

June 14 (R) M.A. Kprime2

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1. Solve the equation

$$5 \tanh x + 7 = 5 \operatorname{sech} x$$

Give each answer in the form $\ln k$ where k is a rational number.

(5)

$$1. 5 \tanh x + 7 = 5 \operatorname{sech} x$$

$$\therefore \frac{5 \sinh x}{\cosh x} - \frac{5}{\cosh x} + 7 = 0$$

 $\times \cosh x$

$$\therefore 5 \sinh x + 7 \cosh x - 5 = 0$$

$$\therefore \frac{5}{2} e^x - \frac{5}{2} e^{-x} + \frac{7}{2} e^x + \frac{7}{2} e^{-x} - 5 = 0$$

$$\therefore 6e^x + e^{-x} - 5 = 0$$

$$\times (e^x) \Rightarrow 6e^{2x} - 5e^x + 1 = 0$$

$$(2e^x - 1)(3e^x - 1) = 0$$

$$\Rightarrow e^x = \frac{1}{2} \quad \therefore x = \ln \frac{1}{2}$$

$$e^x = \frac{1}{3} \quad \therefore x = \ln \frac{1}{3}$$

2.

$$9x^2 + 6x + 5 \equiv a(x + b)^2 + c$$

(a) Find the values of the constants a , b and c .

(3)

Hence, or otherwise, find

(b) $\int \frac{1}{9x^2 + 6x + 5} dx$

(2)

(c) $\int \frac{1}{\sqrt{9x^2 + 6x + 5}} dx$

(2)

$$2(a). \quad 9x^2 + 6x + 5 = 9\left(x^2 + \frac{2}{3}x + \frac{5}{9}\right)$$

$$= 9\left[\left(x + \frac{1}{3}\right)^2 + \frac{4}{9}\right]$$

$$= 9\left(x + \frac{1}{3}\right)^2 + 4$$

$$(b) \quad \frac{1}{9} \int \frac{1}{\left(x + \frac{1}{3}\right)^2 + \frac{4}{9}} dx = \frac{1}{9} \times \frac{3}{2} \arctan\left(\frac{x + \frac{1}{3}}{\frac{2}{3}}\right) + C$$

$$= \frac{1}{6} \arctan\left(\frac{3x + 1}{2}\right) + C$$

$$(c) \quad \frac{1}{9} \int \frac{1}{\sqrt{\left(x + \frac{1}{3}\right)^2 + \frac{4}{9}}} dx = \frac{1}{9} \operatorname{arsinh}\left(\frac{3x + 1}{2}\right) + C$$

3. The curve C has equation

$$y = \frac{1}{2} \ln(\coth x), \quad x > 0$$

(a) Show that

$$\frac{dy}{dx} = -\operatorname{cosech} 2x \quad (3)$$

The points A and B lie on C .

The x coordinates of A and B are $\ln 2$ and $\ln 3$ respectively.

(b) Find the length of the arc AB , giving your answer in the form $p \ln q$, where p and q are rational numbers. (6)

$$3(a) \quad y = \frac{1}{2} \ln(\coth x)$$

$$\therefore \frac{dy}{dx} = \frac{1}{2} \times \frac{\frac{d}{dx}(\coth x)}{\coth x}$$

$$= \frac{1}{2} \times \frac{-\operatorname{cosech}^2 x}{\coth x}$$

$$\begin{aligned} c^2 - s^2 &= 1 \\ \frac{c^2}{s^2} - \frac{s^2}{s^2} &= \frac{1}{s^2} \\ \coth^2 - 1 &= \operatorname{cosech}^2 \end{aligned}$$

$$= \frac{1}{2} \times \frac{1}{\frac{\cosh x}{\sinh x}} \times \frac{\sinh^2 x}{\sinh^2 x}$$

Use: $\frac{1}{2} \sinh 2x = \sinh x \cosh x$

$$= \frac{1}{2} \times \frac{1}{\cosh x \sinh x}$$

$$= \frac{1}{2} \times \frac{1}{\frac{1}{2} \sinh 2x} = -\operatorname{cosech} 2x$$

as required.

