FP3 June 13 (R) M.A Kprime 2

1. The hyperbola H has foci at (5, 0) and (–5, 0) and directrices with equations

$x = \frac{9}{5}$ and $x = -\frac{9}{5}$.

Find a cartesian equation for H.

(7)

1. $\pm ae = \pm 5$

$ae = 5 \Rightarrow a = \frac{5}{e} \qquad e = \frac{5}{a}$

Directrizes $\Rightarrow \frac{9}{5} = \frac{a}{e}$, $-\frac{9}{5} = -\frac{a}{e}$

$\therefore \frac{9}{5} = \frac{a}{5/a} = \frac{a^2}{5}$

$\therefore a^2 = 9 \Rightarrow a = 3$

$\therefore 3e = 5 \Rightarrow e = \frac{5}{3}$

$\therefore \frac{x^2}{9} - \frac{y^2}{16} = 1$ $\quad$ $a=3$, $b=4$

$b^2 = |a^2(1-e^2)| \Rightarrow b^2 = 16 \Rightarrow b = 4$

~~$\therefore b^2 = 9\left(1 - \frac{25}{9}\right) < 0$~~

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2. Two skew lines l_1 and l_2 have equations

$$l_1: \mathbf{r} = (\mathbf{i} - \mathbf{j} + \mathbf{k}) + \lambda(4\mathbf{i} + 3\mathbf{j} + 2\mathbf{k})$$

$$l_2: \mathbf{r} = (3\mathbf{i} + 7\mathbf{j} + 2\mathbf{k}) + \mu(-4\mathbf{i} + 6\mathbf{j} + \mathbf{k})$$

respectively, where λ and μ are real parameters.

(a) Find a vector in the direction of the common perpendicular to l_1 and l_2 **(2)**

(b) Find the shortest distance between these two lines. **(5)**

2(a).

$$\begin{pmatrix}4\\3\\2\end{pmatrix} \times \begin{pmatrix}-4\\6\\1\end{pmatrix} = \times \begin{matrix}4 & 3 & 2 & 4 & 3 & 2\\ -4 & 6 & 1 & -4 & 6 & 1\end{matrix}$$

$$= \begin{pmatrix}-9\\-12\\36\end{pmatrix}$$

$$\therefore \hat{n} = \begin{pmatrix}-9\\-12\\36\end{pmatrix}$$

(b) ~~d=~~

~~d~~= $d = \left| \dfrac{\left(\begin{pmatrix}1\\-1\\1\end{pmatrix} - \begin{pmatrix}3\\7\\2\end{pmatrix}\right) \cdot \begin{pmatrix}-9\\-12\\36\end{pmatrix}}{\left|\begin{pmatrix}-9\\-12\\36\end{pmatrix}\right|} \right|$

$$\begin{pmatrix}-2\\-8\\-1\end{pmatrix}$$

$$\therefore d = \frac{78}{39} \Rightarrow d = 2$$

3. The point P lies on the ellipse E with equation

$$\frac{x^2}{36} + \frac{y^2}{9} = 1$$

N is the foot of the perpendicular from point P to the line $x = 8$

M is the midpoint of PN.

(a) Sketch the graph of the ellipse E, showing also the line $x = 8$ and a possible position for the line PN.

(1)

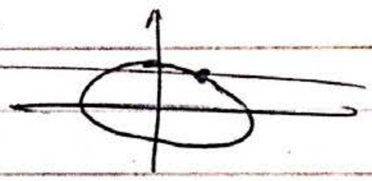
(b) Find an equation of the locus of M as P moves around the ellipse.

(4)

