FP2 De Moivre's Theorem

8 (a) (i) Expand

$$\left(z+\frac{1}{z}\right)\left(z-\frac{1}{z}\right)$$ *(1 mark)*

(ii) Hence, or otherwise, expand

$$\left(z+\frac{1}{z}\right)^4\left(z-\frac{1}{z}\right)^2$$ *(3 marks)*

(b) (i) Use De Moivre's theorem to show that if $z = \cos\theta + \mathrm{i}\sin\theta$ then

$$z^n + \frac{1}{z^n} = 2\cos n\theta$$ *(3 marks)*

(ii) Write down a corresponding result for $z^n - \dfrac{1}{z^n}$. *(1 mark)*

(c) Hence express $\cos^4\theta\sin^2\theta$ in the form

$$A\cos 6\theta + B\cos 4\theta + C\cos 2\theta + D$$

where A, B, C and D are rational numbers. *(4 marks)*

(d) Find $\int \cos^4\theta\sin^2\theta\ \mathrm{d}\theta$. *(2 marks)*

1 Given that $z = 2\mathrm{e}^{\frac{\pi\mathrm{i}}{12}}$ satisfies the equation

$$z^4 = a(1+\sqrt{3}\,\mathrm{i})$$

where a is real:

(a) find the value of a; *(3 marks)*

(b) find the other three roots of this equation, giving your answers in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r > 0$ and $-\pi < \theta \leqslant \pi$. *(5 marks)*

5 (a) Prove by induction that, if n is a positive integer,

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$ *(5 marks)*

(b) Hence, given that

$$z = \cos\theta + i\sin\theta$$

show that

$$z^n + \frac{1}{z^n} = 2\cos n\theta$$ *(3 marks)*

(c) Given further that $z + \frac{1}{z} = \sqrt{2}$, find the value of

$$z^{10} + \frac{1}{z^{10}}$$ *(4 marks)*

7 (a) (i) Express each of the numbers $1 + \sqrt{3}\,i$ and $1 - i$ in the form $re^{i\theta}$, where $r > 0$.
(3 marks)

(ii) Hence express

$$(1 + \sqrt{3}\,i)^8(1 - i)^5$$

in the form $re^{i\theta}$, where $r > 0$. *(3 marks)*

(b) Solve the equation

$$z^3 = (1 + \sqrt{3}\,i)^8(1 - i)^5$$

giving your answers in the form $a\sqrt{2}\,e^{i\theta}$, where a is a positive integer and $-\pi < \theta \leqslant \pi$. *(4 marks)*

8 (a) Express in the form $re^{i\theta}$, where $r > 0$ and $-\pi < \theta \leqslant \pi$:

(i) $4(1 + i\sqrt{3})$;

(ii) $4(1 - i\sqrt{3})$. *(3 marks)*

(b) The complex number z satisfies the equation

$$(z^3 - 4)^2 = -48$$

Show that $z^3 = 4 \pm 4\sqrt{3}\,i$. *(2 marks)*

(c) (i) Solve the equation

$$(z^3 - 4)^2 = -48$$

giving your answers in the form $re^{i\theta}$, where $r > 0$ and $-\pi < \theta \leqslant \pi$. *(5 marks)*

(ii) Illustrate the roots on an Argand diagram. *(3 marks)*

(d) (i) Explain why the sum of the roots of the equation

$$(z^3 - 4)^2 = -48$$

is zero. *(1 mark)*

(ii) Deduce that $\cos\frac{\pi}{9} + \cos\frac{3\pi}{9} + \cos\frac{5\pi}{9} + \cos\frac{7\pi}{9} = \frac{1}{2}$. *(3 marks)*

---

7 (a) (i) Use de Moivre's Theorem to show that

$$\cos 5\theta = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$$

and find a similar expression for $\sin 5\theta$. *(5 marks)*

(ii) Deduce that

$$\tan 5\theta = \frac{\tan\theta(5 - 10\tan^2\theta + \tan^4\theta)}{1 - 10\tan^2\theta + 5\tan^4\theta}$$ *(3 marks)*

(b) Explain why $t = \tan\frac{\pi}{5}$ is a root of the equation

$$t^4 - 10t^2 + 5 = 0$$

and write down the three other roots of this equation in trigonometrical form. *(3 marks)*

(c) Deduce that

$$\tan\frac{\pi}{5}\tan\frac{2\pi}{5} = \sqrt{5}$$ *(5 marks)*

5 Find the smallest positive integer values of p and q for which

$$\frac{\left(\cos\frac{\pi}{8} + \mathrm{i}\sin\frac{\pi}{8}\right)^p}{\left(\cos\frac{\pi}{12} - \mathrm{i}\sin\frac{\pi}{12}\right)^q} = \mathrm{i}$$ *(7 marks)*

8 (a) Write down the five roots of the equation $z^5 = 1$, giving your answers in the form $\mathrm{e}^{\mathrm{i}\theta}$, where $-\pi < \theta \leqslant \pi$. *(1 mark)*

(b) Hence find the four linear factors of

$$z^4 + z^3 + z^2 + z + 1$$ *(3 marks)*

(c) Deduce that

$$z^2 + z + 1 + z^{-1} + z^{-2} = \left(z - 2\cos\frac{2\pi}{5} + z^{-1}\right)\left(z - 2\cos\frac{4\pi}{5} + z^{-1}\right)$$ *(4 marks)*

(d) Use the substitution $z + z^{-1} = w$ to show that $\cos\frac{2\pi}{5} = \frac{\sqrt{5}-1}{4}$. *(6 marks)*

