Solutionbank FP2
Edexcel AS and A Level Modular Mathematics

Exercise A, Question 1

Question:

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6x = 0$$

Solution:

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 0$$

The auxiliary equation is

$$m^2 + 5m + 6 = 0$$

$$\therefore \quad (m + 3)(m + 2) = 0$$

$$\therefore \quad m = -3 \text{ or } -2$$

So the general solution is $y = Ae^{-3x} + Be^{-2x}$.

The auxiliary equation of $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$ is $am^2 + bm^2 + c = 0$. If α and β are roots of this quadratic then $y = Ae^{\alpha x} + Be^{\beta x}$ is the general solution of the differential equation.

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Exercise A, Question 2

Question:

$$\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 12y = 0$$

Solution:

$$\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 12y = 0$$

The auxiliary equation is

$$m^2 - 8m + 12 = 0$$

$\therefore \quad (m-6)(m-2) = 0$

$\therefore \quad m = 2 \text{ or } 6$

So the general solution is $y = Ae^{2x} + Be^{6x}$.

Find the auxiliary equation $am^2 + bm + c = 0$ and solve to give two real roots α and β. General solution is $Ae^{\alpha x} + Be^{\beta x}$.

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Exercise A, Question 3

Question:

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 15y = 0$$

Solution:

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 15y = 0$$

The auxiliary equation is

$$m^2 + 2m - 15 = 0$$

$$\therefore \quad (m + 5)(m - 3) = 0$$

$$\therefore \quad m = -5 \text{ or } 3$$

So the general solution is $y = Ae^{-5x} + Be^{3x}$.

Find the auxiliary equation and solve to give 2 real roots α and β. General solution is $Ae^{\alpha x} + Be^{\beta x}$.

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Edexcel AS and A Level Modular Mathematics

Exercise A, Question 4

Question:

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} - 28y = 0$$

Solution:

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} - 28y = 0$$

Find the auxiliary equation and solve to give 2 real roots α and β. General solution is $Ae^{\alpha x} + Be^{\beta x}$.

The auxiliary equation is

$$m^2 - 3m - 28 = 0$$

$$\therefore \quad (m-7)(m+4) = 0$$

$$\therefore \quad m = 7 \text{ or } -4$$

So the general solution is $y = Ae^{7x} + Be^{-4x}$.

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Exercise A, Question 5

Question:

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 12y = 0$$

Solution:

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 12y = 0$$

The auxiliary equation is

$$m^2 + m - 12 = 0$$

$$\therefore \quad (m + 4)(m - 3) = 0$$

$$\therefore \quad m = -4 \text{ or } 3$$

So the general solution is $y = Ae^{-4x} + Be^{3x}$.

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Edexcel AS and A Level Modular Mathematics

Exercise A, Question 6

Question:

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} = 0$$

Solution:

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} = 0$$

The auxiliary equation is

$$m^2 + 5m = 0$$

$\therefore \quad m(m + 5) = 0$

$\therefore \quad m = 0 \text{ or } -5$

The auxiliary equation has two real roots, but one of them is zero. As $Ae^{0x} = A$, the general solution is $A + Be^{\beta x}$.

So the general solution is

$$y = Ae^{0x} + Be^{-5x}$$
$$= A + Be^{-5x}.$$

NB. There are other methods of solving this differential equation.

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Exercise A, Question 7

Question:

$$3\frac{d^2y}{dx^2}+7\frac{dy}{dx}+2y=0$$

Solution:

$$3\frac{d^2y}{dx^2}+7\frac{dy}{dx}+2y=0$$

The auxiliary equation is

$$3m^2+7m+2=0$$

$\therefore\quad (3m+1)(m+2)=0$

$\therefore\quad m=-\frac{1}{3}$ or -2

$\therefore\quad y=Ae^{-\frac{1}{3}x}+Be^{-2x}$ is the general solution.

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Edexcel AS and A Level Modular Mathematics

Exercise A, Question 8

Question:

$$4\frac{d^2y}{dx^2} - 7\frac{dy}{dx} - 2y = 0$$

Solution:

$$4\frac{d^2y}{dx^2} - 7\frac{dy}{dx} - 2y = 0$$

The auxiliary equation is

$$4m^2 - 7m - 2 = 0$$

$\therefore \quad (4m + 1)(m - 2) = 0$

$\therefore \quad m = -\frac{1}{4} \text{ or } 2$

So the general solution is $y = Ae^{-\frac{1}{4}x} + Be^{2x}$.

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Edexcel AS and A Level Modular Mathematics

Exercise A, Question 9

Question:

$$6\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 0$$

Solution:

$$6\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 0$$

Find the auxiliary equation and solve to give two distinct real roots α and β. The general solution is $y = Ae^{\alpha x} + Be^{\beta x}$.

The auxiliary equation is

$$6m^2 - m - 2 = 0$$

$\therefore \quad (3m - 2)(2m + 1) = 0$

$\therefore \quad m = \frac{2}{3} \text{ or } -\frac{1}{2}$

So the general solution is $y = Ae^{\frac{2}{3}x} + Be^{-\frac{1}{2}x}$.

Solutionbank FP2
Edexcel AS and A Level Modular Mathematics

Exercise A, Question 10

Question:

$$15\frac{d^2y}{dx^2} - 7\frac{dy}{dx} - 2y = 0$$

Solution:

$$15\frac{d^2y}{dx^2} - 7\frac{dy}{dx} - 2y = 0$$

The auxiliary equation is

$$15m^2 - 7m - 2 = 0$$

$$\therefore\ (5m + 1)(3m - 2) = 0$$

$$\therefore\quad m = -\tfrac{1}{5} \text{ or } \tfrac{2}{3}$$

So the general solution is

$$y = Ae^{-\frac{1}{5}x} + Be^{\frac{2}{3}x}.$$

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Edexcel AS and A Level Modular Mathematics

Exercise B, Question 1

Question:

$$\frac{d^2y}{dx^2} + 10\frac{dy}{dx} + 25y = 0$$

Solution:

$$\frac{d^2y}{dx^2} + 10\frac{dy}{dx} + 25y = 0$$

The auxiliary equation is

$$m^2 + 10m + 25 = 0$$

$$\therefore \quad (m + 5)(m + 5) = 0 \quad \text{or} \quad (m + 5)^2 = 0$$

$$\therefore \quad m = -5 \text{ only.}$$

So the general solution is

$$y = (A + Bx)e^{-5x}.$$

The auxiliary equation has repeated roots and so the general solution is of the form $(A + Bx)e^{\alpha x}$, where α is the repeated root.

Solutionbank FP2
Edexcel AS and A Level Modular Mathematics

Exercise B, Question 2

Question:

$$\frac{d^2y}{dx^2} - 18\frac{dy}{dx} + 81y = 0$$

Solution:

$$\frac{d^2y}{dx^2} - 18\frac{dy}{dx} + 81y = 0$$

The auxiliary equation is

$$m^2 - 18m + 81 = 0$$

$$\therefore \quad (m-9)^2 = 0$$

$$\therefore \quad m = 9 \text{ only.}$$

So the general solution is

$$y = (A + Bx)e^{9x}.$$

The auxiliary equation is $m^2 - 18m + 81 = 0$, which has repeated roots and so the general solution is of the form $(A + Bx)e^{\alpha x}$, where α is the repeated root.

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Exercise B, Question 3

Question:

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 0$$

Solution:

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 0$$

The auxiliary equation is

$$m^2 + 2m + 1 = 0$$

$\therefore \quad (m + 1)(m + 1) = 0 \quad \text{or} \quad (m + 1)^2 = 0$

$\therefore \quad m = -1$ only.

So the general solution is

$$y = (A + Bx)e^{-x}.$$

The auxiliary equation is $m^2 + 2m + 1 = 0$, which has repeated roots and so the general solution is of the form $(A + Bx)e^{\alpha x}$, where α is the repeated root.

Solutionbank FP2

Edexcel AS and A Level Modular Mathematics

Exercise B, Question 4

Question:

$$\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 16y = 0$$

Solution:

$$\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 16y = 0$$

The auxiliary equation is

$$m^2 - 8m + 16 = 0$$

$\therefore \quad (m-4)^2 = 0$

$\therefore \quad m = 4$ only.

$\therefore$ The general solution is $y = (A + Bx)e^{4x}$.

The auxiliary equation has repeated roots and so the general solution is of the form $(A + Bx)e^{\alpha x}$, where α is the repeated root.

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Edexcel AS and A Level Modular Mathematics

Exercise B, Question 5

Question:

$$\frac{d^2y}{dx^2} + 14\frac{dy}{dx} + 49y = 0$$

Solution:

$$\frac{d^2y}{dx^2} + 14\frac{dy}{dx} + 49y = 0$$

The auxiliary equation is

$$m^2 + 14m + 49 = 0$$

$\therefore \quad (m + 7)^2 = 0$

$\therefore \quad m = -7$ only.

So the general solution is

$$y = (A + Bx)e^{-7x}.$$

The auxiliary equation has repeated roots and so the general solution is of the form $(A + Bx)e^{\alpha x}$, where α is the repeated root.

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Edexcel AS and A Level Modular Mathematics

Exercise B, Question 6

Question:

$$16\frac{d^2y}{dx^2} + 8\frac{dy}{dx} + y = 0$$

Solution:

$$16\frac{d^2y}{dx^2} + 8\frac{dy}{dx} + y = 0$$

The auxiliary equation is

$$16m^2 + 8m + 1 = 0$$

$\therefore \quad (4m + 1)^2 = 0$

$\therefore \quad m = -\frac{1}{4}$ only.

So the general solution is

$$y = (A + Bx)e^{-\frac{1}{4}x}.$$

The auxiliary equation has repeated roots and so the general solution is of the form $(A + Bx)e^{\alpha x}$, where α is the repeated root.

Solutionbank FP2
Edexcel AS and A Level Modular Mathematics

Exercise B, Question 7

Question:

$$4\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + y = 0$$

Solution:

$$4\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + y = 0$$

The auxiliary equation is

$$4m^2 - 4m + 1 = 0$$

$\therefore \quad (2m - 1)^2 = 0$

$\therefore \quad m = \frac{1}{2}$ only.

So the general solution is

$$y = (A + Bx)e^{\frac{1}{2}x}.$$

The auxiliary equation has repeated roots and so the general solution is of the form $(A + Bx)e^{\alpha x}$, where α is the repeated root.

Solutionbank FP2
Edexcel AS and A Level Modular Mathematics

Exercise B, Question 8

Question:

$$4\frac{d^2y}{dx^2} + 20\frac{dy}{dx} + 25y = 0$$

Solution:

$$4\frac{d^2y}{dx^2} + 20\frac{dy}{dx} + 25y = 0$$

The auxiliary equation is

$$4m^2 + 20m + 25 = 0$$

$$\therefore \quad (2m + 5)^2 = 0$$

$$\therefore \quad m = -2\tfrac{1}{2} = -\tfrac{5}{2} \text{ only.}$$

So the general solution is

$$y = (A + Bx)e^{-\frac{5}{2}x}.$$

The auxiliary equation has repeated roots and so the general solution is of the form $(A + Bx)e^{\alpha x}$, where α is the repeated root.

Solutionbank FP2

Edexcel AS and A Level Modular Mathematics

Exercise B, Question 9

Question:

$$16\frac{d^2y}{dx^2} - 24\frac{dy}{dx} + 9y = 0$$

Solution:

$$16\frac{d^2y}{dx^2} - 24\frac{dy}{dx} + 9y = 0$$

The auxiliary equation is

$$16m^2 - 24m + 9 = 0$$

$\therefore \quad (4m - 3)^2 = 0$

$\therefore \quad m = \frac{3}{4}$ only.

So the general solution is

$$y = (A + Bx)e^{\frac{3}{4}x}.$$

The auxiliary equation has repeated roots and so the general solution is of the form $(A + Bx)e^{\alpha x}$, where α is the repeated root.

