FP2 Complex numbers

3 The complex numbers z_1 and z_2 are given by

$$z_1 = \frac{1+\mathrm{i}}{1-\mathrm{i}} \quad \text{and} \quad z_2 = \frac{1}{2} + \frac{\sqrt{3}}{2}\mathrm{i}$$

(a) Show that $z_1 = \mathrm{i}$. *(2 marks)*

(b) Show that $|z_1| = |z_2|$. *(2 marks)*

(c) Express both z_1 and z_2 in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r > 0$ and $-\pi < \theta \leqslant \pi$. *(3 marks)*

(d) Draw an Argand diagram to show the points representing z_1, z_2 and $z_1 + z_2$. *(2 marks)*

(e) Use your Argand diagram to show that

$$\tan \frac{5}{12}\pi = 2 + \sqrt{3}$$ *(3 marks)*

5 The complex number z satisfies the relation

$$|z + 4 - 4\mathrm{i}| = 4$$

(a) Sketch, on an Argand diagram, the locus of z. *(3 marks)*

(b) Show that the greatest value of $|z|$ is $4(\sqrt{2} + 1)$. *(3 marks)*

(c) Find the value of z for which

$$\arg(z + 4 - 4\mathrm{i}) = \tfrac{1}{6}\pi$$

Give your answer in the form $a + \mathrm{i}b$. *(3 marks)*

4 (a) On one Argand diagram, sketch the locus of points satisfying:

(i) $|z - 3 + 2\mathrm{i}| = 4$; *(3 marks)*

(ii) $\arg(z - 1) = -\frac{1}{4}\pi$. *(3 marks)*

(b) Indicate on your sketch the set of points satisfying both

$$|z - 3 + 2\mathrm{i}| \leqslant 4$$

$$\text{and} \quad \arg(z - 1) = -\tfrac{1}{4}\pi$$ *(1 mark)*

2 (a) Sketch on one diagram:

(i) the locus of points satisfying $|z - 4 + 2i| = 2$; *(3 marks)*

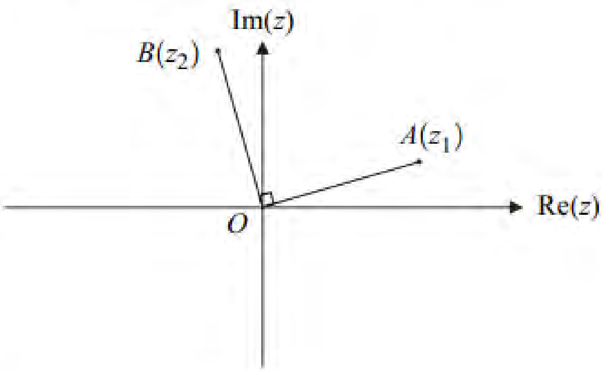
(ii) the locus of points satisfying $|z| = |z - 3 - 2i|$. *(3 marks)*

(b) Shade on your sketch the region in which

both $|z - 4 + 2i| \leqslant 2$

and $|z| \leqslant |z - 3 - 2i|$ *(2 marks)*

5 The sketch shows an Argand diagram. The points A and B represent the complex numbers z_1 and z_2 respectively. The angle $AOB = 90°$ and $OA = OB$.



(a) Explain why $z_2 = iz_1$. *(2 marks)*

(b) On a **single** copy of the diagram, draw:

(i) the locus L_1 of points satisfying $|z - z_2| = |z - z_1|$; *(2 marks)*

(ii) the locus L_2 of points satisfying $\arg(z - z_2) = \arg z_1$. *(3 marks)*

(c) Find, in terms of z_1, the complex number representing the point of intersection of L_1 and L_2. *(2 marks)*

3 A circle C and a half-line L have equations

$$|z - 2\sqrt{3} - \mathrm{i}| = 4$$

and $$\arg(z + \mathrm{i}) = \frac{\pi}{6}$$

respectively.

