/2\\5/4\end{pmatrix}}$$

c) to find the normal of Π_3, we need to 'cross' the normals of Π_1 and Π_2.

This will give us a vector perpendicular to Π_3 which we need.

$$\begin{pmatrix}1\\-2\\-3\end{pmatrix} \times \begin{pmatrix}6\\1\\-4\end{pmatrix} = \begin{pmatrix}11\\-14\\13\end{pmatrix}.$$ (By calculator).

so $r\cdot\begin{pmatrix}11\\-14\\13\end{pmatrix} = \begin{pmatrix}11\\-14\\13\end{pmatrix}\cdot\begin{pmatrix}5/4\\9/2\\5/4\end{pmatrix} = 11(5/4) - 14(9/2) + 13(5/4)$

$$\Rightarrow \boxed{r\cdot\begin{pmatrix}11\\-14\\13\end{pmatrix} = -33}$$

Q6a) $M = \begin{pmatrix} 1 & k & 0 \\ 2 & -2 & 1 \\ -4 & 1 & -1 \end{pmatrix}$

$$\det M = 1\begin{vmatrix} -2 & 1 \\ 1 & -1 \end{vmatrix} - k\begin{vmatrix} 2 & 1 \\ -4 & -1 \end{vmatrix} + \cancel{0\begin{vmatrix} 2 & -2 \\ -4 & 1 \end{vmatrix}}^{0}$$

$$= -2(-1) - 1 \quad - k(-2+4)$$

$$= 2 - 1 + 2k - 4k = \underline{1 - 2k}$$

b) $M^T = \begin{pmatrix} 1 & 2 & -4 \\ k & -2 & 1 \\ 0 & 1 & -1 \end{pmatrix}$ (Transpose)

Matrix of minors $= \begin{pmatrix} \begin{vmatrix} -2 & 1 \\ 1 & -1 \end{vmatrix} & \begin{vmatrix} k & 1 \\ 0 & -1 \end{vmatrix} & \begin{vmatrix} k & -2 \\ 0 & 1 \end{vmatrix} \\ \begin{vmatrix} 2 & -4 \\ 1 & -1 \end{vmatrix} & \begin{vmatrix} 1 & -4 \\ 0 & -1 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} \\ \begin{vmatrix} 2 & -4 \\ -2 & 1 \end{vmatrix} & \begin{vmatrix} 1 & -4 \\ k & 1 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ k & -2 \end{vmatrix} \end{pmatrix}$

$$= \begin{pmatrix} 1 & -k & k \\ 2 & -1 & 1 \\ -6 & 1+4k & -2-2k \end{pmatrix} //$$

changing signs according to rule of alternating signs:

$$\begin{pmatrix} 1 & k & k \\ -2 & -1 & -1 \\ -6 & -1-4k & -2-2k \end{pmatrix}$$ //

& $\det M = 1-2k$

hence $M^{-1} = \frac{1}{1-2k}\begin{pmatrix} 1 & k & k \\ -2 & -1 & -1 \\ -6 & -1-4k & -2-2k \end{pmatrix}$

c) $Tl_1 = l_2$. $l_2: \ r = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 5 \\ 2 \\ 1 \end{pmatrix}$ //

$T^{-1}Tl_1 = T^{-1}l_2$

$l_1 = T^{-1}l_2$

T is just M with $k=0$.

$\therefore T^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & -1 & -1 \\ -6 & -1 & -2 \end{pmatrix}$

$\therefore l_1 = \begin{pmatrix} 1 & 0 & 0 \\ -2 & -1 & -1 \\ -6 & -1 & -2 \end{pmatrix}\begin{pmatrix} 1+5\lambda \\ 2\lambda-2 \\ 3+\lambda \end{pmatrix}$

$$l_1 = \begin{pmatrix} 1+5\lambda+0 \\ -\cancel{2}-10\lambda+\cancel{2}-2\lambda-3-\lambda \\ -6-30\lambda+2-2\lambda-6-2\lambda \end{pmatrix} = \begin{pmatrix} 1+5\lambda \\ -3-13\lambda \\ -10-34\lambda \end{pmatrix}$$

$$l_1 = \begin{pmatrix} 1 \\ -3 \\ -10 \end{pmatrix} + \lambda \begin{pmatrix} 5 \\ -13 \\ -34 \end{pmatrix}$$

cartesian eqn : $\boxed{\frac{x-1}{5} = \frac{y+3}{-13} = \frac{z+10}{-34}}$

Q7a) $I_n = \int_0^{\ln 2} [\cosh^n x]\,dx = \int_0^{\ln 2} [\cosh^{n-2} x \sinh$

$$= \int_0^{\ln 2} [\cosh^{n-2} x][\cosh^2 x]\,dx$$

$\boxed{\begin{array}{l} \cosh^2 x - \sinh^2 x = 1 \\ \therefore \cosh^2 x = 1 + \sinh^2 x \end{array}}$

$$\Rightarrow \int_0^{\ln 2} [\cosh^{n-2} x][1+\sinh^2 x]\,dx$$

$$\Rightarrow \int_0^{\ln 2} [\cosh^{n-2} x]\,dx + \int_0^{\ln 2} [\cosh^{n-2} x \sinh^2 x]\,dx$$

$$\therefore I_n = I_{n-2} + \int_0^{\ln 2} [\cosh^{n-2} x \sinh x][\sinh x]\,dx$$