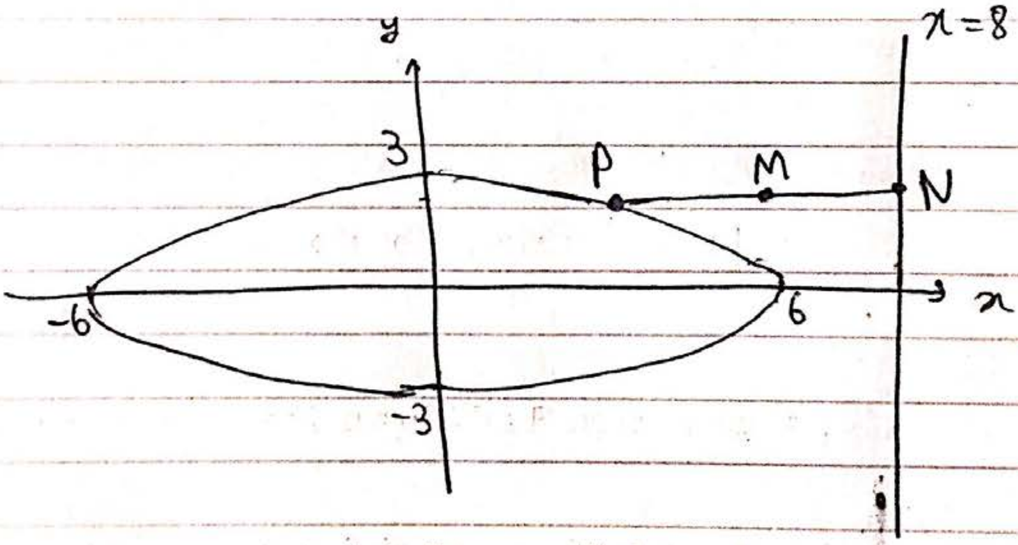
(c) Show that this locus is a circle and state its centre and radius.

(3)

3(a).



3(a).



~~36+e^2~~ -

(b)

~~=36(1-e^2)~~

(b) $\cancel{y=k}$

$P(a\cos\theta, b\sin\theta)$

$W(8, b\sin\theta)$

$\therefore M: \left(\frac{8+a\cos\theta}{2}, b\sin\theta\right)$

$\therefore x = \frac{8+a\cos\theta}{2} \Rightarrow \frac{2x-8}{a} = \cos\theta$

$y = b\sin\theta \Rightarrow \sin\theta = \frac{y}{b}$

$\therefore \sin^2\theta + \cos^2\theta = 1 = \frac{y^2}{b^2} + \frac{(2x-8)^2}{a^2}$

$\therefore \frac{(2x-8)^2}{a^2} + \frac{y^2}{b^2} = 1$

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~~$x^2+y^2 = r^2$~~

~~$a=6$~~ $a^2 = 36$
$b^2 = 9$

$$\therefore \frac{(2x-8)^2}{36} + \frac{y^2}{9} = 1$$

(c) ~~$\frac{1}{9}(x-4)^2 + \frac{1}{9}y^2 = 1$~~

$$\frac{(2x-8)^2}{36} + \frac{y^2}{9} = \frac{[2(x-4)]^2}{36} + \frac{y^2}{9}$$

$$= \frac{4}{36}(x-4)^2 + \frac{y^2}{9}$$

$$= \frac{1}{9}(x-4)^2 + \frac{y^2}{9} = 1$$

$$\therefore (x-4)^2 + y^2 = 9$$

$\therefore$ Centre is $(4,0)$

radius $= 3$.

4. The plane Π_1 has vector equation

$$\mathbf{r} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + s\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix},$$

where s and t are real parameters.

The plane Π_1 is transformed to the plane Π_2 by the transformation represented by the matrix $\mathbf{T}$, where

$$\mathbf{T} = \begin{pmatrix} 2 & 0 & 3 \\ 0 & 2 & -1 \\ 0 & 1 & 2 \end{pmatrix}$$

Find an equation of the plane Π_2 in the form $\mathbf{r}.\mathbf{n} = p$

(9)

Π_1

$$\underset{\frown}{r} = \begin{pmatrix} 1+s+t \\ -1+s+2t \\ 2-2t \end{pmatrix}$$

$$\therefore \begin{pmatrix} 2 & 0 & 3 \\ 0 & 2 & -1 \\ 0 & 1 & 2 \end{pmatrix}\begin{pmatrix} 1+s+t \\ -1+s+2t \\ 2-2t \end{pmatrix}$$

$$= \begin{pmatrix} 2+2s+2t+6-6t \\ -2+2s+4t-2+2t \\ -1+s+2t+4-4t \end{pmatrix}$$

$$= \begin{pmatrix} 8+2s-4t \\ -4+2s+6t \\ 3+s-2t \end{pmatrix}$$

$$= \begin{pmatrix} 8 \\ -4 \\ 3 \end{pmatrix} + s\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} + t\begin{pmatrix} -4 \\ 6 \\ -2 \end{pmatrix}$$

$$\therefore \underset{\sim}{n} = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} \times \begin{pmatrix} -4 \\ 6 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} -10 \\ 0 \\ 20 \end{pmatrix}$$

$$\Rightarrow \underset{\sim}{r} \cdot \underset{\sim}{n} = \begin{pmatrix} 8 \\ -4 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} -10 \\ 0 \\ 20 \end{pmatrix} = -20$$

$$\underset{\sim}{r} \cdot 10 \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} = -20$$

$$\Rightarrow \underset{\sim}{r} \cdot \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} = -2$$