Question 3 continued

$$(b) \quad y = \frac{1}{2} \ln(\coth 2x) \Rightarrow \frac{dy}{dx} = -\operatorname{cosech} 2x$$

$$\therefore \left(\frac{dy}{dx}\right)^2 = \operatorname{cosech}^2 2x$$

$$\therefore s = \int_{\ln 2}^{\ln 3} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

$$\frac{c^2 - s^2 = 1}{\frac{c^2}{s^2} \cdot \frac{s^2}{s^2} = \frac{1}{s^2}}$$

$$\coth^2 - 1 = \operatorname{cosech}^2$$

$$\therefore s = \int_{\ln 2}^{\ln 3} \sqrt{1 + \operatorname{cosech}^2 2x} dx$$

Use: $\operatorname{cosech}^2 2x + 1 = \coth^2 2x$

$$\therefore s = \int_{\ln 2}^{\ln 3} \coth 2x dx = \left[\frac{1}{2} \ln |\sinh 2x| \right]_{\ln 2}^{\ln 3}$$

$$= \frac{1}{2} \ln |\sinh \ln 9| - \frac{1}{2} \ln |\sinh \ln 4|$$

$$= \frac{1}{2} \ln \left| \frac{9 - 9^{-1}}{2} \right| - \frac{1}{2} \ln \left| \frac{4 - 4^{-1}}{2} \right|$$

$$= \ln \left(\frac{40}{9} \right) - \ln \left(\frac{15}{8} \right) = \ln \left(\frac{64}{27} \right) / 2$$

$$= \frac{1}{2} \ln \left(\frac{64}{27} \right) = \ln \left[\frac{(4)^3}{(3)^3} \right] = 3 \ln \left(\frac{4}{3} \right)$$

$$= 3 \ln 4 - 3 \ln 3 = \ln \left[\frac{(4)^3}{(3)^3} \right]$$



$$I_n = \int_0^{\sqrt{3}} (3-x^2)^n dx, \quad n \geq 0$$

(a) Show that, for $n \geq 1$

$$I_n = \frac{6n}{2n+1} I_{n-1} \tag{6}$$

(b) Hence find the exact value of I_4 , giving your answer in the form $k\sqrt{3}$ where k is a rational number to be found. (5)

4(a). $I_n = \int_0^{\sqrt{3}} (3-x^2)^n dx$

\therefore Let $u = (3-x^2)^n$ $u' = n(3-x^2)^{n-1} \cdot x - 2x$
 $= -2nx(3-x^2)^{n-1}$
 $v' = 1$ $v = x$

~~$\therefore I_n = -2nx(3-x^2)^{n-1}$~~

$\therefore I_n = \left[x(3-x^2)^n \right]_0^{\sqrt{3}} + 2n \int_0^{\sqrt{3}} x^2 (3-x^2)^{n-1} dx$

$\therefore I_n = 2n \int_0^{\sqrt{3}} x^2 (3-x^2)^{n-1} dx$

Use $x^2 \equiv 3 - (3-x^2)$

$\therefore I_n = 2n \int_0^{\sqrt{3}} 3(3-x^2)^{n-1} - (3-x^2)^n dx$

$\therefore I_n = 2n (3I_{n-1} - I_n)$

$\therefore I_n = 6nI_{n-1} - 2nI_n$

Question 4 continued

$$\therefore (2n+1)I_n = 6n I_{n-1}$$

$$\Rightarrow I_n = \frac{6n}{2n+1} I_{n-1}$$

as required.

$$(b) I_4 = \frac{8}{3} I_3 = \frac{8}{3} \left(\frac{18}{7} I_2 \right) = \frac{48}{7} \left(\frac{12}{5} I_1 \right)$$

$$= \frac{576}{35} (2 I_0)$$

$$= \frac{1152}{35} \int_0^{\sqrt{3}} (3-x^2)^0 dx$$

$$= \frac{1152}{35} \left[3x - \frac{1}{3}x^3 \right]_0^{\sqrt{3}} = \frac{1152}{35} [x]_0^{\sqrt{3}}$$

$$= \frac{1152}{35} (3\sqrt{3} - \sqrt{3} - 0) = \frac{1152}{35} \sqrt{3}$$

$$\therefore I_4 = \frac{2304\sqrt{3}}{35}$$

$$\therefore I_4 = \frac{1152}{35} \sqrt{3}$$

5. The ellipse E has equation

$$x^2 + 9y^2 = 9$$

The point $P(a \cos \theta, b \sin \theta)$ is a general point on the ellipse E .

