FP3 June 12 M.A. Kerine 2

Leave blank

1. The hyperbola H has equation

$$\frac{x^2}{16} - \frac{y^2}{9} = 1$$

Find

(a) the coordinates of the foci of H, **(3)**

(b) the equations of the directrices of H. **(2)**

(a) $\frac{x^2}{16} - \frac{y^2}{9} = 1 \Rightarrow a = 4$, $b = 3$

Eccentricity: $b^2 = a^2(e^2-1)$

$\therefore \quad 9 = 16(e^2-1) \Rightarrow e = \frac{5}{4}$

$\therefore$ Foci : $(\pm 5, 0)$ $\quad \boxed{\pm(ae, 0)}$

(b) $x = \pm \frac{a}{e} = \pm \frac{4}{5/4} = \pm \frac{16}{5}$

$\therefore \quad x = \pm \frac{16}{5}$

2.

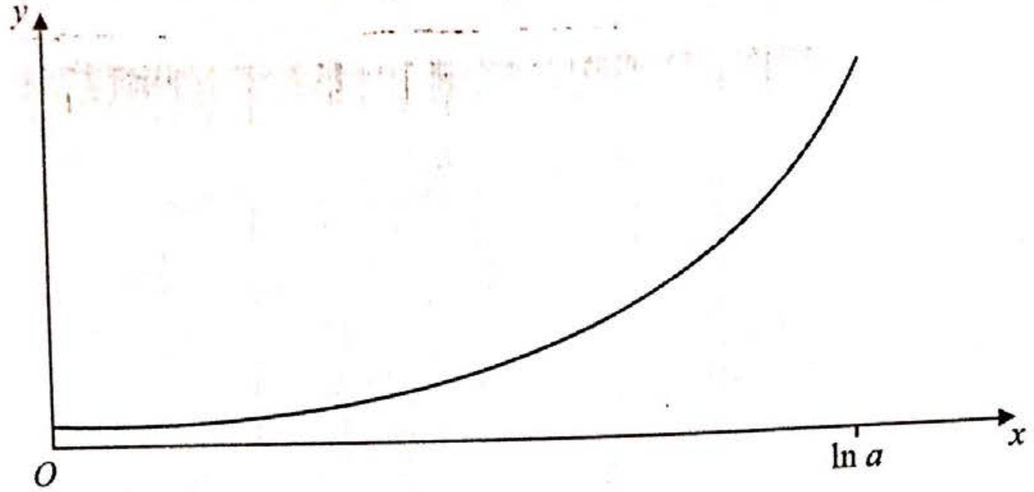


Figure 1

The curve C, shown in Figure 1, has equation

$$y = \frac{1}{3}\cosh 3x, \qquad 0 \leqslant x \leqslant \ln a$$

where a is a constant and $a > 1$

Using calculus, show that the length of curve C is

$$k\left(a^3 - \frac{1}{a^3}\right)$$

and state the value of the constant k.

(6)

2. Arc length $= \displaystyle\int_0^{\ln a} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$

$y = \frac{1}{3}\cosh 3x \Rightarrow \frac{dy}{dx} = \sinh 3x$

$c^2 - s^2 = 1$

$\cosh^2 = 1 + \sinh^2$

$\therefore$ Length $= \displaystyle\int_0^{\ln a} \sqrt{1 + \sinh^2 3x}\, dx$

$= \displaystyle\int_0^{\ln a} \cosh 3x \, dx = \left[\frac{1}{3}\sinh 3x\right]_0^{\ln a}$

Question 2 continued

$$= \frac{1}{3}\sinh(3\ln a) - \frac{1}{3}\sinh 0$$

$$= \frac{1}{3}\,\frac{e^{3\ln a} - e^{-3\ln a}}{2} - \frac{1}{3}\,\frac{e^{0} - e^{-0}}{2}$$

$$= \frac{a^3 - a^{-3}}{6} - 0$$

$$\cancel{= \frac{a^3 - a^{-3}}{6} \times \frac{a^3}{a^3}}$$

$$= \frac{a^3 - a^{-3}}{6} = \frac{1}{6}\left(a^3 - \frac{1}{a^3}\right)$$

as required

$$k = \frac{1}{6}$$

Q2

(Total 6 marks)

Turn over

Leave blank

3. The position vectors of the points A, B and C relative to an origin O are $\mathbf{i}-2\mathbf{j}-2\mathbf{k}$, $7\mathbf{i}-3\mathbf{k}$ and $4\mathbf{i}+4\mathbf{j}$ respectively.