(a) Show that:

(i) the circle C passes through the point where $z = -\mathrm{i}$; *(2 marks)*

(ii) the half-line L passes through the centre of C. *(3 marks)*

(b) On one Argand diagram, sketch C and L. *(4 marks)*

(c) Shade on your sketch the set of points satisfying both

$$|z - 2\sqrt{3} - \mathrm{i}| \leqslant 4$$

and $$0 \leqslant \arg(z + \mathrm{i}) \leqslant \frac{\pi}{6}$$ *(2 marks)*

2 (a) Indicate on an Argand diagram the region for which $|z - 4\mathrm{i}| \leqslant 2$. *(4 marks)*

(b) The complex number z satisfies $|z - 4\mathrm{i}| \leqslant 2$. Find the range of possible values of $\arg z$. *(4 marks)*

6 (a) Two points, A and B, on an Argand diagram are represented by the complex numbers $2 + 3\mathrm{i}$ and $-4 - 5\mathrm{i}$ respectively. Given that the points A and B are at the ends of a diameter of a circle C_1, express the equation of C_1 in the form $|z - z_0| = k$. *(4 marks)*

(b) A second circle, C_2, is represented on the Argand diagram by the equation $|z - 5 + 4\mathrm{i}| = 4$. Sketch on one Argand diagram both C_1 and C_2. *(3 marks)*

(c) The points representing the complex numbers z_1 and z_2 lie on C_1 and C_2 respectively and are such that $|z_1 - z_2|$ has its maximum value. Find this maximum value, giving your answer in the form $a + b\sqrt{5}$. *(5 marks)*

2 (a) On the same Argand diagram, draw:

(i) the locus of points satisfying $|z - 4 + 2\mathrm{i}| = 4$; *(3 marks)*

(ii) the locus of points satisfying $|z| = |z - 2\mathrm{i}|$. *(3 marks)*

(b) Indicate on your sketch the set of points satisfying both

$$|z - 4 + 2\mathrm{i}| \leqslant 4$$

and $$|z| \geqslant |z - 2\mathrm{i}|$$ *(2 marks)*

3 Two loci, L_1 and L_2, in an Argand diagram are given by

$$L_1 : |z + 1 + 3\mathrm{i}| = |z - 5 - 7\mathrm{i}|$$

$$L_2 : \arg z = \frac{\pi}{4}$$

(a) Verify that the point represented by the complex number $2 + 2\mathrm{i}$ is a point of intersection of L_1 and L_2. *(2 marks)*

(b) Sketch L_1 and L_2 on one Argand diagram. *(5 marks)*

(c) Shade on your Argand diagram the region satisfying

both $$|z + 1 + 3\mathrm{i}| \leqslant |z - 5 - 7\mathrm{i}|$$

and $$\frac{\pi}{4} \leqslant \arg z \leqslant \frac{\pi}{2}$$ *(2 marks)*

1 (a) Sketch on an Argand diagram the locus of points satisfying the equation

$$|z - 4 + 3\mathrm{i}| = 5$$ *(3 marks)*

(b) (i) Indicate on your diagram the point P representing z_1, where both

$$|z_1 - 4 + 3\mathrm{i}| = 5 \quad \text{and} \quad \arg z_1 = 0$$ *(1 mark)*

(ii) Find the value of $|z_1|$. *(1 mark)*

1 (a) Draw on the same Argand diagram:

(i) the locus of points for which

$$|z - 2 - 5\mathrm{i}| = 5$$ *(3 marks)*

(ii) the locus of points for which

$$\arg(z + 2\mathrm{i}) = \frac{\pi}{4}$$ *(3 marks)*

(b) Indicate on your diagram the set of points satisfying both

$$|z - 2 - 5\mathrm{i}| \leqslant 5$$

and $$\arg(z + 2\mathrm{i}) = \frac{\pi}{4}$$ *(2 marks)*

2 (a) Draw on an Argand diagram the locus L of points satisfying the equation $\arg z = \frac{\pi}{6}$.

(1 mark)

(b) (i) A circle C, of radius 6, has its centre lying on L and touches the line $\mathrm{Re}(z) = 0$. Draw C on your Argand diagram from part **(a)**. *(2 marks)*

(ii) Find the equation of C, giving your answer in the form $|z - z_0| = k$. *(3 marks)*

(iii) The complex number z_1 lies on C and is such that $\arg z_1$ has its least possible value. Find $\arg z_1$, giving your answer in the form $p\pi$, where $-1 < p \leqslant 1$. *(2 marks)*

2 (a) Draw on the Argand diagram below:

(i) the locus of points for which

$$|z - 2 - 3i| = 2$$ *(3 marks)*

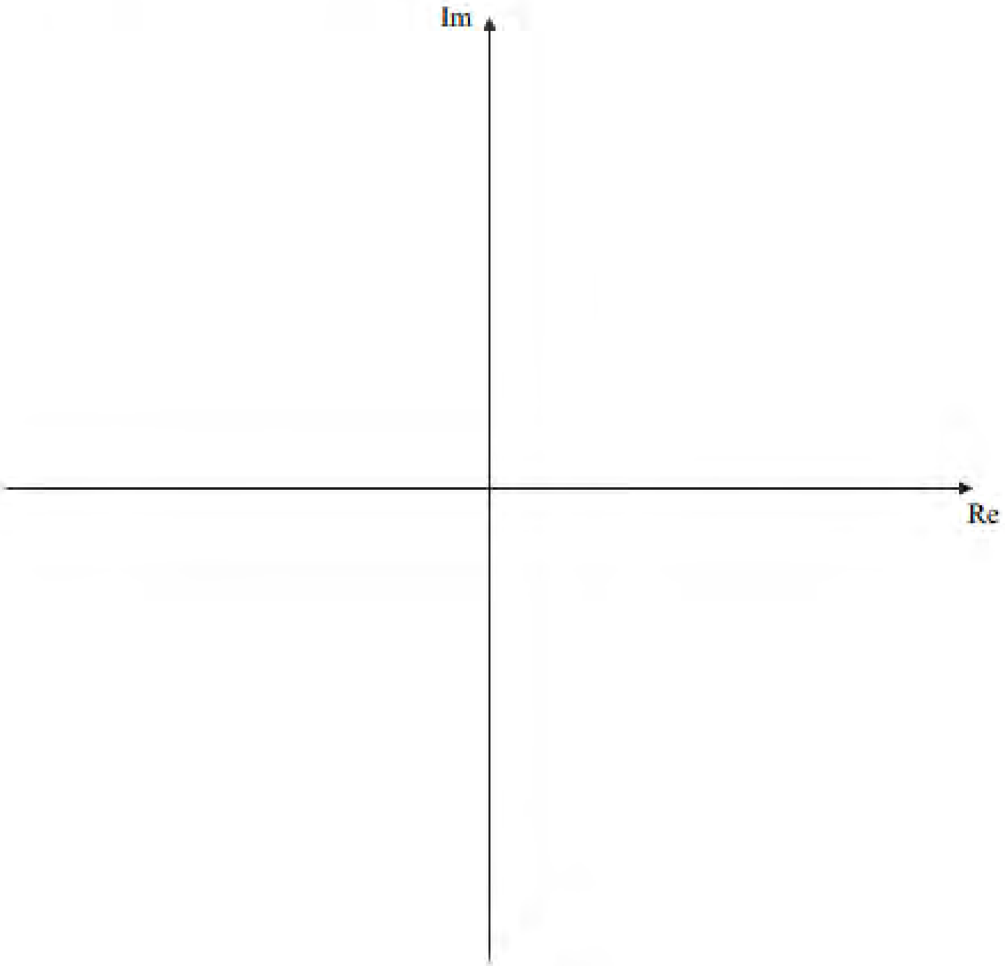
(ii) the locus of points for which

$$|z + 2 - i| = |z - 2|$$ *(3 marks)*

(b) Indicate on your diagram the points satisfying both

$$|z - 2 - 3i| = 2$$

and $$|z + 2 - i| \leqslant |z - 2|$$ *(1 mark)*



2 Two loci, L_1 and L_2, in an Argand diagram are given by

$$L_1 : |z + 6 - 5\mathrm{i}| = 4\sqrt{2}$$

$$L_2 : \quad \arg(z + \mathrm{i}) = \frac{3\pi}{4}$$

The point P represents the complex number $-2 + \mathrm{i}$.

(a) Verify that the point P is a point of intersection of L_1 and L_2. *(2 marks)*

(b) Sketch L_1 and L_2 on one Argand diagram. *(6 marks)*

(c) The point Q is also a point of intersection of L_1 and L_2. Find the complex number that is represented by Q. *(2 marks)*

1 (a) Sketch on an Argand diagram the locus of points satisfying the equation

$$|z - 6\mathrm{i}| = 3$$ *(3 marks)*

(b) It is given that z satisfies the equation $|z - 6\mathrm{i}| = 3$.

(i) Write down the greatest possible value of $|z|$. *(1 mark)*

(ii) Find the greatest possible value of $\arg z$, giving your answer in the form $p\pi$, where $-1 < p \leqslant 1$. *(3 marks)*

2 (a) Sketch, on the Argand diagram below, the locus L of points satisfying

$$\arg(z - 2\mathrm{i}) = \frac{2\pi}{3}$$

[3 marks]

(b) (i) A circle C, of radius 3, has its centre lying on L and touches the line $\mathrm{Im}(z) = 2$. Sketch C on the Argand diagram used in part **(a)**.

