FP3 June 14 M.A. kprime 2



1. The line $l$ passes through the point $P(2,1,3)$ and is perpendicular to the plane $\Pi$ whose vector equation is

$$\mathbf{r}.(\mathbf{i} - 2\mathbf{j} - \mathbf{k}) = 3$$

Find

(a) a vector equation of the line $l$, **(2)**

(b) the position vector of the point where $l$ meets $\Pi$. **(4)**

(c) Hence find the perpendicular distance of $P$ from $\Pi$. **(2)**

1(a). $\underset{\sim}{r} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}$

(b) $\underset{\sim}{r} = \begin{pmatrix} 2+\lambda \\ 1-2\lambda \\ 3-\lambda \end{pmatrix}$ & $x - 2y - z = 3$

$\therefore \quad 2+\lambda-2(1-2\lambda) - (3-\lambda) = 3$

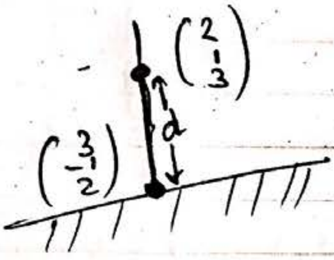
$\therefore \quad 2+\lambda-2+4\lambda-3+\lambda = 3$

$\therefore \quad 6\lambda = 6 \Rightarrow \lambda = 1$

$\therefore$ @ Intersection : $\begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$

Question 1 continued

(c)




$$d = \left| \begin{pmatrix}2\\1\\3\end{pmatrix} - \begin{pmatrix}3\\-1\\2\end{pmatrix} \right|$$

$$\therefore d = \left| \begin{pmatrix}-1\\2\\1\end{pmatrix} \right|$$

$$\therefore d = \sqrt{1^2+2^2+1^2} = \sqrt{6}$$

$$\therefore d = \sqrt{6}$$

Q1

(Total 8 marks)

2.

$$\mathbf{M} = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 4 & 1 \\ 0 & 5 & 0 \end{pmatrix}$$

(a) Show that matrix **M** is not orthogonal.

**(2)**

(b) Using algebra, show that 1 is an eigenvalue of **M** and find the other two eigenvalues of **M**.

**(5)**

(c) Find an eigenvector of **M** which corresponds to the eigenvalue 1

**(2)**

The transformation $M : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is represented by the matrix **M**.

(d) Find a cartesian equation of the image, under this transformation, of the line

$$x = \frac{y}{2} = \frac{z}{-1}$$

**(4)**

2(a). $MM^T = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 4 & 1 \\ 0 & 5 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 5 \\ 2 & 1 & 0 \end{pmatrix}$

$\cancel{= (1+4} = \begin{pmatrix} 5 & 2 & 0 \\ 2 & 17 & 20 \\ 0 & 20 & 25 \end{pmatrix}$

$\neq \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \therefore MM^T \neq I \therefore$ not orthogonal

(b) $M - \lambda I = \begin{pmatrix} 1-\lambda & 0 & 2 \\ 0 & 4-\lambda & 1 \\ 0 & 5 & -\lambda \end{pmatrix}$

$\therefore \det(M - \lambda I) = 0 \Rightarrow (1-\lambda)\begin{vmatrix} 4-\lambda & 1 \\ 5 & -\lambda \end{vmatrix} - 0 + 2\begin{vmatrix} 0 & 4-\lambda \\ 0 & 5 \end{vmatrix} = 0$

$\therefore (1-\lambda)(-\lambda(4-\lambda) - 5) + 2(0) = 0$

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Question 2 continued

$\therefore (1-\lambda)(\lambda^2-4\lambda-5)=0$

$\therefore (1-\lambda)(\lambda-5)(\lambda+1)=0$

$\Rightarrow \lambda=1 \quad \therefore \quad \lambda=1$ is indeed an e. value

$\lambda=-1$, $\lambda=5$ are also e. values

(c) $Mx = x$

$$\therefore \begin{pmatrix} 1 & 0 & 2 \\ 0 & 4 & 1 \\ 0 & 5 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\therefore \begin{pmatrix} x+2z \\ 4y+z \\ 5y \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$\Rightarrow \quad x+2z=x \quad \Rightarrow \quad 2z=0$

