FP2 Roots of Polymonial Equations

2 The cubic equation

$$x^3 + px^2 + qx + r = 0$$

where p, q and r are real, has roots α, β and γ.

(a) Given that

$$\alpha + \beta + \gamma = 4 \qquad \text{and} \qquad \alpha^2 + \beta^2 + \gamma^2 = 20$$

find the values of p and q. *(5 marks)*

(b) Given further that one root is $3 + \mathrm{i}$, find the value of r. *(5 marks)*

5 The cubic equation

$$z^3 - 4\mathrm{i}z^2 + qz - (4 - 2\mathrm{i}) = 0$$

where q is a complex number, has roots α, β and γ.

(a) Write down the value of:

(i) $\alpha + \beta + \gamma$; *(1 mark)*

(ii) $\alpha\beta\gamma$. *(1 mark)*

(b) Given that $\alpha = \beta + \gamma$, show that:

(i) $\alpha = 2\mathrm{i}$; *(1 mark)*

(ii) $\beta\gamma = -(1 + 2\mathrm{i})$; *(2 marks)*

(iii) $q = -(5 + 2\mathrm{i})$. *(3 marks)*

(c) Show that β and γ are the roots of the equation

$$z^2 - 2\mathrm{i}z - (1 + 2\mathrm{i}) = 0$$ *(2 marks)*

(d) Given that β is real, find β and γ. *(3 marks)*

3 The cubic equation

$$z^3 + 2(1 - \mathrm{i})z^2 + 32(1 + \mathrm{i}) = 0$$

has roots α, β and γ.

(a) It is given that α is of the form $k\mathrm{i}$, where k is real. By substituting $z = k\mathrm{i}$ into the equation, show that $k = 4$. *(5 marks)*

(b) Given that $\beta = -4$, find the value of γ. *(2 marks)*

2 The cubic equation

$$z^3 + pz^2 + 6z + q = 0$$

has roots α, β and γ.

(a) Write down the value of $\alpha\beta + \beta\gamma + \gamma\alpha$. *(1 mark)*

(b) Given that p and q are real and that $\alpha^2 + \beta^2 + \gamma^2 = -12$:

(i) explain why the cubic equation has two non-real roots and one real root; *(2 marks)*

(ii) find the value of p. *(4 marks)*

(c) One root of the cubic equation is $-1 + 3\mathrm{i}$.

Find:

(i) the other two roots; *(3 marks)*

(ii) the value of q. *(2 marks)*

4 The cubic equation

$$z^3 + iz^2 + 3z - (1 + i) = 0$$

has roots α, β and γ.

(a) Write down the value of:

(i) $\alpha + \beta + \gamma$; *(1 mark)*

(ii) $\alpha\beta + \beta\gamma + \gamma\alpha$; *(1 mark)*

(iii) $\alpha\beta\gamma$. *(1 mark)*

(b) Find the value of:

(i) $\alpha^2 + \beta^2 + \gamma^2$; *(3 marks)*

(ii) $\alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2$; *(4 marks)*

(iii) $\alpha^2\beta^2\gamma^2$. *(2 marks)*

(c) Hence write down a cubic equation whose roots are α^2, β^2 and γ^2. *(2 marks)*

3 The cubic equation

$$z^3 + qz + (18 - 12i) = 0$$

where q is a complex number, has roots α, β and γ.

(a) Write down the value of:

(i) $\alpha\beta\gamma$; *(1 mark)*

(ii) $\alpha + \beta + \gamma$. *(1 mark)*

(b) Given that $\beta + \gamma = 2$, find the value of:

(i) α; *(1 mark)*

(ii) $\beta\gamma$; *(2 marks)*

(iii) q. *(3 marks)*

(c) Given that β is of the form ki, where k is real, find β and γ. *(4 marks)*

4 It is given that α, β and γ satisfy the equations

$$\begin{aligned} \alpha + \beta + \gamma &= 1 \\ \alpha^2 + \beta^2 + \gamma^2 &= -5 \\ \alpha^3 + \beta^3 + \gamma^3 &= -23 \end{aligned}$$

(a) Show that $\alpha\beta + \beta\gamma + \gamma\alpha = 3$. *(3 marks)*

(b) Use the identity

$$(\alpha + \beta + \gamma)(\alpha^2 + \beta^2 + \gamma^2 - \alpha\beta - \beta\gamma - \gamma\alpha) = \alpha^3 + \beta^3 + \gamma^3 - 3\alpha\beta\gamma$$

to find the value of $\alpha\beta\gamma$. *(2 marks)*

(c) Write down a cubic equation, with integer coefficients, whose roots are α, β and γ. *(2 marks)*

(d) Explain why this cubic equation has two non-real roots. *(2 marks)*

(e) Given that α is real, find the values of α, β and γ. *(4 marks)*

3 The cubic equation

$$z^3 + pz^2 + 25z + q = 0$$

where p and q are real, has a root $\alpha = 2 - 3\mathrm{i}$.

(a) Write down another non-real root, β, of this equation. *(1 mark)*

(b) Find:

(i) the value of $\alpha\beta$; *(1 mark)*

(ii) the third root, γ, of the equation; *(3 marks)*

(iii) the values of p and q. *(3 marks)*

3 The cubic equation

$$2z^3 + pz^2 + qz + 16 = 0$$

where p and q are real, has roots α, β and γ.