- (a) Write down the value of a and the value of b . (1)

The line L is a tangent to E at the point P .

- (b) Show that an equation of the line L is given by

$$3y \sin \theta + x \cos \theta = 3 \quad (3)$$

The line L meets the x -axis at the point Q and meets the y -axis at the point R .

- (c) Show that the area of the triangle OQR , where O is the origin, is given by

$$k \operatorname{cosec} 2\theta$$

where k is a constant to be found.

(3)

The point M is the midpoint of QR .

- (d) Find a cartesian equation of the locus of M , giving your answer in the form $y^2 = f(x)$. (4)

5(a). $\frac{x^2}{9} + y^2 = 1$

$\therefore a = 3$

, $b = 1$

Question 5 continued

$$(b) \frac{dy}{dx} = \frac{-3\sin\theta}{\cos\theta}$$

at P, $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\cos\theta}{-3\sin\theta}$

$$y - y_1 = m(x - x_1)$$

~~$$\therefore y - \sin\theta = -\frac{\cos\theta}{3\sin\theta} (x - 3\cos\theta)$$~~

~~$$\therefore y - \sin\theta = \frac{-\cos\theta}{3\sin\theta} x +$$~~

$$y - \sin\theta = \frac{-\cos\theta}{3\sin\theta} (x - 3\cos\theta)$$

$$\therefore y - \sin\theta = -\frac{\cos\theta}{3\sin\theta} x + \frac{\cos^2\theta}{\sin\theta}$$

$$\textcircled{* 3\sin\theta} \Rightarrow 3y\sin\theta - 3\sin^2\theta = -\cos\theta x + 3\cos^2\theta$$

$$\therefore 3y\sin\theta + x\cos\theta = 3(\cos^2\theta + \sin^2\theta) = 3$$

$$\therefore 3y\sin\theta + x\cos\theta = 1 \quad \text{as required.}$$

Question 5 continued

$$(c) \text{ @ } Q, \quad y=0$$

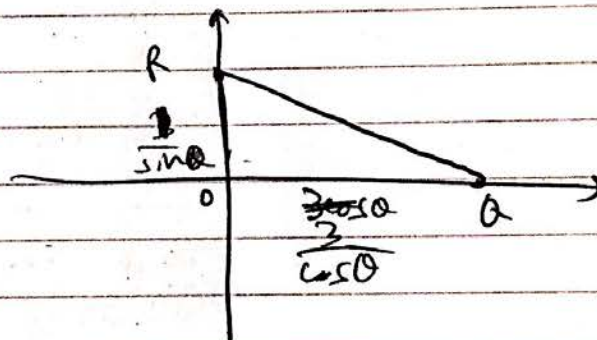
$$\therefore r \cos \theta = 3 \Rightarrow r = \frac{3}{\cos \theta}$$

$$Q \left(\frac{3}{\cos \theta}, 0 \right)$$

$$\text{ @ } R, \quad r=0 \Rightarrow 3y \sin \theta = -3$$

$$\therefore y = \frac{-1}{\sin \theta}$$

$$\therefore R \left(0, \frac{-1}{\sin \theta} \right)$$



$$\therefore \text{Area} = \frac{1}{2} |OQ| \times |OR|$$

$$= \frac{1}{2} \times \frac{3}{\cos \theta} \times \frac{1}{\sin \theta}$$

$$= \frac{3}{2} \times \frac{1}{\sin \theta \cos \theta} = \frac{3}{2} \times \frac{1}{\frac{1}{2} \sin 2\theta}$$

$$\therefore = \frac{3}{2} \times \frac{2}{\sin 2\theta} = 3 \operatorname{cosec} 2\theta$$

$$k=3$$

(Total 11 marks)