Find

(a) $\overrightarrow{AC}\times\overrightarrow{BC}$, **(4)**

(b) the area of triangle ABC, **(2)**

(c) an equation of the plane ABC in the form $\mathbf{r.n}=p$ **(2)**

3(a). $A=\begin{pmatrix}1\\-2\\-2\end{pmatrix}$ $B=\begin{pmatrix}7\\0\\-3\end{pmatrix}$ $C=\begin{pmatrix}4\\4\\0\end{pmatrix}$

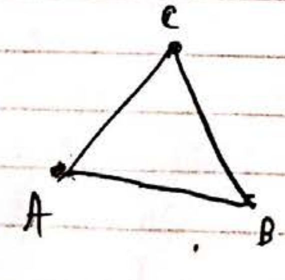
$\overrightarrow{AC}=\begin{pmatrix}4\\4\\0\end{pmatrix}-\begin{pmatrix}1\\-2\\-2\end{pmatrix}=\begin{pmatrix}3\\6\\2\end{pmatrix}$

$\overrightarrow{BC}=\begin{pmatrix}4\\4\\0\end{pmatrix}-\begin{pmatrix}7\\0\\-3\end{pmatrix}=\begin{pmatrix}-3\\4\\3\end{pmatrix}$

$\therefore \overrightarrow{AC}\times\overrightarrow{BC}=\begin{pmatrix}3\\6\\2\end{pmatrix}\times\begin{pmatrix}-3\\4\\3\end{pmatrix}= \begin{matrix}3 & 6 & 2 & 3 & 6 & 2\\ -3 & 4 & 3 & -3 & 4 & 3\end{matrix}$

$=\begin{pmatrix}10\\-15\\30\end{pmatrix}$

(b)



Area $=\frac{1}{2}|AC\times BC|$

$=\frac{1}{2}\left|\begin{pmatrix}10\\-15\\30\end{pmatrix}\right|=\frac{35}{2}$ units^2

Leave blank

Question 3 continued

(c) $\underset{\sim}{n} = \begin{pmatrix} 10 \\ -15 \\ 30 \end{pmatrix}$

$\therefore \underset{\sim}{r} \cdot \underset{\sim}{n} = \begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 10 \\ -15 \\ 30 \end{pmatrix}$

$= 10 + 30 \cancel{*} - 60 \quad = -20$

$\therefore \underset{\sim}{r} \cdot \begin{pmatrix} 10 \\ -15 \\ 30 \end{pmatrix} = -20$

4. $$I_n = \int_0^{\frac{\pi}{4}} x^n \sin 2x \, dx, \qquad n \geqslant 0$$

(a) Prove that, for $n \geqslant 2$,

$$I_n = \frac{1}{4}n\left(\frac{\pi}{4}\right)^{n-1} - \frac{1}{4}n(n-1)I_{n-2}$$

(5)

(b) Find the exact value of I_2 **(4)**

(c) Show that $I_4 = \frac{1}{64}(\pi^3 - 24\pi + 48)$ **(2)**

4(a). $I_n = \int_0^{\pi/4} x^n \sin 2x \, dx$

Let $u = x^n \quad u' = nx^{n-1}$

$v' = \sin 2x \quad v = -\frac{1}{2}\cos 2x$

$\therefore I_n = \left[-\frac{x^n}{2}\cos 2x\right]_0^{\pi/4} + \frac{n}{2}\int_0^{\pi/4} x^{n-1}\cos 2x \, dx$

$\therefore I_n = \frac{n}{2}\int_0^{\pi/4} x^{n-1}\cos 2x \, dx$

Let new $u = x^{n-1} \quad u' = (n-1)x^{n-2}$

new $v' = \cos 2x \quad v = \frac{1}{2}\sin 2x$

$\therefore I_n = \frac{n}{2}\left(\left[\frac{x^{n-1}}{2}\sin 2x\right]_0^{\pi/4} - \frac{(n-1)}{2}\int_0^{\pi/4} x^{n-2}\sin 2x \, dx\right)$

Question 4 continued

$$\therefore I_n = \frac{n}{2}\left(\frac{(\pi/4)^{n-1}}{2} - \frac{(n-1)}{2} I_{n-2}\right)$$

$$\therefore I_n = \frac{1}{4}n\left(\frac{\pi}{4}\right)^{n-1} - \frac{1}{4}n(n-1)I_{n-2}$$

as required.

