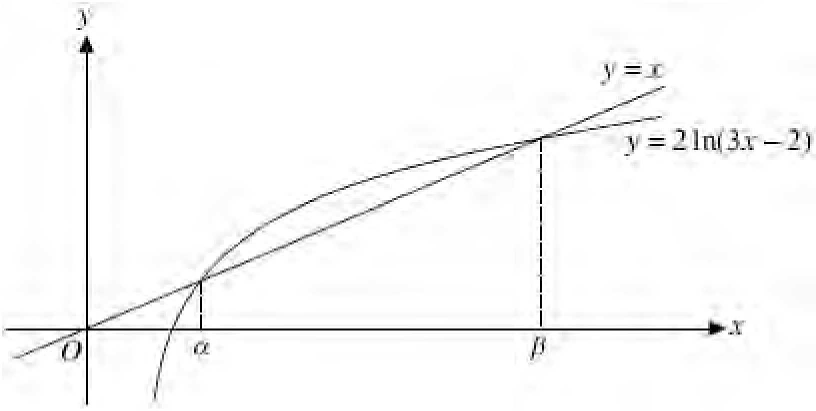
FP2 Numerical Methods

1. June 2010 qu. 7



The line $y = x$ and the curve $y = 2\ln(3x - 2)$ meet where $x = \alpha$ and $x = \beta$, as shown in the diagram.

(i) Use the iteration $x_{n+1} = 2\ln(3x_n - 2)$, with initial value $x_1 = 5.25$, to find the value of β correct to 2 decimal places. Show all your working. [2]

(ii) With the help of a 'staircase' diagram, explain why this iteration will not converge to α, whatever value of x_1 (other than α) is used. [3]

(iii) Show that the equation $x = 2\ln(3x - 2)$ can be rewritten as $x = \frac{1}{3}(e^{\frac{1}{2}x} + 2)$. Use the Newton-Raphson method, with $f(x) = \frac{1}{3}(e^{\frac{1}{2}x} + 2) - x$ and $x_1 = 1.2$, to find α correct to 2 decimal places. Show all your working. [4]

(iv) Given that $x_1 = \ln 36$, explain why the Newton-Raphson method would not converge to a root of $f(x) = 0$. [2]

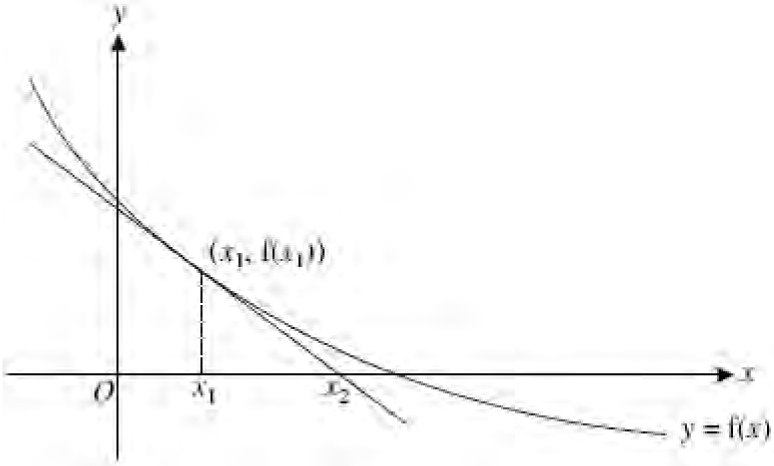
2. Jan 2010 qu.1

It is given that $f(x) = x^2 - \sin x$.

(i) The iteration $x_{n+1} = \sqrt{\sin x_n}$, with $x_1 = 0.875$, is to be used to find a real root, α, of the equation $f(x) = 0$. Find x_2, x_3 and x_4, giving the answers correct to 6 decimal places. [2]

(ii) The error e_n is defined by $e_n = \alpha - x_n$. Given that $\alpha = 0.876\,726$, correct to 6 decimal places, find e_3 and e_4. Given that $g(x) = \sqrt{\sin x}$, use e_3 and e_4 to estimate $g'(\alpha)$. [3]

3. Jan 2010 qu.3



A curve with no stationary points has equation $y = f(x)$. The equation $f(x) = 0$ has one real root α, and the Newton-Raphson method is to be used to find α.
The tangent to the curve at the point $(x_1, f(x_1))$ meets the x-axis where $x = x_2$ (see diagram).