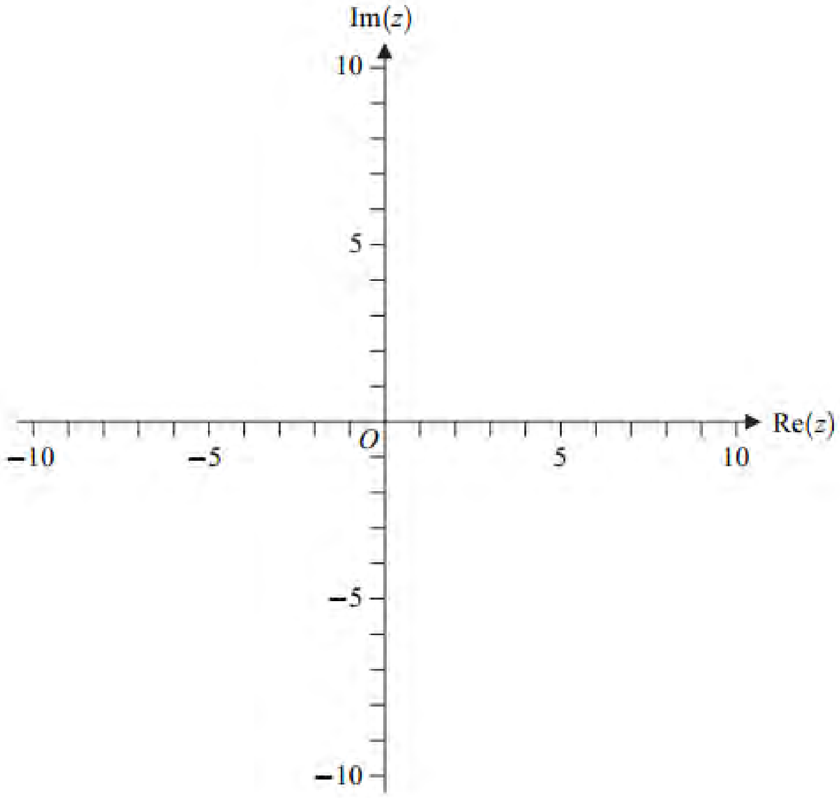
[2 marks]

(ii) Find the centre of C, giving your answer in the form $a + b\mathrm{i}$.

[3 marks]

QUESTION PART REFERENCE

Answer space for question 2



5 The locus of points, L, satisfies the equation

$$|z - 2 + 4\mathrm{i}| = |z|$$

(a) Sketch L on the Argand diagram below.

[3 marks]

(b) The locus L cuts the real axis at A and the imaginary axis at B.

(i) Show that the complex number represented by C, the midpoint of AB, is

$$\frac{5}{2} - \frac{5}{4}\mathrm{i}$$

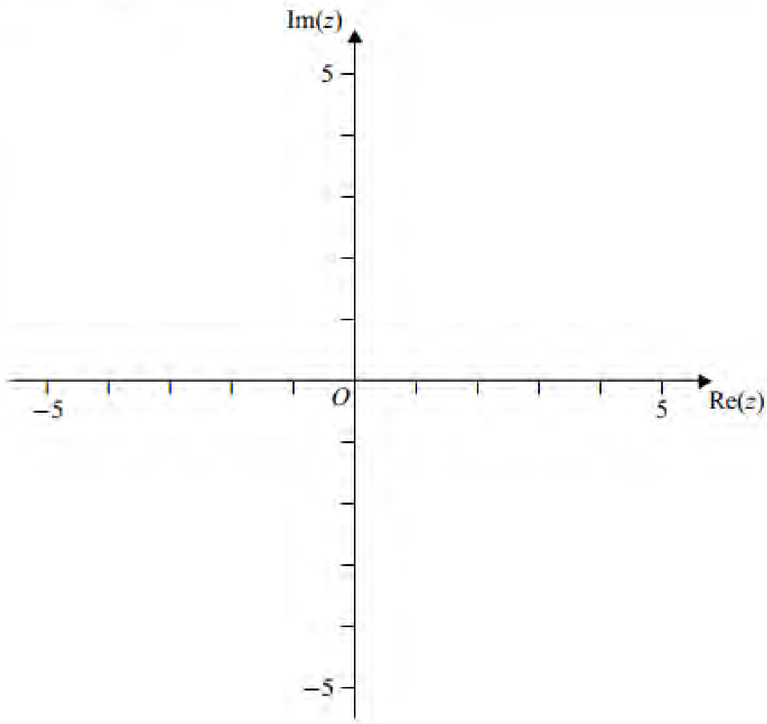
[4 marks]

(ii) The point O is the origin of the Argand diagram. Find the equation of the circle that passes through the points O, A and B, giving your answer in the form $|z - \alpha| = k$.