(b) $I_2 = \frac{1}{2}\left(\frac{\pi}{4}\right) - \frac{1}{2}I_0$

$$= \frac{\pi}{8} - \frac{1}{2}\int_0^{\pi/4} \sin 2x \, dx$$

$$= \frac{\pi}{8} - \frac{1}{2}\left[-\frac{1}{2}\cos 2x\right]_0^{\pi/4}$$

$$= \frac{\pi}{8} - \frac{1}{2}\left(0 - -\frac{1}{2}\right)$$

$$= \frac{\pi}{8} - \frac{1}{4} = \frac{\pi - 2}{8}$$

(c) $I_4 = \left(\frac{\pi}{4}\right)^3 - 3I_2 = \left(\frac{\pi}{4}\right)^3 - 3\left(\frac{\pi-2}{8}\right)$

$$= \frac{\pi^3}{64} - \frac{3\pi - 6}{8} = \frac{\pi^3}{64} - \frac{8(3\pi-6)}{64}$$

$$= \frac{\pi^3}{64} - \frac{24\pi - 48}{64} = \frac{1}{64}\left(\pi^3 - 24\pi + 48\right)$$ as required.

5. (a) Differentiate $x \operatorname{arsinh} 2x$ with respect to x. **(3)**

(b) Hence, or otherwise, find the exact value of

$$\int_0^{\sqrt{2}} \operatorname{arsinh} 2x \, dx$$

giving your answer in the form $A \ln B + C$, where A, B and C are real. **(7)**

5(a). $\frac{\partial}{\partial x}\left(x \operatorname{arsinh} 2x\right) = \operatorname{arsinh} 2x + x \frac{\partial}{\partial x}\left(\operatorname{arsinh} 2x\right)$

Consider $\frac{\partial}{\partial x}\left(\operatorname{arsinh} 2x\right)$

Let $y = \operatorname{arsinh} 2x$

$\sinh y = 2x$ $\qquad c^2 - s^2 = 1$

$\cosh y \frac{\partial y}{\partial x} = 2$

$\therefore \frac{\partial y}{\partial x} = \frac{2}{\cosh y} = \frac{2}{\sqrt{1+\sinh^2 y}}$

$= \frac{2}{\sqrt{1+4x^2}}$

$\therefore \frac{\partial}{\partial x}\left(x \operatorname{arsinh} 2x\right) = \operatorname{arsinh} 2x + \frac{2x}{\sqrt{1+4x^2}}$

Question 5 continued

(b) $\frac{\partial}{\partial x}\left(x\,\mathrm{arsinh}\,2x\right) = \mathrm{arsinh}\,2x + \frac{2x}{\sqrt{1+4x^2}}$

$\therefore\quad \mathrm{arsinh}\,2x = \frac{\partial}{\partial x}\left(x\,\mathrm{arsinh}\,2x\right) - \frac{2x}{\sqrt{1+4x^2}}$

$\therefore\quad \int_0^{\sqrt{2}} \mathrm{arsinh}\,2x\,\partial x = \int_0^{\sqrt{2}} \frac{\partial}{\partial x}\left(x\,\mathrm{arsinh}\,2x\right)\partial x - \int_0^{\sqrt{2}} \frac{2x}{\sqrt{1+4x^2}}\,\partial x$

$= \left[x\,\mathrm{arsinh}\,2x\right]_0^{\sqrt{2}} - \frac{1}{4}\int_0^{\sqrt{2}} 8x\left(1+4x^2\right)^{-1/2}\partial x$

$= \sqrt{2}\,\mathrm{arsinh}\,2\sqrt{2} - \frac{1}{4}\left[2\left(1+4x^2\right)^{1/2}\right]_0^{\sqrt{2}}$

$= \sqrt{2}\,\mathrm{arsinh}\,2\sqrt{2} - \frac{1}{4}\left(6-2\right)$

$= \sqrt{2}\,\mathrm{arsinh}\,2\sqrt{2} - 1$

$= \sqrt{2}\ln\left(2\sqrt{2} + \sqrt{9}\right) - 1$

$= \sqrt{2}\ln\left(3+2\sqrt{2}\right) - 1$

$A = \sqrt{2}$

$B = 3+2\sqrt{2}$

$C = -1$

Leave blank

6. The ellipse E has equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

The line l_1 is a tangent to E at the point P $(a\cos\theta,\ b\sin\theta)$.