$4y+z = y \Rightarrow \quad 3y=-z$

$5y=z$

$z=0 \Rightarrow y=0$, Let $x=1$

$\therefore$ An eigenvector is $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$





Turn over

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Question 2 continued

(d) $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ 2x \\ -x \end{pmatrix} = \cancel{\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}} \; x\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$

let $x = \lambda$

$\therefore$ ~~$\underline{r}$~~

$\underline{r} = \lambda\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$

$\therefore$ ~~$M$~~

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 4 & 1 \\ 0 & 5 & 0 \end{pmatrix}\begin{pmatrix} \lambda \\ 2\lambda \\ -\lambda \end{pmatrix}$$

$$= \begin{pmatrix} -\lambda \\ 7\lambda \\ 10\lambda \end{pmatrix}$$

$\therefore \quad \underline{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -\lambda \\ 7\lambda \\ 10\lambda \end{pmatrix}$

$z = 10x$ ~~$x = \frac{10}{7}$~~

~~$\frac{10}{7}$~~ $= \frac{7}{10}z = 7x$

~~$x = \lambda$~~ $x = -\lambda$

$\Rightarrow$ ~~$x =$~~ $-7x = y = \frac{7}{10}z$

$\Rightarrow -70x = 10y = 7z$



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3. Using calculus, find the exact value of

(a) $\int_1^2 \frac{1}{\sqrt{(x^2 - 2x + 3)}} \, dx$

**(4)**

(b) $\int_0^1 e^{2x} \sinh x \, dx$

**(4)**

3(a).

$$\int_1^2 \frac{1}{\sqrt{(x-1)^2 + 2}} \, \partial x$$

$$= \left[ \text{arsinh} \left( \frac{x-1}{\sqrt{2}} \right) \right]_1^2$$

$$= \text{arsinh} \frac{1}{\sqrt{2}} - \text{arsinh } 0$$

$$= \text{arsinh} \frac{1}{\sqrt{2}} = \ln \left( \frac{1}{\sqrt{2}} + \sqrt{\frac{3}{2}} \right)$$

$$= \ln \left( \frac{\sqrt{6} + \sqrt{2}}{2} \right)$$

blank

Question 3 continued

(b). $\int_0^1 e^{2x} \sinh x \, \partial x$

$= \int_0^1 \frac{e^{2x}(e^{x} - e^{-x})}{2} \, \partial x$

$= \frac{1}{2} \int_0^1 e^{3x} - e^{x} \, \partial x$

$= \frac{1}{2} \left[ \frac{1}{3}e^{3x} - e^{x} \right]_0^1$

$= \frac{1}{2}\left( \frac{1}{3}e^{3} - e - \left(\frac{1}{3} - 1\right)\right)$

$= \frac{1}{2}\left( \frac{1}{3}e^{3} - e + \frac{2}{3}\right) = \frac{1}{6}e^{3} - \frac{1}{2}e + \frac{1}{3}$

4. Using the definitions of hyperbolic functions in terms of exponentials,

(a) show that

$$\operatorname{sech}^2 x = 1 - \tanh^2 x$$

**(3)**

(b) solve the equation

$$4 \sinh x - 3 \cosh x = 3$$

**(4)**

4(a).

$$RHS = 1 - \tanh^2 x = 1 - \frac{\sinh^2 x}{\cosh^2 x} = 1 - \left( \frac{\frac{e^x - e^{-x}}{2}}{\frac{e^x + e^{-x}}{2}} \right)^2$$

$$= 1 - \left( \frac{e^x - e^{-x}}{e^x + e^{-x}} \right)^2 = \left( \frac{e^x + e^{-x}}{e^x + e^{-x}} \right)^2 - \left( \frac{e^x - e^{-x}}{e^x + e^{-x}} \right)^2$$

$$= \frac{(e^x + e^{-x})^2 - (e^x - e^{-x})^2}{(e^x + e^{-x})^2}$$

$$\cancel{= \frac{2e^{-x}}{e^x + e^{-x}} \times \frac{e^x}{e^x} - 2}$$

$$= \frac{e^{2x} + 2 + e^{-2x} - (e^{2x} - 2 + e^{-2x})}{(e^x + e^{-x})^2}$$

$$= \frac{4}{(e^x + e^{-x})^2}$$

Question 4 continued

$$= \left(\frac{2}{e^x+e^{-x}}\right)^2 = \left(\frac{2}{e^x+e^{-x}} \times \frac{1/2}{1/2}\right)^2$$

$$= \left(\frac{1}{\frac{e^x+e^{-x}}{2}}\right)^2 = \left(\frac{1}{\cosh x}\right)^2$$

$$= \operatorname{sech}^2 x = LHS$$

as required.