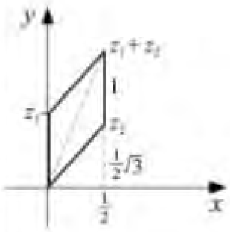
[2 marks]

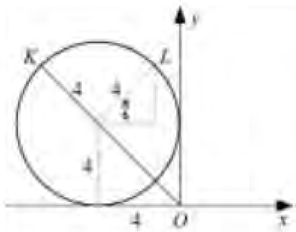
QUESTION PART REFERENCE | **Answer space for question 5**

(a)

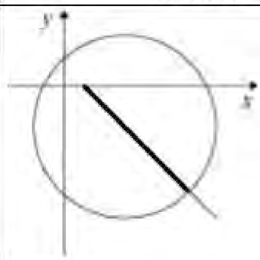


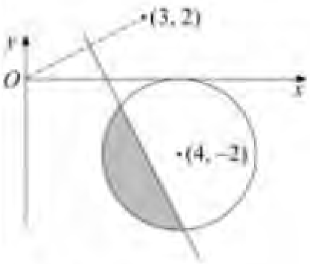
MFP2 (cont)

Q	Solution	Marks	Total	Comments
3(a)	$\frac{1+i}{1-i}=\frac{(1+i)^2}{1-i^2}=i$	M1A1	2	AG
(b)	$\lvert z_2\rvert=\sqrt{\frac{1}{4}+\frac{3}{4}}=1=\lvert z_1\rvert$	M1A1	2	
(c)	$r=1$	B1		PI
	$\theta=\frac{1}{2}\pi,\ \frac{1}{3}\pi$	B1B1	3	Deduct 1 mark if extra solutions
(d)		B2,1F	2	Positions of the 3 points relative to each other, must be approximately correct
(e)	$\operatorname{Arg}(z_1+z_2)=\frac{5}{12}\pi$	B1		**Clearly** shown
	$\tan\frac{5}{12}\pi=\frac{1+\frac{1}{2}\sqrt{3}}{\frac{1}{2}}$	M1		Allow if BO earned
	$=2+\sqrt{3}$	A1	3	AG must earn BO for this

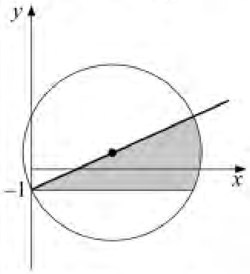
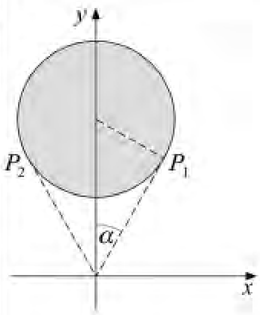
Q	Solution	Marks	Total	Comments
5(a)		B1		Circle
		B1		Correct centre
		B1	3	Touching both axes
(b)	$\lvert z\rvert \max = OK$	M1		Accept $\sqrt{4^2+4^2}+4$ as a method
	$=\sqrt{4^2+4^2}+4$	A1F		Follow through circle in incorrect position
	$=4\left(\sqrt{2}+1\right)$	A1F	3	AG
(c)	Correct position of z, ie L	M1		
	$a=-\left(4-4\cos\frac{1}{6}\pi\right)$			
	$=-\left(4-2\sqrt{3}\right)$	A1F		Follow through circle in incorrect position
	$b=4+4\sin\frac{1}{6}\pi=6$	A1F	3	

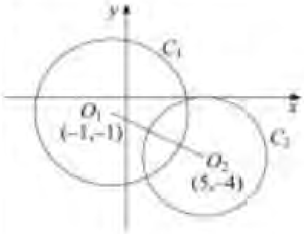
MFP2 (cont)

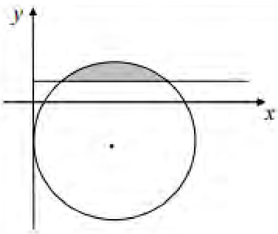
Q	Solution	Marks	Total	Comments
4				
(a)(i)	Circle	B1		
	Correct centre	B1		
	Enclosing the origin	B1	3	
(ii)	Half line	B1		
	Correct starting point	B1		
	Correct angle	B1	3	
(b)	Correct part of the line **indicated**	B1F	1	
	Total		7	