(a) Using calculus, show that an equation for l_1 is

$$\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1$$

(4)

The circle C has equation

$$x^2+y^2=a^2$$

The line l_2 is a tangent to C at the point Q $(a\cos\theta,\ a\sin\theta)$.

(b) Find an equation for the line l_2.

(2)

Given that l_1 and l_2 meet at the point R,

(c) find, in terms of a, b and θ, the coordinates of R.

(3)

(d) Find the locus of R, as θ varies.

(2)

6(a) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, $x=a\cos\theta$ $y=b\sin\theta$

$\frac{dy}{dx}=\frac{b\cos\theta}{-a\sin\theta}=-\frac{b}{a}\cot\theta$

$\therefore$ ~~$y-b\sin\theta=-\frac{b}{a}(x-a\cos\theta)$~~

$\therefore y-b\sin\theta=-\frac{b}{a}\cot\theta(x-a\cos\theta)$

$\therefore y-b\sin\theta=-\frac{b}{a}x\cot\theta+b\frac{\cos^2\theta}{\sin\theta}$

Leave blank

Question 6 continued

$\therefore \times \sin\theta \Rightarrow \quad y\sin\theta - b\sin^2\theta = -\frac{b}{a}x\cos\theta + b\cos^2\theta$

$\div b \Rightarrow \quad \frac{y\sin\theta}{b} - \sin^2\theta = -\frac{x\cos\theta}{a} + \cos^2\theta$

$\therefore \quad \frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = \cos^2\theta + \sin^2\theta$

$\therefore \quad \frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$

as required.

(b) Consider Circle: $\quad x = a\cos\theta \quad y = a\sin\theta$

$\frac{dy}{dx} = \frac{a\cos\theta}{-a\sin\theta} = -\cot\theta$

$\therefore \quad y - a\sin\theta = -\cot\theta(x - a\cos\theta)$

$\therefore \quad y - a\sin\theta = -\frac{\cos\theta}{\sin\theta}x + \frac{a\cos^2\theta}{\sin\theta}$

