FP2 Hyperbolic Functions

1 (a) Express

$$5\sinh x + \cosh x$$

in the form $Ae^{x} + Be^{-x}$, where A and B are integers. *(2 marks)*

(b) Solve the equation

$$5\sinh x + \cosh x + 5 = 0$$

giving your answer in the form $\ln a$, where a is a rational number. *(4 marks)*

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4 (a) Sketch the graph of $y = \tanh x$. *(2 marks)*

(b) Given that $u = \tanh x$, use the definitions of $\sinh x$ and $\cosh x$ in terms of e^{x} and e^{-x} to show that

$$x = \tfrac{1}{2}\ln\left(\frac{1+u}{1-u}\right)$$ *(6 marks)*

(c) (i) Show that the equation

$$3\operatorname{sech}^2 x + 7\tanh x = 5$$

can be written as

$$3\tanh^2 x - 7\tanh x + 2 = 0$$ *(2 marks)*

(ii) Show that the equation

$$3\tanh^2 x - 7\tanh x + 2 = 0$$

has only one solution for x.

Find this solution in the form $\frac{1}{2}\ln a$, where a is an integer. *(5 marks)*

1 (a) Show that

$$9\sinh x - \cosh x = 4e^{x} - 5e^{-x}$$ *(2 marks)*

(b) Given that

$$9\sinh x - \cosh x = 8$$

find the exact value of $\tanh x$. *(7 marks)*

5 (a) Using the identities

$$\cosh^2 t - \sinh^2 t = 1, \quad \tanh t = \frac{\sinh t}{\cosh t} \quad \text{and} \quad \operatorname{sech} t = \frac{1}{\cosh t}$$

show that:

(i) $\tanh^2 t + \operatorname{sech}^2 t = 1$; *(2 marks)*

(ii) $\frac{d}{dt}(\tanh t) = \operatorname{sech}^2 t$; *(3 marks)*

(iii) $\frac{d}{dt}(\operatorname{sech} t) = -\operatorname{sech} t \tanh t$. *(3 marks)*

(b) A curve C is given parametrically by

$$x = \operatorname{sech} t, \quad y = 4 - \tanh t$$

(i) Show that the arc length, s, of C between the points where $t = 0$ and $t = \frac{1}{2}\ln 3$ is given by

$$s = \int_0^{\frac{1}{2}\ln 3} \operatorname{sech} t \, dt$$ *(4 marks)*

(ii) Using the substitution $u = e^{t}$, find the exact value of s. *(6 marks)*

4 (a) Prove that the curve

$$y = 12\cosh x - 8\sinh x - x$$

has exactly one stationary point. *(7 marks)*

(b) Given that the coordinates of this stationary point are (a, b), show that $a + b = 9$. *(4 marks)*

2 (a) Use the definitions of $\cosh\theta$ and $\sinh\theta$ in terms of e^{θ} to show that

$$\cosh x \cosh y - \sinh x \sinh y = \cosh(x - y)$$ *(4 marks)*

(b) It is given that x satisfies the equation

$$\cosh(x - \ln 2) = \sinh x$$

(i) Show that $\tanh x = \frac{5}{7}$. *(4 marks)*

(ii) Express x in the form $\frac{1}{2}\ln a$. *(2 marks)*

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1 (a) Show, by means of a sketch, that the curves with equations

$$y = \sinh x$$

and

$$y = \operatorname{sech} x$$

have exactly one point of intersection. *(4 marks)*

(b) Find the x-coordinate of this point of intersection, giving your answer in the form $a \ln b$. *(4 marks)*

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3 A curve has cartesian equation

$$y = \frac{1}{2}\ln(\tanh x)$$

(a) Show that

$$\frac{dy}{dx} = \frac{1}{\sinh 2x}$$ *(4 marks)*

(b) The points A and B on the curve have x-coordinates $\ln 2$ and $\ln 4$ respectively. Find the arc length AB, giving your answer in the form $p \ln q$, where p and q are rational numbers. *(8 marks)*

1 (a) Sketch the curve $y = \cosh x$. *(1 mark)*

(b) Solve the equation

$$6\cosh^2 x - 7\cosh x - 5 = 0$$

giving your answers in logarithmic form. *(6 marks)*

6 (a) Show that

$$\frac{1}{4}(\cosh 4x + 2\cosh 2x + 1) = \cosh^2 x \cosh 2x$$ *(3 marks)*

(b) Show that, if $y = \cosh^2 x$, then

$$1 + \left(\frac{dy}{dx}\right)^2 = \cosh^2 2x$$ *(3 marks)*

(c) The arc of the curve $y = \cosh^2 x$ between the points where $x = 0$ and $x = \ln 2$ is rotated through 2π radians about the x-axis. Show that the area S of the curved surface formed is given by

$$S = \frac{\pi}{256}(a \ln 2 + b)$$

where a and b are integers. *(7 marks)*

1 (a) Show that

$$12\cosh x - 4\sinh x = 4e^x + 8e^{-x}$$ *(2 marks)*

(b) Solve the equation

$$12\cosh x - 4\sinh x = 33$$

giving your answers in the form $k \ln 2$. *(5 marks)*

5 (a) Using the definition $\tanh y = \dfrac{e^y - e^{-y}}{e^y + e^{-y}}$, show that, for $|x| < 1$,

$$\tanh^{-1} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$$

(3 marks)

(b) Hence, or otherwise, show that $\dfrac{d}{dx}(\tanh^{-1} x) = \dfrac{1}{1-x^2}$. *(3 marks)*

(c) Use integration by parts to show that

$$\int_0^{\frac{1}{2}} 4\tanh^{-1} x \, dx = \ln\left(\frac{3^m}{2^n}\right)$$

where m and n are positive integers. *(5 marks)*

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2 (a) (i) Sketch on the axes below the graphs of $y = \sinh x$ and $y = \cosh x$. *(3 marks)*