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Edexcel AS and A Level Modular Mathematics

Exercise B, Question 10

Question:

$$\frac{d^2y}{dx^2} + 2\sqrt{3}\frac{dy}{dx} + 3y = 0$$

Solution:

$$\frac{d^2y}{dx^2} + 2\sqrt{3}\frac{dy}{dx} + 3y = 0$$

The auxiliary equation is

$$m^2 + 2\sqrt{3}m + 3 = 0$$

$$\therefore \quad (m + \sqrt{3})^2 = 0$$

$$\therefore \quad m = -\sqrt{3}$$

or using quadratic formula:

$$m = \frac{-2\sqrt{3} \pm \sqrt{12 - 12}}{2}$$

$$= -\sqrt{3}$$

So the general solution is

$$y = (A + Bx)e^{-\sqrt{3}x}.$$

The auxiliary equation has repeated roots and so the general solution is of the form $(A + Bx)e^{\alpha x}$, where α is the repeated root.

Solutionbank FP2
Edexcel AS and A Level Modular Mathematics

Exercise C, Question 1

Question:

$$\frac{d^2y}{dx^2} + 25y = 0$$

Solution:

$$\frac{d^2y}{dx^2} + 25y = 0$$

The auxiliary equation is

$m^2 + 25 = 0$

$\therefore \quad m = \pm 5i$

The general solution is

$y = A\cos 5x + B\sin 5x.$

The auxiliary equation has imaginary roots and so the general solution has the form $A \cos \omega x + B \sin \omega x$, where A and B are constants and where $i\omega$ is a solution of the auxiliary equation.

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Exercise C, Question 2

Question:

$$\frac{d^2y}{dx^2} + 81y = 0$$

Solution:

$$\frac{d^2y}{dx^2} + 81y = 0$$

The auxiliary equation is

$$m^2 + 81 = 0$$

$$\therefore \quad m = \pm 9\text{i}$$

The general solution is

$$y = A\cos 9x + B\sin 9x.$$

The auxiliary equation has imaginary roots and so the general solution has the form $A \cos \omega x + B \sin \omega x$, where A and B are constants and where $\text{i}\omega$ is a solution of the auxiliary equation.

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Exercise C, Question 3

Question:

$$\frac{d^2y}{dx^2} + y = 0$$

Solution:

$$\frac{d^2y}{dx^2} + y = 0$$

The auxiliary equation is

$$m^2 + 1 = 0$$

$$\therefore \quad m = \pm i$$

The general solution is

$$y = A\cos x + B\sin x.$$

The auxiliary equation has imaginary roots and so the general solution has the form $A \cos \omega x + B \sin \omega x$, where A and B are constants and where $i\omega$ is a solution of the auxiliary equation.

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Exercise C, Question 4

Question:

$$9\frac{d^2y}{dx^2} + 16y = 0$$

Solution:

$$9\frac{d^2y}{dx^2} + 16y = 0$$

The auxiliary equation is

$$9m^2 + 16 = 0$$

$\therefore \quad m^2 = -\frac{16}{9}$

and $\quad m = \pm\frac{4}{3}\mathrm{i}$

$\therefore$ The general solution is

$$y = A\cos\tfrac{4}{3}x + B\sin\tfrac{4}{3}x.$$

The auxiliary equation has imaginary roots and so the general solution has the form $A\cos\omega x + B\sin\omega x$, where A and B are constants and where $\mathrm{i}\omega$ is a solution of the auxiliary equation.

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Exercise C, Question 5

Question:

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 13y = 0$$

Solution:

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 13y = 0$$

The auxiliary equation is

$$m^2 + 4m + 13 = 0$$

$$\therefore \quad m = \frac{-4 \pm \sqrt{16 - 52}}{2}$$

And $\quad m = -2 \pm 3\text{i}$

The general solution is

$$y = e^{-2x}(A\cos 3x + B\sin 3x).$$

The auxiliary equation has complex roots and so the general solution has the form $e^{px}(A\cos qx + B\sin qx)$, where A and B are constants and where $p \pm iq$ are solutions of the auxiliary equation.

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Exercise C, Question 6

Question:

$$\frac{d^2y}{dx^2} + 8\frac{dy}{dx} + 17y = 0$$

Solution:

$$\frac{d^2y}{dx^2} + 8\frac{dy}{dx} + 17y = 0$$

The auxiliary equation is

$$m^2 + 8m + 17 = 0$$

$$\therefore \quad m = \frac{-8 \pm \sqrt{64 - 4 \times 17}}{2}$$

$$= -4 \pm \tfrac{1}{2}\sqrt{-4}$$

$$= -4 \pm \mathrm{i}$$

The general solution is

$$y = e^{-4x}(A\cos x + B\sin x).$$

The auxiliary equation has complex roots and so the general solution has the form $e^{px}(A\cos qx + B\sin qx)$, where A and B are constants and where $p \pm \mathrm{i}q$ are solutions of the auxiliary equation.

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Exercise C, Question 7

Question:

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 5y = 0$$

Solution:

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 5y = 0$$

The auxiliary equation is

$$m^2 - 4m + 5 = 0$$

$$\therefore \quad m = \frac{4 \pm \sqrt{16 - 20}}{2}$$

$$= 2 \pm \tfrac{1}{2}\sqrt{-4}$$

$$= 2 \pm \mathrm{i}$$

$$\therefore \quad y = e^{2x}(A\cos x + B\sin x).$$

The auxiliary equation has complex roots and so the general solution has the form $e^{px}(A\cos qx + B\sin qx)$, where A and B are constants and where $p \pm \mathrm{i}q$ are solutions of the auxiliary equation.

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Exercise C, Question 8

Question:

$$\frac{d^2y}{dx^2} + 20\frac{dy}{dx} + 109y = 0$$

Solution:

$$\frac{d^2y}{dx^2} + 20\frac{dy}{dx} + 109y = 0$$

The auxiliary equation is

$$m^2 + 20m + 109 = 0$$

$$\therefore \quad m = \frac{-20 \pm \sqrt{400 - 436}}{2}$$

$$= \frac{-20 \pm \sqrt{-36}}{2}$$

$$= -10 \pm 3i$$

The auxiliary equation has complex roots and so the general solution has the form $e^{px}(A \cos qx + B \sin qx)$, where A and B are constants and where $p \pm iq$ are solutions of the auxiliary equation.

$\therefore$ The general solution is

$$y = e^{-10x}(A \cos 3x + B \sin 3x).$$

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Exercise C, Question 9

Question:

$$9\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 5y = 0$$

Solution:

$$9\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 5y = 0$$

The auxiliary equation is

$$9m^2 - 6m + 5 = 0$$

$$\therefore \quad m = \frac{6 \pm \sqrt{36 - 4 \times 9 \times 5}}{2 \times 9}$$

$$= \frac{6 \pm \sqrt{36 - 180}}{18}$$

$$= \frac{6 \pm \sqrt{-144}}{18}$$

$$= \frac{1 \pm 2i}{3}$$

$\therefore$ The general solution is

$$y = e^{\frac{1}{3}x}(A\cos\tfrac{2}{3}x + B\sin\tfrac{2}{3}x).$$

The auxiliary equation has complex roots and so the general solution has the form $e^{px}(A\cos qx + B\sin qx)$, where A and B are constants and where $p \pm iq$ are solutions of the auxiliary equation.

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Exercise C, Question 10

Question:

$$\frac{d^2y}{dx^2} + \sqrt{3}\frac{dy}{dx} + 3y = 0$$

Solution:

$$\frac{d^2y}{dx^2} + \sqrt{3}\frac{dy}{dx} + 3y = 0$$

The auxiliary equation is

$$m^2 + \sqrt{3}m + 3 = 0$$

$$\therefore \quad m = \frac{-\sqrt{3} \pm \sqrt{3 - 4 \times 3}}{2}$$

$$= \frac{-\sqrt{3} \pm \sqrt{-9}}{2}$$

$$= \frac{-\sqrt{3} \pm 3i}{2}$$

$\therefore$ The general solution is

$$y = e^{-\frac{\sqrt{3}}{2}x}(A\cos\tfrac{3}{2}x + B\sin\tfrac{3}{2}x).$$

The auxiliary equation has complex roots and so the general solution has the form $e^{px}(A \cos qx + B \sin qx)$, where A and B are constants and where $p \pm iq$ are solutions of the auxiliary equation.

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Exercise D, Question 1

Question:

$$\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 5y = 10$$

Solution:

$$\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 5y = 10 \quad *$$

First consider

$$\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 5y = 0$$

The auxiliary equation is

$$m^2 + 6m + 5 = 0$$

$$\therefore \quad (m + 5)(m + 1) = 0$$

$$\therefore \quad m = -5 \text{ or } -1$$

So the complementary function is $y = Ae^{-x} + Be^{-5x}$.

The particular integral is λ and so $\frac{dy}{dx} = 0$,

$\frac{d^2y}{dx^2} = 0$ and substituting into $*$ gives

$$5\lambda = 10$$

$$\therefore \quad \lambda = 2$$

The general solution is $y = Ae^{-x} + Be^{-5x} + 2$.

Find the complementary function, which is the solution of $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 5y = 0$, then try a particular integral $y = \lambda$.

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Exercise D, Question 2

Question:

$$\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 12y = 36x$$

Solution:

$$\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 12y = 36x \quad *$$

First consider the equation

$$\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 12y = 0.$$

The auxiliary equation is

$$m^2 - 8m + 12 = 0$$

$\therefore \quad (m - 6)(m - 2) = 0$

$\therefore \quad m = 6$ or 2

So the complementary function is $y = Ae^{6x} + Be^{2x}$.

Try a particular integral of the form $\lambda + \mu x$.

The particular integral is $y = \lambda + \mu x$

so $\quad \frac{dy}{dx} = \mu, \frac{d^2y}{dx^2} = 0$

Substitute into $*$.

Then $\quad -8\mu + 12\lambda + 12\mu x = 36x$.

Comparing coefficients of x: $\quad 12\mu = 36$, and so $\mu = 3$

Comparing constant terms: $-8\mu + 12\lambda = 0$

and as $\mu = 3 \quad \therefore \quad -24 + 12\lambda = 0 \Rightarrow \lambda = 2$

$\therefore \quad 2 + 3x$ is the particular integral.

$\therefore \quad$ The general solution is

$$y = Ae^{6x} + Be^{2x} + 2 + 3x.$$

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Exercise D, Question 3

Question:

$$\frac{d^2y}{dx^2}+\frac{dy}{dx}-12y=12e^{2x}$$

Solution:

$$\frac{d^2y}{dx^2}+\frac{dy}{dx}-12y=12e^{2x} \quad *$$

First consider the equation

$$\frac{d^2y}{dx^2}+\frac{dy}{dx}-12y=0.$$

The auxiliary equation is

$$m^2+m-12=0$$
$$\therefore \quad (m+4)(m-3)=0$$
$$\therefore \quad m=-4 \text{ or } 3$$

So the complementary function is $y=Ae^{-4x}+Be^{3x}$.

The particular integral is $y=\lambda e^{2x}$

Try a particular integral of the form λe^{2x}.

$$\therefore \quad \frac{dy}{dx}=2\lambda e^{2x} \text{ and } \frac{d^2y}{dx^2}=4\lambda e^{2x}$$

Substitute into $*$.

Then $4\lambda e^{2x}+2\lambda e^{2x}-12\lambda e^{2x}=12e^{2x}$

i.e. $-6\lambda e^{2x}=12e^{2x}$

$\therefore \quad \lambda=-2$

$\therefore \quad -2e^{2x}$ is a particular integral.

The general solution is

$$y=Ae^{-4x}+Be^{3x}-2e^{2x}.$$

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Exercise D, Question 4

Question:

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 15y = 5$$

Solution:

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 15y = 5 \quad *$$

First consider the equation

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 15y = 0$$

The auxiliary equation is

$$m^2 + 2m - 15 = 0$$

$\therefore \quad (m + 5)(m - 3) = 0$

$\therefore \quad m = -5 \text{ or } 3$

So the complementary function is $y = Ae^{-5x} + Be^{3x}$.

The particular integral is $y = \lambda$

$\therefore \quad \frac{dy}{dx} = \frac{d^2y}{dx^2} = 0$

Substitute into $*$.