8 (a) Use De Moivre's Theorem to show that, if $z = \cos\theta + \mathrm{i}\sin\theta$, then

$$z^n + \frac{1}{z^n} = 2\cos n\theta$$ *(3 marks)*

(b) (i) Expand $\left(z^2 + \frac{1}{z^2}\right)^4$. *(1 mark)*

(ii) Show that

$$\cos^4 2\theta = A\cos 8\theta + B\cos 4\theta + C$$

where A, B and C are rational numbers. *(4 marks)*

(c) Hence solve the equation

$$8\cos^4 2\theta = \cos 8\theta + 5$$

for $0 \leqslant \theta \leqslant \pi$, giving each solution in the form $k\pi$. *(3 marks)*

(d) Show that

$$\int_0^{\frac{\pi}{2}} \cos^4 2\theta \, \mathrm{d}\theta = \frac{3\pi}{16}$$ *(3 marks)*

8 (a) Express $-4+4\sqrt{3}\,\mathrm{i}$ in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r>0$ and $-\pi<\theta\leqslant\pi$. *(3 marks)*

(b) (i) Solve the equation $z^3=-4+4\sqrt{3}\,\mathrm{i}$, giving your answers in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r>0$ and $-\pi<\theta\leqslant\pi$. *(4 marks)*

(ii) The roots of the equation $z^3=-4+4\sqrt{3}\,\mathrm{i}$ are represented by the points P, Q and R on an Argand diagram.

Find the area of the triangle PQR, giving your answer in the form $k\sqrt{3}$, where k is an integer. *(3 marks)*

(c) By considering the roots of the equation $z^3=-4+4\sqrt{3}\,\mathrm{i}$, show that

$$\cos\frac{2\pi}{9}+\cos\frac{4\pi}{9}+\cos\frac{8\pi}{9}=0$$ *(4 marks)*

---

8 (a) (i) Use de Moivre's theorem to show that

$$\cos 4\theta=\cos^4\theta-6\cos^2\theta\sin^2\theta+\sin^4\theta$$

and find a similar expression for $\sin 4\theta$. *(5 marks)*

(ii) Deduce that

$$\tan 4\theta=\frac{4\tan\theta-4\tan^3\theta}{1-6\tan^2\theta+\tan^4\theta}$$ *(3 marks)*

(b) Explain why $t=\tan\frac{\pi}{16}$ is a root of the equation

$$t^4+4t^3-6t^2-4t+1=0$$

and write down the three other roots in trigonometric form. *(4 marks)*

(c) Hence show that

$$\tan^2\frac{\pi}{16}+\tan^2\frac{3\pi}{16}+\tan^2\frac{5\pi}{16}+\tan^2\frac{7\pi}{16}=28$$ *(5 marks)*

QUESTION | Answer space for question 8

---

1 (a) Express $-9\mathrm{i}$ in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r>0$ and $-\pi<\theta\leqslant\pi$.

[2 marks]

(b) Solve the equation $z^4+9\mathrm{i}=0$, giving your answers in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r>0$ and $-\pi<\theta\leqslant\pi$.

[5 marks]

6 (a) (i) Use De Moivre's Theorem to show that if $z = \cos\theta + \mathrm{i}\sin\theta$, then

$$z^n - \frac{1}{z^n} = 2\mathrm{i}\sin n\theta$$

[3 marks]

(ii) Write down a similar expression for $z^n + \frac{1}{z^n}$.

[1 mark]

(b) (i) Expand $\left(z - \frac{1}{z}\right)^2 \left(z + \frac{1}{z}\right)^2$ in terms of z.

[1 mark]

(ii) Hence show that

$$8\sin^2\theta\cos^2\theta = A + B\cos 4\theta$$

where A and B are integers.

[2 marks]

(c) Hence, by means of the substitution $x = 2\sin\theta$, find the exact value of

$$\int_1^2 x^2\sqrt{4 - x^2}\,\mathrm{d}x$$

[5 marks]

8 The complex number ω is given by $\omega = \cos\frac{2\pi}{5} + \mathrm{i}\sin\frac{2\pi}{5}$.

(a) (i) Verify that ω is a root of the equation $z^5 = 1$.

[1 mark]

(ii) Write down the three other non-real roots of $z^5 = 1$, in terms of ω.

[1 mark]

(b) (i) Show that $1 + \omega + \omega^2 + \omega^3 + \omega^4 = 0$.

[1 mark]

(ii) Hence show that $\left(\omega + \frac{1}{\omega}\right)^2 + \left(\omega + \frac{1}{\omega}\right) - 1 = 0$.

[2 marks]

(c) Hence show that $\cos\frac{2\pi}{5} = \frac{\sqrt{5} - 1}{4}$.

[4 marks]

8 (a) By applying de Moivre's theorem to $(\cos\theta + \mathrm{i}\sin\theta)^4$, where $\cos\theta \neq 0$, show that

$$(1 + \mathrm{i}\tan\theta)^4 + (1 - \mathrm{i}\tan\theta)^4 = \frac{2\cos 4\theta}{\cos^4\theta}$$

[3 marks]

(b) Hence show that $z = \mathrm{i}\tan\frac{\pi}{8}$ satisfies the equation $(1+z)^4 + (1-z)^4 = 0$, and express the three other roots of this equation in the form $\mathrm{i}\tan\phi$, where $0 < \phi < \pi$.

[2 marks]

(c) Use the results from part **(b)** to find the values of:

(i) $\tan^2\frac{\pi}{8}\tan^2\frac{3\pi}{8}$;

[4 marks]

(ii) $\tan^2\frac{\pi}{8} + \tan^2\frac{3\pi}{8}$.