(i) Show that $x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)}$. [3]

(ii) Describe briefly, with the help of a sketch, how the Newton-Raphson method, using an initial approximation $x = x_1$, gives a sequence of approximations approaching α. [2]

(iii) Use the Newton-Raphson method, with a first approximation of 1, to find a second approximation of the root of $x^2 - 2\sinh x + 2 = 0$. [2]

4. June 2009 qu. 7

(i) Sketch the graph of $y = \coth x$, and give the equations of any asymptotes. [3]

(ii) It is given that $f(x) = x\tanh x - 2$. Use the Newton-Raphson method, with a first approximation $x_1 = 2$, to find the next three approximations x_2, x_3 and x_4 to a root of $f(x) = 0$. Give the answers correct to 4 decimal places. [4]

(iii) If $f(x) = 0$, show that $\coth x = \dfrac{1}{2}x$. Hence write down the roots of $f(x) = 0$, correct to 4 decimal places. [3]

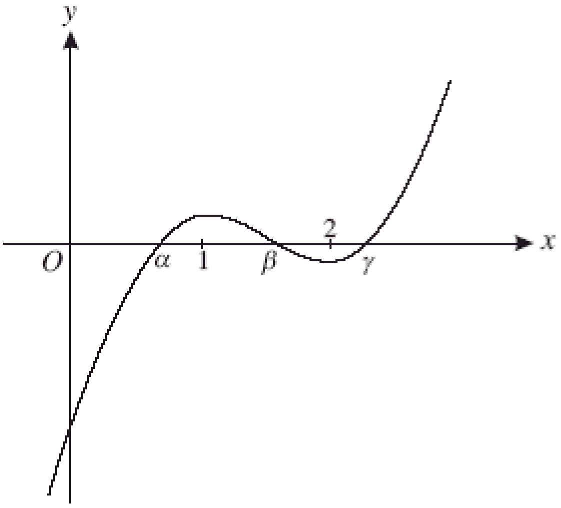
5. Jan 2009 qu.2

It is given that α is the only real root of the equation $x^5 + 2x - 28 = 0$ and that $1.8 < \alpha < 2$.

(i) The iteration $x_{n+1} = \sqrt[5]{28 - 2x_n}$, with $x_1 = 1.9$, is to be used to find α. Find the values of x_2, x_3 and x_4, giving the answers correct to 7 decimal places. [3]

(ii) The error e_n is defined by $e_n = \alpha - x_n$. Given that $\alpha = 1.891\,574\,9$, correct to 7 decimal places, evaluate $\dfrac{e_3}{e_2}$ and $\dfrac{e_4}{e_3}$. Comment on these values in relation to the gradient of the curve with equation $y = \sqrt[5]{28 - 2x}$ at $x = \alpha$. [3]

6. Jan 2009 qu.5



The diagram shows the curve with equation $y = f(x)$, where $f(x) = 2x^3 - 9x^2 + 12x - 4.36$.

The curve has turning points at $x = 1$ and $x = 2$ and crosses the x-axis at $x = \alpha$, $x = \beta$ and $x = \gamma$, where $0 < \alpha < \beta < \gamma$.

(i) The Newton-Raphson method is to be used to find the roots of the equation $f(x) = 0$, with $x_1 = k$.

(a) To which root, if any, would successive approximations converge in each of the cases $k < 0$ and $k = 1$? [2]

(b) What happens if $1 < k < 2$? [2]

(ii) Sketch the curve with equation $y^2 = f(x)$. State the coordinates of the points where the curve crosses the x-axis and the coordinates of any turning points. [4]

7. June 2008 qu. 6

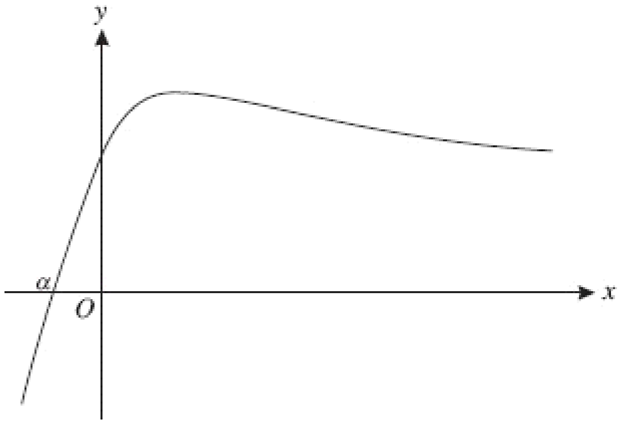
It is given that $f(x) = 1 - \frac{7}{x^2}$.

(i) Use the Newton-Raphson method, with a first approximation $x_1 = 2.5$, to find the next approximations x_2 and x_3 to a root of $f(x) = 0$. Give the answers correct to 6 decimal places.

(ii) The root of $f(x) = 0$ for which x_1, x_2 and x_3 are approximations is denoted by α. Write down the exact value of α. [1]

(iii) The error e_n is defined by $e_n = \alpha - x_n$. Find e_1, e_2 and e_3, giving your answers correct to 5 decimal places. Verify that $e_3 \approx \frac{e_2^3}{e_1^2}$. [3]

8. Jan 2008 qu.5



The diagram shows the curve with equation $y = xe^{-x} + 1$. The curve crosses the x-axis at $x = \alpha$.

(i) Use differentiation to show that the x-coordinate of the stationary point is 1. [2]

α is to be found using the Newton-Raphson method, with $f(x) = xe^{-x} + 1$.