$\therefore$

$y\sin\theta - a\sin^2\theta = -x\cos\theta + a\cos^2\theta$

$\therefore \quad y\sin\theta + x\cos\theta = a(\cos^2\theta + \sin^2\theta)$

$\therefore \quad y\sin\theta + x\cos\theta = a$

Turn over

Leave blank

Question 6 continued

(c) $\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$

& $y\sin\theta + x\cos\theta = a$

$\Rightarrow \quad x\cos\theta = a - y\sin\theta$

$\therefore \quad \frac{a - y\sin\theta}{a} + \frac{y\sin\theta}{b} = 1$

$\therefore$

$\frac{ba - yb\sin\theta}{ab} + \frac{ay\sin\theta}{ab} = 1$

$\therefore \quad ba - yb\sin\theta + ya\sin\theta = ab$

$\therefore \quad \cancel{ba} \quad ab + y\sin\theta(a-b) = ab$

$\therefore \quad y\sin\theta(a-b) = 0$

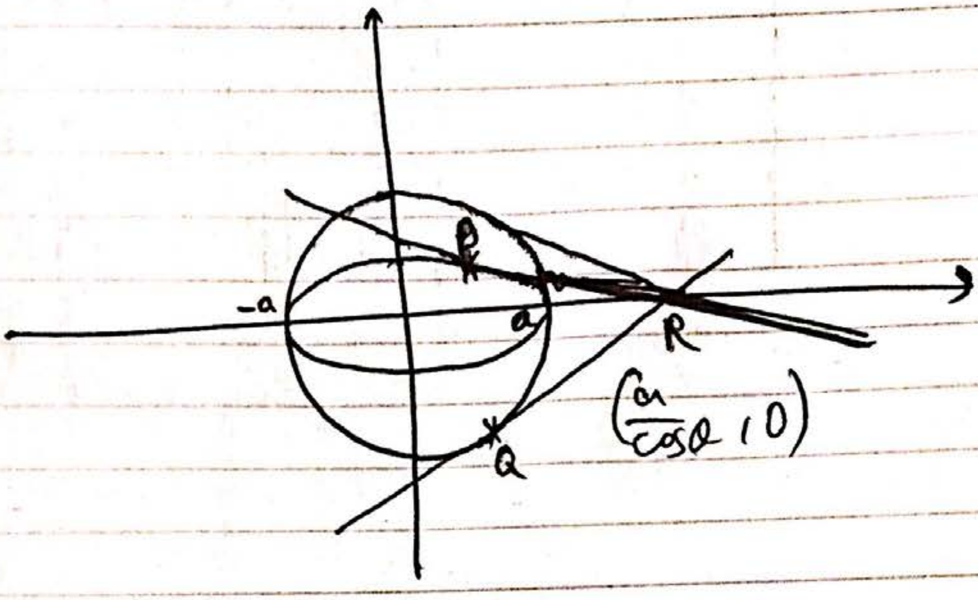
$\Rightarrow \quad y = 0$

$\Rightarrow$

$x\cos\theta = a \Rightarrow x = \frac{a}{\cos\theta}$

$\therefore \quad R : \left(\frac{a}{\cos\theta}, 0\right)$

(d)



~~x_R cannot~~

$x_R \notin [-a, a]$ because

otherwise tangents fall apart.

R lies on x-axis

$\therefore$ $x \geq a$

$x \leq -a$

7. $$\mathrm{f}(x) = 5\cosh x - 4\sinh x, \qquad x \in \mathbb{R}$$

(a) Show that $\mathrm{f}(x) = \frac{1}{2}(\mathrm{e}^{x} + 9\mathrm{e}^{-x})$ **(2)**

Hence

(b) solve $\mathrm{f}(x) = 5$ **(4)**

(c) show that $\int_{\frac{1}{2}\ln 3}^{\ln 3} \frac{1}{5\cosh x - 4\sinh x}\,\mathrm{d}x = \frac{\pi}{18}$ **(5)**

7(a). $f(x) = 5\cosh x - 4\sinh x$

$$= \frac{5e^{x} + 5e^{-x}}{2} - \frac{4e^{x} - 4e^{-x}}{2}$$

$$= \frac{5e^{x} + 5e^{-x} - 4e^{x} + 4e^{-x}}{2}$$

$$= \frac{e^{x} + 9e^{-x}}{2} = \frac{1}{2}(e^{x} + 9e^{-x})$$

as required.

(b) $\frac{1}{2}(e^{x} + 9e^{-x}) = 5$

$\therefore\ e^{x} + 9e^{-x} = 10$

$\therefore\ e^{2x} + 9 = 10e^{x}$

$\therefore\ e^{2x} - 10e^{x} + 9 = 0$

$(e^{x} - 9)(e^{x} - 1) = 0 \Rightarrow e^{x} = 9 \Rightarrow x = \ln 9$

$e^{x} = 1 \Rightarrow x = 0$

Leave blank

Question 7 continued

(c) $$\int_{\frac{1}{2}\ln 3}^{\ln 3} \frac{1}{\frac{1}{2}(e^{x}+9e^{-x})}\,\partial x$$

$$= 2\int_{\frac{1}{2}\ln 3}^{\ln 3} \frac{1}{e^{x}+9e^{-x}}\,\partial x$$

$$\cancel{\frac{e^x}{e^{2x}+9}}$$

$$= 2\int_{\frac{1}{2}\ln 3}^{\ln 3} \frac{e^{x}}{e^{2x}+9}\,\partial x$$

$$\cancel{= 2\int_{\frac{1}{2}\ln 3}^{\ln 3} \frac{2e^{2x}}{\ldots}}$$

Let $u = e^{x}$ $\cancel{u^2 =}$

$$\frac{\partial u}{\partial x} = e^{x}$$

$$\therefore \partial x = \frac{\partial u}{e^{x}}$$

$$\therefore \partial x = \frac{\partial u}{u}$$

$\cancel{\ln 3 = e}$ $u = e^{\ln 3} = 3$

$\cancel{u = e^{\ln\frac{1}{2}\ln 3}}$

$$u = e^{\frac{1}{2}\ln 3} = \sqrt{3}$$

$$= 2\int_{\sqrt{3}}^{3} \frac{u}{u^{2}+9}\cdot\frac{1}{u}\,\partial u$$

$$= 2\int_{\sqrt{3}}^{3} \frac{1}{u^{2}+9}\,\partial u = 2\left[\frac{1}{3}\arctan\frac{u}{3}\right]_{\sqrt{3}}^{3}$$