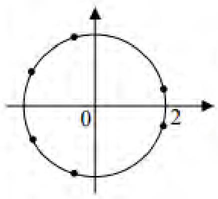
[4 marks]

8(a)(i)	$\left(z+\frac{1}{z}\right)\left(z-\frac{1}{z}\right)=z^2-\frac{1}{z^2}$	B1	1	
(ii)	$\left(z^2-\frac{1}{z^2}\right)^2\left(z+\frac{1}{z}\right)^2$			
	$=\left(z^4-2+\frac{1}{z^4}\right)\left(z^2+2+\frac{1}{z^2}\right)$	M1A1		Alternatives for M1A1: $\left(z^4+4z^2+6+\frac{4}{z^2}+\frac{1}{z^4}\right)\left(z^2-2+\frac{1}{z^2}\right)$ or $\left(z^3-\frac{1}{z^3}\right)^2-2\left(z^3-\frac{1}{z^3}\right)\left(z-\frac{1}{z}\right)+\left(z-\frac{1}{z}\right)^2$
	$=z^6+\frac{1}{z^6}+2\left(z^4+\frac{1}{z^4}\right)-\left(z^2+\frac{1}{z^2}\right)-4$	A1	3	CAO (not necessarily in this form)
(b)(i)	$z^n+\frac{1}{z^n}=\cos n\theta+\mathrm{i}\sin n\theta$ $+\cos(-n\theta)+\mathrm{i}\sin(-n\theta)$	M1A1		
	$=2\cos n\theta$	A1	3	AG SC: if solution is incomplete and $(\cos\theta+\mathrm{i}\sin\theta)^{-n}$ is written as $\cos n\theta-\mathrm{i}\sin n\theta$, award M1A0A1
(ii)	$z^n-z^{-n}=2\mathrm{i}\sin n\theta$	B1	1	
(c)	RHS $=2\cos 6\theta+4\cos 4\theta-2\cos 2\theta-4$	M1 A1F		ft incorrect values in (a)(ii) provided they are cosines
	LHS $=-64\cos^4\theta\sin^2\theta$	M1		
	$\cos^4\theta\sin^2\theta$ $=-\frac{1}{32}\cos 6\theta-\frac{1}{16}\cos 4\theta+\frac{1}{32}\cos 2\theta+\frac{1}{16}$	A1	4	
(d)	$-\frac{\sin 6\theta}{192}-\frac{\sin 4\theta}{64}+\frac{\sin 2\theta}{64}+\frac{\theta}{16}\ (+k)$	M1 A1F	2	ft incorrect coefficients but not letters *A*, *B*, *C*, *D*
	Total		**14**	

1(a)	$z^4=16\mathrm{e}^{\frac{4\pi\mathrm{i}}{12}}$	M1		Allow M1 if $z^4=2\mathrm{e}^{\frac{4\pi\mathrm{i}}{12}}$
	$=16\left(\cos\frac{\pi}{3}+\mathrm{i}\sin\frac{\pi}{3}\right)$	A1		OE could be $2a\mathrm{e}^{\frac{\pi\mathrm{i}}{3}}$ or $2a\left(\cos\frac{\pi}{3}+\mathrm{i}\sin\frac{\pi}{3}\right)$
	$=8+8\sqrt{3}\mathrm{i}$; $a=8$	A1F	3	ft errors in 2^4
(b)	For other roots, $r=2$	B1		for realising roots are of form $2\times\mathrm{e}^{\mathrm{i}\theta}$
	$\theta=\frac{\pi}{12}+\frac{2k\pi}{4}$	M1A1		M1 for strictly correct θ i.e must be $\left(\text{their }\frac{\pi}{3}+2k\pi\right)\times\frac{1}{4}$
	Roots are $2\mathrm{e}^{\frac{7\pi\mathrm{i}}{12}}, 2\mathrm{e}^{\frac{-5\pi\mathrm{i}}{12}}, 2\mathrm{e}^{\frac{-11\pi\mathrm{i}}{12}}$	A2,1,0 F	5	ft error in $\frac{\pi}{12}$ or r $\left[\text{accept } 2\mathrm{e}^{\left(\frac{\pi}{12}+\frac{2k\pi}{4}\right)\mathrm{i}}\quad k=-1,-2,1\right]$
	Total		**8**	

5(a)	$(\cos\theta + \mathrm{i}\sin\theta)^{k+1} =$			
	$(\cos k\theta + \mathrm{i}\sin k\theta)(\cos\theta + \mathrm{i}\sin\theta)$	M1		
	Multiply out	A1		Any form
	$= \cos(k+1)\theta + \mathrm{i}\sin(k+1)\theta$	A1		Clearly shown
	True for $n = 1$ shown	B1		
	$P(k) \Rightarrow P(k+1)$ and $P(1)$ true	E1	5	provided previous 4 marks earned
(b)	$\dfrac{1}{z^n} = \dfrac{1}{\cos n\theta + \mathrm{i}\sin n\theta} = \cos n\theta - \mathrm{i}\sin n\theta$	M1A1		or $z^{-n} = \cos(-n\theta) + \mathrm{i}\sin(-n\theta)$ SC $(\cos\theta + \mathrm{i}\sin\theta)^{-n}$ quoted as $\cos n\theta - \mathrm{i}\sin n\theta$ earns M1A1 only
	$z^n + \dfrac{1}{z^n} = 2\cos n\theta$	A1	3	AG
(c)	$z + \dfrac{1}{z} = \sqrt{2}$			
	$2\cos\theta = \sqrt{2}$	M1		
	$\theta = \dfrac{\pi}{4}$	A1		
	$z^{10} + \dfrac{1}{z^{10}} = 2\cos\left(\dfrac{10\pi}{4}\right)$	M1		M0 for merely writing $z^{10} + \dfrac{1}{z^{10}} = 2\cos 10\theta$
	$= 0$	A1F	4	
	Total		**12**	

