Solutionbank FP2
Edexcel AS and A Level Modular Mathematics

Exercise A, Question 1

Question:

$$\frac{dy}{dx} = 2x$$

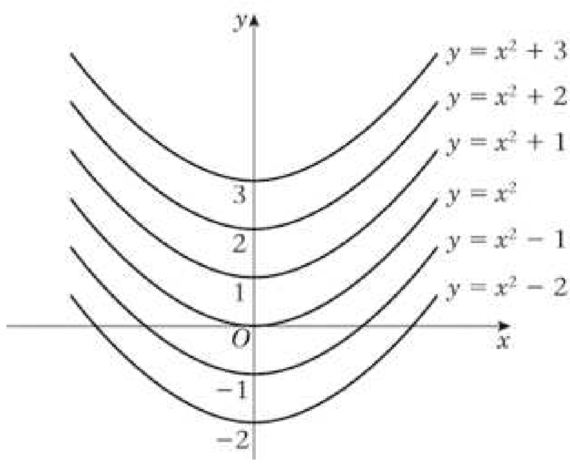
Solution:

$$\frac{dy}{dx} = 2x$$

$\therefore \quad y = \int 2x \, dx$

$\therefore \quad y = x^2 + c \quad$ where c is constant

Integrate and include the constant of integration.
Let the constant take values 1, 2, 3, 0, −1, −2 and draw solution curves.



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Exercise A, Question 2

Question:

$$\frac{dy}{dx} = y$$

Solution:

$$\frac{dy}{dx} = y$$

$$\therefore \int \frac{1}{y}\, dy = \int 1\, dx$$

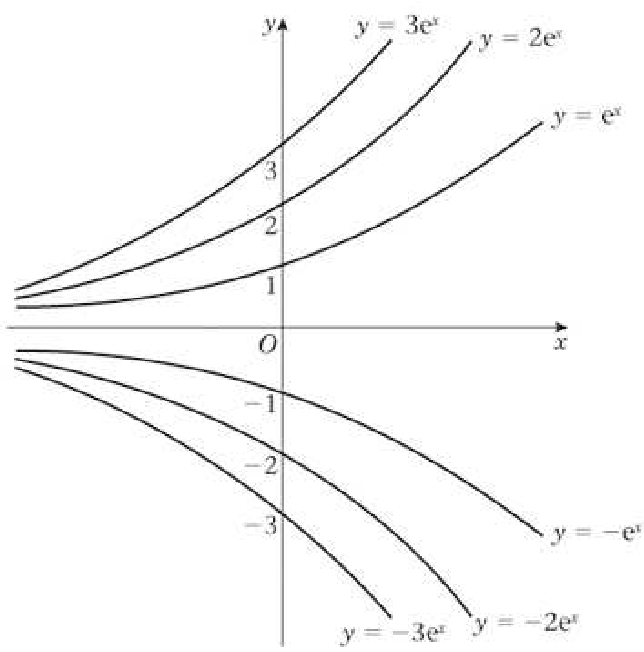
Separate the variables and integrate. Include a constant of integration on one side of the equation.

$$\therefore \quad \ln y = x + c \quad \text{where } c \text{ is constant}$$

$$y = e^{x+c}$$

$$= e^c \times e^x$$

$$y = Ae^x \quad \text{where } A \text{ is constant } (A = e^c)$$



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Exercise A, Question 3

Question:

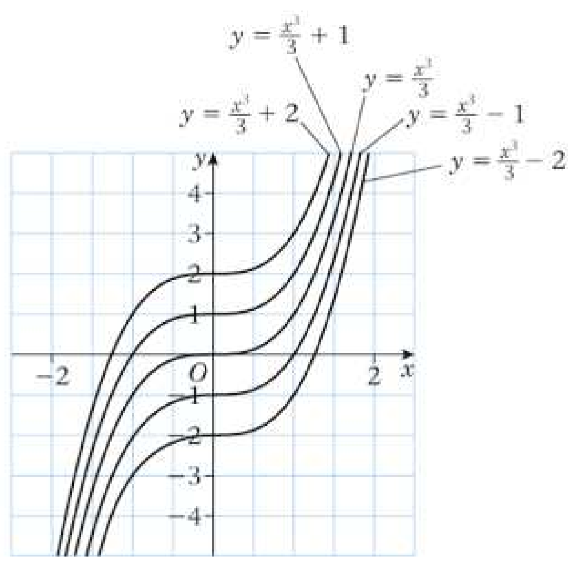
$$\frac{dy}{dx} = x^2$$

Solution:

$$\frac{dy}{dx} = x^2$$

$$y = \int x^2 \, dx$$

$$y = \frac{x^3}{3} + c \quad \text{where } c \text{ is constant}$$



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Exercise A, Question 4

Question:

$$\frac{dy}{dx} = \frac{1}{x}, x > 0$$

Solution:

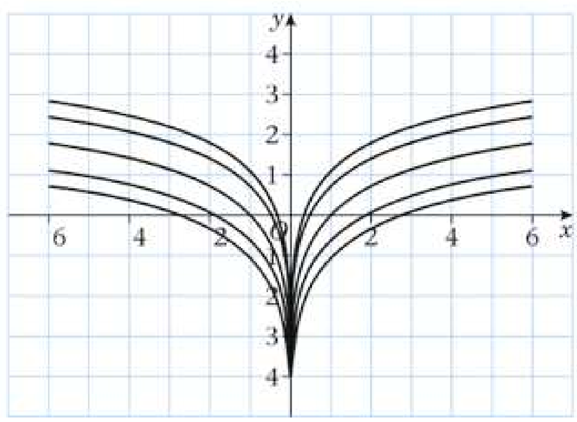
$$\frac{dy}{dx} = \frac{1}{x}$$

$$\therefore \quad y = \int \frac{1}{x}\, dx$$

$$= \ln x + c$$

$$= \ln x + \ln A$$

$$\therefore \quad y = \ln Ax$$



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Exercise A, Question 5

Question:

$$\frac{dy}{dx} = \frac{2y}{x}$$

Solution:

$$\frac{dy}{dx} = \frac{2y}{x}$$

$$\therefore \int \frac{1}{y}\,dy = \int \frac{2}{x}\,dx$$

Separate the variables and integrate.