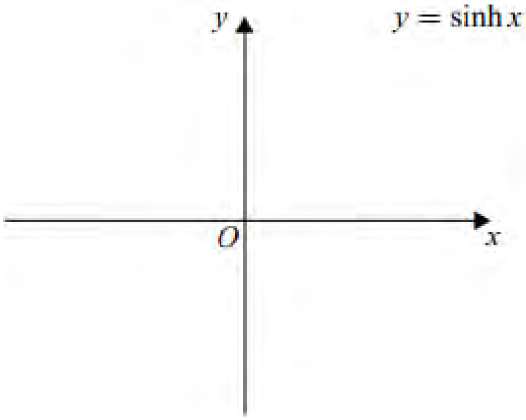
(ii) Use your graphs to explain why the equation

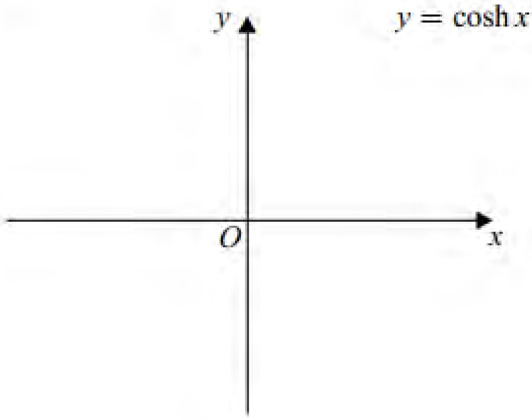
$$(k + \sinh x)\cosh x = 0$$

where k is a constant, has exactly one solution. *(1 mark)*

(b) A curve C has equation $y = 6\sinh x + \cosh^2 x$. Show that C has only one stationary point and show that its y-coordinate is an integer. *(5 marks)*

QUESTION PART REFERENCE	**Answer space for question 2**
(a)(i)	





6 (a) Show that $\dfrac{1}{5\cosh x - 3\sinh x} = \dfrac{e^x}{m + e^{2x}}$, where m is an integer. *(3 marks)*

(b) Use the substitution $u = e^x$ to show that

$$\int_0^{\ln 2} \frac{1}{5\cosh x - 3\sinh x}\,dx = \frac{\pi}{8} - \frac{1}{2}\tan^{-1}\left(\frac{1}{2}\right)$$ *(5 marks)*

7 (a) (i) Show that

$$\frac{d}{du}\left(2u\sqrt{1+4u^2} + \sinh^{-1} 2u\right) = k\sqrt{1+4u^2}$$

where k is an integer. *(4 marks)*

(ii) Hence show that

$$\int_0^1 \sqrt{1+4u^2}\,du = p\sqrt{5} + q\sinh^{-1} 2$$

where p and q are rational numbers. *(2 marks)*

(b) The arc of the curve with equation $y = \frac{1}{2}\cos 4x$ between the points where $x = 0$ and $x = \frac{\pi}{8}$ is rotated through 2π radians about the x-axis.

(i) Show that the area S of the curved surface formed is given by

$$S = \pi\int_0^{\frac{\pi}{8}} \cos 4x\sqrt{1 + 4\sin^2 4x}\,dx$$ *(2 marks)*

(ii) Use the substitution $u = \sin 4x$ to find the exact value of S. *(4 marks)*

5 (a) Using the definition $\sinh\theta = \frac{1}{2}(e^{\theta} - e^{-\theta})$, prove the identity

$$4\sinh^3\theta + 3\sinh\theta = \sinh 3\theta$$

[3 marks]

(b) Given that $x = \sinh\theta$ and $16x^3 + 12x - 3 = 0$, find the value of θ in terms of a natural logarithm.

[4 marks]

(c) Hence find the real root of the equation $16x^3 + 12x - 3 = 0$, giving your answer in the form $2^p - 2^q$, where p and q are rational numbers.

[2 marks]

2 (a) Sketch the graph of $y = \tanh x$ and state the equations of its asymptotes.

[3 marks]

(b) Use the definitions of $\sinh x$ and $\cosh x$ in terms of e^x and e^{-x} to show that

$$\text{sech}^2 x + \tanh^2 x = 1$$

[3 marks]

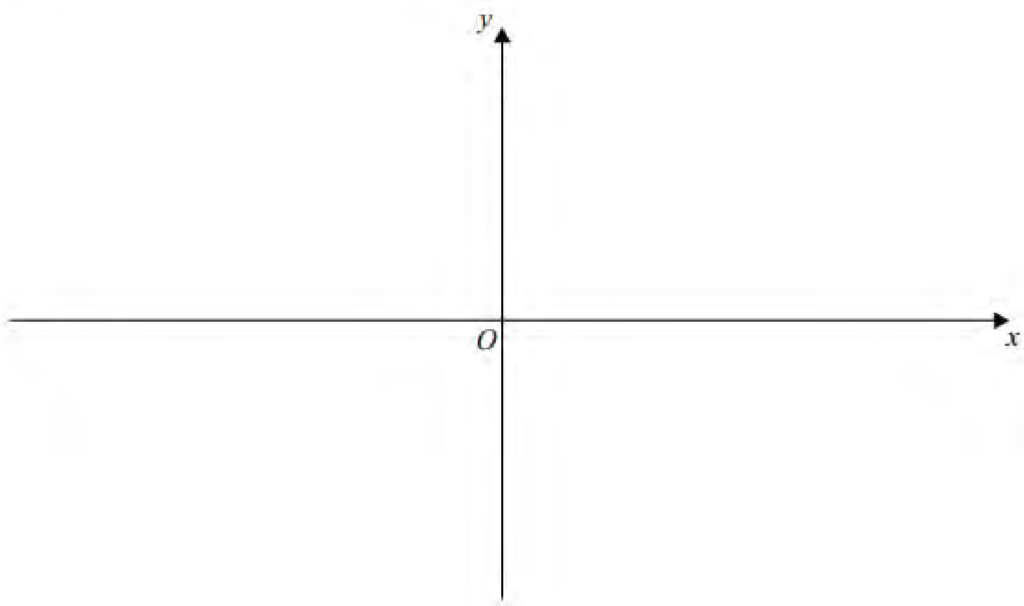
(c) Solve the equation $6\,\text{sech}^2 x = 4 + \tanh x$, giving your answers in terms of natural logarithms.