Q	Solution	Marks	Total	Comments
7(a)(i)	$1 + \sqrt{3}\mathrm{i} = 2\mathrm{e}^{\frac{\pi \mathrm{i}}{3}}$	B1		B1 both correct
	$1 - \mathrm{i} = \sqrt{2}\mathrm{e}^{-\frac{\pi \mathrm{i}}{4}}$	B1B1	3	OE
(ii)	$2^{\frac{21}{2}}$ or equivalent single expression	B1F		No decimals; must include fractional powers
	Raising and adding powers of e	M1		
	$\dfrac{17\pi}{12}$ or equivalent angle	A1F	3	Denominators of angles must be different
(b)	$z = \sqrt[3]{2^{10}\sqrt{2}}\,\mathrm{e}^{\frac{17\pi \mathrm{i}}{36} + \frac{2k\pi \mathrm{i}}{3}}$	M1		
	$\sqrt[3]{2^{10}\sqrt{2}} = 8\sqrt{2}$	B1		CAO
	$\theta = \dfrac{17\pi}{36}, -\dfrac{7\pi}{36}, -\dfrac{31\pi}{36}$	A2,1F	4	Correct answers outside range: deduct 1 mark only
	Total		**10**	

8(a)(i)	$4\left(1+\mathrm{i}\sqrt{3}\right)=8\left(\frac{1}{2}+\mathrm{i}\frac{\sqrt{3}}{2}\right)$	M1		for either $4\left(1+\mathrm{i}\sqrt{3}\right)$ or $4\left(1-\mathrm{i}\sqrt{3}\right)$ used
	$=8\mathrm{e}^{\frac{\pi\mathrm{i}}{3}}$	A1		If either r or θ is incorrect but the same value in both (i) and (ii) allow A1 but for θ only if it is given as $\frac{\pi}{6}$
(ii)	$4\left(1-\mathrm{i}\sqrt{3}\right)=8\mathrm{e}^{\frac{-\pi\mathrm{i}}{3}}$	A1	3	
(b)	$z^3-4=\pm\sqrt{-48}$	M1		taking square root
	$z^3=4\pm4\sqrt{3}\,\mathrm{i}$	A1	2	AG
(c)(i)	$z=2\mathrm{e}^{\frac{\frac{\pi\mathrm{i}}{3}+2k\pi\mathrm{i}}{3}}$ or $z=2\mathrm{e}^{\frac{\frac{-\pi\mathrm{i}}{3}+2k\pi\mathrm{i}}{3}}$	B1F M1		for the 2; ft incorrect 8, but no decimals for either, PI
	$z=2\mathrm{e}^{\frac{\pi\mathrm{i}}{9}},2\mathrm{e}^{\frac{7\pi\mathrm{i}}{9}},2\mathrm{e}^{\frac{5\pi\mathrm{i}}{9}}$ $=2\mathrm{e}^{\frac{-\pi\mathrm{i}}{9}},2\mathrm{e}^{\frac{-7\pi\mathrm{i}}{9}},2\mathrm{e}^{\frac{-5\pi\mathrm{i}}{9}}$	A3,2,1F	5	Allow A1 for any 2 roots not +/– each other Allow A2 for any 3 roots not +/– each other Allow A3 for all 6 correct roots Deduct A1 for each incorrect root in the interval; ignore roots outside the interval ft incorrect r
(ii)	 Radius 2	B1F		clearly indicated; ft incorrect r allow B1 for 3 correct points condone lines
	Plotting roots	B2,1	3	
(d)(i)	Sum of roots $=0$ as coefficient of $z^5=0$	E1	1	OE
(ii)	Use of, say, $\frac{1}{2}\left(\mathrm{e}^{\frac{\pi\mathrm{i}}{9}}+\mathrm{e}^{\frac{-\pi\mathrm{i}}{9}}\right)=\cos\frac{\pi}{9}$	M1		
	$\cos\frac{3\pi}{9}=\frac{1}{2}$ used	A1		
	$\cos\frac{\pi}{9}+\cos\frac{3\pi}{9}+\cos\frac{5\pi}{9}+\cos\frac{7\pi}{9}=\frac{1}{2}$	A1	3	AG

7(a)(i)	$\cos 5\theta + i\sin 5\theta = (\cos\theta + i\sin\theta)^5$	M1		Attempt to expand 3 correct terms
	Expansion in any form	A1		Correct simplification
	Equate real parts:	m1		
	$\cos 5\theta = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$	A1		AG
	Equate imaginary parts:			
	$\sin 5\theta = 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta$	A1	5	CAO
(ii)	$\tan 5\theta = \dfrac{\sin 5\theta}{\cos 5\theta}$	M1		Used
	Division by $\cos^5\theta$ or by $\cos^4\theta$	m1		
	$\tan 5\theta = \dfrac{\tan\theta(5 - 10\tan^2\theta + \tan^4\theta)}{1 - 10\tan^2\theta + 5\tan^4\theta}$	A1	3	AG
(b)	$\theta = \frac{\pi}{5} \Rightarrow \tan 5\theta = 0$	M1		Or for $\tan^4\theta - 10\tan^2\theta + 5 = 0$
	$\therefore \tan\frac{\pi}{5}$ satisfies $t^4 - 10t^2 + 5 = 0$	A1		Or for $\tan 5\theta = 0$
	Other roots $\tan\frac{k\pi}{5}$ $k = 2, 3, 4$	B1	3	OE
(c)	Product of roots = 5	M1		
	$\tan\frac{\pi}{5} = -\tan\frac{4\pi}{5}$	B1		Or $\tan\frac{2\pi}{5} = -\tan\frac{3\pi}{5}$
	$\tan^2\frac{\pi}{5}\tan^2\frac{2\pi}{5} = 5$	A1		
	$\tan\frac{\pi}{5}\tan\frac{2\pi}{5} = +\sqrt{5}$	A1		
	– sign rejected with reason	E1	5	**Alternative (c)** Use of quadratic formula M1 $t^2 = 5 \pm 2\sqrt{5}$ A1 $t = \pm\sqrt{5 \pm 2\sqrt{5}}$ B1 Correct selection of +ve values E1 Multiplied together to get $\sqrt{5}$ A1
	Total		**16**	