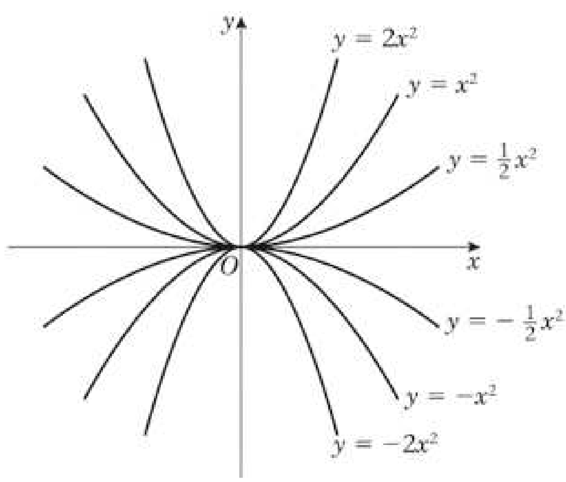
$$\therefore \quad \ln y = 2\ln x + c$$

$$\therefore \quad \ln y = \ln x^2 + \ln A$$

Express the constant of integration as $\ln A$ where A is constant and use laws of logs to simplify your answer.

$$= \ln Ax^2$$

$$\therefore \quad y = Ax^2$$



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Exercise A, Question 6

Question:

$$\frac{dy}{dx} = \frac{x}{y}$$

Solution:

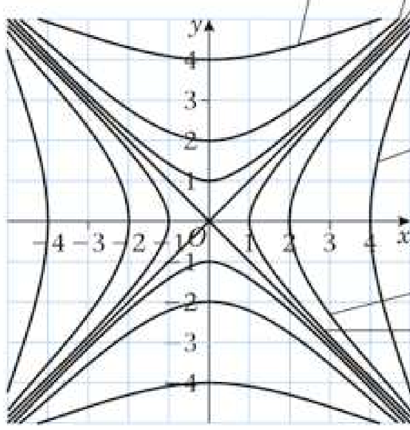
$$\frac{dy}{dx} = \frac{x}{y}$$

$$\therefore \quad \int y \, dy = \int x \, dx$$

$$\therefore \quad \frac{y^2}{2} = \frac{x^2}{2} + c$$

$$\text{or} \quad y^2 - x^2 = 2c$$

$y^2 - x^2 = 0$ is a pair of straight lines.
These are $y = x$ and $y = -x$
$y^2 - x^2 = 2c$, $c \neq 0$ is a hyperbola.



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Exercise A, Question 7

Question:

$$\frac{dy}{dx} = e^y$$

Solution:

$$\frac{dy}{dx} = e^y$$

$$\therefore \int \frac{1}{e^y}\,dy = \int 1\,dx$$

To integrate $\frac{1}{e^y}$, express it as e^{-y}.

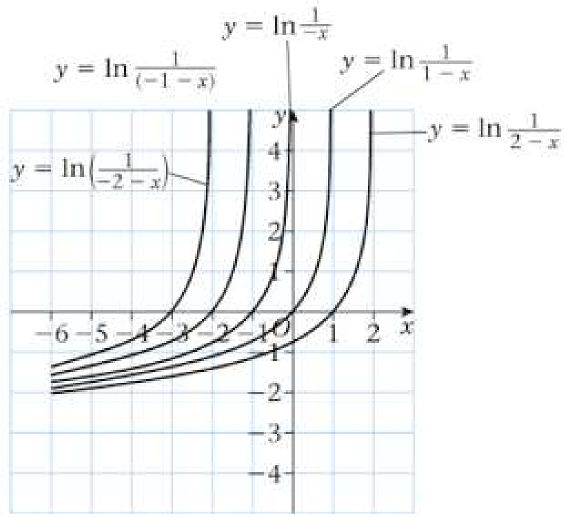
$$\therefore \int e^{-y}\,dy = \int 1\,dx$$

$$\therefore \quad -e^{-y} = x + c$$

$$\therefore \quad -e^{-y} = -x - c$$

$$\therefore \quad -y = \ln[-x - c]$$

$$y = -\ln[-x - c] \text{ or } \ln\frac{1}{(-x - c)}$$



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Exercise A, Question 8

Question:

$$\frac{dy}{dx} = \frac{y}{x(x+1)}, \quad x > 0$$

Solution:

$$\frac{dy}{dx} = \frac{y}{x(x+1)}$$

$$\therefore \int \frac{1}{y}\,dy = \int \frac{1}{x(x+1)}\,dx$$

Separate the variables, then use partial fractions to integrate the function of x.

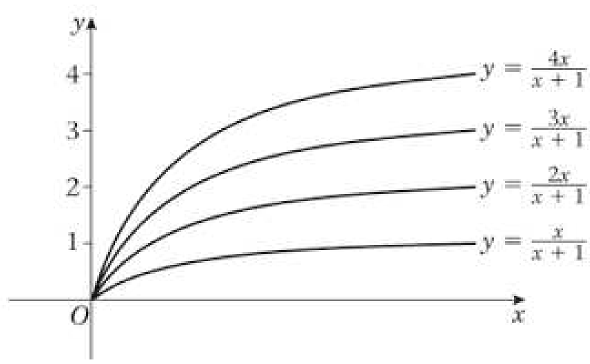
$$\therefore \quad \ln y = \int \left(\frac{1}{x} - \frac{1}{(x+1)}\right) dx$$

$$= \ln x - \ln(x+1) + c$$

$$\therefore \quad \ln y = \ln \frac{x}{x+1} + \ln A$$

$$= \ln \frac{Ax}{x+1}$$

$$\therefore \quad y = \frac{Ax}{x+1} \quad x > 0$$



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Exercise A, Question 9

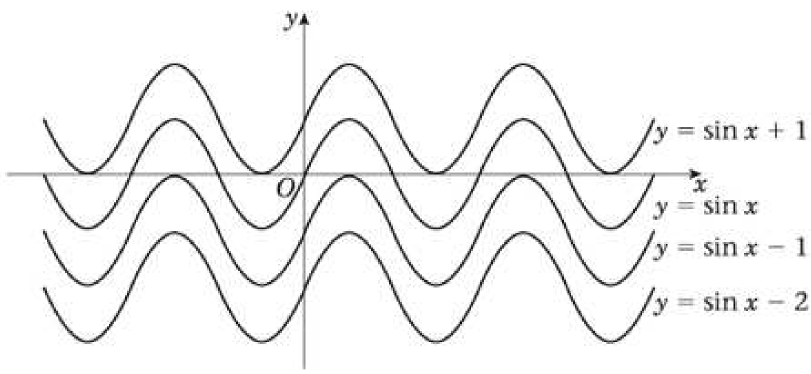
Question:

$$\frac{dy}{dx} = \cos x$$

Solution:

$$\frac{dy}{dx} = \cos x$$

$$\therefore \quad y = \sin x + c$$



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Exercise A, Question 10

Question:

$$\frac{dy}{dx} = y \cot x, \quad 0 < x < \pi$$

Solution:

$$\frac{dy}{dx} = y \cot x \qquad 0 < x < \pi$$

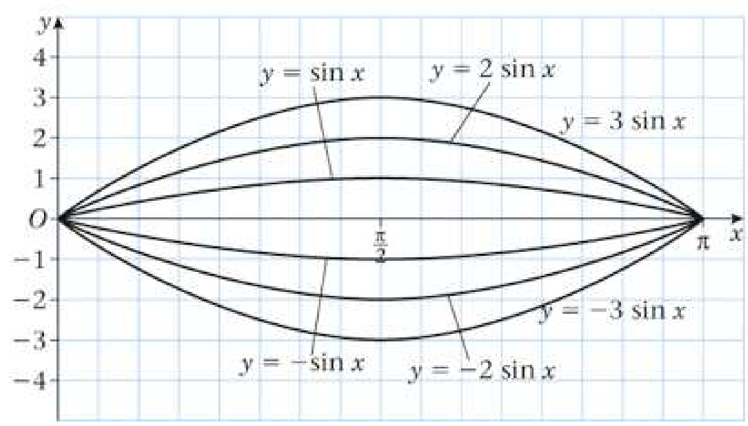
$$\therefore \int \frac{1}{y}\,dy = \int \frac{\cos x}{\sin x}\,dx$$

$$\therefore \quad \ln|y| = \ln|\sin x| + \ln|A|$$

Express the constant of integration as $\ln|A|$ and combine logs to simplify your solution

$$= \ln|A \sin x|$$

$$\therefore \quad y = A \sin x$$



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Exercise A, Question 11

Question:

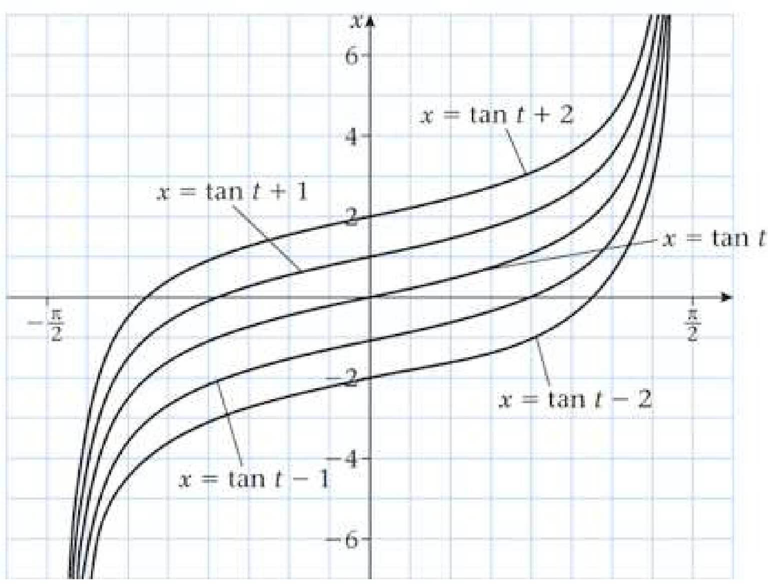
$$\frac{dy}{dx} = \sec^2 t, \ -\frac{\pi}{2} < t < \frac{\pi}{2}$$

Solution:

$$\frac{dx}{dt} = \sec^2 t \qquad -\frac{\pi}{2} < t < \frac{\pi}{2}$$

$$\therefore \quad x = \int \sec^2 t \, dt$$

i.e. $x = \tan t + c$ for $\quad -\frac{\pi}{2} < t < \frac{\pi}{2}$



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Exercise A, Question 12

Question:

$$\frac{dy}{dx} = x(1-x), \quad 0 < x < 1$$

Solution:

$$\frac{dy}{dx} = x(1-x)$$

$$\therefore \int \frac{1}{x(1-x)}\,dx = \int 1\,dt$$

$$\therefore \int \left(\frac{1}{x} + \frac{1}{1-x}\right) dx = \int 1\,dt$$

$$\therefore \ln x - \ln(1-x) = t + c$$

$$\therefore \ln\frac{x}{1-x} = t + c$$

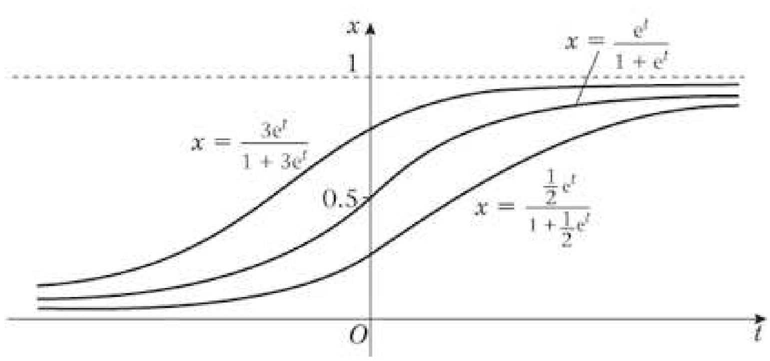
$$\therefore \frac{x}{1-x} = e^{t+c} = Ae^t$$

$0 < x < 1$ implies that A is a positive constant.

$$\therefore x = Ae^t - xAe^t$$

$$\therefore x(1 + Ae^t) = Ae^t$$

$$x = \frac{Ae^t}{1 + Ae^t}$$



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Exercise A, Question 13

Question:

Given that a is an arbitrary constant, show that $y^2 = 4ax$ is the general solution of the differential equation $\frac{dy}{dx} = \frac{y}{2x}$.

a Sketch the members of the family of solution curves for which $a = \frac{1}{4}$, 1 and 4.

b Find also the particular solution, which passes through the point (1, 3), and add this curve to your diagram of solution curves.

Solution:

$$\frac{dy}{dx} = \frac{y}{2x}$$

$$\therefore \int \frac{1}{y}\,dy = \frac{1}{2}\int \frac{1}{x}\,dx$$

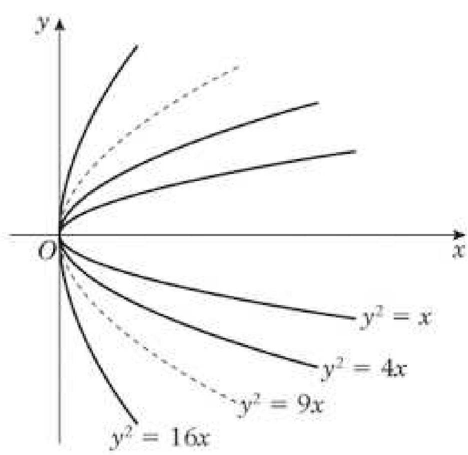
$$\therefore \quad \ln y = \frac{1}{2}\ln x + c$$

$$\text{or} \quad \ln y = \frac{1}{2}\ln x + \ln A$$

$$\therefore \quad \ln y = \ln A\sqrt{x}$$

i.e. $y = A\sqrt{x}$ or $y^2 = A^2 x$ or $y^2 = 4ax$

a Sketch $y^2 = x$, $y^2 = 4x$ and $y^2 = 16x$



b $y^2 = 4ax$ passes through (1, 3)

$\therefore \quad 9 = 4a$

i.e. $a = \frac{9}{4}$ and $y^2 = 9x$

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Exercise A, Question 14

Question:

Given that k is an arbitrary positive constant, show that $y^2 + kx^2 = 9k$ is the general solution of the differential equation $\frac{dy}{dx} = \frac{-xy}{9 - x^2}$ $\quad |x| \leqslant 3$.

a Find the particular solution, which passes through the point (2, 5).

b Sketch the family of solution curves for $k = \frac{1}{9}, \frac{4}{9}, 1$ and include your particular solution in the diagram.

Solution:

$\frac{dy}{dx} = \frac{-xy}{9-x^2}$

$\therefore \int \frac{1}{y}\,dy = -\int \frac{x}{9-x^2}\,dx$

$\therefore \quad \ln y = \frac{1}{2}\ln(9-x^2) + \ln A$

$\therefore \; 2\ln y = \ln A^2(9-x^2)$

$\therefore \quad \ln y^2 = \ln A^2(9-x^2)$

$\therefore \quad y^2 = 9A^2 - A^2x^2$

The solution curves are all ellipses, except when $k = 1$ when the curve is a circle.