2(a)				
(i)	**Circle** Correct centre Correct radius Touching x-axis	 B1 B1 B1	 3	
(ii)	**Line** Point $(3,2)$ indicated Line through $\left(1\frac{1}{2},1\right)$ Perpendicular to $(0,0)\to(3,2)$	 B1 B1√ B1	 3	
(b)	Correct shaded area	B1 B1√	2	For shading inside the circle provided no other area is shaded Must be a circle and a straight line for second B1

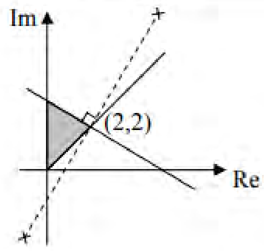
	Total		7	
5(a)	Explanation	E2,1,0	2	E1 for $\mathrm{i}=\mathrm{e}^{\frac{\pi\mathrm{i}}{2}}$ or $\mathrm{i}z_1=-y_1+\mathrm{i}x_1$
(b)(i)	Perpendicular bisector of AB through O	B1 B1	 2	
(ii)	half-line from B parallel to OA	B1 B1 B1	 3	If L_2 is taken to be the line AB give B0
(c)	$(1+\mathrm{i})z_1$	M1A1	2	ft if L_2 taken as line AB

Q	Solution	Marks	Total	Comments
3(a)(i)	$z=-\mathrm{i}$ $\quad \lvert -2\sqrt{3}-2\mathrm{i} \rvert = \sqrt{12+4} = 4$	M1		$\lvert -2\sqrt{3}-2\mathrm{i} \rvert$
		A1	2	4
(ii)	Centre of circle is $2\sqrt{3}+\mathrm{i}$	B1		Do not accept $(2\sqrt{3}, 1)$ unless attempt to solve using trig
	Substitute into line	M1		
	$\arg\left(2\sqrt{3}+2\mathrm{i}\right)=\frac{\pi}{6}$ shown	A1	3	
(b)				
	Circle: centre correct	B1		
	through $(0,-1)$	B1		
	Half line: through $(0,-1)$	B1		
	through centre of circle	B1	4	
(c)	Shading inside circle and below line	B1F		
	Bounded by $y=-1$	B1	2	
	Total		**9**	
2(a)		B1		Circle
		B1		Correct centre
		B1		Correct radius
		B1F	4	Inside shading
(b)	Correct points P_1 and P_2 indicated	B1F		Possibly by tangents drawn ft mirror image of circle in x-axis
	$\sin\alpha=\frac{2}{4}$	M1		
	$\alpha=\frac{\pi}{6}$	A1		
	Range is $\frac{\pi}{3}\leqslant \arg z\leqslant \frac{2\pi}{3}$	A1	4	Deduct 1 for angles in degrees

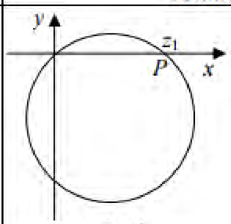
Q	Solution	Marks	Total	Comments
6(a)	Centre $-1-\mathrm{i}$ or $(-1,-1)$	B1		
	Radius 5	M1 A1F		ft incorrect centre if used
	$\lvert z+1+\mathrm{i}\rvert = 5$ or $\lvert z-(-1-i)\rvert = 5$	A1F	4	ft $\lvert z+1+\mathrm{i}\rvert = 10$ earns M0B1
(b)				
	C_1 correct centre, correct radius	B1F		ft errors in (a) but fit circles need to intersect and C_1 enclose $(0,0)$
	C_2 correct centre, correct radius	B1		
	Touching x-axis	B1F	3	error in plotting centre
(c)	$O_1O_2 = 3\sqrt{5}$	M1A1		allow if circles misplaced but O_1O_2 is still $3\sqrt{5}$
	Correct length identified	m1		
	Length is $9+3\sqrt{5}$	M1 A1F	5	ft if r is taken as 10
	Total		9	

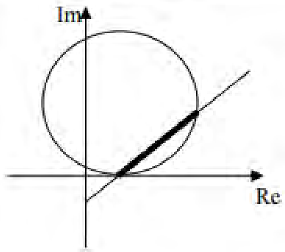
Q	Solution	Marks	Total	Comments
2				
(a)(i)	Circle	B1		
	Correct centre	B1		x-coordinate $\approx -2 \times y$-coordinate in correct quadrant; condone $(4,-2\mathrm{i})$
	Touching y-axis	B1	3	
(ii)	Straight line	B1		
	parallel to x-axis	B1		
	through (0, 1)	B1	3	Assume (0, 1) if distance up y-axis is half distance to top of circle; no other shading outside circle
(b)	Shading: inside circle	B1F		
	above line	B1F	2	Whole question reflected in x-axis loses 2 marks
	Total		8	