5	Numerator $=e^{\frac{p\pi i}{8}}$	B1		
	Denominator $=e^{\frac{-q\pi i}{12}}$	B1		
	Fraction $=e^{\frac{p\pi i}{8}+\frac{q\pi i}{12}}$	M1		allow for attempt to subtract powers
	$=e^{\frac{\pi i}{24}(3p+2q)}$	A1		
	$i=e^{\frac{12\pi i}{24}}$	m1		OE
	$3p+2q=12$	A1F		ft errors of sign or arithmetic slips
	$p=2,\ q=3$	A1	7	CAO
	Alternative 1			
	Numerator $=\cos\frac{p\pi}{8}+i\sin\frac{p\pi}{8}$	(B1)		
	Denominator $=\cos\frac{-q\pi}{12}+i\sin\frac{-q\pi}{12}$	(B1)		needs more than just $\cos\frac{q\pi}{12}-\sin\frac{p\pi}{12}$
	Fraction $=\left(\cos\frac{p\pi}{8}+i\sin\frac{p\pi}{8}\right)\left(\cos\frac{q\pi}{12}+i\sin\frac{q\pi}{12}\right)$	(M1)		
	$=\cos\frac{\pi}{24}(3p+2q)+i\sin\frac{\pi}{24}(3p+2q)$	(A1)		
	$=i$ if $\cos\frac{\pi}{24}(3p+2q)=0$ or $\sin\frac{\pi}{24}(3p+2q)=1$	(m1)		
	$3p+2q=12$	(A1F)		
	$p=2,\ q=3$	(A1)	(7)	CAO
	Alternative 2			
	LHS $\cos\frac{p\pi}{8}+i\sin\frac{p\pi}{8}$	(B1)		
	RHS $i\cos\frac{q\pi}{12}+\sin\frac{q\pi}{12}$	(B1)		
	$\cos\frac{p\pi}{8}=\sin\frac{q\pi}{12}$ or $\sin\frac{p\pi}{8}=\cos\frac{q\pi}{12}$	(M1)		
	Introduction of $\frac{\pi}{2}$	(m1)		
	$\frac{p\pi}{8}=\frac{\pi}{2}-\frac{q\pi}{12}$	(A1)		
	$3p+2q=12$	(A1F)		
	$p=2,\ q=3$	(A1)	(7)	CAO (correct answers, insufficient working 3/7 only)

Q	Solution	Marks	Total	Comments
8(a)	$1,\ e^{\frac{2\pi i}{5}},\ e^{\frac{4\pi i}{5}},\ e^{\frac{-2\pi i}{5}},\ e^{\frac{-4\pi i}{5}}$	B1	1	accept e^0
(b)	$\frac{z^5-1}{z-1}=z^4+z^3+z^2+z+1$	B1		B0 if assumed
	$=\left(z-e^{\frac{2\pi i}{5}}\right)\left(z-e^{\frac{4\pi i}{5}}\right)\left(z-e^{\frac{-2\pi i}{5}}\right)\left(z-e^{\frac{-4\pi i}{5}}\right)$	M1A1	3	accept if $e^{\frac{6\pi i}{5}},\ e^{\frac{8\pi i}{5}}$ used here
(c)	Correct grouping of linear factors	M1		
	$e^{\frac{2\pi i}{5}}+e^{\frac{-2\pi i}{5}}=2\cos\frac{2\pi}{5}$	A1		clearly shown
	$\left(z^2-2\cos\frac{2\pi}{5}z+1\right)\left(z^2-2\cos\frac{4\pi}{5}z+1\right)$	A1		
	$\div z^2$ to give answer	A1	4	AG
(d)	Substitute into LHS to give w^2+w-1	B1		
	RHS $\left(w-2\cos\frac{2\pi}{5}\right)\left(w-2\cos\frac{4\pi}{5}\right)$	B1		
	Solve $w^2+w-1=0$	M1		
	$w=\frac{-1\pm\sqrt{5}}{2}$	A1		
	$\cos\frac{2\pi}{5}=\frac{\sqrt{5}-1}{4}$	A1		
	with reasons for choice	E1	6	
	Total		**14**	