Then $\quad -15\lambda = 5$

i.e. $\quad \lambda = -\frac{1}{3}$

$\therefore \quad -\frac{1}{3}$ is the particular integral.

The general solution is $\quad y = Ae^{-5x} + Be^{3x} - \frac{1}{3}$.

Try a particular integral $y = \lambda$.

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Exercise D, Question 5

Question:

$$\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 16y = 8x + 12$$

Solution:

$$\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 16y = 8x + 12 \quad *$$

First consider the equation

$$\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 16y = 0$$

The auxiliary equation is

$$m^2 - 8m + 16 = 0$$

$\therefore \quad (m-4)^2 = 0$

$\therefore \quad m = 4$ only.

So the complementary function is $y = (A + Bx)e^{4x}$.

The auxiliary equation has a repeated root so the complementary function is of the form $(A + Bx)e^{\alpha x}$.

The particular integral is $y = \lambda + \mu x$

$\therefore \quad \frac{dy}{dx} = \mu$ and $\frac{d^2y}{dx^2} = 0$

Substitute in $*$.

Then $\quad 0 - 8\mu + 16\lambda + 16\mu x = 8x + 12$

Equate coefficients of x: $\quad 16\mu = 8$

$\therefore \quad \mu = \frac{1}{2}$

Equate constant terms: $-8\mu + 16\lambda = 12$

Substitute $\mu = \frac{1}{2} \quad \therefore \quad -4 + 16\lambda = 12$

$\therefore \quad 16\lambda = 16$

and $\quad \lambda = 1$

$\therefore \quad 1 + \frac{1}{2}x$ is a particular integral

The general solution is $y = (A + Bx)e^{4x} + 1 + \frac{1}{2}x$.

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Exercise D, Question 6

Question:

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 25\cos 2x$$

Solution:

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 25\cos 2x \quad *$$

Solve $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 0$

The auxiliary equation is

$$m^2 + 2m + 1 = 0$$

$\therefore \quad (m+1)^2 = 0$

$\therefore \quad m = -1$ only.

So the complementary function is $y = (A + Bx)e^{-x}$.

The particular integral is $y = \lambda \cos 2x + \mu \sin 2x$

$\therefore \quad \frac{dy}{dx} = -2\lambda \sin 2x + 2\mu \cos 2x$

$$\frac{d^2y}{dx^2} = -4\lambda \cos 2x - 4\mu \sin 2x$$

The complementary function is of the form $y = (A + Bx)e^{\alpha x}$.
The particular integral is $\lambda \cos 2x + \mu \sin 2x$.

Substitute in $*$.

Then $(-4\lambda \cos 2x - 4\mu \sin 2x) + 2(-2\lambda \sin 2x + 2\mu \cos 2x)$
$+ (\lambda \cos 2x + \mu \sin 2x) = 25\cos 2x$

Equate coefficients of $\cos 2x$: $\quad -3\lambda + 4\mu = 25 \quad ①$

Equate coefficients of $\sin 2x$: $\quad -3\mu - 4\lambda = 0 \quad ②$

Solve equations ① and ②: $\quad 3 \times ① + 4 \times ② \Rightarrow -25\lambda = 75$

$\therefore \quad \lambda = -3$

Substitute into ① $9 + 4\mu = 25 \quad \therefore \quad \mu = 4$ [check in ②.]

$\therefore$ The particular integral is $y = 4\sin 2x - 3\cos 2x$

$\therefore$ General solution is $y = (A + Bx)e^{-x} + 4\sin 2x - 3\cos 2x$.

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Exercise D, Question 7

Question:

$$\frac{d^2y}{dx^2} + 81y = 15e^{3x}$$

Solution:

$$\frac{d^2y}{dx^2} + 81y = 15e^{3x} \quad *$$

First solve $\frac{d^2y}{dx^2} + 81y = 0$

This has auxiliary equation

$$m^2 + 81 = 0$$

$$\therefore \quad m = \pm 9i$$

The complementary function is $y = A\cos 9x + B\sin 9x$.

The auxiliary equation has imaginary roots, so the complementary function is of the form $A\cos\omega x + B\sin\omega x$.

The particular integral is $y = \lambda e^{3x}$

Then $\frac{dy}{dx} = 3\lambda e^{3x}$ and $\frac{d^2y}{dx^2} = 9\lambda e^{3x}$

Substitute into $*$.

Then $9\lambda e^{3x} + 81\lambda e^{3x} = 15e^{3x}$

$\therefore \quad 90\lambda e^{3x} = 15e^{3x}$

So $\lambda = \frac{15}{90} = \frac{1}{6}$

$\therefore$ The particular integral is $\frac{1}{6}e^{3x}$

$\therefore$ The general solution is $y = A\cos 9x + B\sin 9x + \frac{1}{6}e^{3x}$.

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Exercise D, Question 8

Question:

$$\frac{d^2y}{dx^2} + 4y = \sin x$$

Solution:

$$\frac{d^2y}{dx^2} + 4y = \sin x \quad *$$

First solve $\frac{d^2y}{dx^2} + 4y = 0.$

This has auxiliary equation

$$m^2 + 4 = 0$$

$$\therefore \quad m = \pm 2\mathrm{i}$$

The complementary function is $y = A\cos 2x + B\sin 2x$

The complementary function is of the form $A\cos \omega x + B\sin \omega x$, as the auxiliary equation has imaginary roots.

The particular integral is $y = \lambda \cos x + \mu \sin x$

$$\therefore \quad \frac{dy}{dx} = -\lambda \sin x + \mu \cos x$$

and $$\frac{d^2y}{dx^2} = -\lambda \cos x - \mu \sin x$$

Substitute into $*$.

Then $-\lambda \cos x - \mu \sin x + 4(\lambda \cos x + \mu \sin x) = \sin x$

Equate coefficients of $\cos x$: $3\lambda = 0$

$\therefore$ $\lambda = 0$

Equate coefficients of $\sin x$: $3\mu = 1$

$\therefore$ $\mu = \frac{1}{3}$

So the particular integral is $\frac{1}{3}\sin x$

The general solution is $y = A\cos 2x + B\sin 2x + \frac{1}{3}\sin x$.

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Exercise D, Question 9

Question:

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 5y = 25x^2 - 7$$

Solution:

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 5y = 25x^2 - 7 \quad *$$

First solve $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 5y = 0$

This has auxiliary equation

$$m^2 - 4m + 5 = 0$$

$$\therefore \quad m = \frac{4 \pm \sqrt{16 - 20}}{2}$$

$$= 2 \pm 2i$$

The complementary function is $y = e^{2x}(A\cos 2x + B\sin 2x)$

The particular integral is $y = \lambda + \mu x + \nu x^2$

The P.I is of the form $y = \lambda + \mu x + \nu x^2$

$$\therefore \quad \frac{dy}{dx} = \mu + 2\nu x$$

and $\frac{d^2y}{dx^2} = 2\nu$

Substitute into $*$.

Then $2\nu - 4\mu - 8\nu x + 5\lambda + 5\mu x + 5\nu x^2 = 25x^2 - 7$

Equate coefficients of x^2: $5\nu = 25 \Rightarrow \nu = 5$

coefficients of x: $5\mu - 8\nu = 0 \Rightarrow \mu = 8$

constant terms: $2\nu - 4\mu + 5\lambda = -7$

$$\therefore \quad 10 - 32 + 5\lambda = -7$$

$$\therefore \quad 5\lambda = 15 \Rightarrow \lambda = 3$$

So the particular integral is $3 + 8x + 5x^2$

The general solution is $y = e^{2x}(A\cos 2x + B\sin 2x) + 3 + 8x + 5x^2$.

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Exercise D, Question 10

Question:

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 26y = e^x$$

Solution:

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 26y = e^x \quad *$$

First solve $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 26y = 0$

This has auxiliary equation

$$m^2 - 2m + 26 = 0$$

$$\therefore \quad m = \frac{2 \pm \sqrt{4 - 4 \times 26}}{2}$$

$$= \frac{2 \pm \sqrt{-100}}{2}$$

$$= 1 \pm 5i$$

The auxiliary equation has complex roots and so the complementary function is of the form $e^{px}(A \cos qx + B \sin qx)$.

$\therefore$ the complementary function is $y = e^x(A \cos 5x + B \sin 5x)$.

The particular integral is λe^x, so $\frac{dy}{dx} = \lambda e^x$ and

$$\frac{d^2y}{dx^2} = \lambda e^x$$

Substitute into equation $*$.

Then $\lambda e^x - 2\lambda e^x + 26\lambda e^x = e^x$

i.e. $25\lambda e^x = e^x$

$\therefore \quad \lambda = \frac{1}{25}$

The particular integral is $\frac{1}{25}e^x$.

$\therefore$ The general solution is

$$y = e^x(A \cos 5x + B \sin 5x) + \frac{1}{25}e^x.$$

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Exercise D, Question 11

Question:

a Find the value of λ for which $\lambda x^2 e^x$ is a particular integral for the differential equation

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = e^x$$

b Hence find the general solution.

Solution:

a $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = e^x \quad *$

Given $y = \lambda x^2 e^x$ is a particular integral

$$\frac{dy}{dx} = \lambda x^2 e^x + 2\lambda x e^x$$

$$\frac{d^2y}{dx^2} = \lambda x^2 e^x + 2\lambda x e^x + 2\lambda x e^x + 2\lambda e^x$$

Substitute into $*$.

Then $(\lambda x^2 + 4\lambda x + 2\lambda)e^x - (2\lambda x^2 + 4\lambda x)e^x + \lambda x^2 e^x = e^x$

$\therefore \quad 2\lambda e^x = e^x$

$\therefore \quad \lambda = \frac{1}{2}$

So $y = \frac{1}{2}x^2 e^x$ is a particular integral.

b Now solve $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 0$

This has auxiliary equation $m^2 - 2m + 1 = 0$

$\therefore \quad (m-1)^2 = 0$

$\therefore \quad m = 1$ only

So the complementary function is $(A + Bx)e^x$

The general solution is $y = (A + Bx + \frac{1}{2}x^2)e^x$.

The auxiliary equation has equal roots and so the complementary function has the form $y = (A + Bx)e^{\alpha x}$

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Exercise E, Question 1

Question:

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 12e^x \qquad y = 1 \text{ and } \frac{dy}{dx} = 0 \text{ at } x = 0$$

Solution:

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 12e^x \quad *$$

Find complementary function.

Auxiliary equation is $m^2 + 5m + 6 = 0$

$\therefore \quad (m + 3)(m + 2) = 0$

$\therefore \quad m = -3 \text{ or } -2$

$\therefore$ complementary function is $y = Ae^{-3x} + Be^{-2x}$

Then find particular integral

Let $y = \lambda e^x$

Then $\frac{dy}{dx} = \lambda e^x$ and $\frac{d^2y}{dx^2} = \lambda e^x$

Substitute into $*$. Then $(\lambda + 5\lambda + 6\lambda)e^x = 12e^x$

$\therefore \quad 12\lambda e^x = 12e^x$

$\therefore \quad \lambda = 1$

So particular integral is $y = e^x$

$\therefore$ General solution is $Ae^{-3x} + Be^{-2x} + e^x = y \quad \dagger$

But $y = 1$ when $x = 0 \quad \therefore \quad A + B + 1 = 1$

i.e. $\quad A + B = 0 \quad ①$

$$\frac{dy}{dx} = -3Ae^{-3x} - 2Be^{-2x} + e^x$$

$\frac{dy}{dx} = 0$ when $x = 0 \quad \therefore \quad -3A - 2B + 1 = 0$

$\therefore \quad 3A + 2B = 1 \quad ②$

From ① $B = -A$, substitute into equation ②

$3A - 2A = 1 \Rightarrow A = 1$

$\therefore \quad B = -1$

Substitute these values into $\dagger$

The particuar solution is $y = e^{-3x} - e^{-2x} + e^x$

Solve the equation to find the general solution, then substitute $y = 1$ when $x = 0$ to obtain an equation relating A and B. Obtain a second equation by using $\frac{dy}{dx} = 0$ at $x = 0$, and solve to find A and B.