(b)

$4\sinh x - 3\cosh x = 3$

$$2e^x - 2e^{-x} - \frac{3}{2}(e^x+e^{-x}) = 3$$

$$\therefore \quad \frac{1}{2}e^x - \frac{7}{2}e^{-x} = 3$$

$$\times 2e^x \Rightarrow \quad e^{2x} - 7 = 6e^x$$

$$\therefore e^{2x} - 6e^x - 7 = 0$$

$$\therefore \quad (e^x-7)(e^x+1) = 0$$

$\therefore \; e^x = 7 \Rightarrow x = \ln 7 \quad \Rightarrow \quad x = \ln 7$

$e^x \neq -1 \quad x \neq \ln(-1)$

(Total 7 marks)

Q4



Turn over

5. Given that $y = \text{artanh}\frac{x}{\sqrt{(1+x^2)}}$

show that $\frac{dy}{dx} = \frac{1}{\sqrt{(1+x^2)}}$ **(4)**

5. $y = \text{ar}\tanh \frac{x}{\sqrt{1+x^2}}$

$\therefore \tanh y = \frac{x}{\sqrt{1+x^2}}$

Differentiate :

$\therefore \operatorname{sech}^2 y \cdot \frac{\partial y}{\partial x} = \frac{(1+x^2)^{1/2} - x\left[\frac{1}{2}(1+x^2)^{-1/2} \cdot 2x\right]}{1+x^2}$

$\therefore \operatorname{sech}^2 y \frac{\partial y}{\partial x} = \frac{\sqrt{1+x^2} - x\left(\frac{x}{\sqrt{1+x^2}}\right)}{1+x^2}$

$\operatorname{sech}^2 y = 1 - \tanh^2 y = 1 - \frac{x^2}{1+x^2}$

$\therefore \left(1 - \frac{x^2}{1+x^2}\right)\frac{\partial y}{\partial x} = \frac{1+x^2-x^2}{(1+x^2)^{3/2}}$

Question 5 continued

$$\therefore \frac{1}{1+x^2}\frac{\partial y}{\partial x} = \frac{1}{(1+x^2)^{3/2}}$$

$$\therefore \frac{\partial y}{\partial x} = \frac{1+x^2}{(1+x^2)^{3/2}}$$

$$\therefore \frac{\partial y}{\partial x} = (1+x^2)^{1-3/2} = (1+x^2)^{-1/2}$$

$$\Rightarrow \frac{\partial y}{\partial x} = \frac{1}{\sqrt{1+x^2}}$$

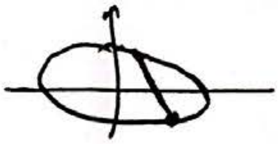
as required.

blank

6. [*In this question you may use the appropriate trigonometric identities on page 6 of the pink Mathematical Formulae and Statistical Tables.*]

The points $P(3\cos\alpha,\ 2\sin\alpha)$ and $Q(3\cos\beta,\ 2\sin\beta)$, where $\alpha \neq \beta$, lie on the ellipse with equation

$$\frac{x^2}{9} + \frac{y^2}{4} = 1$$



(a) Show the equation of the chord $PQ$ is

$$\frac{x}{3}\cos\frac{(\alpha+\beta)}{2} + \frac{y}{2}\sin\frac{(\alpha+\beta)}{2} = \cos\frac{(\alpha-\beta)}{2}$$

**(4)**

(b) Write down the coordinates of the mid-point of $PQ$.

**(1)**

Given that the gradient, $m$, of the chord $PQ$ is a constant,

(c) show that the centre of the chord lies on a line

$$y = -kx$$

expressing $k$ in terms of $m$.