[5 marks]

QUESTION PART REFERENCE

Answer space for question 2

(a)



6 (a) Given that $y = \sinh x$, use the definition of $\sinh x$ in terms of e^x and e^{-x} to show that $x = \ln(y + \sqrt{y^2 + 1})$.

[4 marks]

(b) A curve has equation $y = 6\cosh^2 x + 5\sinh x$.

(i) Show that the curve has a single stationary point and find its x-coordinate, giving your answer in the form $\ln p$, where p is a rational number.

[5 marks]

(ii) The curve lies entirely above the x-axis. The region bounded by the curve, the coordinate axes and the line $x = \cosh^{-1} 2$ has area A.

Show that

$$A = a\cosh^{-1} 2 + b\sqrt{3} + c$$

where a, b and c are integers.

[5 marks]

1(a)	$5\left(\frac{e^x - e^{-x}}{2}\right) + \left(\frac{e^x + e^{-x}}{2}\right)$	M1		M0 if no 2s in denominator
	$= 3e^x - 2e^{-x}$	A1	2	
(b)	$3e^x - 2e^{-x} + 5 = 0$			
	$3e^{2x} + 5e^x - 2 = 0$	M1		ft if 2s missing in (a)
	$(3e^x - 1)(e^x + 2) = 0$	A1F		
	$e^x \neq -2$	E1		any **indication** of rejection
	$e^x = \frac{1}{3}$ $\quad x = \ln\frac{1}{3}$	A1F	4	provided quadratic factorises into real factors
	Total		6	

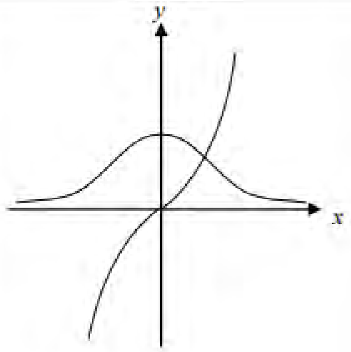
4(a)	Sketch, approximately correct shape	B1		B0 if curve touches asymptotes lines of answer booklet could be used for asymptotes be strict with sketch
	Asymptotes at $y = \pm 1$	B1	2	
(b)	Use of $u = \dfrac{\sinh x}{\cosh x}$	M1		
	$= \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$ or $\dfrac{e^{2x} - 1}{e^{2x} + 1}$	A1		
	$u\left(e^x + e^{-x}\right) = e^x - e^{-x}$	M1		M1 for multiplying up
	$e^{-x}(1+u) = e^x(1-u)$	A1		A1 for factorizing out e's or M1 for attempt at $1+u$ and $1-u$ in terms of e^x
	$e^{2x} = \dfrac{1+u}{1-u}$	m1		
	$x = \dfrac{1}{2}\ln\left(\dfrac{1+u}{1-u}\right)$	A1	6	AG
4(c)(i)	Use of $\tanh^2 x = 1 - \text{sech}^2 x$	M1		
	Printed answer	A1	2	
(ii)	$(3\tanh x - 1)(\tanh x - 2) = 0$	M1		Attempt to factorise
	$\tanh x \neq 2$	E1		Accept tanh $x \neq 2$ written down but not ignored or just crossed out
	$\tanh x = \dfrac{1}{3}$	A1		
	$x = \dfrac{1}{2}\ln 2$	M1 A1F	5	ft
	Total		**15**	

1(a)	$\dfrac{9\left(e^x - e^{-x}\right)}{2} - \dfrac{e^x + e^{-x}}{2}$	M1		M0 if $\cosh x$ mixed up with $\sinh x$
	$= 4e^x - 5e^{-x}$	A1	2	AG
(b)	Attempt to multiply by e^x	M1		
	$4e^{2x} - 8e^x - 5 = 0$	A1		
	$\left(2e^x - 5\right)\left(2e^x + 1\right) = 0$	M1		ft provided quadratic factorises (or use of formula)
	$e^x \neq -\dfrac{1}{2}$	E1F		PI but not ignored
	$e^x = \dfrac{5}{2}$	A1F		
	$\tanh x = \dfrac{\frac{5}{2} - \frac{2}{5}}{\frac{5}{2} + \frac{2}{5}} = \dfrac{21}{29}$	M1 A1F	7	M1 PI for attempt to use $\tanh x = \dfrac{\sinh x}{\cosh x}$ or equivalent fraction
	Total		**9**	