Question 6 continued

$$(d) M: \left(\frac{3}{2\cos\theta}, \frac{1}{2\sin\theta} \right)$$

$$\therefore x = \frac{3}{2\cos\theta} \Rightarrow \cos\theta = \frac{3}{2x}$$

$$y = \frac{1}{2\sin\theta} \Rightarrow \sin\theta = \frac{1}{2y}$$

$$\therefore \sin^2\theta + \cos^2\theta = 1 = \frac{1}{4y^2} + \frac{9}{4x^2}$$

$$\therefore 1 - \frac{9}{4x^2} = \frac{1}{4y^2}$$

$$\therefore \frac{4x^2 - 9}{4x^2} = \frac{1}{4y^2}$$

$$\therefore 4y^2 = \frac{4x^2}{4x^2 - 9}$$

$$\therefore y^2 = \frac{9x^2}{4x^2 - 9} \Rightarrow y^2 = \frac{x^2}{4x^2 - 9}$$

$$y^2 = \frac{9x^2}{4x^2 - 9}$$

Q6

(Total 11 marks)

6. The symmetric matrix M has eigenvectors $\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$, $\begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$ with eigenvalues 5, 2 and -1 respectively.

(a) Find an orthogonal matrix P and a diagonal matrix D such that

$$P^T M P = D \quad (4)$$

Given that $P^{-1} = P^T$

(b) show that

$$M = P D P^{-1} \quad (2)$$

(c) Hence find the matrix M . (5)

Find Normalise e. vectors

$$x_1 = \frac{1}{3} \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$$

$$x_2 = \frac{1}{3} \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}$$

$$x_3 = \frac{1}{3} \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$$

$$\therefore P = \frac{1}{3} \begin{pmatrix} 2 & -2 & 1 \\ 2 & 1 & -2 \\ 1 & 2 & 2 \end{pmatrix}$$

$$D = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Question 6 continued

$$(b) P^{-1} = P^T$$

~~$$P^T M P = D$$~~

$$\therefore P^{-1} M P = D$$

$$(\times P) P P^{-1} M P = P D$$

$$\therefore M P = P D$$

~~$$M P P^{-1} = P D P^{-1} \iff (\times P^{-1})$$~~

$$\therefore M = P D P^{-1} \text{ as required.}$$

$$(c) P^T = P^{-1} = \frac{1}{3} \begin{pmatrix} 2 & 2 & 1 \\ -2 & 1 & 2 \\ 1 & -2 & 2 \end{pmatrix}$$

~~$$M =$$~~
$$M = P D P^{-1}$$

$$P D = \frac{1}{3} \begin{pmatrix} 2 & -2 & 1 \\ 2 & 1 & -2 \\ 1 & 2 & 2 \end{pmatrix} \begin{pmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$= \frac{1}{3} \begin{pmatrix} 10 & -4 & -1 \\ 10 & 2 & 2 \\ 5 & 4 & -2 \end{pmatrix}$$

Question 6 continued

$$\therefore M = \begin{pmatrix} 1 \\ 3 \end{pmatrix}^2 \begin{pmatrix} 10 & -4 & -1 \\ 10 & 2 & 2 \\ 5 & 4 & -2 \end{pmatrix} \begin{pmatrix} 2 & 2 & 1 \\ -2 & 1 & 2 \\ 1 & -2 & 2 \end{pmatrix}$$

$$\therefore M = \frac{1}{9} \begin{pmatrix} 27 & 18 & 0 \\ 18 & 18 & 18 \\ 0 & 18 & 9 \end{pmatrix}$$

$$\therefore M = \begin{pmatrix} 3 & 2 & 0 \\ 2 & 2 & 2 \\ 0 & 2 & 1 \end{pmatrix}$$

7. The curve C has equation

$$y = e^{-x}, \quad x \in \mathbb{R}$$

The part of the curve C between $x = 0$ and $x = \ln 3$ is rotated through 2π radians about the x -axis.

