

June 2012
6669 Further Pure Maths FP3
Mark Scheme

Question Number	Scheme	Marks
1. (a)	Uses formula to obtain $e = \frac{5}{4}$	M1A1
(b)	Uses ae formula	M1 (3)
	Uses other formula $\frac{a}{e}$	M1
	Obtains both Foci are $(\pm 5, 0)$ and Directrices are $x = \pm \frac{16}{5}$ (needs both method marks)	A1 cso (2) (5 marks)

Notes

a1M1: Uses $b^2 = a^2(e^2 - 1)$ to get $e > 1$

a1A1: cao

a2M1: Uses ae

b1M1: Uses $\frac{a}{e}$

b1A1: cso for both foci and both directrices. Must have both of the 2 previous M marks may be implicit.

Question Number	Scheme	Marks
2.	$\frac{dy}{dx} = \sinh 3x$ $\text{so } s = \int \sqrt{1 + \sinh^2 3x} dx$ $\therefore s = \int \cosh 3x dx$ $= \left[\frac{1}{3} \sinh 3x \right]_b^{a}$ $= \frac{1}{3} \sinh 3 \ln a = \frac{1}{6} [e^{3 \ln a} - e^{-3 \ln a}]$ $= \frac{1}{6} \left(a^3 - \frac{1}{a^3} \right) \quad (\text{so } k = 1/6)$	B1 M1 A1 M1 DM1 A1 (6 marks)

Notes

1B1: cao

1M1: Use of arc length formula, need both $\sqrt{\quad}$ and $\left(\frac{dy}{dx}\right)^2$.

1A1: $\int \cosh 3x dx$ cao

2M1: Attempt to integrate, getting a hyperbolic function o.e.

3M1: depends on previous M mark. Correct use of $\ln a$ and 0 as limits. Must see some exponentials.

2A1: cao

Question Number	Scheme	Marks
3. (a)	$\vec{AC} = 3\mathbf{i} + 6\mathbf{j} + 2\mathbf{k}, \quad \vec{BC} = -3\mathbf{i} + 4\mathbf{j} + 3\mathbf{k}$ $\vec{AC} \times \vec{BC} = 10\mathbf{i} - 15\mathbf{j} + 30\mathbf{k}$	B1, B1 M1 A1 (4)
(b)	$\text{Area of triangle } ABC = \frac{1}{2} 10\mathbf{i} - 15\mathbf{j} + 30\mathbf{k} = \frac{1}{2} \sqrt{1225} = 17.5$	M1 A1 (2)
(c)	Equation of plane is $10x - 15y + 30z = -20$ or $2x - 3y + 6z = -4$ So $\mathbf{r} \cdot (2\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}) = -4$ or correct multiple	M1 A1 (2) (8 marks)

Notes

a1B1: $\vec{AC} = 3\mathbf{i} + 6\mathbf{j} + 2\mathbf{k}$ cao, any form

a2B1: $\vec{BC} = -3\mathbf{i} + 4\mathbf{j} + 3\mathbf{k}$ cao, any form

a1M1: Attempt to find cross product, modulus of one term correct.

a1A1: cao, any form.

b1M1: modulus of their answer to (a) – condone missing $\frac{1}{2}$ here. To finding area of triangle by correct method.

b1A1: cao.

c1M1: [Using their answer to (a) to] find **equation** of plane. Look for **a.n** or **b.n** or **c.n** for p.

c1A1: cao

Question Number	Scheme	Marks
4(a)	$I_n = \left[x^n \left(-\frac{1}{2} \cos 2x \right) \right]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} -\frac{1}{2} n x^{n-1} \cos 2x dx$ <p>so</p> $I_n = \left\langle \left[x^n \left(-\frac{1}{2} \cos 2x \right) \right]_0^{\frac{\pi}{4}} \right\rangle + \left[\frac{1}{4} n x^{n-1} \sin 2x \right]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \frac{1}{4} n(n-1) x^{n-2} \sin 2x dx$ <p>i.e. $I_n = \frac{1}{4} n \left(\frac{\pi}{4} \right)^{n-1} - \frac{1}{4} n(n-1) I_{n-2} *$</p>	<p>M1 A1</p> <p>M1 A1</p> <p>A1cso</p> <p>(5)</p>
(b)	$I_0 = \int_0^{\frac{\pi}{4}} \sin 2x dx = \left[-\frac{1}{2} \cos 2x \right]_0^{\frac{\pi}{4}} = \frac{1}{2}$ $I_2 = \frac{1}{4} \times 2 \times \left(\frac{\pi}{4} \right) - \frac{1}{4} \times 2 \times I_0, \text{ so } I_2 = \frac{\pi}{8} - \frac{1}{4}$	<p>M1 A1</p> <p>M1 A1</p> <p>(4)</p>
(c)	$I_4 = \left(\frac{\pi}{4} \right)^3 - \frac{1}{4} \times 4 \times 3 I_2 = \frac{\pi^3}{64} - 3 \left(\frac{\pi}{8} - \frac{1}{4} \right) = \frac{1}{64} (\pi^3 - 24\pi + 48) *$	<p>M1 A1cso</p> <p>(2)</p>