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5.

$$I_n = \int_1^5 x^n (2x-1)^{-\frac{1}{2}}\,dx, \qquad n \geqslant 0$$

(a) Prove that, for $n \geqslant 1$,

$$(2n+1)I_n = nI_{n-1} + 3 \times 5^n - 1$$

(5)

(b) Using the reduction formula given in part (a), find the exact value of I_2

(5)

5(a). $I_n = \int_1^5 x^n (2x-1)^{-1/2}\,\partial x$

Let $u = x^n \qquad u' = nx^{n-1}$

$v' = (2x-1)^{-1/2} \qquad v = \frac{1}{2}(2x-1)^{1/2} \cdot 2$

$= (2x-1)^{1/2}$

$\therefore I_n = \left[x^n(2x-1)^{1/2}\right]_1^5 - n\int_1^5 x^{n-1}(2x-1)^{1/2}\,\partial x$

$\therefore I_n = 3(5^n) - 1 - n\int_1^5 x^{n-1}(2x-1)^{1/2}$

$(2x-1)^{1/2} = (2x-1)^{-1/2}(2x-1)$

$\therefore I_n = 3(5^n) - 1 - n\int_1^5 x^{n-1}(2x-1)(2x-1)^{-1/2}\,\partial x$

$\therefore I_n = 3(5^n) - 1 - n\int 2x^n(2x-1)^{-1/2} - x^{n-1}(2x-1)^{-1/2}\,\partial x$

$\therefore \; I_n = 3(5^n) - 1 - n\left(2I_n - I_{n-1}\right)$

$\therefore \; I_n = 3(5^n) - 1 - 2n I_n + n I_{n-1}$

$\therefore \; (2n+1) I_n = n I_{n-1} + 3(5^n) - 1$

$\Rightarrow \; (2n+1) I_n = n I_{n-1} + 3 \times 5^n - 1$

as required.

b) Using $n = 2 \Rightarrow$

$5 I_2 = 2 I_1 + 74$

Using $n = 1 \Rightarrow$

$3 I_1 = I_0 + 14$

$\therefore \; 3 I_1 = \int_1^5 (2x-1)^{-1/2} \, dx + 14$

$\therefore \; 3 I_1 = \left[(2x-1)^{1/2}\right]_1^5 + 14$

$\therefore \; 3 I_1 = 16 \Rightarrow I_1 = \frac{16}{3}$

$\therefore \; 5 I_2 = 2 \times \frac{16}{3} + 74 \Rightarrow I_2 = \frac{254}{15}$

6. It is given that $\begin{pmatrix}1\\2\\0\end{pmatrix}$ is an eigenvector of the matrix **A**, where

$$\mathbf{A} = \begin{pmatrix}4 & 2 & 3\\ 2 & b & 0\\ a & 1 & 8\end{pmatrix}$$

and a and b are constants.

(a) Find the eigenvalue of **A** corresponding to the eigenvector $\begin{pmatrix}1\\2\\0\end{pmatrix}$. **(3)**

(b) Find the values of a and b. **(3)**

(c) Find the other eigenvalues of **A**. **(5)**

6 (a). $A\underset{\sim}{x} = \lambda \underset{\sim}{x}$

$$\cancel{\therefore \begin{pmatrix}4 & 2 & 3\\ 2 & b & 0\\ a & 1 & 8\end{pmatrix} = \begin{pmatrix}\lambda\\ 2\lambda\\ 0\end{pmatrix}}$$

$\Rightarrow$

$$\therefore \begin{pmatrix}4 & 2 & 3\\ 2 & b & 0\\ a & 1 & 8\end{pmatrix}\begin{pmatrix}1\\2\\0\end{pmatrix} = \begin{pmatrix}\lambda\\ 2\lambda\\ 0\end{pmatrix}$$

$$\therefore \begin{pmatrix}8\\ 2+2b\\ a+2\end{pmatrix} = \begin{pmatrix}\lambda\\ 2\lambda\\ 0\end{pmatrix} \Rightarrow \lambda = 8 \text{ is the corresponding E. Value}$$