It is given that $\alpha = 2 + 2\sqrt{3}\,\mathrm{i}$.

(a) (i) Write down another root, β, of the equation. *(1 mark)*

(ii) Find the third root, γ. *(3 marks)*

(iii) Find the values of p and q. *(3 marks)*

(b) (i) Express α in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r > 0$ and $-\pi < \theta \leqslant \pi$. *(2 marks)*

(ii) Show that

$$(2 + 2\sqrt{3}\,\mathrm{i})^n = 4^n\left(\cos\frac{n\pi}{3} + \mathrm{i}\sin\frac{n\pi}{3}\right)$$ *(2 marks)*

(iii) Show that

$$\alpha^n + \beta^n + \gamma^n = 2^{2n+1}\cos\frac{n\pi}{3} + \left(-\frac{1}{2}\right)^n$$

where n is an integer. *(3 marks)*

4 The roots of the cubic equation

$$z^3 - 2z^2 + pz + 10 = 0$$

are α, β and γ.

It is given that $\alpha^3 + \beta^3 + \gamma^3 = -4$.

(a) Write down the value of $\alpha + \beta + \gamma$. *(1 mark)*

(b) (i) Explain why $\alpha^3 - 2\alpha^2 + p\alpha + 10 = 0$. *(1 mark)*

(ii) Hence show that

$$\alpha^2 + \beta^2 + \gamma^2 = p + 13$$ *(4 marks)*

(iii) Deduce that $p = -3$. *(2 marks)*

(c) (i) Find the real root α of the cubic equation $z^3 - 2z^2 - 3z + 10 = 0$. *(2 marks)*

(ii) Find the values of β and γ. *(3 marks)*

3 (a) Show that $(1 + \mathrm{i})^3 = 2\mathrm{i} - 2$. *(2 marks)*

(b) The cubic equation

$$z^3 - (5 + \mathrm{i})z^2 + (9 + 4\mathrm{i})z + k(1 + \mathrm{i}) = 0$$

where k is a real constant, has roots α, β and γ.

It is given that $\alpha = 1 + \mathrm{i}$.

(i) Find the value of k. *(3 marks)*

(ii) Show that $\beta + \gamma = 4$. *(1 mark)*

(iii) Find the values of β and γ. *(5 marks)*

4 The cubic equation

$$z^3 - 2z^2 + k = 0 \qquad (k \neq 0)$$

has roots α, β and γ.

(a) (i) Write down the values of $\alpha + \beta + \gamma$ and $\alpha\beta + \beta\gamma + \gamma\alpha$. *(2 marks)*

(ii) Show that $\alpha^2 + \beta^2 + \gamma^2 = 4$. *(2 marks)*

(iii) Explain why $\alpha^3 - 2\alpha^2 + k = 0$. *(1 mark)*

(iv) Show that $\alpha^3 + \beta^3 + \gamma^3 = 8 - 3k$. *(2 marks)*

(b) Given that $\alpha^4 + \beta^4 + \gamma^4 = 0$:

(i) show that $k = 2$; *(4 marks)*

(ii) find the value of $\alpha^5 + \beta^5 + \gamma^5$. *(3 marks)*

7 The numbers α, β and γ satisfy the equations

$$\alpha^2 + \beta^2 + \gamma^2 = -10 - 12\text{i}$$
$$\alpha\beta + \beta\gamma + \gamma\alpha = 5 + 6\text{i}$$

(a) Show that $\alpha + \beta + \gamma = 0$. *(2 marks)*

(b) The numbers α, β and γ are also the roots of the equation

$$z^3 + pz^2 + qz + r = 0$$

Write down the value of p and the value of q. *(2 marks)*

(c) It is also given that $\alpha = 3\text{i}$.

(i) Find the value of r. *(3 marks)*

(ii) Show that β and γ are the roots of the equation

$$z^2 + 3\text{i}z - 4 + 6\text{i} = 0$$ *(2 marks)*

(iii) Given that β is real, find the values of β and γ. *(3 marks)*

4 The cubic equation

$$z^3 + pz + q = 0$$

has roots α, β and γ.

(a) (i) Write down the value of $\alpha + \beta + \gamma$. *(1 mark)*

(ii) Express $\alpha\beta\gamma$ in terms of q. *(1 mark)*

(b) Show that

$$\alpha^3 + \beta^3 + \gamma^3 = 3\alpha\beta\gamma$$ *(3 marks)*

(c) Given that $\alpha = 4 + 7\mathrm{i}$ and that p and q are real, find the values of:

(i) β and γ; *(2 marks)*

(ii) p and q. *(3 marks)*

(d) Find a cubic equation with integer coefficients which has roots $\frac{1}{\alpha}$, $\frac{1}{\beta}$ and $\frac{1}{\gamma}$. *(3 marks)*

4 The roots of the equation

$$z^3 - 5z^2 + kz - 4 = 0$$

are α, β and γ.