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Exercise E, Question 2

Question:

$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 12e^{2x}$ $\quad$ $y = 2$ and $\frac{dy}{dx} = 6$ at $x = 0$

Solution:

$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 12e^{2x}$ $\ast$

Find complementary function (c.f.):

Auxiliary equation is $m^2 + 2m = 0$

$\therefore$ $m(m+2) = 0$

$\therefore$ $m = 0$ or -2

$\therefore$ c.f. is $y = Ae^{0x} + Be^{-2x}$

$= A + Be^{-2x}$

Particular integral (p.i.) is of the form $y = \lambda e^{2x}$

$\therefore$ $\frac{dy}{dx} = 2\lambda e^{2x}$, $\frac{d^2y}{dx^2} = 4\lambda e^{2x}$

Substitute into $\ast$.

Then $(4\lambda + 4\lambda)e^{2x} = 12e^{2x}$

i.e. $8\lambda e^{2x} = 12e^{2x} \Rightarrow \lambda = \frac{12}{8} = \frac{3}{2}$

$\therefore$ p.i. is $\frac{3}{2}e^{2x}$

$\therefore$ General solution is $y = A + Be^{-2x} + \frac{3}{2}e^{2x}$ †

The general solution is $y = A + Be^{-2x} + \frac{3}{2}e^{2x}$.

But $y = 2$ when $x = 0$ $\therefore$ $2 = A + B + \frac{3}{2}$

i.e. $A + B = \frac{1}{2}$ ①

$\frac{dy}{dx} = -2Be^{-2x} + 3e^{2x}$

$\frac{dy}{dx} = 6$ when $x = 0$ $\therefore$ $6 = -2B + 3$

$\therefore$ $-2B = 3 \Rightarrow B = -\frac{3}{2}$

Substitute into equation ① $A - \frac{3}{2} = \frac{1}{2}$

$\therefore$ $A = 2$

Substitute A and B into †

$\therefore$ The particular solution is $y = 2 - \frac{3}{2}e^{-2x} + \frac{3}{2}e^{2x}$

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Exercise E, Question 3

Question:

$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 42y = 14$ $\qquad y = 0$ and $\frac{dy}{dx} = \frac{1}{6}$ at $x = 0$

Solution:

$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 42y = 14 \quad *$

Find the general solution, then use the boundary conditions to find the constants A and B.

Find c.f.: The auxiliary equation is

$m^2 - m - 42 = 0$

$\therefore \quad (m-7)(m+6) = 0$

$\therefore \quad m = -6$ or 7

$\therefore$ c.f. is $y = Ae^{-6x} + Be^{7x}$

Find p.i.: The particular integral is $y = \lambda$. Substitute in $*$.

$\therefore \quad -42\lambda = 14$

$\therefore \quad \lambda = -\frac{1}{3}$

$\therefore$ The general solution is $y = Ae^{-6x} + Be^{7x} - \frac{1}{3} \quad \dagger$

When $x = 0, y = 0 \quad \therefore \quad 0 = A + B - \frac{1}{3}$

$\therefore \quad A + B = \frac{1}{3} \quad ①$

$\frac{dy}{dx} = -6Ae^{-6x} + 7Be^{7x}$

When $x = 0$, $\frac{dy}{dx} = \frac{1}{6} \quad \therefore \quad \frac{1}{6} = -6A + 7B$

i.e: $-6A + 7B = \frac{1}{6} \quad ②$

Solve equations ① and ② by forming $6 \times ① + ②$

$\therefore \quad 13B = 2\frac{1}{6}$

$B = \frac{1}{6}$

Substitute into ① $\therefore \quad A + \frac{1}{6} = \frac{1}{3} \Rightarrow A = \frac{1}{6}$

Substitute values of A and B into $\dagger$

$\therefore \quad y = \frac{1}{6}e^{-6x} + \frac{1}{6}e^{7x} - \frac{1}{3}$ is required solution

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Exercise E, Question 4

Question:

$\frac{d^2y}{dx^2} + 9y = 16 \sin x$ $\quad\quad$ $y = 1$ and $\frac{dy}{dx} = 0$ at $x = 0$

Solution:

$$\frac{d^2y}{dx^2} + 9y = 16\sin x \quad *$$

Find c.f.: The auxiliary equation is

$m^2 + 9 = 0$

$\therefore \quad m = \pm 3\text{i}$

$\therefore$ The c.f. is $y = A\cos 3x + B\sin 3x$

The auxiliary equation has imaginary roots and so the complementary function has the form $y = A\cos \omega x + B\sin \omega x$.

Find p.i. use $y = \lambda\cos x + \mu\sin x$

$$\frac{dy}{dx} = -\lambda\sin x + \mu\cos x$$

$$\frac{d^2y}{dx^2} = -\lambda\cos x - \mu\sin x$$

$\therefore$ Substituting into $*$ gives

$$-\lambda\cos x - \mu\sin x + 9\lambda\cos x + 9\mu\sin x = 16\sin x$$

Equating coefficients of $\cos x$: $8\lambda = 0 \Rightarrow \lambda = 0$

$\sin x$: $8\mu = 16 \Rightarrow \mu = 2$

$\therefore$ The particular integral is $y = 2\sin x$

$\therefore$ The general solution is $y = A\cos 3x + B\sin 3x + 2\sin x \quad \dagger$

Given also that $y = 1$ at $x = 0 \quad \therefore \quad 1 = A$

$$\frac{dy}{dx} = -3A\sin 3x + 3B\cos 3x + 2\cos x$$

Using $\frac{dy}{dx} = 8$ at $x = 0 \quad \therefore \quad 8 = 3B + 2 \quad \therefore \quad B = 2$

Substituting A and B into $\dagger$

$y = \cos 3x + 2\sin 3x + 2\sin x$ is the required solution.

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Exercise E, Question 5

Question:

$$4\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 5y = \sin x + 4\cos x \qquad y = 0 \text{ and } \frac{dy}{dx} = 0 \text{ at } x = 0$$

Solution:

$$4\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 5y = \sin x + 4\cos x \quad *$$

The auxiliary equation has complex roots and so the complementary function has the form $y = e^{px}(A\cos qx + B\sin qx)$.

Find c.f.: the auxiliary equation is

$$4m^2 + 4m + 5 = 0$$

$\therefore \quad m = \frac{-4 \pm \sqrt{16 - 80}}{8} = \frac{-4 \pm \sqrt{-64}}{8} = \frac{-4 \pm 8i}{8}$

$\therefore \quad m = -\frac{1}{2} \pm i$

$\therefore$ The c.f. is $y = e^{-\frac{1}{2}x}(A\cos x + B\sin x)$

The p.i. is $y = \lambda\cos x + \mu\sin x$

$\therefore \quad \frac{dy}{dx} = -\lambda\sin x + \mu\cos x$

$\quad \frac{d^2y}{dx^2} = -\lambda\cos x - \mu\sin x$

Substitute into $*$

Then $-4\lambda\cos x - 4\mu\sin x - 4\lambda\sin x + 4\mu\cos x + 5\lambda\cos x + 5\mu\sin x = \sin x + 4\cos x$

Equating coefficients of $\cos x$: $\lambda + 4\mu = 4$ ①

$\sin x$: $\mu - 4\lambda = 1$ ②

Add equation ② to 4 times equation ①

$\therefore \quad 17\mu = 17 \Rightarrow \mu = 1$

Substitute into equation ① $\therefore \lambda + 4 = 4 \Rightarrow \lambda = 0$

$\therefore$ p.i. is $y = \sin x$

$\therefore$ The general solution is

$$y = e^{-\frac{1}{2}x}(A\cos x + B\sin x) + \sin x \quad †$$

As $y = 0$ when $x = 0$

$\therefore \quad 0 = A$

$\therefore \quad y = Be^{-\frac{1}{2}x}\sin x + \sin x$

$\therefore \quad \frac{dy}{dx} = Be^{-\frac{1}{2}x}\cos x - \frac{1}{2}Be^{-\frac{1}{2}x}\sin x + \cos x$

As $\frac{dy}{dx} = 0$ when $x = 0$

$\quad 0 = B + 1 \Rightarrow B = -1$

Substituting these values for A and B into †

$\therefore \quad y = \sin x\,(1 - e^{-\frac{1}{2}x})$ is the required solution.

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Exercise E, Question 6

Question:

$\frac{d^2x}{dt^2} - 3\frac{dx}{dt} + 2x = 2t - 3$ $\quad$ $x = 2$ and $\frac{dx}{dt} = 4$ when $t = 0$

Solution:

$\frac{d^2x}{dt^2} - 3\frac{dx}{dt} + 2x = 2t - 3 \quad *$

This time t is the independent variable, and x the dependent variable. The method of solution is the same as in the questions connecting x and y.

Find c.f.: the auxiliary equation is

$m^2 - 3m + 2 = 0$

$\therefore \quad (m - 2)(m - 1) = 0$

$\therefore \quad m = 1$ or 2

$\therefore$ c.f. is $x = Ae^t + Be^{2t}$

The p.i. is $x = \lambda + \mu t$, $\frac{dx}{dt} = \mu$, $\frac{d^2x}{dt^2} = 0$

Substitute into $*$ to give $-3\mu + 2\lambda + 2\mu t = 2t - 3$

Equate coefficients of t: $\quad 2\mu = 2 \Rightarrow \mu = 1$

Equate constant terms: $\quad 2\lambda - 3\mu = -3 \quad \therefore \quad \lambda = 0$

The particular integral is t.

$\therefore$ The general solution is $x = Ae^t + Be^{2t} + t \quad \dagger$

Given that $x = 2$ when $t = 0 \quad \therefore \quad 2 = A + B \quad ①$

Also $\frac{dx}{dt} = Ae^t + 2Be^{2t} + 1$

As $\frac{dx}{dt} = 4$ when $t = 0 \quad \therefore \quad 4 = A + 2B + 1$

$\therefore \quad A + 2B = 3 \quad ②$

Subtract $\quad ② - ① \Rightarrow B = 1$

Substitute into $\quad \therefore \quad A = 1$

Substituting the values of A and B back into $\dagger$

$x = e^t + e^{2t} + t$

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Exercise E, Question 7

Question:

$\frac{d^2x}{dt^2} - 9x = 10 \sin t$ $\quad\quad x = 2$ and $\frac{dx}{dt} = -1$ when $t = 0$

Solution:

$\frac{d^2x}{dt^2} - 9x = 10\sin t \quad *$

Find c.f.: auxiliary equation is

$m^2 - 9 = 0$

$\therefore \quad m = \pm 3$

$\therefore$ c.f. is $x = Ae^{3t} + Be^{-3t}$

p.i. is of the form $x = \lambda\cos t + \mu\sin t$

The particular integral is of the form $\lambda\cos t + \mu\sin t$.

$\therefore \quad \frac{dx}{dt} = -\lambda\sin t + \mu\cos t$

$\frac{d^2x}{dt^2} = -\lambda\cos t - \mu\sin t$

Substitute into equation $*$.

Then $-\lambda\cos t - \mu\sin t - 9\lambda\cos t - 9\mu\sin t = 10\sin t$

Equate coefficients of $\cos t$: $\therefore \quad -10\lambda = 0 \Rightarrow \lambda = 0$

Equate coefficients of $\sin t$: $\therefore \quad -10\mu = 10 \Rightarrow \mu = -1$

$\therefore$ p.i. is $-\sin t$

$\therefore$ General solution is $x = Ae^{3t} + Be^{-3t} - \sin t \quad †$

When $t = 0, x = 2 \quad \therefore \quad 2 = A + B \quad ①$

$\frac{dx}{dt} = 3Ae^{3t} - 3Be^{-3t} - \cos t$

When $t = 0, \frac{dx}{dt} = -1 \quad \therefore \quad -1 = 3A - 3B - 1$

$\therefore \quad 0 = 3A - 3B \quad ②$

Solving equations ① and ②, $A = B = 1$

$\therefore$ Substitute values of A and B into †

$\therefore \quad x = e^{3t} + e^{-3t} - \sin t$ is the required solution.