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(b) $A x = \lambda x$

$$\Rightarrow \begin{pmatrix} 8 \\ 2+2b \\ a+2 \end{pmatrix} = \begin{pmatrix} 8 \\ 16 \\ 0 \end{pmatrix}$$

$\therefore\ a+2=0 \Rightarrow a=-2$

$2+2b=16 \Rightarrow b=7$

(c) $$A-\lambda I = \begin{pmatrix} 4-\lambda & 2 & 3 \\ 2 & 7-\lambda & 0 \\ -2 & 1 & 8-\lambda \end{pmatrix}$$

$\det(A-\lambda I)=0 \Rightarrow \cancel{(4-\lambda)(7-\lambda)(8-\lambda)-2(2(8-\lambda))+3(}$

$(4-\lambda)(7-\lambda)(8-\lambda) - 2(2(8-\lambda)) + 3(2+2(7-\lambda)) = 0$

$\det(A-\lambda I) = (4-\lambda)(7-\lambda)(8-\lambda) - 4(8-\lambda) + 6(7-\lambda) + 6 = 0$

$\cancel{\lambda=8 \Rightarrow \det(A-\lambda I)=0}$ $\quad 6(7-\lambda+1)$

$\therefore\ \cancel{(4-} \quad \cancel{(8-\lambda)[(4-\lambda)(7-\lambda)-4]+6(7-\lambda)+6=0}$

$\therefore\ (4-\lambda)(7-\lambda)(8-\lambda) - 4(8-\lambda) + 6(8-\lambda) = 0$

$\therefore (8-\lambda)[(4-\lambda)(7-\lambda)+2]$ $\quad \lambda^2-11\lambda+30=0$

$\therefore\ (8-\lambda)(\lambda^2-11\lambda+30)=0 \Rightarrow$

$\lambda^2-11\lambda+30 \Rightarrow (\lambda-5)(\lambda-6)=0$ ~~Via~~ $\rightarrow \therefore \cancel{\lambda = \frac{11 \pm \sqrt{\ }}{2}}$

$\therefore\ \lambda=5,\ \lambda=6$ ~~Quadratic Formula~~

7.

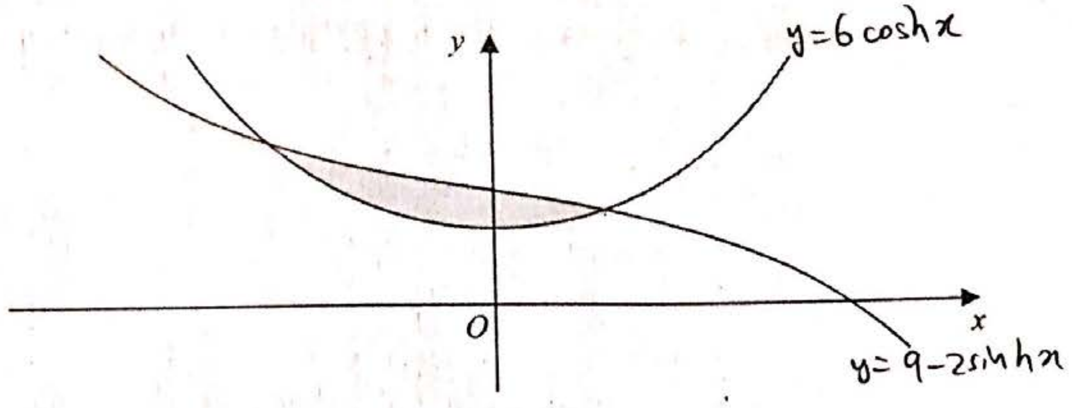

Figure 1

The curves shown in Figure 1 have equations

$$y = 6\cosh x \text{ and } y = 9 - 2\sinh x$$

(a) Using the definitions of $\sinh x$ and $\cosh x$ in terms of e^x, find exact values for the x-coordinates of the two points where the curves intersect. **(6)**

The finite region between the two curves is shown shaded in Figure 1.