(a) (i) Write down the value of $\alpha + \beta + \gamma$ and the value of $\alpha\beta\gamma$. *(2 marks)*

(ii) Hence find the value of $\alpha^2\beta\gamma + \alpha\beta^2\gamma + \alpha\beta\gamma^2$. *(2 marks)*

(b) The value of $\alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2$ is -4.

(i) Explain why α, β and γ cannot all be real. *(1 mark)*

(ii) By considering $(\alpha\beta + \beta\gamma + \gamma\alpha)^2$, find the possible values of k. *(4 marks)*

5 The cubic equation

$$z^3 + pz^2 + qz + 37 - 36\mathrm{i} = 0$$

where p and q are constants, has three complex roots, α, β and γ.

It is given that $\beta = -2 + 3\mathrm{i}$ and $\gamma = 1 + 2\mathrm{i}$.

(a) (i) Write down the value of $\alpha\beta\gamma$. *(1 mark)*

(ii) Hence show that $(8 + \mathrm{i})\alpha = 37 - 36\mathrm{i}$. *(2 marks)*

(iii) Hence find α, giving your answer in the form $m + n\mathrm{i}$, where m and n are integers. *(3 marks)*

(b) Find the value of p. *(1 mark)*

(c) Find the value of the complex number q. *(2 marks)*

4 The roots of the equation

$$z^3 + 2z^2 + 3z - 4 = 0$$

are α, β and γ.

(a) (i) Write down the value of $\alpha + \beta + \gamma$ and the value of $\alpha\beta + \beta\gamma + \gamma\alpha$.

[2 marks]

(ii) Hence show that $\alpha^2 + \beta^2 + \gamma^2 = -2$.

[2 marks]

(b) Find the value of:

(i) $(\alpha + \beta)(\beta + \gamma) + (\beta + \gamma)(\gamma + \alpha) + (\gamma + \alpha)(\alpha + \beta)$;

[3 marks]

(ii) $(\alpha + \beta)(\beta + \gamma)(\gamma + \alpha)$.

[4 marks]

(c) Find a cubic equation whose roots are $\alpha + \beta$, $\beta + \gamma$ and $\gamma + \alpha$.

[3 marks]

7 The cubic equation $27z^3 + kz^2 + 4 = 0$ has roots α, β and γ.

(a) Write down the values of $\alpha\beta + \beta\gamma + \gamma\alpha$ and $\alpha\beta\gamma$.

[2 marks]

(b) (i) In the case where $\beta = \gamma$, find the roots of the equation.

[5 marks]

(ii) Find the value of k in this case.

[1 mark]

(c) (i) In the case where $\alpha = 1 - \mathrm{i}$, find α^2 and α^3.

[2 marks]

(ii) Hence find the value of k in this case.

[2 marks]

(d) In the case where $k = -12$, find a cubic equation with integer coefficients which has roots $\frac{1}{\alpha} + 1$, $\frac{1}{\beta} + 1$ and $\frac{1}{\gamma} + 1$.

[5 marks]

2 The cubic equation $3z^3 + pz^2 + 17z + q = 0$, where p and q are real, has a root $\alpha = 1 + 2\mathrm{i}$.

(a) (i) Write down the value of another non-real root, β, of this equation.

[1 mark]

(ii) Hence find the value of $\alpha\beta$.

[1 mark]

(b) Find the value of the third root, γ, of this equation.

[3 marks]

(c) Find the values of p and q.

[3 marks]

2(a)	$p = -4$	B1		
	$(\alpha+\beta+\gamma)^2 = \Sigma\alpha^2 + 2\Sigma\alpha\beta$	M1		
	$16 = 20 + 2\Sigma\alpha\beta$	A1		
	$\Sigma\alpha\beta = -2$	A1F		
	$q = -2$	A1F	5	
(b)	$3 - \mathrm{i}$ is a root	B1		
	Third root is -2	B1F		
	$\alpha\beta\gamma = (3+\mathrm{i})(3-\mathrm{i})(-2)$	M1		
	$= -20$	A1F		Real $\alpha\beta\gamma$
	$r = +20$	A1F	5	Real r
	Alternative to (b) Substitute 3 + i into equation	M1		
	$(3+\mathrm{i})^2 = 8 + 6\mathrm{i}$	B1		
	$(3+\mathrm{i})^3 = 18 + 26\mathrm{i}$	B1		
	$r = 20$	A2,1,0		Provided r is real