Notes

a1M1: Use of integration by parts, integrating $\sin 2x$, differentiating x^n .

a1A1: cao

a2M1: Second application of integration by parts, integrating $\cos 2x$, differentiating x^{n-1} .

a2A1: cao

a3A1: cso Including correct use of $\frac{\pi}{4}$ and 0 as limits.

b1M1: Integrating to find I_0 or setting up parts to find I_2 .

b1A1: cao (Accept $I_0 = \frac{1}{2}$ here for both marks)

b2M1: Finding I_2 in terms of π . If 'n's left in M0

b2A1: cao

c1M1: Finding I_4 in terms of I_2 then in terms of π . If 'n's left in M0

c1A1: cso

Question Number	Scheme	Marks
5. (a)	$\operatorname{ar sinh} 2x, +x \frac{2}{\sqrt{1+4x^2}}$	M1A1, A1 (3)
(b)	$\begin{aligned} \therefore \int_0^{\sqrt{2}} \operatorname{arsinh} 2x dx &= [x \operatorname{ar sinh} 2x]_0^{\sqrt{2}} - \int_0^{\sqrt{2}} \frac{2x}{\sqrt{1+4x^2}} dx \\ &= [x \operatorname{ar sinh} 2x]_0^{\sqrt{2}} - \left[\frac{1}{2} (1+4x^2)^{\frac{1}{2}} \right]_0^{\sqrt{2}} \\ &= \sqrt{2} \operatorname{ar sinh} 2\sqrt{2} - \left[\frac{3}{2} - \frac{1}{2} \right] \\ &= \sqrt{2} \ln(3+2\sqrt{2}) - 1 \end{aligned}$	1M1 1A1ft 2M1 2A1 3DM1 4M1 3A1 (7) (10 marks)

Notes

a1M1: Differentiating getting an arsinh term **and** a term of the form $\frac{px}{\sqrt{1 \pm qx^2}}$

a1A1: cao $\operatorname{ar sinh} 2x$

a2A1: cao $+ \frac{2x}{\sqrt{1+4x^2}}$

b1M1: rearranging their answer to (a). **OR** setting up parts

b1A1: ft from their (a) **OR** setting up parts correctly

b2M1: Integrating getting an arsinh or arcosh term **and** a $(1 \pm ax^2)^{\frac{1}{2}}$ term o.e..

b2A1: cao

b3DM1: depends on previous M, correct use of $\sqrt{2}$ and 0 as limits.

b4M1: converting to log form.

b3A1: cao depends on all previous M marks.

Question Number	Scheme	Marks
6(a)	$\frac{2x}{a^2} + \frac{2y}{b^2} \frac{dy}{dx} = 0 \quad \text{and so} \quad \frac{dy}{dx} = -\frac{xb^2}{ya^2} = -\frac{b \cos \theta}{a \sin \theta}$ $\therefore y - b \sin \theta = -\frac{b \cos \theta}{a \sin \theta} (x - a \cos \theta)$ <p>Uses $\cos^2 \theta + \sin^2 \theta = 1$ to give $\frac{x \cos \theta}{a} + \frac{y \sin \theta}{b} = 1$ *</p>	<p>M1 A1</p> <p>M1</p> <p>A1cso</p> <p>(4)</p>
(b)	<p>Gradient of circle is $-\frac{\cos \theta}{\sin \theta}$ and equation of tangent is</p> $y - a \sin \theta = -\frac{\cos \theta}{\sin \theta} (x - a \cos \theta) \quad \text{or sets } a = b \text{ in previous answer}$ <p>So $y \sin \theta + x \cos \theta = a$</p>	<p>M1</p> <p>A1</p> <p>(2)</p>
(c)	<p>Eliminate x or y to give $y \sin \theta (\frac{a}{b} - 1) = 0$ or $x \cos \theta (\frac{b}{a} - 1) = b - a$</p> <p>$l_1$ and l_2 meet at $(\frac{a}{\cos \theta}, 0)$</p>	<p>M1</p> <p>A1, B1</p> <p>(3)</p>
(d)	<p>The locus of R is part of the line $y = 0$, such that $x \geq a$ and $x \leq -a$</p> <p>Or clearly labelled sketch.</p> <p>Accept "real axis"</p>	<p>B1, B1</p> <p>(2)</p> <p>(11 marks)</p>

Notes

a1M1: Finding gradient in terms of θ . Must use calculus.

a1A1: cao

a2M1: Finding equation of tangent

a2A1: cso (answer given). Need to get $\cos^2 \theta + \sin^2 \theta$ on the same side.

b1M1: Finding gradient and equation of tangent, **or** setting $a = b$.

b1A1: cao need not be simplified.

c1M1: As scheme

c1A1: $x = \frac{a}{\cos \theta}$, need not be simplified.

c1B1: $y = 0$, need not be simplified.

d1B1: Identifying locus as $y = 0$ or real/'x' axis.

d2B1: Depends on previous B mark, identifies correct parts of $y = 0$. Condone use of strict inequalities.