5(a)(i)	Divide $\cosh^2 t - \sinh^2 t = 1$ by $\cosh^2 t$	M1		Or $\frac{\sinh^2 t}{\cosh^2 t} + \frac{1}{\cosh^2 t}$
	Rearrange	A1	2	AG If solved back to front with no conclusion ending $\cosh^2 t - \sinh^2 t = 1$ B1 only
(ii)	$\frac{d}{dt}\left(\frac{\sinh t}{\cosh t}\right) = \frac{\cosh^2 t - \sinh^2 t}{\cosh^2 t}$	M1A1		
	$= \operatorname{sech}^2 t$	A1	3	AG
(iii)	$\frac{d}{dt}(\operatorname{sech} t) = -(\cosh t)^{-2} \sinh t$	M1A1		Allow A1 if negative sign missing
	$= -\operatorname{sech} t \tanh t$	A1	3	AG
(b)(i)	$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \operatorname{sech}^4 t + \operatorname{sech}^2 t \tanh^2 t$	M1		Allow slips of sign before squaring for this M1
	Use of $\tanh^2 t + \operatorname{sech}^2 t = 1$	m1		Correct formula only for m1
	$= \operatorname{sech}^2 t$	A1		
	$\therefore s = \int_0^{\frac{1}{2}\ln 3} \operatorname{sech} t \, dt$	A1	4	AG (including limits)
(ii)	$u = e^t \quad du = e^t \, dt$	B1		
	$\int \operatorname{sech} t \, dt = \int \frac{2}{u^2 + 1} du$	M1A1		CAO M1 for putting integrand in terms of u (<u>no</u> sech ($\ln u$))
	$\left[2\tan^{-1} u\right]$	A1		Or $2\tan^{-1} e^t$
	Change limits correctly or change back to t	m1		At some stage
	$= \frac{2\pi}{3} - \frac{2\pi}{4} = \frac{\pi}{6}$	A1	6	CAO
	Total		**18**	

Q	Solution	Marks	Total	Comments
4(a)	$\frac{dy}{dx} = 12\sinh x - 8\cosh x - 1$	B1		The B1 and M1 could be in reverse order if put in terms of e first
	$12\frac{(e^x - e^{-x})}{2} - 8\frac{(e^x + e^{-x})}{2} - 1 = 0$	M1		M0 if $\sinh x$ and $\cosh x$ in terms of e^x are interchanged
	$2e^{2x} - e^x - 10 = 0$	A1F		ft slips of sign
	$(2e^x - 5)(e^x + 2) = 0$	M1A1F		ft provided quadratic factorises
	$e^x \neq -2$	E1		some indication of rejection needed
	$x = \ln\frac{5}{2}$ one stationary point	A1F	7	Condone $e^x = \frac{5}{2}$ with statement provided quadratic factorises **Special Case** If $\frac{dy}{dx} = 12\sinh x - 8\cosh x$ B0 For substitution in terms of e^x M1 leading to $e^{2x} = 5$ A1 Then M0
(b)	$b = 12\frac{\left(\frac{5}{2}+\frac{2}{5}\right)}{2} - 8\frac{\left(\frac{5}{2}-\frac{2}{5}\right)}{2} - \ln\frac{5}{2}$	M1A1F		for substitution into original equation
	$= \frac{174}{10} - \frac{84}{10} - \ln\frac{5}{2}$	A1		CAO
	$= 9 - a$	A1	4	AG; accept $b = 9 - a$
	Total		11	

2(a)	$\frac{(e^x+e^{-x})}{2}\frac{(e^y+e^{-y})}{2}-\frac{(e^x-e^{-x})}{2}\frac{(e^y-e^{-y})}{2}$	M1A1		M0 if sinh and cosh confused M1 for formula quoted correctly
	Correct expansions	A1		Use of e^{xy} A0
	$=\frac{1}{2}\left(e^{x-y}+e^{-(x-y)}\right)=\cosh(x-y)$	A1	4	AG
				Alternative:
(b)(i)	$\cosh(x-\ln 2)=\cosh x\cosh(\ln 2)$ $-\sinh x\sinh(\ln 2)$	M1		$\frac{e^{x-\ln 2}+e^{-x+\ln 2}}{2}=\frac{e^x-e^{-x}}{2}$ M1
	$\left.\begin{array}{l}\cosh(\ln 2)=\frac{5}{4}\\ \sinh(\ln 2)=\frac{3}{4}\end{array}\right\}$ any method	B1		Both correct $e^{x-\ln 2}=\frac{e^x}{2}$ or $e^{-x+\ln 2}=2e^{-1}$ used B1
	$\frac{5}{4}\cosh x=\frac{7}{4}\sinh x$	A1F		$e^x=\sqrt{6}$ A1
	$\tanh x=\frac{5}{7}$	A1	4	AG $\tanh x=\frac{5}{7}$ A1
(ii)	$x=\frac{1}{2}\ln\left(\frac{1+\frac{5}{7}}{1-\frac{5}{7}}\right)$ or $\frac{e^x-e^{-x}}{e^x+e^{-x}}=\frac{5}{7}$	M1		Could be embedded in (b)(i)
	$=\frac{1}{2}\ln 6$	A1	2	
	Total		**10**	

1(a)				
	Sketch $y = \sinh x$	B1		gradient > 0 at (0, 0); no asymptotes
	Sketch $y = \operatorname{sech} x$:			
	Symmetry about $x = 0$ with max point	B1		
	Asymptote $y = 0$	B1		must not cross x-axis
	Point (0, 1) marked or implied	B1	4	
(b)	$\sinh x = \dfrac{1}{\cosh x}$	M1		
	$\sinh 2x = 2$	M1		use of double angle formula
	Use of ln	m1		dependent on previous M2
	$x = \frac{1}{2}\ln\left(2+\sqrt{5}\right)$	A1	4	
	or			
	$\frac{1}{2}\left(e^{2x} - e^{-2x}\right) = 2$ OE	(M1)		incorrect $\sinh x$, $\cosh x$ M0 (no marks)
	$e^{4x} - 4e^{2x} - 1 = 0$	(M1)		ie multiply by e^{2x} and rewrite
	Correct use of formula	(m1)		
	Result	(A1)	(4)	
	Total		**8**	