5(a)(i)	$\alpha+\beta+\gamma = 4\mathrm{i}$	B1	1	
(ii)	$\alpha\beta\gamma = 4 - 2\mathrm{i}$	B1	1	
(b)(i)	$\alpha + \alpha = 4\mathrm{i}$, $\alpha = 2\mathrm{i}$	B1	1	AG
(ii)	$\beta\gamma = \dfrac{4-2\mathrm{i}}{2\mathrm{i}} = -2\mathrm{i} - 1$	M1		Some method must be shown, eg $\frac{2}{\mathrm{i}} - 1$
		A1	2	AG
(iii)	$q = \alpha\beta + \beta\gamma + \gamma\alpha$	M1		
	$= \alpha(\beta+\gamma) + \beta\gamma$	M1		Or $\alpha^2 + \beta\gamma$, ie suitable grouping
	$= 2\mathrm{i}.2\mathrm{i} - 2\mathrm{i} - 1 = -2\mathrm{i} - 5$	A1	3	AG
(c)	Use of $\beta+\gamma = 2\mathrm{i}$ and $\beta\gamma = -2\mathrm{i} - 1$	M1		Elimination of say γ to arrive at
	$z^2 - 2\mathrm{i}z - (1+2\mathrm{i}) = 0$	A1	2	$\beta^2 - 2\mathrm{i}\beta - (1+2\mathrm{i}) = 0$ M1A0 unless also some reference to γ being a root AG
(d)	$\mathrm{f}(-1) = 1 + 2\mathrm{i} - 1 - 2\mathrm{i} = 0$	M1		For any correct method
	$\beta = -1$, $\gamma = 1 + 2\mathrm{i}$	A1A1	3	A1 for each answer

Q	Solution	Marks	Total	Comments
3(a)	$-k^3\mathrm{i}+2(1-\mathrm{i})(-k^2)+32(1+\mathrm{i})=0$	M1		Any form
	Equate real and imaginary parts:			
	$-k^3+2k^2+32=0$	A1		
	$-2k^2+32=0$	A1		
	$k=\pm 4$	A1		
	$k=+4$	E1	5	AG
(b)	Sum of roots is $-2(1-\mathrm{i})$	M1		Or $\alpha\beta\gamma=-(32+32\mathrm{i})$ Must be correct for M1
	Third root $2-2\mathrm{i}$	A1√	2	
	Total			
2(a)	$\sum\alpha\beta=6$	B1	1	
(b)(i)	Sum of squares <0 ∴ not all real	E1		
	Coefficients real ∴ conjugate pair	E1	2	
(ii)	$\left(\sum\alpha\right)^2=\sum\alpha^2+2\sum\alpha\beta$	M1A1		A1 for numerical values inserted
	$\left(\sum\alpha\right)^2=0$	A1F		
	$p=0$	A1F	4	cao
(c)(i)	$-1-3\mathrm{i}$ is a root	B1		
	Use of appropriate relationship			
	eg $\sum\alpha=0$	M1		M0 if $\sum\alpha^2$ used unless the root 2 is checked
	Third root 2	A1F	3	incorrect p√
(ii)	$q=-(-1-3\mathrm{i})(-1+3\mathrm{i})2$	M1		allow even if sign error
	$=-20$	A1F	2	ft incorrect 3rd root
	Total		11	
4(a)(i)	$\sum\alpha=-\mathrm{i}$	B1	1	
(ii)	$\sum\alpha\beta=3$	B1	1	
(iii)	$\alpha\beta\gamma=1+\mathrm{i}$	B1	1	
(b)(i)	$\sum\alpha^2=\left(\sum\alpha\right)^2-2\sum\alpha\beta$ used	M1		Allow if sign error or 2 missing
	$=(-\mathrm{i})^2-2\times 3$	A1F		
	$=-7$	A1F	3	ft errors in (a)
(ii)	$\sum\alpha^2\beta^2=\left(\sum\alpha\beta\right)^2-2\sum\alpha\beta\cdot\beta\gamma$	M1		Allow if sign error in 2 missing
	$=\left(\sum\alpha\beta\right)^2-2\alpha\beta\gamma\sum\alpha$	A1		
	$=9-2(1+\mathrm{i})(-\mathrm{i})$	A1F		ft errors in (a)
	$=7+2\mathrm{i}$	A1F	4	ft errors in (a)
(iii)	$\alpha^2\beta^2\gamma^2=(1+\mathrm{i})^2=2\mathrm{i}$	M1		
		A1F	2	ft sign error in $\alpha\beta\gamma$
(c)	$z^3+7z^2+(7+2\mathrm{i})z-2\mathrm{i}=0$	B1F		Correct numbers in correct places
		B1F	2	Correct signs

Q	Solution	Marks	Total	Comments
3(a)(i)	$\alpha\beta\gamma = -18+12\text{i}$	B1	1	accept $-(18-12\text{i})$
(ii)	$\alpha+\beta+\gamma=0$	B1	1	
(b)(i)	$\alpha=-2$	B1F	1	
(ii)	$\beta\gamma=\dfrac{\alpha\beta\gamma}{\alpha}=9-6\text{i}$	M1 A1F	2	ft sign errors in (a) or (b)(i) or slips such as miscopy
(iii)	$q=\sum\alpha\beta=\alpha(\beta+\gamma)+\beta\gamma$ $=-2\times2+9-6\text{i}$ $=5-6\text{i}$	M1 A1F A1F	3	ft incorrect $\beta\gamma$ or α
(c)	$\beta=k\text{i},\quad \gamma=2-k\text{i}$ $k\text{i}(2-k\text{i})=9-6\text{i}$ $2k=-6\quad (k^2=9)\quad k=-3$ $\beta=-3\text{i},\quad \gamma=2+3\text{i}$	B1 M1 m1 A1	4	imaginary parts
	Total		[illegible]	