Let $A^2 = k$

Then $y^2 + kx^2 = 9k$

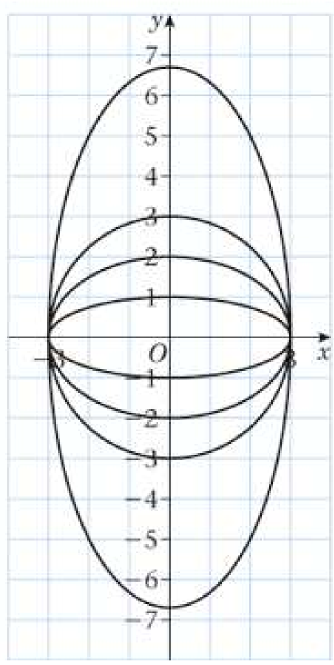
a If this curve passes through (2, 5) then

$25 + 4k = 9k$

$\therefore \quad 25 = 5k \rightarrow k = 5$

i.e. $y^2 + 5x^2 = 45$

b When $y = 0$ $x = \pm 3$, when $x = 0$ $y = \pm\sqrt{9k}$



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Exercise B, Question 1

Question:

$$x\frac{dy}{dx} + y = \cos x$$

Solution:

$$x\frac{dy}{dx} + y = \cos x$$

So $\frac{d}{dx}(xy) = \cos x$

$$\therefore \quad xy = \int \cos x \, dx$$

$$= \sin x + c$$

Remember to add the constant of integration when you integrate – not at the end of the process.

$$\therefore \quad y = \frac{1}{x}\sin x + \frac{c}{x}$$

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Exercise B, Question 2

Question:

$$e^{-x}\frac{dy}{dx} - e^{-x}y = xe^{x}$$

Solution:

$$e^{-x}\frac{dy}{dx} - e^{-x}y = xe^{x}$$

$$\therefore \quad \frac{d}{dx}(e^{-x}y) = xe^{x}$$

$$\therefore \quad e^{-x}y = \int xe^{x}\,dx$$

Use integration by parts to integrate xe^{x}.

$$= xe^{x} - \int e^{x}\,dx$$

$$= xe^{x} - e^{x} + c$$

$$\therefore \quad y = xe^{2x} - e^{2x} + ce^{x}$$

Multiply integration by e^{x}.

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Exercise B, Question 3

Question:

$$\sin x \frac{dy}{dx} + y \cos x = 3$$

Solution:

$$\sin x \frac{dy}{dx} + y \cos x = 3$$

$$\therefore \quad \frac{d}{dx}(y \sin x) = 3$$

$$\therefore \quad y \sin x = \int 3 \, dx$$

$$\therefore \quad y \sin x = 3x + c$$

$$\therefore \quad y = \frac{3x}{\sin x} + \frac{c}{\sin x}$$

$$= 3x \operatorname{cosec} x + c \operatorname{cosec} x$$

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Exercise B, Question 4

Question:

$$\frac{1}{x}\frac{dy}{dx} - \frac{1}{x^2}y = e^x$$

Solution:

$$\frac{1}{x}\frac{dy}{dx} - \frac{1}{x^2}y = e^x$$

$$\therefore \frac{d}{dx}\left(\frac{1}{x}y\right) = e^x$$

$$\therefore \quad \frac{1}{x}y = \int e^x \, dx$$

$$= e^x + c$$

$$\therefore \quad y = xe^x + cx$$

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Exercise B, Question 5

Question:

$$x^2 e^y \frac{dy}{dx} + 2xe^y = x$$

Solution:

$$x^2 e^y \frac{dy}{dx} + 2xe^y = x$$

This time the left hand side is $\frac{d}{dx}(x^2 f(y))$ not just $\frac{d}{dx}(x^2 y)$.

$$\therefore \quad \frac{d}{dx}(x^2 e^y) = x$$

$$\therefore \quad x^2 e^y = \int x \, dx$$

$$= \frac{x^2}{2} + c$$

$$\therefore \quad e^y = \frac{1}{2} + \frac{c}{x^2}$$

or $\quad y = \ln\left[\frac{1}{2} + \frac{c}{x^2}\right]$

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Exercise B, Question 6

Question:

$$4xy\frac{dy}{dx} + 2y^2 = x^2$$

Solution:

$$4xy\frac{dy}{dx} + 2y^2 = x^2$$

Again the left hand side of the equation can be written $\frac{d}{dx}(2x\,f(y))$.

$$\therefore \quad \frac{d}{dx}(2xy^2) = x^2$$

$$\therefore \quad 2xy^2 = \int x^2\,dx$$

$$= \tfrac{1}{3}x^3 + c$$

$$\therefore \quad y^2 = \tfrac{1}{6}x^2 + \frac{c}{2x}$$

Divide both sides by $2x$.

$$\text{or} \quad y = \pm\sqrt{\left(\tfrac{1}{6}x^2 + \frac{c}{2x}\right)}$$

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Exercise B, Question 7

Question:

a Find the general solution of the differential equation

$$x^2\frac{dy}{dx} + 2xy = 2x + 1.$$

b Find the three particular solutions which pass through the points with coordinates $(-\frac{1}{2}, 0)$, $(-\frac{1}{2}, 3)$ and $(-\frac{1}{2}, 19)$ respectively and sketch their solution curves for $x < 0$.

Solution:

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Exercise B, Question 8

Question:

a Find the general solution of the differential equation

$$\ln x \frac{dy}{dx} + \frac{y}{x} = \frac{1}{(x+1)(x+2)}, \qquad x > 1.$$

b Find the specific solution which passes through the point (2, 2).

Solution:

a $$\ln x \frac{dy}{dx} + \frac{y}{x} = \frac{1}{(x+1)(x+2)}$$

$$\therefore \frac{d}{dx}(\ln x \times y) = \frac{1}{(x+1)(x+2)}$$

$$\therefore \quad y \ln x = \int \frac{1}{(x+1)(x+2)}\,dx$$

You will need to use partial fractions to do the integration.

$$= \int \left(\frac{1}{x+1} - \frac{1}{x+2}\right) dx$$

$$= \ln(x+1) - \ln(x+2) + c$$

$$\therefore \quad y = \frac{\ln(x+1) - \ln(x+2) + \ln A}{\ln x}$$

$$\therefore \quad y = \frac{\ln \frac{A(x+1)}{(x+2)}}{\ln x}$$ is the general solution

b When $x = 2$, $y = 2$

$$\therefore \quad 2 = \frac{\ln \frac{3}{4}A}{\ln 2}$$

$$\therefore \quad \ln \tfrac{3}{4}A = 2\ln 2 = \ln 4$$

$$\therefore \quad A = \tfrac{16}{3}$$

So $$y = \frac{\ln \frac{16(x+1)}{3(x+2)}}{\ln x}$$

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Exercise C, Question 1

Question:

$$\frac{dy}{dx} + 2y = e^x$$

Solution:

$$\frac{dy}{dx} + 2y = e^x$$

The integrating factor is $e^{\int 2dx} = e^{2x}$

Find the integral factor $e^{\int p dx}$ and multiply the differential equation by it to give an exact equation.