(a) Show that the area S of the curved surface generated is given by

$$S = 2\pi \int_0^{\ln 3} e^{-x} \sqrt{1 + e^{-2x}} \, dx \tag{3}$$

(b) Use the substitution $e^{-x} = \sinh u$ to show that

$$S = 2\pi \int_{\operatorname{arsinh} \alpha}^{\operatorname{arsinh} \beta} \cosh^2 u \, du$$

where α and β are constants to be determined.

(5)

(c) Show that

$$2 \int \cosh^2 u \, du = \frac{1}{2} \sinh 2u + u + k$$

where k is an arbitrary constant.

(2)

(d) Hence find the value of S , giving your answer to 3 decimal places.

(2)

7(a). $y = e^{-x} \Rightarrow \frac{dy}{dx} = -e^{-x} \Rightarrow \left(\frac{dy}{dx}\right)^2 = e^{-2x}$
 Limits are from $x=0$ to $x = \ln 3$

$$\therefore S = 2\pi \int_0^{\ln 3} y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx$$

$$\Rightarrow S = 2\pi \int_0^{\ln 3} e^{-x} \sqrt{1 + e^{-2x}} \, dx$$

as required.

$$\frac{1}{e^x} = \sinh u$$

$$(b) e^{-x} = \sinh u \Rightarrow$$

$$\therefore \frac{dx}{du} \cdot -e^{-x} = \cosh u$$

$$\therefore -\frac{dx}{du} \frac{1}{e^x} = \cosh u$$

$$\therefore dx = -e^x \cosh u du$$

$$\therefore dx = -\frac{\cosh u}{\sinh u} du$$

$$e^x = \frac{1}{\sinh u}$$

$$e^{-x} \sqrt{1+e^{-2x}} \equiv \sinh u \sqrt{1+\sinh^2 u} = \sinh u \sqrt{\cosh^2 u}$$

$$\therefore e^{-x} \sqrt{1+e^{-2x}} \equiv \sinh u \cosh u$$

Consider limits: from $x=0$ to $x=\ln 3$

~~if~~

$$\underline{x=0}: x=0 \Rightarrow e^{-0} = \sinh u$$

$$\therefore 1 = \sinh u \Rightarrow u = \operatorname{arsinh}(1)$$

$$\underline{x=\ln 3}: x=\ln 3 \Rightarrow e^{-\ln 3} = \sinh u$$

$$\therefore \frac{1}{3} = \sinh u \Rightarrow u = \operatorname{arsinh}\left(\frac{1}{3}\right)$$

\therefore New limits are from $u = \operatorname{arsinh}(1)$
to $u = \operatorname{arsinh}\left(\frac{1}{3}\right)$



Question 7 continued

Now:

$$dn = -\frac{\cosh u}{\sinh u} du$$

$$\& e^{-n} \sqrt{1+e^{-2n}} \equiv \sinh u \cosh u$$

$$\therefore \int = 2\pi \int_0^{\ln 3} e^{-n} \sqrt{1+e^{-2n}} dn = 2\pi \int_{\operatorname{arsinh} 1}^{\operatorname{arsinh} \frac{1}{3}} \sinh u \cosh u \cdot -\frac{\cosh u}{\sinh u} du$$

$$\therefore \int = -2\pi \int_{\operatorname{arsinh} 1}^{\operatorname{arsinh} \frac{1}{3}} \cosh^2 u du = 2\pi \int_{\operatorname{arsinh} \frac{1}{3}}^{\operatorname{arsinh} 1} \cosh^2 u du$$

as required

$$\beta = 1 \quad \alpha = \frac{1}{3}$$

(C) $\cosh^2 u + \sinh^2 u = \cosh 2u$

$$\therefore 2 \cosh^2 u - 1 = \cosh 2u$$

$$\therefore \cosh^2 u = \frac{1}{2} (\cosh 2u + 1)$$

$$\therefore 2 \int \cosh^2 u du = \left(2 \times \frac{1}{2}\right) \int \cosh 2u + 1 du$$

$$= \frac{1}{2} \sinh 2u + u + k$$

as required.