Q	Solution	Marks	Total	Comments
4(a)	Use of $\left(\sum\alpha\right)^2=\sum\alpha^2+2\sum\alpha\beta$ $1=-5+2\sum\alpha\beta$ $\sum\alpha\beta=3$	M1 A1 A1	3	AG
(b)	$1(-5-3)=-23-3\alpha\beta\gamma$ $\alpha\beta\gamma=-5$	M1 A1	2	For use of identity
(c)	$z^3-z^2+3z+5=0$	M1 A1F	2	For correct signs and "= 0"
(d)	$\alpha^2+\beta^2+\gamma^2<0\Rightarrow$ non real roots Coefficients real $\therefore$ conjugate pair	B1 B1	2	
(e)	$\text{f}(-1)=0\Rightarrow z+1$ is a factor $(z+1)(z^2-2z+5)=0$ $z=-1,\ 1\pm2\text{i}$	M1A1 A1 A1	4	
	Total		[illegible]	

Q	Solution	Marks	Total	Comments
3(a)	$2+3\text{i}$	B1	1	
(b)(i)	$\alpha\beta=13$	B1	1	
(ii)	$\alpha\beta+\beta\gamma+\gamma\alpha=25$ $\gamma(\alpha+\beta)=12$ $\gamma=3$	M1 A1F A1F	3	M1A0 for -25 (no ft) ft error in $\alpha\beta$
(iii)	$p=-\sum\alpha=-7$ $q=-\alpha\beta\gamma=-39$	M1 A1F A1F	3	M1 for a correct method for either p or q ft from previous errors p and q must be real for sign errors in p and q allow M1 but A0
	Alternative for (b)(ii) and (iii):			
(ii)	Attempt at $(z-2+3\text{i})(z-2-3\text{i})$ $z^2-4z+13$ cubic is $(z^2-4z+13)(z-3)\therefore\gamma=3$	(M1) (A1) (A1)	(3)	
(iii)	Multiply out or pick out coefficients $p=-7,\ q=-39$	(M1) (A1, A1)	(3)	
	Total		[illegible]	

Q	Solution	Marks	Total	Comments
3(a)(i)	$\beta = 2 - 2\sqrt{3}\,\mathrm{i}$	B1	1	
(ii)	$\alpha\beta\gamma = -8$	M1		Allow for +8 but not ±16
	$\alpha\beta = 16$	B1		
	$\gamma = -\frac{1}{2}$	A1	3	
(iii)	Either $\frac{-p}{2} = \alpha + \beta + \gamma$ or $\frac{q}{2} = \alpha\beta + \beta\gamma + \gamma\alpha$	M1		**SC** if failure to divide by 2 throughout, allow M1A1 for either p or q correct ft
	$p = -7,\quad q = 28$	A1F, A1F	3	ft incorrect γ
	Alternative to (a)(ii) and (a)(iii):			
	$(z^2 - 4z + 16)(az + b)$	(M1)		
	$\alpha\beta = 16$	(B1)		
	$a = 2,\ b = +1,\ \gamma = -\frac{1}{2}$	(A1)		
	Equating coefficients	(M1)		
	$p = -7$	(A1F)		
	$q = 28$	(A1F)		
(b)(i)	$r = 4,\quad \theta = \frac{\pi}{3}$	B1,B1	2	
(ii)	$(2 + 2\sqrt{3}\,\mathrm{i})^n = \left(4e^{\frac{\pi\mathrm{i}}{3}}\right)^n$	M1		
	$= 4^n\left(\cos\frac{n\pi}{3} + \mathrm{i}\sin\frac{n\pi}{3}\right)$	A1	2	AG
(iii)	$(2 - 2\sqrt{3}\,\mathrm{i})^n = 4^n\left(\cos\frac{n\pi}{3} - \mathrm{i}\sin\frac{n\pi}{3}\right)$	B1		
	$\alpha^n + \beta^n + \gamma^n = 4^n\left(\cos\frac{n\pi}{3} + \mathrm{i}\sin\frac{n\pi}{3}\right)$ $+ 4^n\left(\cos\frac{n\pi}{3} - \mathrm{i}\sin\frac{n\pi}{3}\right) + \left(-\frac{1}{2}\right)^n$	M1		
	$= 2^{2n+1}\cos\frac{n\pi}{3} + \left(-\frac{1}{2}\right)^n$	A1	3	AG
	Total		14	

4(a)	$\alpha+\beta+\gamma=2$	B1	1	
(b)(i)	α is a root and so satisfies the equation	E1	1	
(ii)	$\sum\alpha^3-2\sum\alpha^2+p\sum\alpha+30=0$	M1A1		
	Substitution for $\sum\alpha^3$ and $\sum\alpha$	m1		
	$\sum\alpha^2=p+13$	A1	4	AG
(iii)	$\left(\sum\alpha\right)^2=\sum\alpha^2+2\sum\alpha\beta$ used	M1		do not allow this M mark if used in (b)(ii)
	$p=-3$	A1	2	AG
(c)(i)	$\mathrm{f}(-2)=0$	M1		
	$\alpha=-2$	A1	2	
(ii)	$(z+2)(z^2-4z+5)=0$	M1		For attempting to find quadratic factor
	$z=\dfrac{4\pm\sqrt{-4}}{2}$	m1		Use of formula or completing the square m0 if roots are not complex
	$=2\pm\mathrm{i}$	A1	3	CAO