(b) Using calculus, find the area of the shaded region, giving your answer in the form $a \ln b + c$, where a, b and c are integers. **(6)**

7(a). $y = 6\cosh x$ & $y = 9 - 2\sinh x$ @ intersection

$\therefore \quad 6\cosh x = 9 - 2\sinh x$

$\Rightarrow \cancel{6} \ \cancel{3(e} \ 3e^x + 3e^{-x} = 9 - (e^x - e^{-x})$

$\therefore \quad 3e^x + 3e^{-x} = 9 - e^x + e^{-x}$

$\therefore \quad 4e^x + 2e^{-x} - 9 = 0$

$\times e^x \therefore \quad 4e^{2x} - 9e^x + 2 = 0$

$(e^x - 2)(4e^x - 1) = 0$

$\Rightarrow e^x = 2 \Rightarrow x = \ln 2$

$e^x = 1/4 \Rightarrow x = \ln \frac{1}{4}$

(b) Area $= \int_{\ln 1/4}^{\ln 2} 9 - 2\sinh x \, dx - \int_{\ln 1/4}^{\ln 2} 6\cosh x \, dx$

$\therefore$ Area $= \int_{\ln 1/4}^{\ln 2} 9 - 2\sinh x - 6\cosh x \, dx$

$= \left[9x - 2\cosh x - 6\sinh x \right]_{\ln 1/4}^{\ln 2}$

$= 9\ln 2 - 2\cosh[\ln(2)] - 6\sinh[\ln 2]$

$\cancel{\frac{1}{4} = (2^2)^{-1}}$

$- 9\ln 1/4 + 2\cosh(\ln \frac{1}{4}) + 6\sinh(\ln \frac{1}{4})$

$\frac{1}{4} = (2^2)^{-1}$

~~Use $\ln(\frac{1}{4}) = -2\ln(2)$~~

$\therefore$ ~~$= 9\ln 2 - 2\cosh(\ln 2) - 6\sinh(\ln 2)$~~

$= 9\ln 2 - 2 \cdot \frac{2 + 2^{-1}}{2} - \frac{6}{2}(2 - 2^{-1}) - 9\ln\frac{1}{4} + \frac{4}{4} + 4$

$+ 3(1/4 - 4)$

$= 9\ln 2 - 2 - \frac{1}{2} - 6 + \frac{3}{2} - 9\ln\frac{1}{4} + \frac{1}{4} + 4 + \frac{3}{4} - 12$

$= -14 + 9\ln 8$

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8.

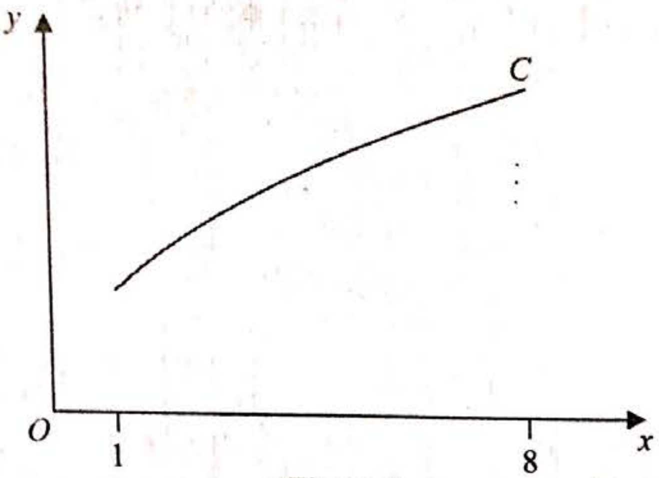

Figure 2

The curve C, shown in Figure 2, has equation

$$y = 2x^{\frac{1}{2}}, \qquad 1 \leqslant x \leqslant 8$$

(a) Show that the length s of curve C is given by the equation

$$s = \int_1^8 \sqrt{\left(1 + \frac{1}{x}\right)}\,\mathrm{d}x$$

(2)

(b) Using the substitution $x = \sinh^2 u$, or otherwise, find an exact value for s.

Give your answer in the form $a\sqrt{2} + \ln(b + c\sqrt{2})$ where a, b and c are integers.

(9)

8(a) $y = 2x^{1/2}$ $\therefore$ $\frac{dy}{dx} = x^{-1/2} = \frac{1}{\sqrt{x}}$ $\therefore$ $\left(\frac{dy}{dx}\right)^2 = \frac{1}{x} = \left(\frac{1}{\sqrt{x}}\right)^2$

sub in $\left(\frac{dy}{dx}\right)^2$

$$s = \int_1^8 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_1^8 \sqrt{1 + \frac{1}{x}}\, dx$$

as required.