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Exercise E, Question 8

Question:

$$\frac{d^2x}{dt^2} - 4\frac{dx}{dt} + 4x = 3te^{2t} \qquad x = 0 \text{ and } \frac{dx}{dt} = 1 \text{ when } t = 0$$

Solution:

$$\frac{d^2x}{dt^2} - 4\frac{dx}{dt} + 4x = 3te^{2t} \quad *$$

The complementary function has the form $x = (A + Bt)e^{\alpha t}$.

Find c.f.: auxiliary equation is

$m^2 - 4m + 4 = 0$

$\therefore \quad (m - 2)^2 = 0$

$\therefore \quad m = 2$ only

$\therefore$ c.f. is $x = (A + Bt)e^{2t}$

Find p.i.: Let p.i. be $x = \lambda t^3 e^{2t}$

Then $\frac{dx}{dt} = 2\lambda t^3 e^{2t} + 3\lambda t^2 e^{2t}$

$$\frac{d^2x}{dt^2} = 4\lambda t^3 e^{2t} + 6\lambda t^2 e^{2t} + 6\lambda t^2 e^{2t} + 6\lambda te^{2t}$$

Substitute into $*$.

Then $(4\lambda t^3 + 12\lambda t^2 + 6\lambda t - 8\lambda t^3 - 12\lambda t^2 + 4\lambda t^3)e^{2t} = 3te^{2t}$

$\therefore \quad 6\lambda = 3 \Rightarrow \lambda = \frac{1}{2}$

$\therefore$ p.i. is $x = \frac{1}{2}t^3 e^{2t}$

$\therefore$ General solution is $x = \left((A + Bt) + \frac{1}{2}t^3\right)e^{2t} \quad \dagger$

But $x = 0$ when $t = 0 \quad \therefore \quad 0 = A$

$$\frac{dx}{dt} = 2\left[A + Bt + \frac{1}{2}t^3\right]e^{2t} + \left[B + \frac{3}{2}t^2\right]e^{2t}$$

As $\frac{dx}{dt} = 1$ when $t = 0$ and $A = 0$

$\therefore \quad 1 = B$

Substitute $A = 0$ and $B = 1$ into $\dagger$

Then $x = \left(t + \frac{1}{2}t^3\right)e^{2t}$ is the required solution.

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Exercise E, Question 9

Question:

$25\frac{d^2x}{dt^2} + 36x = 18$ $x = 1$ and $\frac{dx}{dt} = 0.6$ when $t = 0$

Solution:

$25\frac{d^2x}{dt^2} + 36x = 18$ *

Find c.f.: auxiliary equation is

$25m^2 + 36 = 0$

$\therefore \quad m^2 = -\frac{36}{25}$ and $m = \pm\frac{6}{5}\text{i}$

The auxiliary equation has imaginary roots and so $x = A\cos\omega t + B\sin\omega t$ is the form of the complementary function.

$\therefore$ c.f. is $x = A\cos\frac{6}{5}t + B\sin\frac{6}{5}t$

Let p.i. be $x = \lambda$. Substitute into *

Then $36\lambda = 18$

$\therefore \quad \lambda = \frac{18}{36} = \frac{1}{2}$

$\therefore$ General solution is $x = A\cos\frac{6}{5}t + B\sin\frac{6}{5}t + \frac{1}{2}$ †

When $t = 0, x = 1 \quad \therefore \quad 1 = A + \frac{1}{2} \Rightarrow A = \frac{1}{2} = 0.5$

$$\frac{dx}{dt} = -\frac{6}{5}A\sin\frac{6}{5}t + \frac{6}{5}B\cos\frac{6}{5}t$$

When $t = 0, \frac{dx}{dt} = 0.6 \quad \therefore \quad 0.6 = \frac{6}{5}B$

$\therefore \quad B = 0.5 = \frac{1}{2}$

Substitute values for A and B into †

Then $x = \frac{1}{2}\left(\cos\frac{6}{5}t + \sin\frac{6}{5}t + 1\right)$

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Exercise E, Question 10

Question:

$\frac{d^2x}{dt^2} - 2\frac{dx}{dt} + 2x = 2t^2$ $\qquad x = 1$ and $\frac{dx}{dt} = 3$ when $t = 0$

Solution:

$\frac{d^2x}{dt^2} - 2\frac{dx}{dt} + 2x = 2t^2$ ✱

Find c.f.: auxiliary equation is

$m^2 - 2m + 2 = 0$

$\therefore \quad m = \frac{2 \pm \sqrt{4-8}}{2} = 1 \pm i$

$\therefore$ c.f. is $x = e^t (A \cos t + B \sin t)$

The particular integral has the form $x = \lambda + \mu t + \nu t^2$.

Let p.i. be $x = \lambda + \mu t + \nu t^2$

then $\frac{dx}{dt} = \mu + 2\nu t$

$\frac{d^2x}{dt^2} = 2\nu$

Substitute into ✱

Then $2\nu - 2(\mu + 2\nu t) + 2(\lambda + \mu t + \nu t^2) = 2t^2$

Equate coefficients of t^2: $2\nu = 2 \Rightarrow \nu = 1$

coefficients of t: $-4\nu + 2\mu = 0 \Rightarrow \mu = 2$

constants: $2\nu - 2\mu + 2\lambda = 0 \Rightarrow \lambda = 1$

$\therefore$ p.i. is $x = 1 + 2t + t^2$

$\therefore$ General solution is $x = e^t (A \cos t + B \sin t) + 1 + 2t + t^2$ †

But $x = 1$ when $t = 0$ $\therefore$ $1 = A + 1$ $\therefore$ $A = 0$

As $x = Be^t \sin t + 1 + 2t + t^2$

$\frac{dx}{dt} = Be^t \cos t + Be^t \sin t + 2 + 2t$

As $\frac{dx}{dt} = 3$ when $t = 0$

$\therefore \quad 3 = B + 2$

$\therefore \quad B = 1$

Substitute $A = 0$ and $B = 1$ into the general solution †

$\therefore \quad x = e^t \sin t + 1 + 2t + t^2$ or $x = e^t \sin t + (1 + t)^2$

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Exercise F, Question 1

Question:

$$x^2\frac{d^2y}{dx^2} + 6x\frac{dy}{dx} + 4y = 0$$

Solution:

$$x^2\frac{d^2y}{dx^2} + 6x\frac{dy}{dx} + 4y = 0 \quad *$$

First express $x\frac{dy}{dx}$ as $\frac{dy}{du}$ and $x\frac{d^2y}{dx^2}$ as $\frac{d^2y}{du^2} - \frac{dy}{du}$.

As $x = e^u$, $\frac{dx}{du} = e^u = x$

From the chain rule $\frac{dy}{du} = \frac{dy}{dx} \times \frac{dx}{du}$

$\therefore \quad \frac{dy}{du} = x\frac{dy}{dx} \quad$ ①

Also $\frac{d^2y}{du^2} = \frac{d}{du}\left(x\frac{dy}{dx}\right)$

$$= \frac{dx}{du} \times \frac{dy}{dx} + x\frac{d^2y}{dx^2} \times \frac{dx}{du}$$

$$= \frac{dy}{du} + x^2\frac{d^2y}{dx^2}$$

$\therefore \quad x^2\frac{d^2y}{dx^2} = \frac{d^2y}{du^2} - \frac{dy}{du} \quad$ ②

Use the results ① and ② to change the variable in $*$

$\therefore \quad \frac{d^2y}{du^2} - \frac{dy}{du} + 6\frac{dy}{du} + 4y = 0$

i.e. $\quad \frac{d^2y}{du^2} + 5\frac{dy}{du} + 4y = 0 \quad$ †

This has auxiliary equation

$$m^2 + 5m + 4 = 0$$

$\therefore \quad (m + 4)(m + 1) = 0$

i.e. $\quad m = -4$ or -1

$\therefore$ The solution of the differential equation † is

$$y = Ae^{-4u} + Be^{-u}$$

But $\quad e^u = x$

$\therefore \quad e^{-u} = x^{-1} = \frac{1}{x}$

and $e^{-4u} = x^{-4} = \frac{1}{x^4}$

$\therefore \quad y = \frac{A}{x^4} + \frac{B}{x}$

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Exercise F, Question 2

Question:

$$x^2\frac{d^2y}{dx^2} + 5x\frac{dy}{dx} + 4y = 0$$

Solution:

$$x^2\frac{d^2y}{dx^2} + 5x\frac{dy}{dx} + 4y = 0 \quad *$$

As $x = e^u$, $x\frac{dy}{dx} = \frac{dy}{du}$ and $x^2\frac{d^2y}{dx^2} = \frac{d^2y}{du^2} - \frac{dy}{du}$

(See solution to question 1 for proof of this.)

Use these results to change the variable in $*$.

$\therefore \quad \frac{d^2y}{du^2} - \frac{dy}{du} + 5\frac{dy}{du} + 4y = 0.$

$\therefore \quad \frac{d^2y}{du^2} + 4\frac{dy}{du} + 4y = 0 \quad †$

This has auxiliary equation

$m^2 + 4m + 4 = 0$

$\therefore \quad (m+2)^2 = 0$

$\therefore \quad m = -2$ only

The solution of the differential equation † is thus

$$y = (A + Bu)e^{-2u}$$

As $x = e^u \quad \therefore \quad e^{-2u} = x^{-2} = \frac{1}{x^2}$

and $\quad u = \ln x$

$\therefore \quad y = (A + B\ln x) \times \frac{1}{x^2}$

Use $x\frac{dy}{dx} = \frac{dy}{du}$ and $x^2\frac{d^2y}{dx^2} = \frac{d^2y}{du^2} - \frac{dy}{du}$.

Ensure that you can prove these two results.

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Exercise F, Question 3

Question:

$$x^2\frac{d^2y}{dx^2} + 6x\frac{dy}{dx} + 6y = 0$$

Solution:

$$x^2\frac{d^2y}{dx^2} + 6x\frac{dy}{dx} + 6y = 0 \quad *$$

As $x = e^u$, $x\frac{dy}{dx} = \frac{dy}{du}$ and $x^2\frac{d^2y}{dx^2} = \frac{d^2y}{du^2} - \frac{dy}{du}$

(See solution to question 1 for proof of this.)

Use these results to change the variable in $*$.

$$\therefore \quad \frac{d^2y}{du^2} - \frac{dy}{du} + 6\frac{dy}{du} + 6y = 0$$

$$\therefore \quad \frac{d^2y}{du^2} + 5\frac{dy}{du} + 6y = 0 \quad \dagger$$

This has auxiliary equation

$$m^2 + 5m + 6 = 0$$

$$\therefore \quad (m+2)(m+3) = 0$$

$$\therefore \quad m = -2 \text{ or } -3$$

The solution of the differential equation $\dagger$ is thus

$$y = Ae^{-2u} + Be^{-3u}$$

As $\quad x = e^u$, $e^{-2u} = x^{-2} = \frac{1}{x^2}$

and $\quad e^{-3u} = x^{-3} = \frac{1}{x^3}$

$$\therefore \quad y = \frac{A}{x^2} + \frac{B}{x^3}$$

Use $x\frac{dy}{dx} = \frac{dy}{du}$ and $x^2\frac{d^2y}{dx^2} = \frac{d^2y}{du^2} - \frac{dy}{du}$.

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Exercise F, Question 4

Question:

$$x^2\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} - 28y = 0$$

Solution:

$$x^2\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} - 28y = 0 \quad *$$

As $x = e^u$, $x\frac{dy}{dx} = \frac{dy}{du}$ and $x^2\frac{d^2y}{dx^2} = \frac{d^2y}{du^2} - \frac{dy}{du}$

Use $x\frac{dy}{dx} = \frac{dy}{du}$ and $x^2\frac{d^2y}{dx^2} = \frac{d^2y}{du^2} - \frac{dy}{du}$.