	Total		6	
3(a)	$(1+\mathrm{i})^2=2\mathrm{i}$ or $(1+\mathrm{i})=\sqrt{2}\,\mathrm{e}^{\frac{\pi}{4}}$	B1		
	$2\mathrm{i}(1+\mathrm{i})=2\mathrm{i}-2$	B1	2	AG **Alternative method:** $(1+\mathrm{i})^3=1+3\mathrm{i}+3\mathrm{i}^2+\mathrm{i}^3$ B1 $=2\mathrm{i}-2$ B1
(b)(i)	Substitute $z=1+\mathrm{i}$	M1		
	Correct expansion	A1		allow for correctly picking out either the real or the imaginary parts
	$k=-5$	A1	3	
(ii)	$\beta+\gamma=5+\mathrm{i}-\alpha=4$	B1	1	AG
(iii)	$\alpha\beta\gamma=5(1+\mathrm{i})$	M1		allow if sign error
	$\beta\gamma=5$	A1F		ft incorrect k
	$z^2-4z+5=0$	M1		
	Use of formula or $(z-2)^2=-1$	A1F		No ft for real roots if error in k
	$z=2\pm\mathrm{i}$ NB allow marks for (b) in whatever order they appear	A1F	5	

	Total		6	
4(a)(i)	$\sum\alpha = 2$	B1		
	$\sum\alpha\beta = 0$	B1	2	
(ii)	$\sum\alpha^2 = \left(\sum\alpha\right)^2 - 2\sum\alpha\beta$	M1		Used. Watch $\sum\alpha = -2$ (M1A0)
	$= 4$	A1	2	AG
(iii)	Clear explanation	E1	1	eg α satisfies the cubic equation since it is a root. Accept $z = \alpha$
(iv)	$\sum\alpha^3 = 2\sum\alpha^2 - 3k$	M1		Or $\sum\alpha^3 = \left(\sum\alpha\right)^3 - 3\sum\alpha\sum\alpha\beta + 3\alpha\beta\gamma$
	$= 8 - 3k$	A1	2	AG
(b)(i)	$\alpha^4 = 2\alpha^3 - k\alpha$	B1		
	$\sum\alpha^4 = 2\sum\alpha^3 - k\sum\alpha$	M1		Or $\sum\alpha^4 = \left(\sum\alpha^2\right)^2 - 2\left(\sum\alpha\beta\right)^2 + 4\alpha\beta\gamma\sum\alpha$
	$= 2(8 - 3k) - 2k$	A1		ft on $\sum\alpha = -2$
	$k = 2$	A1	4	AG
(ii)	$\sum\alpha^5 = 2\sum\alpha^4 - k\sum\alpha^2$	M1		
	Substitution of values	A1		
	$= -8$	A1	3	

	Total		8	
7(a)	Use of $\left(\Sigma\alpha\right)^2 = \Sigma\alpha^2 + 2\Sigma\alpha\beta$	M1		
		A1	2	AG
(b)	$p = 0,\ q = 5 + 6\mathrm{i}$	B1,B1	2	
(c)(i)	Substitute $3\mathrm{i}$ for z **or** use $3\mathrm{i}\beta\gamma = -r$	M1		allow for $3\mathrm{i}\beta\gamma = r$
	$-27\mathrm{i} + 15\mathrm{i} - 18 + r = 0$ **or** $\beta\gamma = 5 + 6\mathrm{i} + \alpha^2$	A1		any form
	$r = 18 + 12\mathrm{i}$	A1F	3	one error
(ii)	Cubic is $(z - 3\mathrm{i})(z^2 + 3\mathrm{i}z - 4 + 6\mathrm{i})$ or use of $\beta\gamma$ and $\beta + \gamma$	M1A1	2	clearly shown
(iii)	$\mathrm{f}(-2) = 0$ or equate imaginary parts	M1		
	$\beta = -2,\ \gamma = 2 - 3\mathrm{i}$	A1,A1F	3	correct answers no working and no check B1 only

	Total		6	
4(a)(i)	$\alpha+\beta+\gamma=0$	B1	1	
(ii)	$\alpha\beta\gamma=-q$	B1	1	
(b)	$\alpha^3+p\alpha+q=0$	M1		
	$\sum\alpha^3+p\sum\alpha+3q=0$	m1		
	$\alpha^3+\beta^3+\gamma^3=3\alpha\beta\gamma$	A1	3	AG
	Alternative to (b) Use of $\left(\sum\alpha\right)^3=\left(\sum\alpha^3\right)+6\alpha\beta\gamma+3\left(\sum\alpha\sum\alpha\beta-3\alpha\beta\gamma\right)$	(M1)		
	Substitution of $\sum\alpha=0$	(m1)		
	Result	(A1)		
(c)(i)	$\beta=4-7\text{i},\ \gamma=-8$	B1,B1	2	
(ii)	Attempt at either p or q	M1		
	$p=1$	A1F		
	$q=520$	A1F	3	ft incorrect roots provided p and q are real
(d)	Replace z by $\frac{1}{z}$ in cubic equation	M1 A1F		or $\sum\frac{1}{\alpha}=-\frac{p}{q},\sum\frac{1}{\alpha\beta}=0,\frac{1}{\alpha\beta\gamma}=-\frac{1}{q}$ ft on incorrect p and/or q
	$520z^3+z^2+1=0$ coefficients must be integers	A1	3	CAO
	Total		13	