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(b) $x = \sinh^2 u$

First sort out limits:

$8 = \sinh^2 u \Rightarrow \quad \sinh u = \sqrt{8}$

$\therefore \quad u = \operatorname{arsinh}(\sqrt{8}) = \cancel{\ln(\sqrt{8} + \ )}$
$\ln(\sqrt{8} + 3)$

$1 = \sinh^2 u \Rightarrow \sinh u = 1$

$\therefore \quad u = \operatorname{arsinh}(1) = \ln(1 + \sqrt{2})$

New limits are $\displaystyle\int_{\ln(1+\sqrt{2})}^{\ln(3+\sqrt{8})}$

$\sqrt{1 + \frac{1}{x}} = \sqrt{1 + \frac{1}{\sinh^2 u}} = \sqrt{\frac{\sinh^2 u}{\sinh^2 u} + \frac{1}{\sinh^2 u}}$

$c^2 - s^2 = 1$

$= \sqrt{\frac{\sinh^2 u + 1}{\sinh^2 u}} = \sqrt{\frac{\cosh^2 u}{\sinh^2 u}}$

$= \coth(u)$

$\therefore \quad \sqrt{1 + \frac{1}{x}} = \coth u$

Question 8 continued

$x = \sinh^2 u$

$\therefore \frac{dx}{du} = 2\sinh u \cosh u$

$\therefore dx = 2\sinh u \cosh u \quad du$

Now:

$\therefore \sqrt{1+\frac{1}{x}} = \coth u \quad \& \quad dx = 2\sinh u \cosh u \, du$

$\therefore s = \int_1^8 \sqrt{1+\frac{1}{x}} \, dx \equiv 2\int_{\ln(1+\sqrt{2})}^{\ln(3+\sqrt{8})} \frac{\cosh u}{\sinh u} \cdot \sinh u \cdot \cosh u \, du$

$= 2\int_{\ln(1+\sqrt{2})}^{\ln(3+\sqrt{8})} \cosh^2 u \, du$

$~~c^2~~$

$c^2 + s^2 = \cosh 2x$

$~~c^2+1~~ \quad c^2 + c^2 - 1$

$c^2 - s^2 = 1 \quad 2c^2 - 1$

$c^2 - 1$

$= \cancel{2}\int_{\ln(1+\sqrt{2})}^{\ln(3+\sqrt{8})} \cosh(2x) + 1 \, d\cancel{x}u$

$= \left[\frac{1}{2}\sinh 2x + x\right]_{\ln(1+\sqrt{2})}^{\ln(3+\sqrt{8})}$

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Question 8 continued

$$= \frac{1}{2}\sinh\left(2\ln(3+\sqrt{8})\right) + \ln(3+\sqrt{8}) - \frac{1}{2}\sinh\left(\ln(1+\sqrt{2})^2\right) - \ln(1+\sqrt{2})$$

$$= \ln(1+\sqrt{2}) + \frac{1}{2}\sinh\left(2\ln(3+\sqrt{8})\right) - \frac{1}{2}\sinh\left(\ln(1+\sqrt{2})^2\right)$$

$$= \ln(1+\sqrt{2}) + \frac{1}{2}\left(\frac{e^{\ln[(3+\sqrt{8})^2]} - e^{-\ln[(3+\sqrt{8})^2]}}{2}\right)$$

$$- \frac{1}{2}\left(\frac{e^{\ln(1+\sqrt{2})^2} - e^{-\ln(1+\sqrt{2})^2}}{2}\right)$$

$$= \ln(1+\sqrt{2}) + \frac{1}{2}\left(\frac{(3+\sqrt{8})^2 - (3+\sqrt{8})^{-2}}{2}\right)$$

$$- \frac{1}{2}\left(\frac{(1+\sqrt{2})^2 - (1+\sqrt{2})^{-2}}{2}\right)$$

7-95

Question 8 continued

$$= \ln(1+\sqrt{2}) + \frac{1}{4}\left(17+12\sqrt{2} - 17 + 12\sqrt{2}\right)$$

$$- \frac{1}{4}\left(3+2\sqrt{2} - 3 + 2\sqrt{2}\right)$$

$\frac{1}{+\sqrt{2})^{2}}$

$$= \ln(1+\sqrt{2}) + 6\sqrt{2} - \sqrt{2}$$

$$= 5\sqrt{2} + \ln(1+\sqrt{2})$$