Substitute these results into equation $*$

$\therefore \quad \frac{d^2y}{du^2} - \frac{dy}{du} + 4\frac{dy}{du} - 28y = 0$

$\therefore \quad \frac{d^2y}{du^2} + 3\frac{dy}{du} - 28y = 0 \quad \dagger$

This has auxiliary equation:

$m^2 + 3m - 28 = 0$

$\therefore \quad (m + 7)(m - 4) = 0$

$\therefore \quad m = -7$ or 4

$\therefore \quad y = Ae^{-7u} + Be^{4u}$ is the solution to $\dagger$.

As $x = e^u$, $\therefore \quad e^{-7u} = \frac{1}{x^7}$

and $e^{4u} = x^4$

$\therefore \quad y = \frac{A}{x^7} + Bx^4$

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Exercise F, Question 5

Question:

$$x^2\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} - 14y = 0$$

Solution:

$$x^2\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} - 14y = 0 \quad *$$

As $x = e^u$, $x\frac{dy}{dx} = \frac{dy}{du}$ and $x^2\frac{d^2y}{dx^2} = \frac{d^2y}{du^2} - \frac{dy}{du}$

Substituting these results into $*$ gives

$$\frac{d^2y}{du^2} - \frac{dy}{du} - 4\frac{dy}{du} - 14y = 0$$

i.e. $$\frac{d^2y}{du^2} - 5\frac{dy}{du} - 14y = 0 \quad \dagger$$

This has auxiliary equation:

$$m^2 - 5m - 14 = 0$$

i.e. $(m-7)(m+2) = 0$

$\therefore \quad m = 7$ or -2

$\therefore$ The solution of the differential equation $\dagger$ is

$$y = Ae^{7u} + Be^{-2u}$$

But $\quad x = e^u, \quad \therefore \quad e^{7u} = x^7$

and $\quad e^{-2u} = x^{-2} = \frac{1}{x^2}$

$\therefore \quad y = Ax^7 + \frac{B}{x^2}$

Use $x\frac{dy}{dx} = \frac{dy}{du}$ and $x^2\frac{d^2y}{dx^2} = \frac{d^2y}{du^2} - \frac{dy}{du}$.

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Exercise F, Question 6

Question:

$$x^2\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} + 2y = 0$$

Solution:

$$x^2\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} + 2y = 0 \quad *$$

As $x = e^u$, $x\frac{dy}{dx} = \frac{dy}{du}$ and $x^2\frac{d^2y}{dx^2} = \frac{d^2y}{du^2} - \frac{dy}{du}$

Substitute these results into $*$ to give:

$$\frac{d^2y}{du^2} - \frac{dy}{du} + 3\frac{dy}{du} + 2y = 0$$

i.e. $\frac{d^2y}{du^2} + 2\frac{dy}{du} + 2y = 0 \quad †$

This has auxiliary equation:

$$m^2 + 2m + 2 = 0$$

$$\therefore \quad m = \frac{-2 \pm \sqrt{4-8}}{2}$$

$$= -1 \pm i$$

The solution of the differential equation † is thus

$$y = e^{-u}[A\cos u + B\sin u]$$

As $x = e^u$, $e^{-u} = x^{-1} = \frac{1}{x}$

and $u = \ln x$

$$\therefore \quad y = \frac{1}{x}[A\cos\ln x + B\sin\ln x]$$

Use $x\frac{dy}{dx} = \frac{dy}{du}$ and $x^2\frac{d^2y}{dx^2} = \frac{d^2y}{du^2} - \frac{dy}{du}$.
A proof of these results is given in the book in Section 5.6.

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Exercise F, Question 7

Question:

Use the substitution $y = \frac{z}{x}$ to transform the differential equation

$x\frac{d^2y}{dx^2} + (2 - 4x)\frac{dy}{dx} - 4y = 0$ into the equation $\frac{d^2z}{dx^2} - 4\frac{dz}{dx} = 0$.

Hence solve the equation $x\frac{d^2y}{dx^2} + (2 - 4x)\frac{dy}{dx} - 4y = 0$, giving y in terms of x.

Solution:

Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in terms of $\frac{dz}{dx}$ and $\frac{d^2z}{dx^2}$.

$y = \frac{z}{x}$ implies $xy = z$

$\therefore \quad x\frac{dy}{dx} + y = \frac{dz}{dx}$

Also $\quad x\frac{d^2y}{dx^2} + \frac{dy}{dx} + \frac{dy}{dx} = \frac{d^2z}{dx^2}$

$\therefore$ The equation $\quad x\frac{d^2y}{dx^2} + (2 - 4x)\frac{dy}{dx} - 4y = 0$

becomes $\quad \frac{d^2z}{dx^2} - 4\left(\frac{dz}{dx} - y\right) - 4y = 0$

i.e. $\quad \frac{d^2z}{dx^2} - 4\frac{dz}{dx} = 0 \quad *$

The equation $*$ has auxiliary equation

$m^2 - 4m = 0$

$\therefore \quad m(m - 4) = 0$

i.e. $\quad m = 0$ or 4

$\therefore \quad z = A + Be^{4x}$ is the solution of $*$

But $\quad z = xy$

$\therefore \quad xy = A + Be^{4x}$

$\therefore \quad y = \frac{A}{x} + \frac{B}{x}e^{4x}$

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Exercise F, Question 8

Question:

Use the substitution $y = \frac{z}{x^2}$ to transform the differential equation

$x^2\frac{d^2y}{dx^2} + 2x(x + 2)\frac{dy}{dx} + 2(x + 1)^2y = e^{-x}$ into the equation $\frac{d^2z}{dx^2} + 2\frac{dz}{dx} + 2z = e^{-x}$.

Hence solve the equation $x^2\frac{d^2y}{dx^2} + 2x(x + 2)\frac{dy}{dx} + 2(x + 1)^2y = e^{-x}$, giving y in terms of x.

Solution:

Express $\frac{dz}{dx}$ and $\frac{d^2z}{dx^2}$ in terms of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ respectively.

$y = \frac{z}{x^2}$ implies $z = yx^2$ or $x^2y = z$

$\therefore \quad x^2\frac{dy}{dx} + 2xy = \frac{dz}{dx}$ ①

Also $x^2\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} + 2x\frac{dy}{dx} + 2y = \frac{d^2z}{dx^2}$ ②

The differential equation:

$$x^2\frac{d^2y}{dx^2} + 2x(x+2)\frac{dy}{dx} + 2(x+1)^2y = e^{-x} \text{ can be written}$$

$$\left(x^2\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + 2y\right) + \left(2x^2\frac{dy}{dx} + 4xy\right) + 2x^2y = e^{-x}$$

Using the results ① and ②

$$\frac{d^2z}{dx^2} + 2\frac{dz}{dx} + 2z = e^{-x} \quad \dagger$$

This has auxiliary equation

$m^2 + 2m + 2 = 0$

$\therefore \quad m = \frac{-2 \pm \sqrt{4-8}}{2}$

$m = -1 \pm i$

$\therefore \quad z = e^{-x}(A\cos x + B\sin x)$ is the complementary function

A particular integral of † is $z = \lambda e^{-x}$

$\therefore \quad \frac{dz}{dx} = -\lambda e^{-x}$ and $\frac{d^2z}{dx^2} = \lambda e^{-x}$

Substituting into †

$(\lambda - 2\lambda + 2\lambda)e^{-x} = e^{-x}$

$\therefore \quad \lambda = 1$

So $z = e^{-x}$ is a particular integral.

$\therefore$ The general solution of † is

$z = e^{-x}(A\cos x + B\sin x + 1)$

But $z = x^2y$

$\therefore \quad y = \frac{e^{-x}}{x^2}(A\cos x + B\sin x + 1)$ is the general solution of the given differential equation.

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Exercise F, Question 9

Question:

Use the substitution $z = \sin x$ to transform the differential equation

$\cos x\frac{d^2y}{dx^2} + \sin x\frac{dy}{dx} - 2y\cos^3 x = 2\cos^5 x$ into the equation $\frac{d^2y}{dz^2} - 2y = 2(1 - z^2)$.

Hence solve the equation $\cos x\frac{d^2y}{dx^2} + \sin x\frac{dy}{dx} - 2y\cos^3 x = 2\cos^5 x$, giving y in terms of x.

Solution:

Find $\frac{dy}{dx}$ in terms of $\frac{dy}{dx}$ and find $\frac{d^2y}{dx^2}$ in terms of $\frac{d^2y}{dz^2}$ and $\frac{dy}{dz}$.

$z = \sin x$ implies $\frac{dz}{dx} = \cos x$

$\therefore \quad \frac{dy}{dx} = \frac{dy}{dz} \times \cos x$

And $\frac{d^2y}{dx^2} = \frac{d^2y}{dz^2}\cos^2 x - \frac{dy}{dz}\sin x$

$\therefore$ The equation $\cos x \frac{d^2y}{dx^2} + \sin x \frac{dy}{dx} - 2y\cos^3 x = 2\cos^5 x$

becomes $\cos^3 x \frac{d^2y}{dz^2} - \cos x \sin x \frac{dy}{dz} + \cos x \sin x \frac{dy}{dz} - 2y\cos^3 x = 2\cos^5 x$

$\therefore$ Divide by $\cos^3 x$ gives:

$$\frac{d^2y}{dz^2} - 2y = 2\cos^2 x$$
$$= 2(1 - z^2) \dagger \qquad [\text{as } \cos^2 x = 1 - \sin^2 x = 1 - z^2]$$

First solve $\frac{d^2y}{dz^2} - 2y = 0$

This has auxiliary equation

$m^2 - 2 = 0$

$\therefore \quad m = \pm\sqrt{2}$

$\therefore$ The complementary function is $y = Ae^{\sqrt{2}z} + Be^{-\sqrt{2}z}$.

Let $y = \lambda z^2 + \mu z + \nu$ be a particular integral of the differential equation $\dagger$.

Then $\frac{dy}{dz} = 2\lambda z + \mu$ and $\frac{d^2y}{dz^2} = 2\lambda$

Substitute into $\dagger$

Then $2\lambda - 2(\lambda z^2 + \mu z + \nu) = 2(1 - z^2)$

Compare coefficients of z^2: $-2\lambda = -2 \quad \therefore \quad \lambda = 1$

Compare coefficients of z: $-2\mu = 0 \quad \therefore \quad \mu = 0$

Compare constants: $2\lambda - 2\nu = 2 \quad \therefore \quad \nu = 0$

$\therefore$ z^2 is the particular integral.

$\therefore$ The general solution of $\dagger$ is

$y = Ae^{\sqrt{2}z} + Be^{-\sqrt{2}z} + z^2$.

But $z = \sin x$

$\therefore \quad y = Ae^{\sqrt{2}\sin x} + Be^{-\sqrt{2}\sin x} + \sin^2 x$

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Exercise G, Question 1

Question:

Find the general solution of the differential equation $\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = 0$

Solution:

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = 0$$

Auxiliary equation is

$$m^2 + m + 1 = 0$$

$$\therefore \quad m = \frac{-1 \pm \sqrt{1-4}}{2}$$

$$= -\tfrac{1}{2} \pm \frac{\sqrt{3}}{2}\mathrm{i}$$

$\therefore$ The solution of the equation is

$$y = e^{-\frac{1}{2}x}\left(A\cos\tfrac{\sqrt{3}}{2}x + B\sin\tfrac{\sqrt{3}}{2}x\right)$$

The auxiliary equation has complex roots and so the solution is of the form $y = e^{px}(A\cos qx + B\sin qx)$.

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Exercise G, Question 2

Question:

Find the general solution of the differential equation $\frac{d^2y}{dx^2} - 12\frac{dy}{dx} + 36y = 0$

Solution:

$$\frac{d^2y}{dx^2} - 12\frac{dy}{dx} + 36y = 0$$

The auxiliary equation is

$m^2 - 12m + 36 = 0$

$\therefore \quad (m - 6)^2 = 0$

$\therefore \quad m = 6$ only

$\therefore$ The solution of the equation is

$y = (A + Bx)e^{6x}$.

The auxiliary equation has a repeated solution and so the solution is of the form $y = (A + Bx)e^{\alpha x}$.