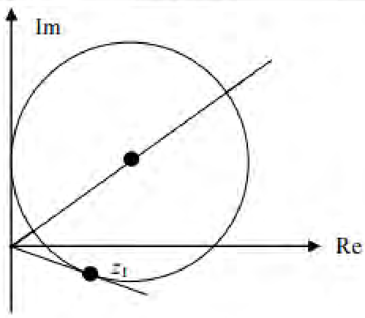
MFP2 (cont)

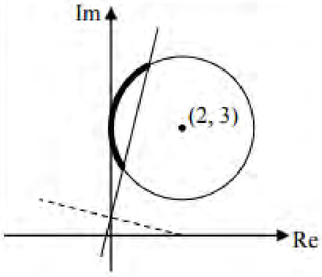
Q	Solution	Marks	Total	Comments
3				
(a)	$\lvert 2+2\mathrm{i}+1+3\mathrm{i}\rvert = \lvert 2+2\mathrm{i}-5-7\mathrm{i}\rvert$	B1		Clearly shown do not allow $\lvert 3+5\mathrm{i}\rvert = \lvert -3-5\mathrm{i}\rvert$ without comment
	$\arg(2+2\mathrm{i}) = \frac{\pi}{4}$	B1	2	Clearly shown
(b)	L_1 : straight line with negative gradient	B1		
	perpendicular to line joining $(-1,-3)$ to $(5,7)$	B1		
	through $(2,2)$	B1		The point (2,2) must be shown either by (2,2) or 2+2i or with numbered axes
	L_2 : half line through O	B1		
	through $(2,2)$	B1	5	
(c)	Shading between $\frac{\pi}{4}$ and $\frac{\pi}{2}$	B1		No marks for shading if circles drawn in (b)
	Below L_1	B1	2	

MFP2

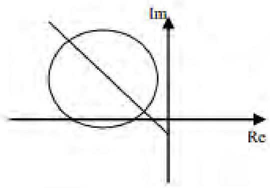
Q	Solution	Marks	Total	Comments
1(a)	 Circle	B1		
	correct centre	B1		
	through (0, 0)	B1	3	
(b)(i)	z_1 correctly chosen	B1F	1	ft if circle encloses (0, 0)
(ii)	$\lvert z_1\rvert = 8$	B1F	1	ft if centre misplotted
	Total		5	

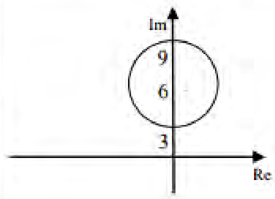
Q	Solution	Marks	Total	Comments
1(a)				Use average of whole question if 2 diagrams used
(i)	Circle correct centre touching x-axis	B1 B1 B1F	3	Circle in any position Must be shown ft incorrect centre
(ii)	half-line through (0, –2) through point of contact of circle with x-axis	B1 B1 B1	3	 Can be inferred
(b)	Inside circle On line	B1 B1F	2	 ft errors in position of line and circle

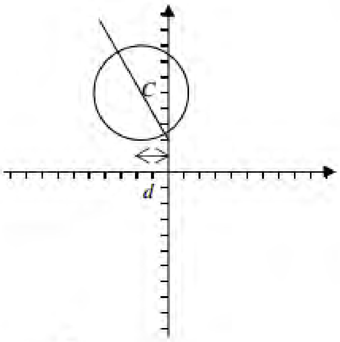
Q	Solution	Marks	Total	Comments
2(a)	 Half-line with gradient < 1	B1	1	condone a short line, ie it stops at or inside circle
(b)(i)	Circle centre on L, x-coord 6 indicated touching $\text{Re } z = 0$ not at (0, 0)	B1 B1	2	 not touching Re axis
(ii)	y-coord of centre is $2\sqrt{3}$ or $\dfrac{6}{\sqrt{3}}$ $z_0 = 6 + 2\sqrt{3}\,\text{i}$, $k = 6$	B1 B1F, B1	 3	OE; PI ft error in coords of centre
(iii)	Point z_1 shown $\arg z_1 = -\dfrac{1}{6}$	B1 B1	 2	PI