**(5)**

6 (a). Gradient of Chord $PQ = \dfrac{2\sin\beta - 2\sin\alpha}{3\cos\beta - 3\cos\alpha}$

$\therefore y - y_1 = m(x - x_1)$

$\therefore y - 2\sin\alpha = \dfrac{2\sin\beta - 2\sin\alpha}{3\cos\beta - 3\cos\alpha}(x - 3\cos\alpha)$

$\therefore \cancel{y - 2\sin\alpha = \dfrac{2\sin\beta - 2\sin\alpha}{3\cos\beta - 3\cos\alpha}x - 2\sin\beta\cos\alpha}$

Leave blank

Question 6 continued

$$\therefore y - 2\sin\alpha = \frac{2\sin\beta - 2\sin\alpha}{3\cos\beta - 3\cos\alpha}\, x - \frac{2\sin\beta\cos\alpha - 2\sin\alpha\cos\alpha}{\cos\beta - \cos\alpha}$$

$\times \cos\beta - \cancel{\cos\alpha}$ $\quad \div 2 \quad \frac{y}{2}$

$$\therefore \frac{y}{2} - \sin\alpha = \frac{x}{3}\,\frac{\sin\beta - \sin\alpha}{\cos\beta - \cos\alpha} - \frac{\sin\beta\cos\alpha - \sin\alpha\cos\alpha}{\cos\beta - \cos\alpha}$$

use identities:

$$\sin(\beta) - \sin(\alpha) \equiv 2\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\beta-\alpha}{2}\right)$$

$$\cos\beta - \cos\alpha = -2\sin\frac{\alpha+\beta}{2}\sin\frac{\beta-\alpha}{2}$$

Sub in

$$\therefore \frac{y}{2} - \sin\alpha = \frac{x}{3}\,\frac{2\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\beta-\alpha}{2}\right)}{-2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\beta-\alpha}{2}\right)} - \frac{\cos\alpha(\sin\beta - \sin\alpha)}{\cos\beta - \cos\alpha}$$

$$\therefore \frac{y}{2} - \sin\alpha = -\frac{x}{3}\,\frac{\cos\left(\frac{\alpha+\beta}{2}\right)}{\sin\left(\frac{\alpha+\beta}{2}\right)} + \cos\alpha\left(\frac{\cos\left(\frac{\alpha+\beta}{2}\right)}{\sin\left(\frac{\alpha+\beta}{2}\right)}\right)$$

$\times \sin\left(\frac{\alpha+\beta}{2}\right) \Rightarrow$

blank

Question 6 continued

$\times \sin\left(\frac{\alpha+\beta}{2}\right)$

$\therefore \frac{y}{2}\sin\left(\frac{\alpha+\beta}{2}\right) - \sin\alpha \sin\left(\frac{\alpha+\beta}{2}\right) = -\frac{x}{3}\cos\left(\frac{\alpha+\beta}{2}\right)$
$+\cos\alpha\left(\cos\left(\frac{\alpha+\beta}{2}\right)\right)$

$\therefore$
$\frac{x}{3}\cos\left(\frac{\alpha+\beta}{2}\right) + \frac{y}{2}\sin\left(\frac{\alpha+\beta}{2}\right) = \cos\alpha\cos\left(\frac{\alpha+\beta}{2}\right) + \sin\alpha\left(\sin\frac{\alpha+\beta}{2}\right)$

$RHS = \cos\alpha\cos\left(\frac{\alpha+\beta}{2}\right) + \sin\alpha\sin\left(\frac{\alpha+\beta}{2}\right)$

$= \cos\left(\alpha - \frac{(\alpha+\beta)}{2}\right) = \cos\frac{2\alpha-\alpha-\beta}{2}$

$= \cos\left(\frac{\alpha-\beta}{2}\right)$

$\therefore \frac{x}{3}\cos\left(\frac{\alpha+\beta}{2}\right) + \frac{y}{2}\sin\left(\frac{\alpha+\beta}{2}\right) = \cos\left(\frac{\alpha-\beta}{2}\right)$

as required.