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Exercise G, Question 3

Question:

Find the general solution of the differential equation $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} = 0$

Solution:

$$\frac{d^2y}{dx^2} - \frac{4\,dy}{dx} = 0$$

The auxiliary equation is

$$m^2 - 4m = 0$$

$\therefore \quad m(m-4) = 0$

$\therefore \quad m = 0 \text{ or } 4$

$\therefore$ The solution of the equation is

$$y = Ae^{0x} + Be^{4x}$$
$$= A + Be^{4x}$$

The auxiliary equation has two distinct roots, but one of them is zero.

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Exercise G, Question 4

Question:

Find y in terms of k and x, given that $\frac{d^2y}{dx^2} + k^2y = 0$ where k is a constant, and $y = 1$ and $\frac{dy}{dx} = 1$ at $x = 0$.

Solution:

$$\frac{d^2y}{dx^2} + k^2y = 0$$

The auxiliary equation is

$m^2 + k^2 = 0$

$\therefore \quad m = \pm\, ik$

The solution of the equation is

$y = A\cos kx + B\sin kx$. [This is the general solution.]

But $\quad y = 1$ when $x = 0$

$\therefore \quad 1 = A + 0 \Rightarrow A = 1$

$\therefore \quad y = \cos kx + B\sin kx$

$$\frac{dy}{dx} = -k\sin kx + Bk\cos kx$$

Also $\frac{dy}{dx} = 1$ when $x = 0$

$\therefore \quad 1 = Bk \Rightarrow B = \frac{1}{k}$

$\therefore \quad y = \cos kx + \frac{1}{k}\sin kx.$

The auxiliary equation has imaginary solutions and so the general solution has the form $y = A\cos\omega x + B\sin\omega x$. A and B can be found by using the boundary conditions.

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Exercise G, Question 5

Question:

Find the solution of the differential equation $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 10y = 0$ for which $y = 0$ and $\frac{dy}{dx} = 3$ at $x = 0$.

Solution:

$$\frac{d^2y}{dx^2} - \frac{2dy}{dx} + 10y = 0$$

This has auxiliary equation

$$m^2 - 2m + 10 = 0$$

$$\therefore \quad m = \frac{2 \pm \sqrt{4 - 40}}{2}$$

$$= 1 \pm 3i$$

The general solution of the equation is

$$y = e^x (A \cos 3x + B \sin 3x)$$

As $y = 0$ when $x = 0$,

$\therefore \quad 0 = A + 0 \Rightarrow A = 0$

$\therefore \quad y = Be^x \sin 3x$

$$\frac{dy}{dx} = 3Be^x \cos 3x + Be^x \sin 3x$$

Also $\frac{dy}{dx} = 3$ when $x = 0$

$\therefore \quad 3 = 3B + 0 \Rightarrow B = 1$

$\therefore \quad y = e^x \sin 3x$ is the required solution.

The auxiliary equation has complex roots and so the general solution is of the form $y = e^{px} (A \cos qx + B \sin qx)$.

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Exercise G, Question 6

Question:

Given that the differential equation $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 13y = e^{2x}$ has a particular integral of the form ke^{2x}, determine the value of the constant k and find the general solution of the equation.

Solution:

$$\frac{d^2y}{dx^2} - \frac{4\,dy}{dx} + 13y = e^{2x} \quad *$$

Use the fact that the general solution = complementary function + particular integral.

First find the complementary function (c.f.):

the auxiliary equation is

$m^2 - 4m + 13 = 0$

$\therefore \quad m = \frac{4 \pm \sqrt{16 - 52}}{2}$

$= 2 \pm 3i$

$\therefore$ The c.f. is $y = e^{2x}(A\cos 3x + B\sin 3x)$

Let the particular integral (p.i.) be $y = ke^{2x}$

Then $\frac{dy}{dx} = 2ke^{2x}$ and $\frac{d^2y}{dx^2} = 4ke^{2x}$.

Substitute in $*$ to give

$(4k - 8k + 13k)e^{2x} = e^{2x}$

i.e. $9ke^{2x} = e^{2x}$

$\therefore \quad k = \frac{1}{9}$

$\therefore$ The general solution of $*$ is y = c.f. + p.i.

i.e. $y = e^{2x}(A\cos 3x + B\sin 3x) + \frac{1}{9}e^{2x}$.

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Exercise G, Question 7

Question:

Given that the differential equation $\frac{d^2y}{dx^2} - y = 4e^x$ has a particular integral of the form kxe^x, determine the value of the constant k and find the general solution of the equation.

Solution:

$\frac{d^2y}{dx^2} - y = 4e^x$ $*$

First find the c.f.

Use general solution = complementary function + particular integral.

The auxiliary equation is

$m^2 - 1 = 0$

$\therefore \quad m = \pm 1$

$\therefore$ The c.f. is $y = Ae^x + Be^{-x}$

Let the p.i. be $y = kxe^x$

Then $\frac{dy}{dx} = kxe^x + ke^x$

$\frac{d^2y}{dx^2} = kxe^x + ke^x + ke^x$

Substitute into $*$.

Then $kxe^x + 2ke^x - kxe^x = 4e^x$

$\therefore \quad k = 2$

So the p.i. is $y = 2xe^x$

The general solution is $y = \text{c.f.} + \text{p.i.}$

$\therefore \quad y = Ae^x + Be^{-x} + 2xe^x.$

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Exercise G, Question 8

Question:

The differential equation $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 4e^{2x}$ is to be solved.

a Find the complementary function.

b Explain why **neither** λe^{2x} **nor** $\lambda x e^{2x}$ can be a particular integral for this equation.

c Determine the value of the constant k and find the general solution of the equation.

Solution:

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 4e^{2x} \quad *$$

a First find the c.f.

The auxiliary equation is

$m^2 - 4m + 4 = 0$

$\therefore \quad (m-2)^2 = 0$

i.e. $\quad m = 2$ only

$\therefore$ The c.f. is $y = (A + Bx)e^{2x}$

The auxiliary equation has a repeated root and so the c.f. is of the form $y = (A + Bx)e^{\alpha x}$.

b Ae^{2x} and Bxe^{2x} are part of the c.f. so satisfy the equation $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 0$.

The p.i. must satisfy $*$.

c Let $\quad y = kx^2e^{2x}$

$$\frac{dy}{dx} = 2kx^2e^{2x} + 2kxe^{2x}$$

$$\frac{d^2y}{dx^2} = 4kx^2e^{2x} + 4kxe^{2x} + 2kx \times 2e^{2x} + 2ke^{2x}$$

Substitute into $*$

$\therefore \quad (4kx^2 + 8kx + 2k - 8kx^2 - 8kx + 4kx^2)e^{2x} = 4e^{2x}$

$\therefore \quad 2ke^{2x} = 4e^{2x}$

$\therefore \quad k = 2$

So the p.i. is $2x^2e^{2x}$

$\therefore$ The general solution is $y = (A + Bx + 2x^2)e^{2x}$.

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Exercise G, Question 9

Question:

Given that the differential equation $\frac{d^2y}{dt^2} + 4y = 5\cos 3t$ has a particular integral of the form $k\cos 3t$, determine the value of the constant k and find the general solution of the equation.

Find the solution which satisfies the initial conditions that when $t = 0$, $y = 1$ and $\frac{dy}{dt} = 2$.

Solution:

$$\frac{d^2y}{dt^2} + 4y = 5\cos 3t \quad *$$

The p.i. is $y = k\cos 3t$.

$$\frac{dy}{dt} = -3k\sin 3t$$

$$\frac{d^2y}{dt^2} = -9k\cos 3t$$

Substitute into $*$

Then $-9k\cos 3t + 4k\cos 3t = 5\cos 3t$

$\therefore \quad -5k\cos 3t = 5\cos 3t$

$\therefore \quad k = -1$

$\therefore$ The p.i. is $-\cos 3t$.

The c.f. is found next.

The auxiliary equation is $m^2 + 4 = 0$.

$\therefore \quad m = \pm 2\text{i}$

$\therefore$ The c.f. is $y = A\cos 2t + B\sin 2t$

$\therefore$ The general solution is $y = A\cos 2t + B\sin 2t - \cos 3t$

The auxiliary equation has imaginary roots so the c.f. is $y = A\cos\omega t + B\sin\omega t$. '$t$' is the independent variable in this question.

When $t = 0$, $y = 1$ $\quad \therefore \quad 1 = A - 1 \Rightarrow A = 2$

$$\frac{dy}{dt} = -2A\sin 2t + 2B\cos 2t + 3\sin 3t$$

When $t = 0$, $\frac{dy}{dt} = 2$ $\quad \therefore \quad 2 = 2B \Rightarrow B = 1$

$\therefore \quad y = 2\cos 2t + \sin 2t - \cos 3t.$

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Exercise G, Question 10

Question:

Given that the differential equation $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 4x + e^{2x}$ has a particular integral of the form $\lambda + \mu x + kxe^{2x}$, determine the values of the constants λ, μ and k and find the general solution of the equation.

Solution:

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 4x + e^{2x} \quad *$$

Find the complementary function and add to the particular integral to give the general solution.

P.I. is $y = \lambda + \mu x + kxe^{2x}$

$\therefore \quad \frac{dy}{dx} = \mu + 2kxe^{2x} + ke^{2x}$

$\frac{d^2y}{dx^2} = 2kx \times 2e^{2x} + 2ke^{2x} + 2ke^{2x}$

Substitute into $*$.

Then $(4kx + 4k)e^{2x} - 3\mu - (6kx + 3k)e^{2x} + 2\lambda + 2\mu x + 2kxe^{2x} = 4x + e^{2x}$

$\therefore \quad ke^{2x} + (2\lambda - 3\mu) + 2\mu x = 4x + e^{2x}$.

Equating coefficients of e^{2x}: $k = 1$

x: $2\mu = 4 \Rightarrow \mu = 2$

constants: $2\lambda - 3\mu = 0 \Rightarrow \lambda = 3$

$\therefore \quad y = 3 + 2x + xe^{2x}$ is the particular integral.

The auxiliary equation for $*$ is

$m^2 - 3m + 2 = 0$

$\therefore \quad (m - 2)(m - 1) = 0$

$\therefore \quad m = 1$ or 2

$\therefore \quad$ The c.f. is $y = Ae^x + Be^{2x}$

$\therefore \quad$ The general solution is $y = Ae^x + Be^{2x} + 3 + 2x + xe^{2x}$.

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Exercise G, Question 11

Question:

Find the solution of the differential equation $16\frac{d^2y}{dx^2} + 8\frac{dy}{dx} + 5y = 5x + 23$ for which $y = 3$ and $\frac{dy}{dx} = 3$ at $x = 0$. Show that $y \approx x + 3$ for large values of x.

Solution:

$$16\frac{d^2y}{dx^2} + 8\frac{dy}{dx} + 5y = 5x + 23 \qquad ①$$

The particular integral is of the form $y = \lambda x + \mu$.

The auxiliary equation is

$16m^2 + 8m + 5 = 0$

$$\therefore \quad m = \frac{-8 \pm \sqrt{64 - 320}}{32}$$

$$= -\tfrac{1}{4} \pm \frac{\sqrt{-256}}{32}$$

$$= -\tfrac{1}{4} \pm \tfrac{1}{2}i$$

$\therefore$ The c.f. is $y = e^{-\frac{1}{4}x}(A\cos\frac{1}{2}x + B\sin\frac{1}{2}x)$

Let the p.i. be $y = \lambda x + \mu$.

$$\therefore \quad \frac{dy}{dx} = \lambda, \quad \frac{d^2y}{dx^2} = 0$$

Substitute into ①

$\therefore$ $8\lambda + 5\lambda x + 5\mu = 5x + 23$

Equate coefficients of x: $\therefore$ $5\lambda = 5 \Rightarrow \lambda = 1$

constant terms: $8\lambda + 5\mu = 23 \Rightarrow \mu = 3$

$\therefore$ The p.i. is $y = x + 3$

The general solution is c.f. + p.i.

i.e. $y = e^{-\frac{1}{4}x}(A\cos\frac{1}{2}x + B\sin\frac{1}{2}x) + x + 3$.