Question 7 continued

$$= 2\left(\frac{1}{3}\arctan 1 - \frac{1}{3}\arctan\frac{\sqrt{3}}{3}\right)$$

$$= 2\left(\frac{\pi}{12} - \frac{\pi}{18}\right)$$

$$= 2 \times \frac{\pi}{36} = \frac{\pi}{18}$$

as required.

8. The matrix **M** is given by

$$\mathbf{M} = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ -1 & 0 & 4 \end{pmatrix}$$

(a) Show that 4 is an eigenvalue of **M**, and find the other two eigenvalues. **(5)**

(b) For the eigenvalue 4, find a corresponding eigenvector. **(3)**

The straight line l_1 is mapped onto the straight line l_2 by the transformation represented by the matrix **M**.

The equation of l_1 is $(\mathbf{r}-\mathbf{a})\times\mathbf{b}=\mathbf{0}$, where $\mathbf{a}=3\mathbf{i}+2\mathbf{j}-2\mathbf{k}$ and $\mathbf{b}=\mathbf{i}-\mathbf{j}+2\mathbf{k}$.

(c) Find a vector equation for the line l_2. **(5)**

8(a). ~~M =~~

$$M - \lambda I = \begin{pmatrix} 2-\lambda & 1 & 0 \\ 1 & 2-\lambda & 0 \\ -1 & 0 & 4-\lambda \end{pmatrix}$$

$$\therefore \det(M-\lambda I) = (2-\lambda)(2-\lambda)(4-\lambda) - (4-\lambda)$$

$$= (4-\lambda)((2-\lambda)^2 - 1)$$

$$= (4-\lambda)(\lambda^2 - 4\lambda + 3)$$

$$= (4-\lambda)(\lambda-3)(\lambda-1) = 0$$

$\therefore (4-\lambda)=0 \Rightarrow \lambda = 4$ is indeed an E. Value

$\lambda=3$ $\lambda=1$ are also eigenvalues.

Question 8 continued

(b) $Mx = 4x$

$$\therefore \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ -1 & 0 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4x \\ 4y \\ 4z \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 2x + y \\ x + 2y \\ -x + 4z \end{pmatrix} = \begin{pmatrix} 4x \\ 4y \\ 4z \end{pmatrix}$$

$\therefore \quad 2x + y = 4x \Rightarrow 2x = y$

$x + 2y = 4y \Rightarrow 2y = x$

$-x + 4z = 4z \Rightarrow x = 0$

$x = 0 \Rightarrow y = 0$

let $z = 1$

$\therefore$ An e-vector is $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

(c)

(c) eqn of l_1 $(\underline{r} - \underline{a}) \times \underline{b} = 0$

$$\therefore \underline{r} \times \underline{b} = \underline{a} \times \underline{b}$$

$$\therefore \underline{r} = \underline{a} + \lambda \underline{b}$$

$$\underline{r} = \begin{pmatrix} 3 \\ 2 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

$$\underline{r} = \begin{pmatrix} 3+\lambda \\ 2-\lambda \\ -2+2\lambda \end{pmatrix}$$

$M_{l_1} = l_2$

$$\therefore \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ -1 & 0 & 4 \end{pmatrix} \begin{pmatrix} 3+\lambda \\ 2-\lambda \\ -2+2\lambda \end{pmatrix} =$$

$$= \begin{pmatrix} 6+2\lambda+2-\lambda \\ 3+\lambda+4-2\lambda \\ -3-\lambda-8+8\lambda \end{pmatrix} = \begin{pmatrix} 8+\lambda \\ 7-\lambda \\ -11+7\lambda \end{pmatrix}$$

$$\therefore \underline{r} = \begin{pmatrix} 8 \\ 7 \\ -11 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 7 \end{pmatrix}$$

$$\therefore \left(\underline{r} - \begin{pmatrix} 8 \\ 7 \\ -11 \end{pmatrix} \right) \times \begin{pmatrix} 1 \\ -1 \\ 7 \end{pmatrix} = 0$$