$$\therefore e^{2x}\frac{dy}{dx} + 2e^{2x}y = e^{3x}$$

$$\therefore \quad \frac{d}{dx}(e^{2x}y) = e^{3x}$$

$$\therefore \quad e^{2x}y = \int e^{3x}\,dx$$

$$= \frac{1}{3}e^{3x} + c$$

$$\therefore \quad y = \frac{1}{3}e^x + ce^{-2x}$$

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Exercise C, Question 2

Question:

$$\frac{dy}{dx} + y\cot x = 1$$

Solution:

$$\frac{dy}{dx} + y\cot x = 1$$

The integrating factor is $e^{\int P dx} = e^{\int \cot x\, dx}$

$= e^{\ln \sin x}$

$= \sin x$

The integrating factor $e^{\ln f(x)}$ can be simplified to $f(x)$.

Multiply differential equation by $\sin x$.

$\therefore \sin x \frac{dy}{dx} + y\cos x = \sin x$

$\therefore \quad \frac{d}{dx}(y\sin x) = \sin x$

$\therefore \quad y\sin x = \int \sin x\, dx$

$= -\cos x + c$

$\therefore \quad y = -\cot x + c\,\text{cosec}\, x$

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Exercise C, Question 3

Question:

$$\frac{dy}{dx} + y\sin x = e^{\cos x}$$

Solution:

$$\frac{dy}{dx} + y\sin x = e^{\cos x}$$

The integrating factor is $e^{\int \sin x\,dx} = e^{-\cos x}$

$\therefore\ e^{-\cos x}\frac{dy}{dx} + y\sin x e^{-\cos x} = 1$

$\therefore\ \frac{d}{dx}(ye^{-\cos x}) = 1$

$\therefore\ ye^{-\cos x} = x + c$

$\therefore\ y = xe^{\cos x} + ce^{\cos x}$

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Exercise C, Question 4

Question:

$$\frac{dy}{dx} - y = e^{2x}$$

Solution:

$$\frac{dy}{dx} - y = e^{2x}$$

The integrating factor is $e^{\int -1\,dx} = e^{-x}$

Remember that $P(x) = -1$ and the minus sign is important.

$$\therefore e^{-x}\frac{dy}{dx} - ye^{-x} = e^{2x} \times e^{-x}$$

$$\therefore \quad \frac{d}{dx}(ye^{-x}) = e^{x}$$

$$\therefore \quad ye^{-x} = \int e^{x}\,dx$$

$$= e^{x} + c$$

$$\therefore \quad y = e^{2x} + ce^{x}$$

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Exercise C, Question 5

Question:

$$\frac{dy}{dx} + y\tan x = x\cos x$$

Solution:

$$\frac{dy}{dx} + y\tan x = x\cos x$$

The integrating factor is $e^{\int \tan x\, dx} = e^{\ln\sec x}$
$= \sec x$

Find the integrating factor and simplify $e^{\ln f(x)}$ to give $f(x)$.

$$\therefore \sec x\frac{dy}{dx} + y\sec x\tan x = x$$

$$\therefore \quad \frac{d}{dx}(y\sec x) = x$$

$$\therefore \quad y\sec x = \int x\, dx$$

$$= \tfrac{1}{2}x^2 + c$$

$$\therefore \quad y = \left(\tfrac{1}{2}x^2 + c\right)\cos x$$

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Exercise C, Question 6

Question:

$$\frac{dy}{dx} + \frac{y}{x} = \frac{1}{x^2}$$

Solution:

$$\frac{dy}{dx} + \frac{y}{x} = \frac{1}{x^2}$$

The integrating factor is $e^{\int \frac{1}{x} dx} = e^{\ln x} = x$

$$\therefore x\frac{dy}{dx} + y = \frac{1}{x}$$

$$\therefore \frac{d}{dx}(xy) = \frac{1}{x}$$

$$\therefore \quad xy = \int \frac{1}{x} dx$$

$$= \ln x + c$$

$$\therefore \quad y = \frac{1}{x}\ln x + \frac{c}{x}$$

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Exercise C, Question 7

Question:

$$x^2 \frac{dy}{dx} - xy = \frac{x^3}{x+2} \quad x > -2$$

Solution:

$$x^2 \frac{dy}{dx} - xy = \frac{x^3}{x+2}$$

Divide by x^2

First divide the equation through by x^2, to give the correct form of equation.

$$\therefore \frac{dy}{dx} - \frac{1}{x} y = \frac{x}{x+2}$$

The integrating factor is $e^{\int -\frac{1}{x} dx} = e^{-\ln x} = e^{\ln \frac{1}{x}} = \frac{1}{x}$

Multiply the new equation by $\frac{1}{x}$

$$\therefore \frac{1}{x} \frac{dy}{dx} - \frac{1}{x^2} y = \frac{1}{x+2}$$

$$\therefore \quad \frac{d}{dx}\left(\frac{1}{x} y\right) = \frac{1}{x+2}$$

$$\therefore \quad \frac{1}{x} y = \int \frac{1}{x+2} dx$$

$$= \ln(x+2) + c$$

$$\therefore \quad y = x \ln(x+2) + cx$$

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Exercise C, Question 8

Question:

$3x\frac{dy}{dx}+y=x$

Solution:

$3x\frac{dy}{dx}+y=x$

$\therefore \frac{dy}{dx}+\frac{1}{3x}y=\frac{1}{3}$ *

First divide equation through by $3x$, to get an equation of the correct form.

The integrating factor is $e^{\int\frac{1}{3x}dx}=e^{\frac{1}{3}\ln x}$

$=e^{\ln x^{\frac{1}{3}}}=x^{\frac{1}{3}}$

Multiply equation * by $x^{\frac{1}{3}}$

$\therefore x^{\frac{1}{3}}\frac{dy}{dx}+\frac{1}{3}x^{-\frac{2}{3}}y=\frac{1}{3}x^{\frac{1}{3}}$

$\therefore \frac{d}{dx}\left(x^{\frac{1}{3}}y\right)=\frac{1}{3}x^{\frac{1}{3}}$

$\therefore x^{\frac{1}{3}}y=\int\frac{1}{3}x^{\frac{1}{3}}dx$

$=\frac{1}{4}x^{\frac{4}{3}}+c$

$\therefore y=\frac{1}{4}x+cx^{-\frac{1}{3}}$

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Exercise C, Question 9

Question:

$(x + 2)\frac{dy}{dx} - y = (x + 2)$

Solution:

$(x + 2)\frac{dy}{dx} - y = (x + 2)$

$\therefore \frac{dy}{dx} - \frac{1}{(x + 2)}y = 1 \quad *$

Divide equation by $(x + 2)$ before finding integrating factor.