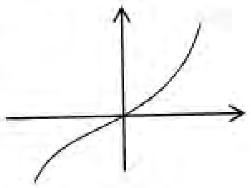
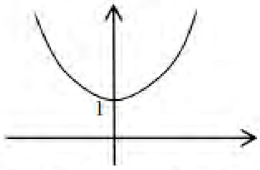
3(a)	$\dfrac{dy}{dx} = \dfrac{1}{2\tanh x}$	B1		
	$\times \operatorname{sech}^2 x$	B1		
	$= \dfrac{1}{2\sinh x \cosh x}$	M1		for expressing in terms of $\sinh x$ and $\cosh x$
	$= \dfrac{1}{\sinh 2x}$	A1	4	AG; PI by previous line
(b)	$\sqrt{1+\left(\dfrac{dy}{dx}\right)^2} = \sqrt{1+\dfrac{1}{\sinh^2 2x}}$	M1		use of formula; accept √ inserted at any stage
	$= \sqrt{\dfrac{\cosh^2 2x}{\sinh^2 2x}}$	m1		relevant use of $\cosh^2 - \sinh^2 = 1$
	$= \dfrac{\cosh 2x}{\sinh 2x}$	A1		OE
	Integral is $\frac{1}{2}\ln \sinh 2x$	M1A1		M1 for ln sinh
	$\sinh(2\ln 4) = \dfrac{255}{32}$ $\quad \sinh(2\ln 2) = \dfrac{15}{8}$	B1B1		PI
	$s = \frac{1}{2}\ln\left(\frac{17}{4}\right)$	A1F	8	ft error in $\frac{1}{2}$
	Total		**12**	

1(a)	Sketch of $y = \cosh x$	B1	1	approximately correct with minimum point above the x-axis, symmetrical about y-axis
(b)	Attempt to factorise	M1		or complete square or use (correct unsimplified) formula
	$(3\cosh x - 5)(2\cosh x + 1) = 0$	A1		
	$\cosh x \neq -\frac{1}{2}$	E1		indicated or stated (not merely neglected)
	$x = \ln\left(\frac{5}{3} + \sqrt{\frac{25}{9} - 1}\right)$	M1 A1F		evidence of use of formula. Must see -1 or equivalent ft incorrect factorisation
	$= \pm \ln 3$	A1F	6	A1 for $\pm$
	Alternative: $3\left(\frac{e^x + e^{-x}}{2}\right) = 5$			
	$3e^{2x} - 10e^x + 3 = 0$	(M1)		
	$(3e^x - 1)(e^x - 3) = 0$	(A1F)		Correct factors
	$x = \ln\frac{1}{3}$ or $\ln 3$	(A1F)		for both
	NB if $\cosh x = \frac{e^x + e^{-x}}{2}$ used initially, M0 until quartic in e^x is factorised			M1 for e^x-3 is a factor A1 if correct M1 for $3e^x-1$ is a factor A1 if correct A1 for $x = \pm \ln 3$ E1 for showing remaining quadratic has no real roots
	Total		**7**	

Q	Solution	Marks	Total	Comments
6(a)	Use of $\cosh 2x = 2\cosh^2 x - 1$	M1		or $\cosh 4x = 2\cosh^2 2x - 1$
	$\text{RHS} = \frac{1}{2}\cosh 2x + \frac{1}{2}\cosh^2 2x$	A1		
	$= \frac{1}{4}(1 + 2\cosh 2x + \cosh 4x)$	A1	3	
	If substituted for both cosh $4x$ and cosh $2x$ in LHS M1 only, until corrected If RHS is put in terms of e^x M1 for correct substitution A1 for correct expansion A1 for correct result			
(b)	$\frac{dy}{dx} = 2\cosh x \sinh x = \sinh 2x$	M1A1		allow A1 for $1 + \left(\frac{dy}{dx}\right)^2 = 1 - 4\cosh^2 x + 4\cosh^4 x$ Incorrect form for $\cosh^2 x$ in terms of cosh $2x$ M1 only
	Or $y = \left(\frac{e^x + e^{-x}}{2}\right)^2 = \frac{e^{2x} + 2 + e^{-2x}}{4}$			
	$\frac{dy}{dx} = \frac{2e^{2x} - 2e^x}{4}$	(M1)		
	= sinh $2x$	(A1)		
	$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \sinh^2 2x = \cosh^2 2x$	A1	3	AG
(c)	$S = 2\pi \int_{(0)}^{(\ln 2)} \cosh^2 x \cosh 2x \, dx$	M1A1		allow even if limits missing
	$= 2\pi \int_0^{\ln 2} \frac{1}{4}(1 + 2\cosh 2x + \cosh 4x)\, dx$	m1		
	$= \frac{2\pi}{4}\left[x + \frac{2\sinh 2x}{2} + \frac{\sinh 4x}{4}\right]$	A1		Integrated correctly
	Correct use of limits	m1		
	$a = 128$, $b = 495$	A1,A1	7	accept correct answers written down with no working. Only one A1 if 2π not used

Q	Solution	Marks	Total	Comments
1(a)	$\cosh x = \frac{1}{2}(e^x + e^{-x})$			*or* $12\cosh x = 6(e^x + e^{-x})$
	or $\sinh x = \frac{1}{2}(e^x - e^{-x})$	M1		*or* $4\sinh x = 2(e^x - e^{-x})$
	$12\cosh x - 4\sinh x =$ $6(e^x + e^{-x}) - 2(e^x - e^{-x})$			
	$12\cosh x - 4\sinh x = 4e^x + 8e^{-x}$	A1 **cso**	2	**AG**
(b)	$4e^x + 8e^{-x} = 33$			
	$\Rightarrow 4e^{2x} - 33e^x + 8 \quad (=0)$	M1		attempt to multiply by e^x to form quadratic in e^x
	$\Rightarrow (e^x - 8)(4e^x - 1) \quad (=0)$	m1		factorisation attempt (see below) or correct use of formula
	$\Rightarrow (e^x =)\ 8, \ (e^x =)\ \frac{1}{4}$	A1		correct roots
	$(x =)\ 3\ln 2$	A1		
	$(x =)\ -2\ln 2$	A1	5	
	Total		**7**	