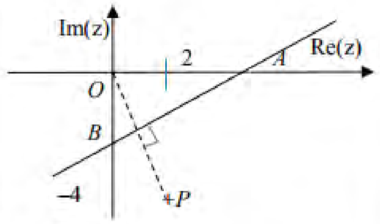
Q	Solution	Marks	Total	Comments
2(a)				
(i)	Circle	B1		Convex loop
	Correct centre	B1		Some indication of position of centre
	Touching Im axis	B1	3	
(ii)	Straight line well to left of centre	B1		$\frac{1}{2}$ line through $(0, \frac{1}{2})$ B0
	through $(0, \frac{1}{2})$	B1		Point approximately between 0 and 1
	$\perp$ to line joining $(-2,1)$ and $(2,0)$	B1	3	
	NB 0/3 for line parallel to x-axis			
	0/3 for line joining the two points (–2, 1) and (2,0)			
	0/3 for line joining (0,0) to centre of circle			
(b)	Minor arc indicated	B1F	1	ft incorrect position of line or circle

MFP2 (cont)

Q	Solution	Marks	Total	Comments
2(a)	$\|4-4\text{i}\| = \sqrt{16+16} = \sqrt{32} = 4\sqrt{2}$	B1		verification that $\|-2+\text{i}+6-5\text{i}\| = 4\sqrt{2}$
	$\arg(-2+2\text{i}) = \pi - \tan^{-1}(1) = \frac{3\pi}{4}$	B1	2	verification that $\arg(z+\text{i}) = \frac{3\pi}{4}$
				
(b)	Circle	M1		freehand circle sketched
	Centre at $-6+5\text{i}$	A1		clear from diagram or centre stated
	Cutting Re axis but not cutting Im axis	A1		
	"Straight" line	M1		freehand line
	Half line from $0-\text{i}$	A1		not horizontal or vertical but end point at $0-\text{i}$ must be clear from diagram/stated
	gradient -1 (approx)	A1	6	making $45°$ to negative Re axis and positive Im axis
(c)	Calculation based on fact that L_2 passes through centre of L_1	M1		idea of vector $\begin{bmatrix}-4\\4\end{bmatrix}$ from centre
	Q represents $-10+9\text{i}$	A1	2	must write as a complex number

Q	Solution	Marks	Total	Comments
1(a)				
	Circle	M1		freehand circle
	Centre at 6i	A1		6 marked on Im axis as centre
	Radius 3 & cutting positive Im axis twice	A1	3	radius of 3 clearly indicated with circle in position shown
(b)(i)	(Max $\|z\|$ is) 9	B1	1	
(ii)	Tangent from O to circle	M1		FT their circle position
	Angle of $\frac{\pi}{6}$ or $\frac{\pi}{3}$ ***correctly*** marked	A1		PI ; condone degrees for first A1
	(Max $\arg z$ is) $\frac{2\pi}{3}$	A1cso	3	exactly this
	Total		7	

Q	Solution	Mark	Total	Comment
2(a)	Straight line	**M1**		
	Half line from 2 on Im axis	**A1**		not vertical or horizontal
	Making approx. 30° to positive Im axis & 60° to negative Re axis	**A1**	**3**	
(b)(i)	Circle with centre on 'their' L	**M1**		
	Circle correct and touching Im $z = 2$	**A1**	**2**	lowest point of circle at approx 2
				
(b)(ii)	$d = 3\tan\dfrac{\pi}{6}$	**M1**		any correct expression for distance or $\dfrac{b-2}{a} = -\sqrt{3}$ for **M1**
	$a = -\sqrt{3}$	**A1**		condone -1.73 or better
	$b = 5$	**B1**	**3**	centre is $-\sqrt{3} + 5\mathrm{i}$

Q5	Solution	Mark	Total	Comment
(a)				Ignore the line *OP* drawn in full or circles drawn as part of construction for locus *L*.
	Straight line	M1		
	Through midpoint of *OP*, *P* correct	A1		*P* represents $2-4\text{i}$
	Perpendicular to *OP*, *P* correct	A1	3	
(b)(i)	$(x-2)^2+(y+4)^2=x^2+y^2$	M1		
	$2y-x+5=0$	A1		
	$A(5,0)$ & $B(0,-2.5)$	A1		may have $5+0\text{i}$ and $0-2.5\text{i}$
	$C\left(\frac{5}{2},-\frac{5}{4}\right) \Rightarrow$ complex num $=\frac{5}{2}-\frac{5}{4}\text{i}$	A1	4	
(ii)	***either*** $\alpha=\frac{5}{2}-\frac{5}{4}\text{i}$ ***or*** $k=\frac{5\sqrt{5}}{4}$	M1		allow statement with correct value for centre or radius of circle
	$\left\lvert z-\frac{5}{2}+\frac{5}{4}\text{i}\right\rvert=\frac{5\sqrt{5}}{4}$	A1	2	must have exact surd form