The integrating factor is $e^{\int \frac{-1}{(x+2)}dx} = e^{-\ln(x+2)} = e^{\ln\frac{1}{x+2}}$

$= \frac{1}{x + 2}$

Multiply differential equation $*$ by integrating factor.

$\therefore \frac{1}{(x + 2)}\frac{dy}{dx} - \frac{1}{(x + 2)^2}y = \frac{1}{(x + 2)}$

$\therefore \quad \frac{d}{dx}\left[\frac{1}{(x + 2)}y\right] = \frac{1}{x + 2}$

$\therefore \quad \frac{1}{(x + 2)}y = \int \frac{1}{x + 2}dx$

$= \ln(x + 2) + c$

$\therefore \quad y = (x + 2)\ln(x + 2) + c(x + 2)$

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Exercise C, Question 10

Question:

$$x\frac{dy}{dx}+4y=\frac{e^x}{x^2}$$

Solution:

$$x\frac{dy}{dx}+4y=\frac{e^x}{x^2}$$

Divide throughout by x

Then $\frac{dy}{dx}+\frac{4}{x}y=\frac{e^x}{x^3}$ *

The integrating factor is $e^{\int\frac{4}{x}dx}=e^{4\ln x}=e^{\ln x^4}=x^4$

$\therefore x^4\frac{dy}{dx}+4x^3y=xe^x$ [having multiplied * by x^4]

Integrate xe^x using integration by parts.

$\therefore \quad \frac{d}{dx}(x^4y)=xe^x$

$\therefore \quad x^4y=\int xe^x\,dx$

$=xe^x-\int e^x\,dx$

$=xe^x-e^x+c$

$\therefore \quad y=\frac{1}{x^3}e^x-\frac{1}{x^4}e^x+\frac{c}{x^4}$

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Exercise C, Question 11

Question:

Find y in terms of x given that

$x\frac{dy}{dx}+2y=e^x$ and that $y=1$ when $x=1$.

Solution:

$$x\frac{dy}{dx}+2y=e^x$$

Divide throughout by x

Then $\frac{dy}{dx}+\frac{2}{x}y=\frac{1}{x}e^x$ $*$

The integrating factor is $e^{\int\frac{2}{x}dx}=e^{2\ln x}=e^{\ln x^2}=x^2$

Multiply equation $*$ by x^2

Then $x^2\frac{dy}{dx}+2xy=xe^x$

$\therefore \quad \frac{d}{dx}(x^2y)=xe^x$

$\therefore \quad x^2y=\int xe^x\,dx$

$=xe^x-\int e^x\,dx$

$=xe^x-e^x+c$

$\therefore \quad y=\frac{1}{x}e^x-\frac{1}{x^2}e^x+\frac{c}{x^2}$

Solve the differential equation then use the boundary condition $y=1$ when $x=1$ to find the constant of integration.

Given also that $y=1$ when $x=1$

Then $1=e-e+c$

$\therefore \quad c=1$

$\therefore \quad y=\frac{1}{x}e^x-\frac{1}{x^2}e^x+\frac{1}{x^2}$

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Exercise C, Question 12

Question:

Solve the differential equation, giving y in terms of x, where
$x^3 \frac{dy}{dx} - x^2y = 1$ and $y = 1$ at $x = 1$.

Solution:

$$x^3 \frac{dy}{dx} - x^2y = 1$$

Divide throughout by x^3

$$\frac{dy}{dx} - \frac{1}{x}y = \frac{1}{x^3} \quad *$$

The integrating factor is $e^{-\int \frac{1}{x} dx} = e^{-\ln x} = e^{\ln \frac{1}{x}} = \frac{1}{x}$

Multiply equation $*$ by $\frac{1}{x}$

Then $\frac{1}{x}\frac{dy}{dx} - \frac{1}{x^2}y = \frac{1}{x^4}$

$\therefore \quad \frac{d}{dx}\left(\frac{1}{x}y\right) = \frac{1}{x^4}$

$\therefore \quad \frac{1}{x}y = \int \frac{1}{x^4} dx$

$= \int x^{-4} dx$

$= -\frac{1}{3}x^{-3} + c$

$\therefore \quad y = -\frac{1}{3}x^{-2} + cx$

So $\quad y = -\frac{1}{3x^2} + cx$

But $y = 1$, when $x = 1$

$\therefore \quad 1 = -\frac{1}{3} + c$

$\therefore \quad c = \frac{4}{3}$

$\therefore \quad y = -\frac{1}{3x^2} + \frac{4x}{3}$

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Exercise C, Question 13

Question:

Find the general solution of the differential equation

$$\left(x+\frac{1}{x}\right)\frac{dy}{dx}+2y=2(x^2+1)^2,$$

giving y in terms of x.

Find the particular solution which satisfies the condition that $y = 1$ at $x = 1$.

Solution:

$$\left(x+\frac{1}{x}\right)\frac{dy}{dx}+2y=2\,(x^2+1)^2$$

Divide equation by $\left(x+\frac{1}{x}\right)$.

$$\therefore \frac{dy}{dx}+\frac{2}{\left(x+\frac{1}{x}\right)}y=\frac{2(x^2+1)^2}{\left(x+\frac{1}{x}\right)}$$

i.e. $\frac{dy}{dx}+\frac{2x}{x^2+1}\times y=2x(x^2+1)$ $*$

The integrating factor is $e^{\int\frac{2x}{x^2+1}dx}=e^{\ln(x^2+1)}=(x^2+1)$

Multiply $*$ by (x^2+1)

Then $(x^2+1)\frac{dy}{dx}+2xy=2x(x^2+1)^2$

$$\therefore \quad \frac{d}{dx}\left[(x^2+1)\,y\right]=2x\,(x^2+1)^2$$

$$\therefore \quad y\,(x^2+1)=\int 2x\,(x^2+1)^2\,dx$$

$$=\tfrac{1}{3}\,(x^2+1)^3+c$$

$$\therefore \quad y=\tfrac{1}{3}\,(x^2+1)^2+\frac{c}{(x^2+1)}$$

But $y = 1$, when $x = 1$

$\therefore \quad 1=\frac{1}{3}\times 4+\frac{1}{2}c$

$\therefore \quad c=-\frac{2}{3}$

$\therefore \quad y=\frac{1}{3}\,(x^2+1)^2-\frac{2}{3(x^2+1)}$

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Exercise C, Question 14

Question:

Find the general solution of the differential equation

$\cos x \frac{dy}{dx} + y = 1, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}.$

Find the particular solution which satisfies the condition that $y = 2$ at $x = 0$.