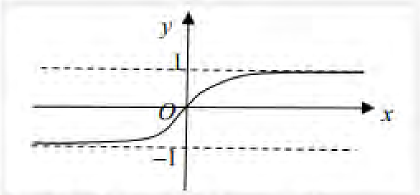
5(a)	$x=\tanh y=\dfrac{e^{y}-e^{-y}}{e^{y}+e^{-y}}$			
	$xe^{y}+xe^{-y}=e^{y}-e^{-y}$	M1		*or* $xe^{2y}+x=e^{2y}-1$
	$\Rightarrow(x+1)e^{-y}=e^{y}(1-x)$			
	$\Rightarrow(x+1)=e^{2y}(1-x)$	A1		
	$e^{2y}=\dfrac{1+x}{1-x}\Rightarrow y=\dfrac{1}{2}\ln\left(\dfrac{1+x}{1-x}\right)$	A1cso	3	**AG**
(b)	$y=\dfrac{1}{2}\ln(1+x)-\dfrac{1}{2}\ln(1-x)$	M1		
	$\dfrac{dy}{dx}=\dfrac{1}{2(1+x)}+\dfrac{1}{2(1-x)}$	A1		
	$=\dfrac{1-x+1+x}{2(1+x)(1-x)}=\dfrac{2}{2(1-x^{2})}=\dfrac{1}{1-x^{2}}$	A1cso	3	**AG**
				Alternative 1 $\dfrac{dy}{dx}=\dfrac{1}{2}\times\dfrac{(1-x)}{(1+x)}\times\dfrac{d}{dx}\left(\dfrac{1+x}{1-x}\right)$ M1 $\dfrac{dy}{dx}=\dfrac{1}{2}\times\dfrac{(1-x)}{(1+x)}\times\dfrac{(1-x)+(1+x)}{(1-x)^{2}}$ A1 $\Rightarrow\dfrac{dy}{dx}=\dfrac{1}{1-x^{2}}$ A1 cso
(c)	$\int 4\tanh^{-1}x\,dx=4x\tanh^{-1}x-\int\dfrac{4x}{1-x^{2}}\,dx$	M1		
	$4x\tanh^{-1}x+2\ln(1-x^{2})$	A1		
	$\tanh^{-1}\frac{1}{2}=\frac{1}{2}\ln 3$	B1		must simplify logarithm to ln3
	Value of integral = $\ln 3+2\ln\frac{3}{4}$	A1		any correct form
	$\ln\left(\dfrac{3^{3}}{2^{4}}\right)$	A1cso	5	all working must be correct
	Total		**11**	

2(a)(i)	$\sinh x$ graph 	M1		 shape - curve through O, in 1st and 3rd quadrants
	$\cosh x$ graph 	M1		 shape - curve all above x-axis
	Gradient of $\sinh x > 0$ at origin **and** $\cosh x$ minimum at $(0,1)$	A1	3	
(ii)	$\cosh x = 0$ has ***no*** solutions **and** $\sinh x = -k$ has **one** solution (hence equation has exactly one solution)	E1	1	or $\cosh x > 0$ etc (since $y = -k$ cuts $y = \sinh x$ exactly once)
(b)	$\dfrac{\mathrm{d}y}{\mathrm{d}x} = 6\cosh x + 2\cosh x \sinh x$	M1 A1		one term correct all correct - may have $6\cosh x + \sinh 2x$
	$(2)\cosh x(3 + \sinh x) = 0$ therefore C has only one stationary point	E1√		putting = 0, factorising **and** concluding statement (may be later)
	$\Rightarrow \sinh x = -3$	m1		finding $\sinh x$ from "their" equation
	$\cosh^2 x = 10$			
	$y\,(= -18 + 10) = -8$	A1	5	answer must be integer so do not accept calculator approximation rounded to –8
	Total		**9**	

Q	Solution	Marks	Total	Comments
6(a)	$(5\cosh x - 3\sinh x)$			
	$= \frac{5}{2}(e^x + e^{-x}) - \frac{3}{2}(e^x - e^{-x})$	M1		$\cosh x$ **and** $\sinh x$ correct in terms of e^x
	$= e^x + 4e^{-x}$	A1		may be seen as denominator
	$\frac{1}{5\cosh x - 3\sinh x} = \frac{e^x}{4 + e^{2x}}$	A1cso	3	** must have left hand-side ; $m = 4$
(b)	$u = e^x \Rightarrow du = e^x dx$	M1		or $\frac{du}{dx} = e^x$
	$\Rightarrow \int \frac{1}{4+u^2}(du)$	A1√		FT "their" m from part(a) $\Rightarrow \int \frac{1}{m+u^2} du$
	$= \frac{1}{2}\tan^{-1}\frac{u}{2}$	A1√		FT "their" $\frac{1}{\sqrt{m}}\tan^{-1}\frac{u}{\sqrt{m}}$
	$x = 0 \Rightarrow u = 1 \quad x = \ln 2 \Rightarrow u = 2$			
	$\frac{1}{2}\tan^{-1}1 - \frac{1}{2}\tan^{-1}\frac{1}{2}$	A1√		FT "their" $\frac{1}{\sqrt{m}}\left(\tan^{-1}\frac{2}{\sqrt{m}} - \tan^{-1}\frac{1}{\sqrt{m}}\right)$
	$= \frac{\pi}{8} - \frac{1}{2}\tan^{-1}\frac{1}{2}$	A1cso	5	**AG**
	Total		**8**	

