General Certificate of Education
January 2009
Advanced Level Examination

MATHEMATICS **MFP3**
Unit Further Pure 3

Wednesday 21 January 2009 1.30 pm to 3.00 pm

For this paper you must have:
- a 12-page answer book
- the blue AQA booklet of formulae and statistical tables.

You may use a graphics calculator.

Time allowed: 1 hour 30 minutes

Instructions
- Use black ink or black ball-point pen. Pencil should only be used for drawing.
- Write the information required on the front of your answer book. The *Examining Body* for this paper is AQA. The *Paper Reference* is MFP3.
- Answer **all** questions.
- Show all necessary working; otherwise marks for method may be lost.

Information
- The maximum mark for this paper is 75.
- The marks for questions are shown in brackets.

Advice
- Unless stated otherwise, you may quote formulae, without proof, from the booklet.

Answer **all** questions.

1 The function $y(x)$ satisfies the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{f}(x, y)$$

where $$\mathrm{f}(x, y) = \frac{x^2 + y^2}{x + y}$$

and $$y(1) = 3$$

(a) Use the Euler formula

$$y_{r+1} = y_r + h\mathrm{f}(x_r, y_r)$$

with $h = 0.2$, to obtain an approximation to $y(1.2)$. *(3 marks)*

(b) Use the improved Euler formula

$$y_{r+1} = y_r + \tfrac{1}{2}(k_1 + k_2)$$

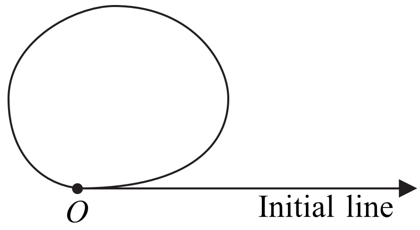
where $k_1 = h\mathrm{f}(x_r, y_r)$ and $k_2 = h\mathrm{f}(x_r + h, y_r + k_1)$ and $h = 0.2$, to obtain an approximation to $y(1.2)$, giving your answer to four decimal places. *(5 marks)*

2 (a) Show that $\frac{1}{x^2}$ is an integrating factor for the first-order differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{2}{x}y = x$$ *(3 marks)*

(b) Hence find the general solution of this differential equation, giving your answer in the form $y = \mathrm{f}(x)$. *(4 marks)*

3 The diagram shows a sketch of a loop, the pole O and the initial line.



The polar equation of the loop is

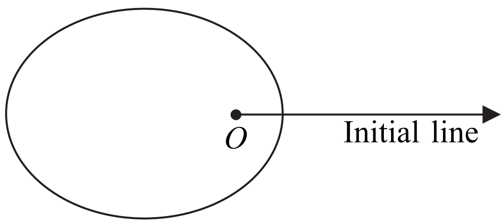
$$r = (2 + \cos\theta)\sqrt{\sin\theta}, \quad 0 \leqslant \theta \leqslant \pi$$

Find the area enclosed by the loop. *(6 marks)*

4 (a) Use integration by parts to show that $\int \ln x \, dx = x \ln x - x + c$, where c is an arbitrary constant. *(2 marks)*

(b) Hence evaluate $\int_0^1 \ln x \, dx$, showing the limiting process used. *(4 marks)*

5 The diagram shows a sketch of a curve C, the pole O and the initial line.



The curve C has polar equation

$$r = \frac{2}{3 + 2\cos\theta}, \quad 0 \leqslant \theta \leqslant 2\pi$$

(a) Verify that the point L with polar coordinates $(2, \pi)$ lies on C. *(1 mark)*

(b) The circle with polar equation $r = 1$ intersects C at the points M and N.

(i) Find the polar coordinates of M and N. *(3 marks)*

(ii) Find the area of triangle LMN. *(4 marks)*

(c) Find a cartesian equation of C, giving your answer in the form $9y^2 = f(x)$. *(5 marks)*

Turn over for the next question

6 The function f is defined by $\mathrm{f}(x) = \mathrm{e}^{2x}(1+3x)^{-\frac{2}{3}}$.

(a) (i) Use the series expansion for e^x to write down the first four terms in the series expansion of e^{2x}. *(2 marks)*

(ii) Use the binomial series expansion of $(1+3x)^{-\frac{2}{3}}$ and your answer to part (a)(i) to show that the first three non-zero terms in the series expansion of $\mathrm{f}(x)$ are $1 + 3x^2 - 6x^3$. *(5 marks)*

(b) (i) Given that $y = \ln(1 + 2\sin x)$, find $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$. *(4 marks)*

(ii) By using Maclaurin's theorem, show that, for small values of x,

$$\ln(1 + 2\sin x) \approx 2x - 2x^2$$ *(2 marks)*

(c) Find

$$\lim_{x \to 0} \frac{1 - \mathrm{f}(x)}{x \ln(1 + 2\sin x)}$$ *(3 marks)*

7 (a) Given that $x = \mathrm{e}^t$ and that y is a function of x, show that

$$x^2 \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{\mathrm{d}^2y}{\mathrm{d}t^2} - \frac{\mathrm{d}y}{\mathrm{d}t}$$ *(7 marks)*

(b) Hence show that the substitution $x = \mathrm{e}^t$ transforms the differential equation

$$x^2 \frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 4x\frac{\mathrm{d}y}{\mathrm{d}x} = 10$$

into

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} - 5\frac{\mathrm{d}y}{\mathrm{d}t} = 10$$ *(2 marks)*

(c) Find the general solution of the differential equation $\dfrac{\mathrm{d}^2y}{\mathrm{d}t^2} - 5\dfrac{\mathrm{d}y}{\mathrm{d}t} = 10$. *(5 marks)*

(d) Hence solve the differential equation $x^2 \dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} - 4x\dfrac{\mathrm{d}y}{\mathrm{d}x} = 10$, given that $y = 0$ and $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 8$ when $x = 1$. *(5 marks)*

END OF QUESTIONS