Leave blank

**Question 6 continued**

(b)

Midpoint : $\left(\frac{3}{2}(\cos\alpha+\cos\beta),\ \sin\alpha+\sin\beta\right)$

Leave blank

Question 3 continued

(c) Gradient $= \frac{2}{3}\ \frac{\sin\beta - \sin\alpha}{\cos\beta - \cos\alpha} = -\frac{2}{3}\cot\left(\frac{\alpha+\beta}{2}\right) = m$

Mid cont. $x = \frac{3}{2}(\cos\alpha + \cos\beta) = \frac{3}{2}\left(2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}\right)$

$y = \sin\alpha + \sin\beta = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$

$$\frac{y}{x} = \frac{2\sin\left(\frac{\alpha+\beta}{2}\right)\cancel{\cos\left(\frac{\alpha-\beta}{2}\right)}}{3\cos\frac{\alpha+\beta}{2}\cancel{\cos\left(\frac{\alpha-\beta}{2}\right)}}$$

$\therefore \frac{y}{x} = \frac{2}{3}\tan\left(\frac{\alpha+\beta}{2}\right)$

If $-\frac{2}{3}\cot\left(\frac{\alpha+\beta}{2}\right) = m$

$\Rightarrow \tan\left(\frac{\alpha+\beta}{2}\right) = -\frac{2}{3m}$

$\therefore \frac{y}{x} = \frac{2}{3} \times -\frac{2}{3m} = \frac{-4}{9m}$

$\therefore y = -\frac{4}{9m}x \Rightarrow k = \frac{4}{9m}$

Q3

**(Total 8 marks)**





Turn over

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7. A circle $C$ with centre $O$ and radius $r$ has cartesian equation $x^2 + y^2 = r^2$ where $r$ is a constant.

(a) Show that $1 + \left(\frac{dy}{dx}\right)^2 = \frac{r^2}{r^2 - x^2}$ **(3)**

(b) Show that the surface area of the sphere generated by rotating $C$ through $\pi$ radians about the $x$-axis is $4\pi r^2$. **(5)**

(c) Write down the length of the arc of the curve $y = \sqrt{(1 - x^2)}$ from $x = 0$ to $x = 1$ **(1)**

7(a). $x^2 + y^2 = r^2$

$\therefore 2x + 2y\frac{\partial y}{\partial x} = 0$

$\therefore x + y\frac{\partial y}{\partial x} = 0$

$\therefore y\frac{\partial y}{\partial x} = -x$

$\therefore \frac{\partial y}{\partial x} = -\frac{x}{y} \Rightarrow \left(\frac{\partial y}{\partial x}\right)^2 = \frac{x^2}{y^2}$

$RHS = \frac{r^2}{r^2 - x^2} = \frac{x^2+y^2}{r^2-(r^2-y^2)} = \frac{x^2+y^2}{y^2}$

$= \frac{y^2}{y^2} + \frac{x^2}{y^2} = 1 + \frac{x^2}{y^2}$

$= 1 + \left(\frac{\partial y}{\partial x}\right)^2 = LHS$

as required.

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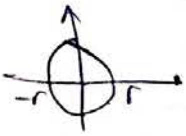
**Question 7 continued**

$-\frac{1}{2}(1-x^2)^{-1/2}$

(b) $S = 2\pi \int_{-r}^{r} y\sqrt{1+\left(\frac{\partial y}{\partial x}\right)^2}\,\partial x$

$= 2\pi \int_{-r}^{r} \sqrt{r^2-x^2}\sqrt{\frac{r^2}{r^2-x^2}}\,\partial x$




$= 2\pi \int_{-r}^{r} r\,\partial x$

$= 2\pi\left[rx\right]_{-r}^{r} = 2\pi\left(r^2 - -r^2\right)$

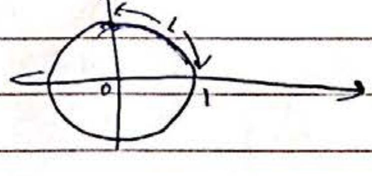
$= 2\pi\left(2r^2\right)$

$= 4\pi r^2$

as required.

(c) $y^2 = 1 - x^2$

$\Rightarrow x^2 + y^2 = 1$




Circumference $= 2\pi r = 2\pi$

$\therefore$ Arc length $= \frac{1}{4} \times 2\pi$

$= \frac{\pi}{2}$

8. The position vectors of the points $A$, $B$ and $C$ from a fixed origin $O$ are

$$\mathbf{a} = \mathbf{i} - \mathbf{j},\ \ \mathbf{b} = \mathbf{i} + \mathbf{j} + \mathbf{k},\ \ \mathbf{c} = 2\mathbf{j} + \mathbf{k}$$

respectively.