7(a)(i)	$\frac{\mathrm{d}}{\mathrm{d}u}\left(2u\sqrt{1+4u^2}\right)=\frac{8u^2}{\sqrt{1+4u^2}}+2\sqrt{1+4u^2}$	M1 A1		M1 for clear use of product rule (condone one error in one term) correct unsimplified
	$\frac{\mathrm{d}}{\mathrm{d}u}\left(\sinh^{-1}2u\right)=\frac{2}{\sqrt{1+4u^2}}$	B1		
	$\frac{8u^2+2}{\sqrt{1+4u^2}}=\frac{2(1+4u^2)}{\sqrt{1+4u^2}}=2\sqrt{1+4u^2}$			be convinced – must see this line OE
	$\frac{\mathrm{d}}{\mathrm{d}u}\left(2u\sqrt{1+4u^2}+4\sinh^{-1}2u\right)=4\sqrt{1+4u^2}$	A1cso	4	all working must be correct (not enough to just say $k=4$)
(ii)	$\frac{1}{\text{“their”}k}\left[2u\sqrt{1+4u^2}+\sinh^{-1}2u\right]_0^1$	M1		anti differentiation
	$=\frac{\sqrt{5}}{2}+\frac{1}{4}\sinh^{-1}2$	A1√	2	FT “their” k or even use of k
(b)(i)	$y=\frac{1}{2}\cos 4x$ and $\frac{\mathrm{d}y}{\mathrm{d}x}=A\sin 4x$			$\frac{\mathrm{d}y}{\mathrm{d}x}=-2\sin 4x$
	substituted into $\int K\, y\left(1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right)(\mathrm{d}x)$	M1		clear attempt to use formula for CSA
	$(S=)\ \int_0^{\frac{\pi}{8}} 2\pi\times\frac{1}{2}\cos 4x\sqrt{1+4\sin^2 4x}\,\mathrm{d}x$ = printed answer (combining $2\times\frac{1}{2}$)	A1cso	2	**AG** $\frac{\mathrm{d}y}{\mathrm{d}x}=-2\sin 4x$ **and** $2\times\frac{1}{2}$ **and** $\mathrm{d}x$ must be seen to award A1cso
(ii)	$u=\sin 4x\Rightarrow \mathrm{d}u=4\cos 4x\,\mathrm{d}x$	M1		condone $\mathrm{d}u=B\cos 4x\,\mathrm{d}x$ for M1
	$(S=)\ \frac{\pi}{4}\int_0^1\sqrt{1+4u^2}\,(\mathrm{d}u)$	A1		condone limits seen later
		m1		use of their result from (a)(ii) **correctly** FT “their” B
	$(S=)\ \frac{\pi\sqrt{5}}{8}+\frac{\pi}{16}\sinh^{-1}2$	A1cso	4	OE
	Total		**12**	

Q	Solution	Mark	Total	Comment
5(a)	$\left(e^{\theta}-e^{-\theta}\right)^3 = e^{3\theta}-3e^{\theta}+3e^{-\theta}-e^{-3\theta}$ OE	B1		correct expansion; terms need not be combined
	$4\sinh^3\theta+3\sinh\theta=$ $\frac{4}{8}\left(e^{3\theta}-3e^{\theta}+3e^{-\theta}-e^{-3\theta}\right)+\frac{1}{2}\left(3e^{\theta}-3e^{-\theta}\right)$	M1		correct expression for $\sinh\theta$ and attempt to expand $\left(e^{\theta}-e^{-\theta}\right)^3$
	$=\frac{1}{2}\left(e^{3\theta}-e^{-3\theta}\right)=\sinh 3\theta$	A1	3	**AG** identity proved
(b)	$16\sinh^3\theta+12\sinh\theta-3=0$ $\Rightarrow 4\sinh 3\theta-3=0$	M1		attempt to use previous result
	$\sinh 3\theta=\frac{3}{4}$	A1		
	$(3\theta=)\ln\left(\frac{3}{4}+\sqrt{\frac{9}{16}+1}\right)$	m1		correct ln form of $\sinh^{-1}$ for "their" $\frac{3}{4}$
	$\theta=\frac{1}{3}\ln 2$	A1	4	
(c)	$x=\sinh\theta=\frac{1}{2}\left(2^{\frac{1}{3}}-2^{-\frac{1}{3}}\right)$	M1		correctly substituting their expression for θ into $\sinh\theta$ removing any ln terms
	$2^{-\frac{2}{3}}-2^{-\frac{4}{3}}$	A1	2	
	Total		9	

Q2	Solution	Mark	Total	Comment
(a)				
	Graph roughly correct through O	**M1**		condone infinite gradient at O for **M1**
	Correct behaviour as $x \to \pm\infty$ & grad at O	**A1**		
	Asymptotes have equations $y = 1$ & $y = -1$	**B1**	**3**	must state equations
(b)	$\operatorname{sech} x = \dfrac{2}{e^x + e^{-x}}$; $\tanh x = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$	**B1**		**both** correct **ACF** or **correct** squares of these expressions seen
	$(\operatorname{sech}^2 x + \tanh^2 x =)\ \dfrac{2^2 + \left(e^x - e^{-x}\right)^2}{\left(e^x + e^{-x}\right)^2}$	**M1**		attempt to combine their squared terms with correct single denominator
	$\operatorname{sech}^2 x + \tanh^2 x = \dfrac{e^{2x} + 2 + e^{-2x}}{e^{2x} + 2 + e^{-2x}} = 1$	**A1**	**3**	**AG** valid proof convincingly shown to equal 1 **including LHS seen**
(c)	$6(1 - \tanh^2 x) = 4 + \tanh x$	**B1**		correct use of identity from part **(b)**
	$6\tanh^2 x + \tanh x - 2 \quad (= 0)$	**M1**		forming quadratic in $\tanh x$
	$\tanh x = \dfrac{1}{2}$, $\tanh x = -\dfrac{2}{3}$	**A1**		obtained from correct quadratic
	$\tanh x = k \Rightarrow x = \dfrac{1}{2}\ln\left(\dfrac{1+k}{1-k}\right)$	**A1F**		FT a value of k provided $\lvert k \rvert < 1$
	$x = \dfrac{1}{2}\ln 3$, $x = \dfrac{1}{2}\ln\dfrac{1}{5}$	**A1**	**5**	both solutions correct and no others any equivalent form involving ln
	Total		**11**	