8(a)	Use of $(\cos\theta + \text{i}\sin\theta)^n = \cos n\theta + \text{i}\sin n\theta$	M1		Stated or used
	$\cos(-n\theta) + \text{i}\sin(-n\theta) = \cos n\theta - \text{i}\sin n\theta$	A1		allow $\frac{2}{3}$ if this line is assumed allow if complex conjugate used
	$z^n + \frac{1}{z^n} = 2\cos n\theta$	A1	3	AG
(b)(i)	$z^8 + 4z^4 + 6 + 4z^{-4} + z^{-8}$	B1	1	allow in retrospect
(ii)	$z^2 + \frac{1}{z^2} = 2\cos 2\theta$ used	B1		Can be implied from (b)(i)
	$(2\cos 2\theta)^4 = 2\cos 8\theta + 8\cos 4\theta + 6$	M1A1		M1 for RHS A1 for whole line
	$\cos^4 2\theta = \frac{1}{8}\cos 8\theta + \frac{1}{2}\cos 4\theta + \frac{3}{8}$	A1F	4	ft coefficients on previous line
	Alternative to (b)(ii)			
	$\cos^4 2\theta = \left(\frac{1+\cos 4\theta}{2}\right)^2$	(M1) (A1)		
	$\cos^2 4\theta = \frac{1}{2}(1+\cos 8\theta)$	(B1)		
	Final result	(A1)		
(c)	$8\cos^4 2\theta = \cos 8\theta + 5 \rightarrow \cos 4\theta = \frac{1}{2}$	M1 A1F		ft provided simplifies to $\cos 4\theta = p$
	$k = \frac{1}{12}, \frac{5}{12}, \frac{7}{12}, \frac{11}{12}$	A1	3	CAO
(d)	$\int_0^{\frac{\pi}{2}} \cos^4 2\theta \, \text{d}\theta =$			
	$\left[\frac{\sin 8\theta}{64} + \frac{\sin 4\theta}{8} + \frac{3}{8}\theta\right]_0^{\frac{\pi}{2}}$	M1 A1F		ie their $\cos^4 2\theta$
	$= \frac{3\pi}{16}$	A1	3	AG
	Total		**14**	

8(a)	$r = 8$	B1		
	$\tan^{-1} \pm \frac{4\sqrt{3}}{4}$ or $\pm\frac{\pi}{3}$ seen	M1		or $\frac{\pi}{6}$ marked as angle to Im axis with "vector" in second quadrant on Arg diag
	$\Rightarrow \theta = \frac{2\pi}{3}$	A1	3	$-4 + 4\sqrt{3}\mathrm{i} = 8\,\mathrm{e}^{\mathrm{i}\frac{2\pi}{3}}$
(b)(i)	modulus of each root = 2	B1√		
		M1		use of De Moivre – dividing argument by 3
	$\Rightarrow \theta = -\frac{4\pi}{9}, \frac{2\pi}{9}, \frac{8\pi}{9}$	A2	4	A1 if 3 "correct" values not all in requested interval $2\,\mathrm{e}^{-\mathrm{i}\frac{4\pi}{9}},\ 2\,\mathrm{e}^{\mathrm{i}\frac{2\pi}{9}},\ 2\,\mathrm{e}^{\mathrm{i}\frac{8\pi}{9}}$
(ii)	Area $= 3 \times \frac{1}{2} \times PO \times OR \times \sin\frac{2\pi}{3}$	M1		Correct expression for area of triangle PQR
	$= 3 \times \frac{1}{2} \times 2 \times 2 \times \sin\frac{2\pi}{3}$	A1		correct values of lengths in formula
	$= 3\sqrt{3}$	A1cso	3	
(c)	Sum of roots (of cubic) = 0	E1		must be stated explicitly
	Sum of 3 roots including Im terms	M1		in form $r(\cos\theta + i\sin\theta)$
	$2\left(\cos\frac{(-)4\pi}{9} + \cos\frac{2\pi}{9} + \cos\frac{8\pi}{9}\right)$	A1		isolating real terms ; correct and with "2"
	$\mathrm{e}^{-\mathrm{i}\frac{4\pi}{9}} = \cos\frac{4\pi}{9} - \mathrm{i}\sin\frac{4\pi}{9}$ seen earlier			*or* $\cos\frac{-4\pi}{9} = \cos\frac{4\pi}{9}$ explicitly stated to earn final A1 mark
	$\cos\frac{2\pi}{9} + \cos\frac{4\pi}{9} + \cos\frac{8\pi}{9} = 0$	A1cso	4	**AG**

8(a)(i)	$\cos 4\theta + \mathrm{i}\sin 4\theta = (\cos\theta + \mathrm{i}\sin\theta)^4$	M1		De Moivre & attempt to expand RHS
	$\cos^4\theta + 4\mathrm{i}\cos^3\theta\sin\theta + 6\mathrm{i}^2\cos^2\theta\sin^2\theta$ $+ 4\mathrm{i}^3\cos\theta\sin^3\theta + \mathrm{i}^4\sin^4\theta$	A1		any correct expansion
	Equating "their" real parts	m1		or imaginary parts
	$\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$	A1		**AG** be convinced
	$\sin 4\theta = 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta$	B1	5	correct
(ii)	$\tan 4\theta = \dfrac{\text{"their expression for" } \sin 4\theta}{\text{"their expression for" } \cos 4\theta}$	M1		
	Division by $\cos^4\theta$	m1		
	$\tan 4\theta = \dfrac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}$	A1	3	**AG** be convinced
(b)	$(\tan 4\theta = 1 \Rightarrow)$ $\quad 1 = \dfrac{4t - 4t^3}{1 - 6t^2 + t^4}$	M1		when $\theta = \dfrac{\pi}{16}$
	$1 - 6t^2 + t^4 = 4t - 4t^3$ $\Rightarrow t^4 + 4t^3 - 6t^2 - 4t + 1 = 0$	A1		**AG** be convinced
	$\theta = \dfrac{\pi}{16}$ *satisfies* $\tan 4\theta = 1$ $\Rightarrow \tan\dfrac{\pi}{16}$ is root of quartic equation	E1		both statements required
	(other roots are) $\tan\dfrac{5\pi}{16}, \tan\dfrac{9\pi}{16}, \tan\dfrac{13\pi}{16}$	B1	4	or equivalent tan expressions
(c)	$\sum\alpha = -4$ **and** $\sum\alpha\beta = -6$	B1		watch for minus signs
	$\sum\alpha^2 = \left(\sum\alpha\right)^2 - 2\sum\alpha\beta$	M1		correct formula
	$(= 16 + 12) = 28$	A1cso		
	$\tan\dfrac{9\pi}{16} = -\tan\dfrac{7\pi}{16}$, $\tan\dfrac{13\pi}{16} = -\tan\dfrac{3\pi}{16}$	B1		explicitly seen
	$\tan^2\dfrac{\pi}{16} + \tan^2\dfrac{3\pi}{16} + \tan^2\dfrac{5\pi}{16} + \tan^2\dfrac{7\pi}{16} = 28$	A1cso	5	**AG** must earn previous 4 marks
	Total		17	