(a) Using vector products, find the area of the triangle $ABC$. **(4)**

(b) Show that $\frac{1}{6}\mathbf{a}.(\mathbf{b} \times \mathbf{c}) = 0$ **(3)**

(c) Hence or otherwise, state what can be deduced about the vectors **a**, **b** and **c**. **(1)**

8(a). $AB = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$

$\cancel{B}AC = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix}$

$\therefore$ Area $= \frac{1}{2} \left| AB \times AC \right|$

$AB \times AC = \begin{matrix} 0 & 2 & 1 & 0 & 2 & 1 \\ -1 & 3 & 1 & -1 & 3 & 1 \end{matrix}$

$= \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}$

$\therefore$ Area $= \frac{1}{2} \left| \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix} \right| = \frac{1}{2}\sqrt{1+1+4}$

$= \frac{\sqrt{6}}{2}$ //

**Question 8 continued**

(b) $\underline{b} \times \underline{c} = \begin{pmatrix}1\\1\\1\end{pmatrix} \times \begin{pmatrix}2\\0\\1\end{pmatrix}$

$= \begin{matrix}1\\2\end{matrix} \quad \begin{matrix}1\\0\end{matrix} \times \begin{matrix}1\\1\end{matrix} \times \begin{matrix}1\\2\end{matrix} \times \begin{matrix}1\\0\end{matrix} \quad \begin{matrix}1\\1\end{matrix}$

$= \begin{pmatrix}1\\1\\-2\end{pmatrix}$

$\therefore \frac{1}{6}\underline{a} \cdot (\underline{b} \times \underline{c}) = \frac{1}{6}\begin{pmatrix}1\\-1\\0\end{pmatrix} \cdot \begin{pmatrix}1\\1\\-2\end{pmatrix}$

$= \frac{1}{6}(1 - 1 + 0)$

$= 0$

as required.

(c) They lie on the same plane

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9. $$I_n = \int (x^2+1)^{-n}\,dx, \quad n > 0$$

(a) Show that, for $n > 0$

$$I_{n+1} = \frac{x(x^2+1)^{-n}}{2n} + \frac{2n-1}{2n} I_n$$

**(5)**

(b) Find $I_2$

**(3)**

$I_n = \int (x^2+1)^{-n}\,\partial x$

Let $u = (x^2+1)^{-n}$ $\quad \cancel{u' = 2xn(x^2+1)^{-n-1}}$

$u' = -2nx(x^2+1)^{-n-1}$

$v' = 1 \qquad v = x$

$\therefore I_n = x(x^2+1)^{-n} + 2n\int x^2(x^2+1)^{-n-1}\,\partial x$

$I_n = x(x^2+1)^{-n} + 2n\int x^2(x^2+1)^{-n}(x^2+1)^{-1}\,\partial x$

$I_n = x(x^2+1)^{-n} + 2n\int \frac{x^2}{x^2+1}(x^2+1)^{-n}\,\partial x$

$I_n = x(x^2+1)^{-n} + 2n\int \frac{x^2+1-1}{x^2+1}(x^2+1)^{-n}\,\partial x$

$I_n = x(x^2+1)^{-n} + 2n\int \left(1 - \frac{1}{x^2+1}\right)(x^2+1)^{-n}\,\partial x$

$I_n = x(x^2+1)^{-n} + 2n\int (x^2+1)^{-n} - (x^2+1)^{-n-1}\,\partial x$

$$I_n = x(x^2+1)^{-n} + 2n\left(I_n - I_{n+1}\right)$$

$$I_n = x\cdot(x^2+1)^{-n} + 2n I_n - 2n I_{n+1}$$

$\therefore$ $\div 2n$ $$\frac{I_n}{2n} = \frac{x(x^2+1)^{-n}}{2n} + I_n - I_{n+1}$$

$$\therefore \quad I_{n+1} = \frac{x(x^2+1)^{-n}}{2n} + \left(1 - \frac{1}{2n}\right) I_n$$

$$\therefore I_{n+1} = \frac{x(x^2+1)^{-n}}{2n} + \frac{2n-1}{2n} I_n$$

as required.

Question 9 continued

(b) $I_2 = \dfrac{x(x^2+1)^{-1}}{2} + \dfrac{1}{2} I_1$

$= \cancel{x(} \dfrac{x}{2x^2+2} + \dfrac{1}{2} \int \dfrac{1}{(x^2+1)} \, dx$

$I_2 = \dfrac{x}{2x^2+2} + \dfrac{1}{2} \arctan x + C$