By Parts

$\left[\frac{dv}{dx} = \cosh^{n-2} x \sinh x \rightarrow v = \frac{\cosh^{n-1} x}{n-1}\right]$ (Pattern!) $\left[\begin{array}{l} u = \sinh x \\ \therefore u' = \cosh x \end{array}\right]$

$$\therefore I_n = I_{n-2} + \left[\frac{\sinh x \cosh^{n-1} x}{n-1}\right]_0^{\ln 2} - \frac{1}{n-1}\int_0^{\ln 2} (\cosh^{n-1} x \cosh x)\,dx$$

$$I_n = I_{n-2} + \left[\frac{\frac{3}{4}\left(\frac{5}{4}\right)^{n-1}}{n-1}\right] - \frac{1}{n-1}\int_0^{\ln 2} [\cosh^n x]\,dx$$

$$\therefore I_n = I_{n-2} + \frac{3\left(\frac{5}{4}\right)^{n-1}}{4(n-1)} - \frac{1}{n-1} I_n .$$

$$I_n\left(1 + \frac{1}{n-1}\right) = I_{n-2} + \frac{3\left(\frac{5}{4}\right)^{n-1}}{4(n-1)}$$

$$\left(\frac{n-1+1}{n-1}\right) I_n = \left(\frac{n}{n-1}\right) I_n = I_{n-2} + \frac{3\left(\frac{5}{4}\right)^{n-1}}{4(n-1)}$$

$$\times\frac{(n-1)}{n}: \quad I_n = \left(\frac{n-1}{n}\right) I_{n-2} + \frac{3\left(\frac{5}{4}\right)^{n-1}}{4n}$$

$$I_n = \left(\frac{n-1}{n}\right) I_{n-2} + \frac{3(5)^{n-1}}{(4)^{n-1+1} n}$$

$$I_n = \left(\frac{n-1}{n}\right) I_{n-2} + \frac{3(5)^{n-1}}{(4^n)n}$$

b) $\int_0^{\ln 2} \cosh^4 x \, dx = I_4$

$$I_4 = \frac{(4-1)}{4} I_2 + \frac{3(5)^{4-1}}{4(4)^4}$$

$$I_4 = \frac{3}{4} I_2 + \frac{375}{1024}$$

$$I_2 = \frac{1}{2} I_0 + \frac{3(5)}{4^2(2)} = \frac{1}{2} I_0 + \frac{15}{32}$$

$$I_0 = \int_0^{\ln 2} [1] \, dx = [x]_0^{\ln 2} = \ln 2 \,//$$

$$\therefore I_2 = \frac{1}{2}\ln 2 + \frac{15}{32}$$

$$\therefore I_4 = \frac{3}{4}\left(\frac{1}{2}\ln 2 + \frac{15}{32}\right) + \frac{375}{1024}$$

$$I_4 = \frac{3}{8}\ln 2 + \frac{45}{128} + \frac{375}{1024}$$

$$I_4 = \frac{3}{8}\ln 2 + \frac{735}{1024}$$

Q8a) $y = \ln\left(\frac{e^x+1}{e^x-1}\right)$

let $t = \frac{e^x+1}{e^x-1} = (e^x+1)(e^x-1)^{-1}$

then $\frac{dt}{dx} = -(e^x+1)(e^x-1)^{-2}(e^x) + (e^x)(e^x-1)^{-1}$

$= (e^x-1)^{-2}\left[-e^x(e^x+1) + e^x(e^x-1)\right]$

$= (e^x-1)^{-2}\left[-e^{2x} - e^x + e^{2x} - e^x\right]$

$= (e^x-1)^{-2}\left[-2e^x\right] = \frac{-2e^x}{(e^x-1)^2} = \frac{dt}{dx}$

$y = \ln(t)$

$\frac{dy}{dt} = \frac{1}{t}$

$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{1}{t} \times \frac{-2e^x}{(e^x-1)^2}$

$= \left(\frac{\cancel{e^x-1}}{e^x+1}\right)\left(\frac{-2e^x}{(e^x-1)^{\cancel{2}}}\right)$

$= \frac{-2e^x}{(e^x+1)(e^x-1)} = \boxed{\frac{-2e^x}{e^{2x}-1}}$

□

b) $$\text{length} = \int_{\ln 2}^{\ln 3} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

$$1+\left(\frac{dy}{dx}\right)^2 = 1+\frac{4e^{2x}}{(e^{2x}-1)^2} = \frac{(e^{2x}-1)^2+4e^{2x}}{(e^{2x}-1)^2}$$

$$= \frac{e^{4x}-2e^{2x}+1+4e^{2x}}{(e^{2x}-1)^2}$$

$$= \frac{e^{4x}+2e^{2x}+1}{(e^{2x}-1)^2}$$

$$= \frac{(e^{2x}+1)^2}{(e^{2x}-1)^2}$$

//

$$\therefore \text{length} = \int_{\ln 2}^{\ln 3} \sqrt{\frac{(e^{2x}+1)^2}{(e^{2x}-1)^2}}\, dx = \int_{\ln 2}^{\ln 3} \left(\frac{e^{2x}+1}{e^{2x}-1}\right) dx$$

$$\frac{\cosh x}{\sinh x} = \frac{\frac{e^x+e^{-x}}{\cancel{2}}}{\frac{e^x-e^{-x}}{\cancel{2}}} = \frac{e^x+e^{-x}}{e^x-e^{-x}} = \frac{e^{2x}+1}{e^{2x}-1}$$

//

$$\text{hence length} = \int_{\ln 2}^{\ln 3} [\coth x]\, dx = \Big[\ln \sinh x\Big]_{\ln 2}^{\ln 3}$$

$$= [\ln \sinh(\ln 3)] - [\ln \sinh(\ln 2)]$$

$$= \ln\frac{4}{3} - \ln\frac{3}{4} = \boxed{\ln\frac{16}{9}}$$