Solution:

$$\cos x \frac{dy}{dx} + y = 1$$

Divide throughout by $\cos x$

$$\therefore \quad \frac{dy}{dx} + \sec x \, y = \sec x$$

The integrating factor is $e^{\int \sec x \, dx} = e^{\ln(\sec x + \tan x)}$

$= \sec x + \tan x$

$\int \sec x \, dx = \ln(\sec x + \tan x)$

$$\therefore (\sec x + \tan x)\frac{dy}{dx} + (\sec^2 x + \sec x \tan x)\, y = \sec^2 x + \sec x \tan x$$

$$\therefore \quad \frac{d}{dx}\left[(\sec x + \tan x)y\right] = \sec^2 x + \sec x \tan x$$

$$\therefore \quad (\sec x + \tan x)y = \int \sec^2 x + \sec x \tan x \, dx$$

$$= \tan x + \sec x + c$$

$$\therefore \quad y = 1 + \frac{c}{\sec x + \tan x}$$

Given also that $y = 2$, when $x = 0$

$$\therefore \quad 2 = 1 + \frac{c}{1 + 0}$$

$$\therefore \quad c = 1$$

So $\quad y = 1 + \frac{1}{\sec x + \tan x}$ or $y = 1 + \frac{\cos x}{1 + \sin x}$

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Exercise D, Question 1

Question:

$$\frac{dy}{dx} = \frac{y}{x} + \frac{x}{y}, \quad x > 0, y > 0$$

Solution:

$$z = \frac{y}{x} \quad \Rightarrow \quad y = xz$$

$$\therefore \quad \frac{dy}{dx} = z + x\frac{dz}{dx}$$

Use the given substitution to express $\frac{dy}{dx}$ in terms of z, x and $\frac{dz}{dx}$.

Substitute into the equation:

$$\frac{dy}{dx} = \frac{y}{x} + \frac{x}{y}$$

$$\therefore \quad z + x\frac{dz}{dx} = z + \frac{1}{z}$$

$$\therefore \quad x\frac{dz}{dx} = \frac{1}{z}$$

Separate the variables:

Then $\int z\, dz = \int \frac{1}{x}\, dx$

$$\therefore \quad \frac{z^2}{2} = \ln x + c$$

$$\therefore \quad \frac{y^2}{2x^2} = \ln x + c, \text{ as } z = \frac{y}{x}$$

$$\therefore \quad y^2 = 2x^2(\ln x + c)$$

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Exercise D, Question 2

Question:

$$\frac{dy}{dx} = \frac{y}{x} + \frac{x^2}{y^2}, \quad x > 0$$

Solution:

As $z = \frac{y}{x}$, $y = zx$ and $\frac{dy}{dx} = z + x\frac{dz}{dx}$

$\therefore \frac{dy}{dx} = \frac{y}{x} + \frac{x^2}{y^2} \Rightarrow z + x\frac{dz}{dx} = z + \frac{1}{z^2}$

$\therefore \quad x\frac{dz}{dx} = \frac{1}{z^2}$

Separate the variables:

Then $\int z^2 \, dz = \int \frac{1}{x} \, dx$

$\therefore \quad \frac{z^3}{3} = \ln x + c$

But $z = \frac{y}{x}$

$\therefore \frac{y^3}{3x^3} = \ln x + c$

$\therefore \quad y^3 = 3x^3 (\ln x + c)$

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Exercise D, Question 3

Question:

$$\frac{dy}{dx} = \frac{y}{x} + \frac{y^2}{x^2}, \quad x > 0$$

Solution:

As $z = \frac{y}{x}$, $y = zx$ and $\frac{dy}{dx} = z + x\frac{dz}{dx}$

$\therefore \quad \frac{dy}{dx} = \frac{y}{x} + \frac{y^2}{x^2} \Rightarrow z + x\frac{dz}{dx} = z + z^2$

$\therefore \quad x\frac{dz}{dx} = z^2$

Separate the variables:

$\therefore \int \frac{1}{z^2}\, dz = \int \frac{1}{x}\, dx$

$\therefore \quad -\frac{1}{z} = \ln x + c$

$\therefore \quad z = \frac{-1}{\ln x + c}$

But $z = \frac{y}{x}$

$\therefore \quad y = \frac{-x}{\ln x + c}$

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Exercise D, Question 4

Question:

$$\frac{dy}{dx} = \frac{x^3 + 4y^3}{3xy^2}, x > 0$$

Solution:

$$z = \frac{y}{x} \Rightarrow y = zx \text{ and } \frac{dy}{dx} = z + x\frac{dz}{dx}$$

$$\therefore \frac{dy}{dx} = \frac{x^3 + 4y^3}{3xy^2} \Rightarrow z + x\frac{dz}{dx} = \frac{x^3 + 4z^3x^3}{3xz^2x^2}$$

$$\therefore \quad x\frac{dz}{dx} = \frac{1 + 4z^3}{3z^2} - z$$

$$= \frac{1 + z^3}{3z^2}$$

Separate the variables:

$$\therefore \int \frac{3z^2}{1 + z^3}\,dz = \int \frac{1}{x}\,dx$$

$\therefore$ $\ln(1 + z^3) = \ln x + \ln A$, where A is constant

$\therefore$ $\ln(1 + z^3) = \ln Ax$

So $1 + z^3 = Ax$

And $z^3 = Ax - 1$. But $z = \frac{y}{x}$

$$\therefore \quad \frac{y^3}{x^3} = Ax - 1$$

$\therefore$ $y^3 = x^3(Ax - 1)$, where A is a positive constant

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Exercise D, Question 5

Question:

Use the substitution $z = y^{-2}$ to transform the differential equation

$$\frac{dy}{dx} + (\tfrac{1}{2}\tan x)\, y = -(2\sec x)y^3, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$$

into a differential equation in z and x. By first solving the transformed equation, find the general solution of the original equation, giving y in terms of x.

Solution:

Given $z = y^{-2}$ $\therefore$ $y = z^{-\frac{1}{2}}$

and $\frac{dy}{dx} = -\frac{1}{2} z^{-\frac{3}{2}} \frac{dz}{dx}$

Find $\frac{dy}{dx}$ in terms of $\frac{dz}{dx}$ and z.

$\therefore$ $\frac{dy}{dx} + \left(\frac{1}{2}\tan x\right) y = -(2\sec x)\, y^3$

$\Rightarrow -\frac{1}{2} z^{-\frac{3}{2}} \frac{dz}{dx} + \left(\frac{1}{2}\tan x\right) z^{-\frac{1}{2}} = -2\sec x\, z^{-\frac{3}{2}}$

$\therefore$ $\frac{dz}{dx} - z\tan x = 4\sec x$ $\ast$

This is a first order equation which can be solved by using an integrating factor.

The integrating factor is $e^{-\int \tan x\, dx} = e^{\ln\cos x}$

$= \cos x$

The equation that you obtain needs an integrating factor to solve it.

Multiply the equation $\ast$ by $\cos x$

Then $\cos x \times \frac{dz}{dx} - z\sin x = 4$

$\therefore$ $\frac{d}{dx}(z\cos x) = 4$

$\therefore$ $z\cos x = \int 4\, dx$

$= 4x + c$

$\therefore$ $z = \frac{4x + c}{\cos x}$

As $y = z^{-\frac{1}{2}}$, $y = \sqrt{\frac{\cos x}{4x + c}}$

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Exercise D, Question 6

Question:

Use the substitution $z = x^{\frac{1}{2}}$ to transform the differential equation

$$\frac{dx}{dt} + t^2x = t^2x^{\frac{1}{2}}$$

into a differential equation in z and t. By first solving the transformed equation, find the general solution of the original equation, giving x in terms of t.