Q3	Solution	Mark	Total	Comment
(a)	$\frac{dy}{dx}=\frac{2x}{(1-x^2)}$	**B1**		
	$1+\left(\frac{dy}{dx}\right)^2=1+\frac{(2x)^2}{(1-x^2)^2}$	**M1**		FT their $\frac{dy}{dx}$
	$\frac{1-2x^2+x^4+4x^2}{(1-x^2)^2}=\frac{(1+x^2)^2}{(1-x^2)^2}$	**m1**		Allow **m1** if sign error in $\frac{dy}{dx}$
	$s=\int\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$			
	$s=\int_0^{\frac{3}{4}}\left(\frac{1+x^2}{1-x^2}\right)dx$	**A1cso**	**4**	**AG** must have dx and limits on final line
(b)	$\frac{1+x^2}{1-x^2}=\frac{A}{1-x^2}+B$	**M1**		and attempt to find constants $B\neq 0$
	$\frac{1+x^2}{1-x^2}=\frac{2}{1-x^2}-1$	**A1**		
	$\left(\frac{A}{2}\ln\left(\frac{1+x}{1-x}\right)\text{ or } A\tanh^{-1}x\right)+Bx$	**m1**		**FT** integral of their $\frac{A}{1-x^2}+B$
	$\ln\left(\frac{1+x}{1-x}\right)-x$	**A1**		**or** $2\tanh^{-1}x-x$ correct
	$\ln\left(\frac{1+\frac{3}{4}}{1-\frac{3}{4}}\right)-\frac{3}{4}$ **OE**	**A1**		**PI** by next **A1** **or** $(s=)2\tanh^{-1}\left(\frac{3}{4}\right)-\frac{3}{4}$
	$-\frac{3}{4}+\ln 7$	**A1**	**6**	or $(s)=\ln 7-\frac{3}{4}$
	Alternative			
	$\frac{1+x^2}{1-x^2}=\frac{C}{1+x}+\frac{D}{1-x}+E$	**(M1)**		and attempt to find constants $E\neq 0$
	$\frac{1+x^2}{1-x^2}=\frac{1}{1+x}+\frac{1}{1-x}-1$	**(A1)**		
	$C\ln(1+x)-D\ln(1-x)+Ex$	**(m1)**		**FT** integral of their $\frac{C}{1+x}+\frac{D}{1-x}+E$
	$=\ln(1+x)-\ln(1-x)-x$	**(A1)**		correct
	$(s=)\ \ln\frac{7}{4}-\ln\frac{1}{4}-\frac{3}{4}$ **OE**	**(A1)**		correct unsimplified
	$(s)=\ln 7-\frac{3}{4}$	**(A1)**	**(6)**	
	Total		**10**	

Q6	Solution	Mark	Total	Comment
(a)	$y=\frac{1}{2}\left(e^x-e^{-x}\right)$			allow $e^{2x}-2ye^x=1$ for **M1** if attempting to complete square
	$\Rightarrow e^{2x}-2ye^x-1\quad(=0)$	M1		terms all on one side
	$\left(e^x=\right)\ \frac{2y\pm\sqrt{4y^2+4}}{2}$	A1		or $e^x-y=\pm\sqrt{y^2+1}$ after completing square
	$e^x>0$ so reject negative root	E1		any correct explanation for rejection
	$e^x=y+\sqrt{y^2+1}\Rightarrow x=\ln\left(y+\sqrt{y^2+1}\right)$	A1	4	**AG** must earn previous **A1**
(b)(i)	$\frac{dy}{dx}=6\times 2\cosh x\sinh x$ (**not** 6 sinh2x)	B1		directly or via $3\cosh 2x+3$
	$+5\cosh x$	B1		
	$\cosh x=0$ gives no solution	E1		Not simply cancelling $\cosh x$
	(only stationary point when) $\sinh x=-\frac{5}{12}$	M1		**FT** "their" $\sinh x$ from equation of form $A\cosh x\sinh x+B\cosh x$
	$x=\ln\left(-\frac{5}{12}+\sqrt{1+\frac{25}{144}}\right)$			**or M1** for using exponentials obtaining $e^x=\frac{2}{3}$ or $-\frac{3}{2}$ OE
	$=\ln\left(\frac{2}{3}\right)$	A1	5	accept $\ln\left(\frac{8}{12}\right)$ **OE**
(ii)	$Area=\int_0^{\cosh^{-1}2}\left(6\cosh^2 x+5\sinh x\right)dx$			
	$6\cosh^2 x=3+3\cosh 2x$	B1		or $6\cosh^2 x=\frac{3}{2}\left(e^{2x}+2+e^{-2x}\right)$
	$Ax+B\sinh 2x$ or $Cx+D(e^{2x}-e^{-2x})$	M1		correct **FT** "their" $\int 6\cosh^2 x\,dx$
	$3x+\frac{3}{2}\sinh 2x+5\cosh x$	A1		integration all correct (may be in e^x form)
	$3\cosh^{-1}2+\frac{3}{2}\sinh(2\cosh^{-1}2)+10\quad -5$	m1		$F(\cosh^{-1}2)-F(0)$ correct substitution of limits into **their** expression
	$(Area=)3\cosh^{-1}2+6\sqrt{3}+5$	A1	5	
	Total		**14**	