Question Number	Scheme	Marks
7(a)	$f(x) = 5\cosh x - 4\sinh x = 5 \times \frac{1}{2}(e^x + e^{-x}) - 4 \times \frac{1}{2}(e^x - e^{-x})$ $= \frac{1}{2}(e^x + 9e^{-x}) \quad *$	M1 A1cso (2)
(b)	$\frac{1}{2}(e^x + 9e^{-x}) = 5 \Rightarrow e^{2x} - 10e^x + 9 = 0$ So $e^x = 9$ or 1 and $x = \ln 9$ or 0	M1 A1 M1 A1 (4)
(c)	Integral may be written $\int \frac{2e^x}{e^{2x} + 9} dx$ This is $\frac{2}{3} \arctan\left(\frac{e^x}{3}\right)$ Uses limits to give $\left[\frac{2}{3} \arctan 1 - \frac{2}{3} \arctan\left(\frac{1}{\sqrt{3}}\right)\right] = \left[\frac{2}{3} \times \frac{\pi}{4} - \frac{2}{3} \times \frac{\pi}{6}\right] = \frac{\pi}{18} *$	B1 M1 A1 DM1 A1cso (5) (11 marks)

Notes

a1M1: Replacing both $\cosh x$ and $\sinh x$ by terms in e^x and e^{-x} condone sign errors here.

a1A1: cso (answer given)

b1M1: Getting a three term quadratic in e^x

b1A1: cao

b2M1: solving to $x =$

b2A1: cao need $\ln 9$ (o.e) and 0 (not $\ln 1$)

c1B1: cao getting into suitable form, may substitute first.

c1M1: Integrating to give term in \arctan

c1A1: cao

c2M1: Depends on previous M mark. Correct use of $\ln 3$ and $\frac{1}{2} \ln 3$ as limits.

c2A1: cso must see them subtracting two terms in π .

Question Number	Scheme	Marks
8. (a)	$\begin{vmatrix} 2-\lambda & 1 & 0 \\ 1 & 2-\lambda & 0 \\ -1 & 0 & 4-\lambda \end{vmatrix} = 0 \therefore (2-\lambda)(2-\lambda)(4-\lambda) - (4-\lambda) = 0$ <p>(4 - λ) = 0 verifies λ = 4 is an eigenvalue (can be seen anywhere)</p> <p>∴ (4 - λ){4 - 4λ + λ² - 1} = 0 ∴ (4 - λ){λ² - 4λ + 3} = 0</p> <p>∴ (4 - λ)(λ - 1)(λ - 3) = 0 and 3 and 1 are the other two eigenvalues</p>	M1 M1 A1 M1 A1 (5)
(b)	<p>Set $\begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ -1 & 0 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 4 \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ or $\begin{pmatrix} -2 & 1 & 0 \\ 1 & -2 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$</p> <p>Solve -2x + y = 0 and x - 2y = 0 and -x = 0 to obtain x = 0, y = 0, z = k</p> <p>Obtain eigenvector as k (or multiple)</p>	M1 M1 A1 (3)
(c)	<p>l_1 has equation which may be written $\begin{pmatrix} 3+\lambda \\ 2-\lambda \\ -2+2\lambda \end{pmatrix}$</p> <p>So l_2 is given by $\mathbf{r} = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ -1 & 0 & 4 \end{pmatrix} \begin{pmatrix} 3+\lambda \\ 2-\lambda \\ -2+2\lambda \end{pmatrix}$</p> <p>i.e. $\mathbf{r} = \begin{pmatrix} 8+\lambda \\ 7-\lambda \\ -11+7\lambda \end{pmatrix}$</p> <p>So $(\mathbf{r} - \mathbf{c}) \times \mathbf{d} = \mathbf{0}$ where $\mathbf{c} = 8\mathbf{i} + 7\mathbf{j} - 11\mathbf{k}$ and $\mathbf{d} = \mathbf{i} - \mathbf{j} + 7\mathbf{k}$</p>	B1 M1 M1 A1 A1ft (5) (13 marks)

Notes

a1M1: Condone missing = 0. (They might expand the determinant using any row or column)

a2M1: Shows λ = 4 is an eigenvalue. Some working needed need to see = 0 at some stage.

a1A1: Three term quadratic factor cao, may be implicit (this A depends on 1st M only)

a2M1: Attempt at factorisation (usual rules), solving to λ = .

a2A1: cao. If they state λ = 1 and 3 please give the marks.

b1M1: Using $\mathbf{Ax} = 4\mathbf{x}$ o.e.

b2M1: Getting a pair of correct equations.

b1A1: cao

c1B1: Using **a** and **b**.

c1M1: Using $\mathbf{r} = \mathbf{M} \times$ their matrix in **a** and **b**.

c2M1: Getting an expression for l_2 with at least one component correct.

c1A1: cao all three components correct

c2A1ft: ft their vector, must have $\mathbf{r} =$ or $(\mathbf{r} - \mathbf{c}) \times \mathbf{d} = \mathbf{0}$ need both equation and r.