Q	Solution	Marks	Total	Comments
4(a)(i)	$\alpha+\beta+\gamma=5$	B1		
	$\alpha\beta\gamma=4$	B1	2	
(ii)	$\alpha\beta\gamma^2+\alpha\beta^2\gamma+\alpha^2\beta\gamma=\alpha\beta\gamma(\alpha+\beta+\gamma)$	M1		
	$=5\times4=20$	A1√	2	FT their results from (a)(i)
(b)(i)	If α,β,γ are all real then $\alpha^2\beta^2+\beta^2\gamma^2+\gamma^2\alpha^2\geqslant0$ Hence α,β,γ cannot all be real	E1	1	argument must be sound
(ii)	$\alpha\beta+\beta\gamma+\gamma\alpha=k$	B1		$\sum\alpha\beta=k$ PI
	$(\alpha\beta+\beta\gamma+\gamma\alpha)^2$ $=\sum\alpha^2\beta^2+2(\alpha\beta\gamma^2+\alpha\beta^2\gamma+\alpha^2\beta\gamma)$	M1		correct identity for $\left(\sum\alpha\beta\right)^2$
	$=-4+2(20)$	A1√		substituting their result from (a)(ii)
	$k=\pm6$	A1 cso	4	must see $k=\ldots$

Q	Solution	Marks	Total	Comments
5(a)(i)	$(\alpha\beta\gamma =)\quad -37+36\mathrm{i}$	B1	1	
(ii)	$(\beta\gamma =)\quad (-2+3\mathrm{i})(1+2\mathrm{i}) = -2+3\mathrm{i}-4\mathrm{i}-6$	M1		correct unsimplified but must simplify i^2
	$(-8-\mathrm{i})\ \alpha = -37+36\,\mathrm{i}$			
	$\Rightarrow (8+\mathrm{i})\ \alpha = 37-36\,\mathrm{i}$	A1cso	2	**AG** be convinced
(iii)	$\Rightarrow \alpha = \dfrac{37-36\mathrm{i}}{8+\mathrm{i}} \times \dfrac{8-\mathrm{i}}{8-\mathrm{i}}$	M1		
	$= \dfrac{296-37\mathrm{i}-288\mathrm{i}-36}{65}$	A1		correct unsimplified
	$= \dfrac{260-325\mathrm{i}}{65}$			
	$= 4-5\mathrm{i}$	A1cao	3	
				Alternative $(8+\mathrm{i})\,(m+n\,\mathrm{i}) = 37-36\,\mathrm{i}$ $8m-n=37;\ m+8n=-36$ M1 *either* $m=4$ *or* $n=-5$ A1 $\alpha = 4-5\mathrm{i}$ A1
(b)	$\alpha+\beta+\gamma = -p$			
	$-2+3\mathrm{i}+1+2\mathrm{i}+4-5\mathrm{i} = 3$			
	$(\Rightarrow p =)\ -3$	B1	1	
(c)	$\alpha\beta+\beta\gamma+\gamma\alpha = q$			$q=\sum\alpha\beta$ and attempt to evaluate three
	$(7+22\mathrm{i}) + (-8-\mathrm{i}) + (14+3\mathrm{i}) = q$	M1		products FT "their" α
	$q = 13+24\mathrm{i}$	A1cao	2	

Q	Solution	Mark	Total	Comment
4 (a) (i)	$\alpha+\beta+\gamma=-2$	B1		
	$\alpha\beta+\beta\gamma+\gamma\alpha=3$	B1	2	
(ii)	$\alpha^2+\beta^2+\gamma^2$			
	$=(\alpha+\beta+\gamma)^2-2(\alpha\beta+\beta\gamma+\gamma\alpha)$	M1		correct formula
	$=4-6=-2$	A1cso	2	**AG** be convinced; must see $4-6$ **A0** if $\alpha+\beta+\gamma$ or $\alpha\beta+\beta\gamma+\gamma\alpha$ not correct
(b) (i)	$\sum(\alpha+\beta)(\beta+\gamma)=\sum\alpha^2+3\sum\alpha\beta$	M1		or may use $12+4\sum\alpha+\sum\alpha\beta$
	$=-2+9$	m1		ft their $\alpha\beta+\beta\gamma+\gamma\alpha$
	$=7$	A1	3	
(ii)	$\alpha\beta\gamma=4$	B1		**PI** when earning **m1** later
	$(\alpha+\beta)(\beta+\gamma)(\gamma+\alpha)$ $=\sum\alpha\sum\alpha\beta-\alpha\beta\gamma$	M1		or $(-2-\alpha)(-2-\beta)(-2-\gamma)$ $=-8-4\sum\alpha-2\sum\alpha\beta-\alpha\beta\gamma$
	$=-6-4$	m1		Sub their $\sum\alpha$, $\sum\alpha\beta$ & $\alpha\beta\gamma$
	$=-10$	A1	4	
(c)	Sum of new roots $=2\sum\alpha=-4$	B1		***or* NMS** coefficient of z^2 written as $+4$
	$z^3\pm4z^2+\text{"their7"}z-\text{"their}-10\text{"}\ (=0)$	M1		correct sub of their results from part **(b)**
	New equation $z^3+4z^2+7z+10=0$	A1	3	**Alternative** $y=-2-z$ **B1** $(-2-y)^3+2(-2-y)^2+3(-2-y)-4=0$ **M1** $y^3+4y^2+7y+10=0$ **A1** **NB** candidate may do this first and then obtain results for part (b)