(ii) Explain why this method will not converge to α if an initial approximation x_1 is chosen such that $x_1 > 1$. [2]

(iii) Use this method, with a first approximation $x_1 = 0$, to find the next three approximations x_2, x_3 and x_4. Find α, correct to 3 decimal places. [5]

9. June 2007 qu. 8

The iteration $x_{n+1} = \dfrac{1}{(x_n + 2)^2}$, with $x_1 = 0.3$, is to be used to find the real root, α, of the equation $x(x + 2)^2 = 1$.

(i) Find the value of α, correct to 4 decimal places. You should show the result of each step of the iteration process. [4]

(ii) Given that $f(x) = \dfrac{1}{(x+2)^2}$, show that $f'(\alpha) \neq 0$. [2]

(iii) The difference, δ_r, between successive approximations is given by $\delta_r = x_{r+1} - x_r$. Find δ_3. [1]

(iv) Given that $\delta_{r+1} \approx f'(\alpha)\delta_r$, find an estimate for δ_{10}. [3]

10. Jan 2007 qu.2

It is given that $f(x) = x^2 - \tan^{-1}x$.

(i) Show by calculation that the equation $f(x) = 0$ has a root in the interval $0.8 < x < 0.9$. [2]

(ii) Use the Newton-Raphson method, with a first approximation 0.8, to find the next approximation to this root. Give your answer correct to 3 decimal places. [4]

11. June 2006 qu. 8

The curve with equation $y = \dfrac{\sinh x}{x^2}$, for $x > 0$, has one turning point.

(i) Show that the x-coordinate of the turning point satisfies the equation $x - 2\tanh x = 0$. [3]

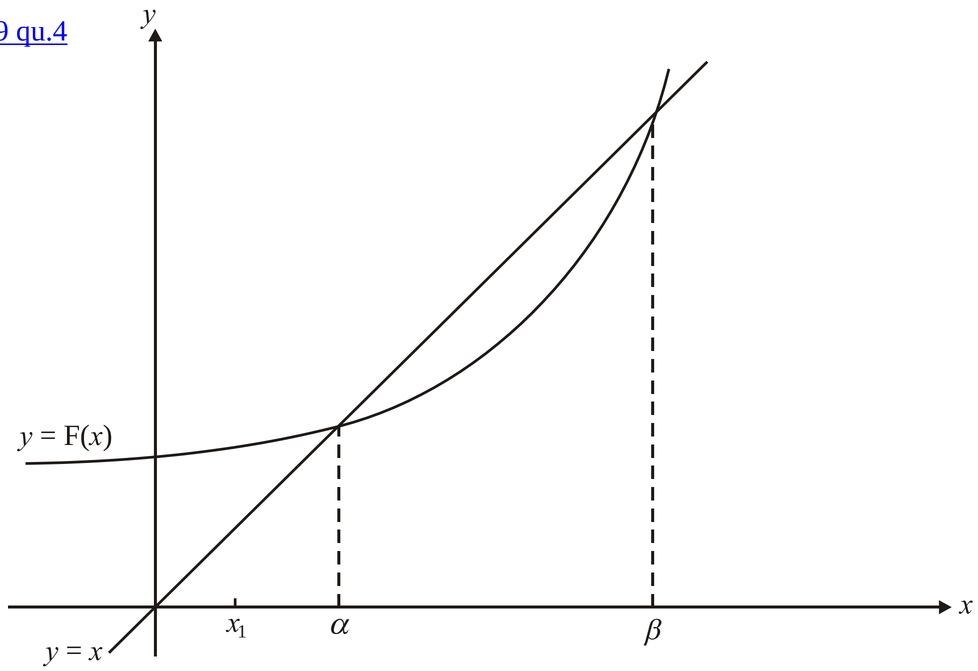
(ii) Use the Newton-Raphson method, with a first approximation $x_1 = 2$, to find the next two approximations, x_2 and x_3, to the positive root of $x - 2\tanh x = 0$. [5]

(iii) By considering the approximate errors in x_1 and x_2, estimate the error in x_3. (You are not expected to evaluate x_4) [3]

12. Jan 2006 qu.2

Use the Newton-Raphson method to find the root of the equation $e^{-x} = x$ which is close to $x = 0.5$. Give the root correct to 3 decimal places. [5]

13. Jan 2009 qu.4



The sketch shows the curve with equation $y = F(x)$ and the line $y = x$. The equation $x = F(x)$ has roots $x = \alpha$ and $x = \beta$ as shown.

(i) Show how an iteration of the form $x_{n+1} = F(x_n)$, with starting value x_1 such that $0 < x_1 < \alpha$ as shown, converges to the root $x = \alpha$.

....

.... [3]

(ii) State what happens in the iteration in the following two cases.

(a) x_1 is chosen such that $\alpha < x_1 < \beta$.

....

(b) x_1 is chosen such that $x_1 > \beta$.

.... [3]