Question 7 continued

$$S = 2\pi \int_{\operatorname{arcsinh} \frac{1}{3}}^{\operatorname{arcsinh} 1} \cosh^2 u \, du = \pi \left[\frac{1}{2} \sinh 2u + u \right]_{\operatorname{arcsinh} \frac{1}{3}}^{\operatorname{arcsinh} 1}$$

$\operatorname{arcsinh} 1 = \ln(1+\sqrt{2})$
 $\operatorname{arcsinh} \frac{1}{3} = \ln \left(\frac{1+\sqrt{10}}{3} \right)$

$$= \pi \left(\frac{1}{2} \sinh [2 \ln(1+\sqrt{2})] + \ln(1+\sqrt{2}) - \frac{1}{2} \sinh \left[2 \ln \frac{1+\sqrt{10}}{3} \right] - \ln \frac{1+\sqrt{10}}{3} \right)$$

$$= \pi \left(1.414\dots + \ln(1+\sqrt{2}) - 0.3513\dots - \ln \frac{1+\sqrt{10}}{3} \right)$$

$$= \pi \times 1.616\dots$$

$$= 5.079 \text{ (3 s.f.)}$$

Q7

(Total 12 marks)



8. The plane Π_1 has vector equation $\mathbf{r} \cdot \begin{pmatrix} 12 \\ 1 \\ 3 \end{pmatrix} = 5$

The plane Π_2 has vector equation $\mathbf{r} \cdot \begin{pmatrix} -1 \\ 2 \\ 4 \end{pmatrix} = 7$

- (a) Find a vector equation for the line of intersection of Π_1 and Π_2 , giving your answer in the form $\mathbf{r} = \mathbf{a} + \lambda \mathbf{b}$ where \mathbf{a} and \mathbf{b} are constant vectors and λ is a scalar parameter. (6)

The plane Π_3 has cartesian equation

$$x - y + 2z = 31$$

- (b) Using your answer to part (a), or otherwise, find the coordinates of the point of intersection of the planes Π_1 , Π_2 and Π_3 . (3)

8) a) $\Pi_1 : 2x + y + 3z = 5$

$\Pi_2 : -x + 2y + 4z = 7$

Solve simultaneously:

$\Rightarrow x = 2y + 4z - 7$

$\therefore 4y + 8z - 14 + y + 3z = 5$

$\therefore 5y + 11z = 19$

$\therefore y = \frac{19}{5} - \frac{11}{5}z$

$\therefore x = \frac{38}{5} - \frac{22}{5}z + 4z - 7$

$\therefore x = \frac{3}{5} - \frac{2}{5}z$



Question 8 continued

$$\therefore x = \frac{3}{5} - \frac{2}{5}z$$

$$y = \frac{19}{5} - \frac{11}{5}z$$

$$z = z$$

$$\text{Let } z = \lambda$$

$$\Rightarrow \underline{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3/5 \\ 19/5 \\ 0 \end{pmatrix} + z \begin{pmatrix} -2/5 \\ -11/5 \\ 1 \end{pmatrix}$$

$$\therefore \underline{r} = \begin{pmatrix} 3/5 \\ 19/5 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -2/5 \\ -11/5 \\ 1 \end{pmatrix}$$

 ~~$\frac{1}{5}$~~
 λ is arbitrary.

$$\text{or } \underline{r} = \begin{pmatrix} 3/5 \\ 19/5 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ -11 \\ 5 \end{pmatrix}$$

(b)

~~$x - 2y + z = 31$~~

$$x - y + 2z = 31$$

$$\therefore \frac{3}{5} - \frac{2}{5}\lambda - \frac{19}{5} + \frac{11}{5}\lambda + 2\lambda = 31$$

$$\therefore \frac{19}{5}\lambda = \frac{171}{5}$$

$$\Rightarrow \lambda = 9$$

$$\Rightarrow \text{at intersection } \underline{r} = \begin{pmatrix} -3 \\ -16 \\ 9 \end{pmatrix}$$