Q	Solution	Mark	Total	Comment
1 (a)	$r = 9$	B1		
	$\theta = -\frac{\pi}{2}$	B1		condone $-1.57\ldots$ here only
			2	$-9\text{i} = 9\text{e}^{-\text{i}\frac{\pi}{2}}$
(b)	$r = \sqrt{3}$	B1$\surd$		follow through $(\textit{their } r)^{\frac{1}{4}}$; accept $9^{\frac{1}{4}}$ etc
	$(\textit{their }\theta)/4$	M1		generous
	$\theta = -\frac{5\pi}{8}, -\frac{\pi}{8}, \frac{3\pi}{8}, \frac{7\pi}{8}$	A1		two angles correct in correct interval
		A1		**exactly four** angles correct mod 2π
	$\sqrt{3}\,\text{e}^{-\frac{\text{i}5\pi}{8}}, \sqrt{3}\,\text{e}^{-\frac{\text{i}\pi}{8}}, \sqrt{3}\,\text{e}^{\frac{\text{i}3\pi}{8}}, \sqrt{3}\,\text{e}^{\frac{\text{i}7\pi}{8}}$	A1	5	four correct roots in correct interval and in given form; accept $3^{\frac{1}{2}}$ for $\sqrt{3}$
	Total		**7**	

1(a)	Accept correct values of r and θ for full marks without candidates actually writing $9\text{e}^{-\text{i}\frac{\pi}{2}}$. Do not accept angles outside the required interval. **Example** "$\theta = -\frac{\pi}{2}$ or $\theta = \frac{3\pi}{2}$" scores **B0**
(b)	Condone $r = 1.73\ldots$ for **B1** only. Do not follow through a negative value of r for **B1**$\surd$. **Example** $\theta = \frac{3\pi}{8}, \frac{7\pi}{8}, \frac{11\pi}{8}, \frac{15\pi}{8}$ scores **M1 A1 A1** **Example** $\sqrt{3}\,\text{e}^{-\frac{\text{i}\pi}{8}+\frac{\text{i}k\pi}{2}}$ scores **B1 M1** then $k = -1, 0, 1, 2$ scores **A1 A1** with final **A1** only earned when four roots are written in given form

(a)(i)	$(\omega^5 =)\cos 2\pi + i\sin 2\pi = 1$ So ω is a root of $z^5 = 1$	B1	1	must have conclusion plus verification that $\omega^5 = 1$
(ii)	$\omega^2, \omega^3, \omega^4$.	B1	1	**OE** powers mod 5 (must not include 1)
(b)(i)	$1+\omega+\omega^2+\omega^3+\omega^4 = \dfrac{1-\omega^5}{1-\omega} = 0$	B1	1	or clear statement that sum of roots (of $z^5 - 1 = 0$) is zero
(ii)	$\left(\omega+\dfrac{1}{\omega}\right)^2 + \left(\omega+\dfrac{1}{\omega}\right) - 1$			
	$= \omega^2 + 2 + \dfrac{1}{\omega^2} + \omega + \dfrac{1}{\omega} - 1$	M1		correct expansion
	$= \dfrac{1+\omega+\omega^2+\omega^3+\omega^4}{\omega^2} = 0$	A1	2	**AG** correctly shown to = 0 do not allow simply multiplying by ω^2
(c)	$\dfrac{1}{\omega} = \cos\dfrac{2\pi}{5} - i\sin\dfrac{2\pi}{5}$	M1		
	$\Rightarrow \omega + \dfrac{1}{\omega} = 2\cos\dfrac{2\pi}{5}$	A1		**SC1** if result merely stated
	Solving quadratic $\left(\omega+\dfrac{1}{\omega}=\right)$ $\dfrac{-1\pm\sqrt{5}}{2}$	M1		must see both values
	Rejecting negative root since $\cos\dfrac{2\pi}{5} > 0$			must see this line for final **A1**
	Hence $\cos\dfrac{2\pi}{5} = \dfrac{\sqrt{5}-1}{4}$	A1	4	
				It is possible to score **SC1 M1 A1**
	Total		9	