Q7	Solution	Mark	Total	Comment
(a)	$\alpha\beta+\beta\gamma+\gamma\alpha=0$	B1		
	$\alpha\beta\gamma=-\frac{4}{27}$	B1	2	
(b)(i)	$\alpha\beta+\alpha\beta+\beta^2=0$; $\alpha\beta^2=-\frac{4}{27}$	B1		May use γ instead of β throughout **(b)(i)**
		M1		Clear attempt to eliminate either α or β from "their" equations
	$\alpha^3=-\frac{1}{27}$ **or** $\beta^3=\frac{8}{27}$	A1		correct
	either $\alpha=-\frac{1}{3}$ or $\beta=\frac{2}{3}$	A1		
	$\alpha=-\frac{1}{3}$, $\beta=\frac{2}{3}$, $\gamma=\frac{2}{3}$	A1	5	all 3 roots clearly stated
(ii)	$\left(\sum\alpha=1=-\frac{k}{27}\ \Rightarrow\right)$ $k=-27$	B1	1	or substituting correct root into equation
(c)(i)	$\alpha^2=-2\mathrm{i}$	B1		
	$\alpha^3=-2-2\mathrm{i}$	B1	2	
(ii)	$27(-2-2\mathrm{i})-2\mathrm{i}k+4=0$	M1		correctly substituting "their" $\alpha^2=-2\mathrm{i}$ and "their" $\alpha^3=-2-2\mathrm{i}$
	$k=-27+25\mathrm{i}$	A1	2	
(d)	$y=\frac{1}{z}+1\Rightarrow z=\frac{1}{y-1}$	B1		may use any letter instead of y
	$\frac{27}{(y-1)^3}-\frac{12}{(y-1)^2}+4=0$	M1		sub their z into cubic equation
	$27-12(y-1)+4(y-1)^3=0$	A1		removing denominators correctly
	$27-12y+12+4(y^3-3y^2+3y-1)=0$	A1		correct and $(y-1)^3$ expanded correctly
	$4y^3-12y^2+35=0$	A1	5	
	Alternative: $\sum\alpha'=3+\frac{\alpha\beta+\beta\gamma+\gamma\alpha}{\alpha\beta\gamma}=3$	(B1)		sum of new roots =3
	$\sum\alpha'\beta'=3+\frac{2(\alpha\beta+\beta\gamma+\gamma\alpha)+\alpha+\beta+\gamma}{\alpha\beta\gamma}$	(M1)		**M1** for either of the other two formulae correct in terms of $\alpha\beta\gamma$, $\alpha\beta+\beta\gamma+\gamma\alpha$ and $\alpha+\beta+\gamma$
	$=0$	(A1)		
	$\prod=1+\frac{\alpha\beta+\beta\gamma+\gamma\alpha+1+\alpha+\beta+\gamma}{\alpha\beta\gamma}$			
	$=\frac{-35}{4}$	(A1)		
	$4y^3-12y^2+35=0$	(A1)	(5)	may use any letter instead of y
	Total		17	

Q2	Solution	Mark	Total	Comment
(a)(i)	$1-2\mathrm{i}$	B1	1	
(ii)	$(\alpha\beta = 1+4=)\quad 5$	B1	1	
(b)	$\sum \alpha\beta = \frac{17}{3}$	B1		**PI** by next line
	$\alpha\gamma+\beta\gamma+\text{"their"}5=\text{"their"}\frac{17}{3}$	M1		**FT** "their" $\alpha\beta$ and $\sum\alpha\beta$ values
	$\Rightarrow \gamma = \frac{1}{3}$	A1	3	**Alternative** $z^3+\frac{p}{3}z^2+\frac{17}{3}z+\frac{q}{3}$ quadratic factor z^2-2z+5 **B1** $(z^2-2z+5)(z-\gamma)$ comparing coefficient of z: $5+2\gamma=\frac{17}{3}$ **M1** $\Rightarrow \gamma=\frac{1}{3}$ **A1 (3)**
(c)	$\alpha+\beta+\gamma=\frac{-p}{3}$, $\alpha\beta\gamma=\frac{-q}{3}$	M1		Either of these expressions correct **PI** by correct p or q
	$p=-7$	A1		
	$q=-5$	A1	3	**Alternative** comparing coefficients either $-5\gamma=\frac{q}{3}$ or $-\gamma-2=\frac{p}{3}$ **M1** $p=-7$ **A1**; $q=-5$ **A1 (3)**