As $y = 3$, when $x = 0$

$\therefore$ $3 = A + 3 \Rightarrow A = 0$

$\therefore$ $y = Be^{-\frac{1}{4}x}\sin\frac{1}{2}x + x + 3$

$$\therefore \quad \frac{dy}{dx} = \tfrac{1}{2}Be^{-\frac{1}{4}x}\cos\tfrac{1}{2}x - \tfrac{1}{4}Be^{-\frac{1}{4}x}\sin\tfrac{1}{2}x + 1$$

As $\frac{dy}{dx} = 3$ when $x = 0$

$3 = \frac{1}{2}B + 1 \Rightarrow B = 4$

$\therefore$ $y = 4e^{-\frac{1}{4}x}\sin\frac{1}{2}x + x + 3$

As $x \to \infty$, $e^{-\frac{1}{4}x} \to 0$; $\therefore$ $y \to x + 3$

$\therefore$ $y \approx x + 3$ for large values of x.

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Exercise G, Question 12

Question:

Find the solution of the differential equation $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 6y = 3\sin 3x - 2\cos 3x$ for which $y = 1$ at $x = 0$ and for which y remains finite for large values of x.

Solution:

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 6y = 3\sin 3x - 2\cos 3x \quad *$$

The auxiliary equation is

$m^2 - m - 6 = 0$

$\therefore \quad (m-3)(m+2) = 0$

$\therefore \quad m = 3 \text{ or } -2$

$\therefore$ The c.f. is $y = Ae^{3x} + Be^{-2x}$.

Let the particular integral be $y = \lambda \sin 3x + \mu \cos 3x$.

The particular inegral is $y = \lambda \sin 3x + \mu \cos 3x$.

Then $\dfrac{dy}{dx} = 3\lambda \cos 3x - 3\mu \sin 3x$

$\dfrac{d^2y}{dx^2} = -9\lambda \sin 3x - 9\mu \cos 3x$

Substitute into $*$

Then $-9\lambda \sin 3x - 9\mu \cos 3x - 3\lambda \cos 3x + 3\mu \sin 3x - 6\lambda \sin 3x - 6\mu \cos 3x$
$= 3\sin 3x - 2\cos 3x$.

Equate coefficients of $\sin 3x$:

$-9\lambda + 3\mu - 6\lambda = 3$ i.e. $3\mu - 15\lambda = 3$ ①

Equate coefficients of $\cos 3x$:

$-9\mu - 3\lambda - 6\mu = -2$ i.e. $-15\mu - 3\lambda = -2$ ②

Solve equations ① and ② to give $\lambda = -\frac{1}{6} \quad \mu = \frac{1}{6}$

$\therefore$ P.I. is $y = \frac{1}{6}(\cos 3x - \sin 3x)$

$\therefore$ The general solution is

$y = Ae^{3x} + Be^{-2x} + \frac{1}{6}(\cos 3x - \sin 3x)$

As $y = 1$ when $x = 0$, $1 = A + B + \frac{1}{6}$

$\therefore \quad A + B = \frac{5}{6}$

As y remains finite for large values of x,

$A = 0$

$\therefore \quad B = \frac{5}{6}$

$\therefore \quad y = \frac{5}{6}e^{-2x} + \frac{1}{6}(\cos 3x - \sin 3x)$

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Exercise G, Question 13

Question:

Find the general solution of the differential equation $\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 10x = 27\cos t - 6\sin t$.

The equation is used to model water flow in a reservoir. At time t days, the level of the water above a fixed level is x m. When $t = 0$, $x = 3$ and the water level is rising at 6 metres per day.

a Find an expression for x in terms of t.

b Show that after about a week, the difference between the lowest and highest water level is approximately 6 m.

Solution:

a $\dfrac{d^2x}{dt^2} + 2\dfrac{dx}{dt} + 10x = 27\cos t - 6\sin t \quad *$

The auxiliary equation is

$m^2 + 2m + 10 = 0$

$\therefore \quad m = \dfrac{-2 \pm \sqrt{4 - 40}}{2}$

$= -1 \pm 3i$

$\therefore$ The c.f. is $x = e^{-t}(A\cos 3t + B\sin 3t)$

The p.i. is $x = \lambda\cos t + \mu\sin t$

$\dfrac{dx}{dt} = -\lambda\sin t + \mu\cos t$

$\dfrac{d^2x}{dt^2} = -\lambda\cos t - \mu\sin t$

Substitute into $*$

$\therefore \quad -\lambda\cos t - \mu\sin t - 2\lambda\sin t + 2\mu\cos t + 10\lambda\cos t + 10\mu\sin t = 27\cos t - 6\sin t$

Equate coefficients of $\cos t$: $9\lambda + 2\mu = 27$ ①

$\sin t$: $9\mu - 2\lambda = -6.$ ②

Solve equations ① and ② to give $\lambda = 3$, $\mu = 0$.

$\therefore$ The p.i. is $x = 3\cos t$.

$\therefore$ The general solution is $x = 3\cos t + e^{-t}(A\cos 3t + B\sin 3t)$

But $x = 3$ when $t = 0$: $\therefore \quad 3 = 3 + A \Rightarrow A = 0$

$\therefore \quad x = 3\cos t + Be^{-t}\sin 3t$

$\therefore \quad \dfrac{dx}{dt} = -3\sin t + 3Be^{-t}\cos 3t - Be^{-t}\sin 3t$

When $t = 0$, $\dfrac{dx}{dt} = 6$

$\therefore \quad 6 = 3B \Rightarrow B = 2$

$\therefore \quad x = 3\cos t + 2e^{-t}\sin 3t.$

b After a week $t \approx 7$ days. $\therefore \quad e^{-t} \to 0.$

$\therefore \quad x \approx 3\cos t$

The distance between highest and lowest water level is $3 - (-3) = 6$ m.

In part **b** if t is large, then $e^{-t} \to 0$.

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Exercise G, Question 14

Question:

a Find the general solution of the differential equation

$$x^2\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + 2y = \ln x, \qquad x > 0,$$

using the substitution $x = e^u$, where u is a function of x.

b Find the equation of the solution curve passing through the point (1, 1) with gradient 1.

Solution:

a Let $x = e^u$, then $\frac{dx}{du} = e^u$

and $\frac{dy}{du} = \frac{dy}{dx} \times \frac{dx}{du} = e^u \frac{dy}{dx} = x\frac{dy}{dx}$

$$\frac{d^2y}{du^2} = \frac{dx}{du} \times \frac{dy}{dx} + x\frac{d^2y}{dx^2} \times \frac{dx}{du}$$

$$= x\frac{dy}{dx} + x^2\frac{d^2y}{dx^2}$$

Find $\frac{dy}{du}$ in terms of x and $\frac{dy}{dx}$, and show that $\frac{d^2y}{du^2} = x\frac{dy}{dx} + x^2\frac{d^2y}{dx^2}$ then substitute into the differential equation.

$\therefore \quad x^2\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + 2y = \ln x \Rightarrow \frac{d^2y}{du^2} + 3\frac{dy}{du} + 2y = \ln x = u \quad *$

The auxiliary equation is

$$m^2 + 3m + 2 = 0$$

$\therefore \quad (m + 2)(m + 1) = 0$

$\Rightarrow \quad m = -1$ or -2

$\therefore$ The c.f. is $y = Ae^{-u} + Be^{-2u}$

Let the p.i. be $y = \lambda u + \mu \Rightarrow \frac{dy}{du} = \lambda, \frac{d^2y}{du^2} = 0$

Substitute into $*$

$\therefore \quad 3\lambda + 2\lambda u + 2\mu = u$

Equate coefficients of u: $2\lambda = 1 \Rightarrow \lambda = \frac{1}{2}$

constants: $3\lambda + 2\mu = 0 \quad \therefore \quad \mu = -\frac{3}{4}$

$\therefore$ The p.i. is $y = \frac{1}{2}u - \frac{3}{4}$

The general solution is $y = Ae^{-u} + Be^{-2u} + \frac{1}{2}u - \frac{3}{4}$.

But $x = e^u \rightarrow u = \ln x$.

Also $e^{-u} = x^{-1} = \frac{1}{x}$ and $e^{-2u} = x^{-2} = \frac{1}{x^2}$

$\therefore$ The general solution of the original equation is $y = \frac{A}{x} + \frac{B}{x^2} + \frac{1}{2}\ln x - \frac{3}{4}$.

b But $y = 1$ when $x = 1$

$\therefore \quad 1 = A + B - \frac{3}{4} \Rightarrow A + B = 1\frac{3}{4}$ ①

$$\frac{dy}{dx} = -\frac{A}{x^2} - \frac{2B}{x^3} + \frac{1}{2x}$$

When $x = 1$, $\frac{dy}{dx} = 1$

$\therefore \quad 1 = -A - 2B + \frac{1}{2} \Rightarrow A + 2B = -\frac{1}{2}$ ②

Solve the simultaneous equations ① and ② to give $B = -2\frac{1}{4}$ and $A = 4$.

$\therefore$ The equation of the solution curve described is $y = \frac{4}{x} - \frac{9}{4x^2} + \frac{1}{2}\ln x - \frac{3}{4}$.

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Exercise G, Question 15

Question:

Solve the equation $\frac{d^2y}{dx^2} + \tan x \frac{dy}{dx} + y\cos^2 x = \cos^2 x\, e^{\sin x}$, by putting $z = \sin x$, finding the solution for which $y = 1$ and $\frac{dy}{dx} = 3$ at $x = 0$.

Solution:

$z = \sin x \quad \therefore \quad \frac{dz}{dx} = \cos x$ and $\frac{dy}{dx} = \frac{dy}{dz} \times \cos x$

$$\therefore \quad \frac{d^2y}{dx^2} = -\frac{dy}{dz}\sin x + \cos x\,\frac{d^2y}{dz^2} \times \frac{dz}{dx}$$

$$= -\frac{dy}{dz}\sin x + \cos^2 x\,\frac{d^2y}{dz^2}$$

$$\therefore \quad \frac{d^2y}{dx^2} + \tan x\,\frac{dy}{dx} + y\cos^2 x = \cos^2 x\, e^{\sin x} \quad \dagger$$

$$\Rightarrow \cos^2 x\,\frac{d^2y}{dz^2} - \sin x\,\frac{dy}{dz} + \tan x\cos x\,\frac{dy}{dz} + y\cos^2 x = \cos^2 x\, e^z$$

$$\Rightarrow \frac{d^2y}{dz^2} + y = e^z \quad *$$

The auxiliary equation is $m^2 + 1 = 0 \Rightarrow m = \pm i$

$\therefore$ The c.f. is $y = A\cos z + B\sin z$

The p.i. is $y = \lambda e^z \Rightarrow \frac{dy}{dz} = \lambda e^z$ and $\frac{d^2y}{dz^2} = \lambda e^z$

Substitute in $*$ to give

$$2\lambda e^z = e^z \Rightarrow \lambda = \tfrac{1}{2}$$

$\therefore$ The general solution of $*$ is $y = A\cos z + B\sin z + \frac{1}{2}e^z$.

The original equation $\dagger$ has solution

$$y = A\cos(\sin x) + B\sin(\sin x) + \tfrac{1}{2}e^{\sin x}$$

But $\quad y = 1$ when $x = 0$

$$\therefore \quad 1 = A + \tfrac{1}{2} \Rightarrow A = \tfrac{1}{2}$$

$$\frac{dy}{dx} = \cos x\,(-A\sin(\sin x)) + \cos x(B\cos(\sin x)) + \tfrac{1}{2}\cos x e^{\sin x}$$

As $\quad \frac{dy}{dx} = 3$ when $x = 0$

$$\therefore \quad 3 = B + \tfrac{1}{2} \Rightarrow B = 2\tfrac{1}{2}$$

$$\therefore \quad y = \tfrac{1}{2}\cos(\sin x) + \tfrac{5}{2}\sin(\sin x) + \tfrac{1}{2}e^{\sin x}$$