6(a)(i)	$z^n = \cos n\theta + \mathrm{i}\sin n\theta$	M1		
	$z^{-n} = \cos(-n\theta) + \mathrm{i}\sin(-n\theta)$ $= \cos n\theta - \mathrm{i}\sin n\theta$	E1		or $\frac{1}{\cos n\theta + \mathrm{i}\sin n\theta} \times \frac{\cos n\theta - \mathrm{i}\sin n\theta}{\cos n\theta - \mathrm{i}\sin n\theta} = \ldots$ shown – not just stated
	$z^n - \frac{1}{z^n} = 2\mathrm{i}\sin n\theta$	A1	3	AG
(ii)	$\left(z^n + \frac{1}{z^n} =\right) \quad 2\cos n\theta$	B1	1	
(b)(i)	$\left(z - \frac{1}{z}\right)^2 \left(z + \frac{1}{z}\right)^2 = z^4 - 2 + \frac{1}{z^4}$	B1	1	*or* $z^4 - 2 + z^{-4}$
(ii)	$(2\mathrm{i}\sin\theta)^2(2\cos\theta)^2 = 2\cos 4\theta - 2$ $-16\sin^2\theta\cos^2\theta = 2\cos 4\theta - 2$	M1		using previous results
	$8\sin^2\theta\cos^2\theta = 1 - \cos 4\theta$	A1cso	2	
(c)	$x = 2\sin\theta \Rightarrow \mathrm{d}x = 2\cos\theta\,\mathrm{d}\theta$	M1		$x = 2\sin\theta \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}\theta} = k\cos\theta$
	$\int x^2\sqrt{4 - x^2}\,\mathrm{d}x = \int 16\sin^2\theta\cos^2\theta\,\mathrm{d}\theta$	A1		
	$= \int (2 - 2\cos 4\theta)\,(\mathrm{d}\theta)$	m1		correct ***or*** **FT** their (b)(ii) result
	$= 2\theta - \frac{1}{2}\sin 4\theta$	B1√		**FT** integrand of form $k(1 - \cos 4\theta)$
	$= \left[\pi - \frac{1}{2}\sin 2\pi\right] - \left[\frac{\pi}{3} - \frac{1}{2}\sin\frac{2\pi}{3}\right]$			$x = 1 \Rightarrow \theta = \frac{\pi}{6}; \quad x = 2 \Rightarrow \theta = \frac{\pi}{2};$
	$= \frac{2\pi}{3} + \frac{\sqrt{3}}{4}$	A1cso	5	
	Total		**12**	

Q8	Solution	Mark	Total	Comment
(a)	$(\cos\theta + \mathrm{i}\sin\theta)^4 = \cos 4\theta + \mathrm{i}\sin 4\theta$	B1		
	$(\cos\theta - \mathrm{i}\sin\theta)^4 = \cos 4\theta - \mathrm{i}\sin 4\theta$ $(\mathrm{c}+\mathrm{is})^4 + (\mathrm{c}-\mathrm{is})^4 = 2\cos 4\theta$	M1		
	Divide throughout by $\cos^4\theta$			
	$(1+\mathrm{i}\tan\theta)^4 + (1-\mathrm{i}\tan\theta)^4 = \dfrac{2\cos 4\theta}{\cos^4\theta}$	A1cso	3	**AG –** must see both sides equated penalise poor notation/brackets for **A1cso**
(b)	$\theta = \dfrac{\pi}{8} \Rightarrow \cos 4\theta = 0$			**or** $\cos 4\theta = 0 \Rightarrow \theta = \dfrac{\pi}{8}$
	$\Rightarrow\ z = \mathrm{i}\tan\dfrac{\pi}{8}$ is root or satisfies equation $((1+z)^4 + (1-z)^4 = 0)$	E1		**AG** be convinced: must have statement must mention $\mathrm{i}\tan\dfrac{\pi}{8}$ **but** may be listed with other 3 roots
	other roots are $\mathrm{i}\tan\dfrac{3\pi}{8}, \mathrm{i}\tan\dfrac{5\pi}{8}, \mathrm{i}\tan\dfrac{7\pi}{8},$	B1	2	
(c)(i)	$\alpha\beta\gamma\delta = \mathrm{i}\tan\dfrac{\pi}{8}\,\mathrm{i}\tan\dfrac{3\pi}{8}\,\mathrm{i}\tan\dfrac{5\pi}{8}\,\mathrm{i}\tan\dfrac{7\pi}{8}$	M1		product of their 4 roots
	$\tan\dfrac{5\pi}{8} = -\tan\dfrac{3\pi}{8}$ and $\tan\dfrac{7\pi}{8} = -\tan\dfrac{\pi}{8}$	B1		May earn this mark in part **(c)(ii)** if not earned here
	$(1+z)^4 + (1-z)^4 = 2z^4 + 12z^2 + 2$	B1		or $z^4 + 6z^2 + 1\quad (=0)$ seen
	$\alpha\beta\gamma\delta = 1 \quad \Rightarrow \tan^2\dfrac{\pi}{8}\tan^2\dfrac{3\pi}{8} = 1$	A1cso	4	must see i^4 become 1 for final **A1 cso**
(ii)	$\left(\sum\alpha\right)^2 = \sum\alpha^2 + 2\sum\alpha\beta$	M1		
	$\sum\alpha = 0 \Rightarrow \sum\alpha^2 = -2\sum\alpha\beta = -12$	A1		using $z^4 + 6z^2 + 1 = 0$
	$\mathrm{i}^2\left(\tan^2\dfrac{\pi}{8} + \tan^2\dfrac{3\pi}{8} + \tan^2\dfrac{5\pi}{8} + \tan^2\dfrac{7\pi}{8}\right) = -12$	A1		$\tan^2\dfrac{\pi}{8} + \tan^2\dfrac{3\pi}{8} + \tan^2\dfrac{5\pi}{8} + \tan^2\dfrac{7\pi}{8} = 12$ **OE**
	$\tan^2\dfrac{\pi}{8} + \tan^2\dfrac{3\pi}{8} = 6$	A1cso	4	must see i^2 become -1 for final **A1 cso**
	Total		13	