Solution:

Given that $z = x^{\frac{1}{2}}$, $x = z^2$ and $\frac{dx}{dt} = 2z\frac{dz}{dt}$

$\therefore$ The equation $\frac{dx}{dt} + t^2x = t^2x^{\frac{1}{2}}$ becomes

$$2z\frac{dz}{dt} + t^2z^2 = t^2z$$

Divide through by $2z$

Then $\frac{dz}{dt} + \frac{1}{2}t^2z = \frac{1}{2}t^2$

The integrating factor is $e^{\int \frac{1}{2}t^2\,dt} = e^{\frac{1}{6}t^3}$

$\therefore\ e^{\frac{1}{6}t^3}\frac{dz}{dt} + \frac{1}{2}t^2e^{\frac{1}{6}t^3}z = \frac{1}{2}t^2e^{\frac{1}{6}t^3}$

$\therefore\ \frac{d}{dt}\left(ze^{\frac{1}{6}t^3}\right) = \frac{1}{2}t^2e^{\frac{1}{6}t^3}$

$\therefore\ ze^{\frac{1}{6}t^3} = \int \frac{1}{2}t^2e^{\frac{1}{6}t^3}\,dt$

$= e^{\frac{1}{6}t^3} + c$

$\therefore\ z = 1 + ce^{-\frac{1}{6}t^3}$

But $x = z^2 \quad \therefore x = \left(1 + ce^{-\frac{1}{6}t^3}\right)^2$

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Exercise D, Question 7

Question:

Use the substitution $z = y^{-1}$ to transform the differential equation

$$\frac{dy}{dx} - \frac{1}{x}y = \frac{(x+1)^3}{x}y^2$$

into a differential equation in z and x. By first solving the transformed equation, find the general solution of the original equation, giving y in terms of x.

Solution:

Let $z = y^{-1}$, then $y = z^{-1}$ and $\frac{dy}{dx} = -z^{-2}\frac{dz}{dx}$

So $\frac{dy}{dx} - \frac{1}{x}y = \frac{(x+1)^3}{x}y^2$ becomes:

$$-z^{-2}\frac{dz}{dx} - \frac{1}{x}z^{-1} = \frac{(x+1)^3}{x}z^{-2}$$

Multiply through by $-z^2$

Then $\frac{dz}{dx} + \frac{1}{x}z = -\frac{(x+1)^3}{x}$

The integrating factor is $e^{\int P\,dx} = e^{\int \frac{1}{x}dx} = e^{\int \ln x} = x$

$$\therefore\ x\frac{dz}{dx} + z = -(x+1)^3$$

i.e. $\frac{d}{dx}(xz) = -(x+1)^3$

$$\therefore\quad xz = -\int (x+1)^3\,dx$$

$$= -\tfrac{1}{4}(x+1)^4 + c$$

$$\therefore\quad z = -\frac{1}{4x}(x+1)^4 + \frac{c}{x}$$

$$\therefore\quad y = -\frac{4x}{4c - (x+1)^4}$$

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Exercise D, Question 8

Question:

Use the substitution $z = y^2$ to transform the differential equation

$$2(1 + x^2)\frac{dy}{dx} + 2xy = \frac{1}{y}$$

into a differential equation in z and x. By first solving the transformed equation,

a find the general solution of the original equation, giving y in terms of x.

b Find the particular solution for which $y = 2$ when $x = 0$.

Solution:

a Given that $z = y^2$, and so $y = z^{\frac{1}{2}}$ and $\frac{dy}{dx} = \frac{1}{2}z^{-\frac{1}{2}}\frac{dz}{dx}$

The equation $2(1 + x^2)\frac{dy}{dx} + 2xy = \frac{1}{y}$ becomes

$$2(1 + x^2) \times \frac{1}{2}z^{-\frac{1}{2}}\frac{dz}{dx} + 2xz^{\frac{1}{2}} = z^{-\frac{1}{2}}$$

Multiply the equation by $\frac{z^{\frac{1}{2}}}{1 + x^2}$

Then $\frac{dz}{dx} + \frac{2x}{1 + x^2}z = \frac{1}{1 + x^2}$

The integrating factor is $e^{\int \frac{2x}{1 + x^2}dx} = e^{\ln(1 + x^2)} = 1 + x^2$

$$\therefore (1 + x^2)\frac{dz}{dx} + 2xz = 1$$

$$\therefore \quad \frac{d}{dx}[(1 + x^2)z] = 1$$

$$\therefore \quad (1 + x^2)z = \int 1\, dx$$

$$= x + c$$

$$\therefore \quad z = \frac{x + c}{(1 + x^2)}$$

As $y = z^{\frac{1}{2}}$, $\quad y = \sqrt{\frac{x + c}{(1 + x^2)}}$

b When $x = 0$, $y = 2$ $\quad \therefore 2 = \sqrt{c} \Rightarrow c = 4$

$$\therefore y = \sqrt{\frac{x + 4}{1 + x^2}}$$

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Exercise D, Question 9

Question:

Show that the substitution $z = y^{-(n-1)}$ transforms the general equation

$\frac{dy}{dx} + Py = Qy^n$,

where P and Q are functions of x, into the linear equation $\frac{dz}{dx} - P(n-1)z = -Q(n-1)$

(Bernoulli's equation)

Solution:

Given $z = y^{-(n-1)}$

$\therefore \quad y = z^{-\frac{1}{(n-1)}}$

$\frac{dy}{dx} = \frac{-1}{n-1} z^{-\frac{1}{n-1}-1} \frac{dz}{dx}$

$= \frac{-1}{n-1} z^{-\frac{n}{n-1}} \frac{dz}{dx}$

$\therefore \frac{dy}{dx} + Py = Q y^n$ becomes

$\frac{-1}{n-1} z^{-\frac{n}{n-1}} \frac{dz}{dx} + P z^{-\frac{1}{n-1}} = Q z^{-\frac{n}{n-1}}$

Multiply each term by $-(n-1) z^{\frac{n}{n-1}}$

Then $\frac{dz}{dz} - P(n-1) z^{\frac{n}{n-1}} z^{-\frac{1}{n-1}} = -Q(n-1) z^{\frac{n}{n-1}} z^{-\frac{n}{n-1}}$

i.e. $\frac{dz}{dz} - P(n-1) z = -Q(n-1)$

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Exercise D, Question 10

Question:

Use the substitution $u = y + 2x$ to transform the differential equation

$$\frac{dy}{dx} = \frac{-(1 + 2y + 4x)}{1 + y + 2x}$$

into a differential equation in u and x. By first solving this new equation, show that the general solution of the original equation may be written $4x^2 + 4xy + y^2 + 2y + 2x = k$, where k is a constant

Solution:

Given $u = y + 2x$ and so $y = u - 2x$ and $\frac{dy}{dx} = \frac{du}{dx} - 2$

$\therefore$ the differential equation $\frac{dy}{dx} = -\frac{(1 + 2y + 4x)}{1 + y + 2x}$ becomes

Rearrange the given substitution to give y in terms of u and x, and $\frac{dy}{dx}$ in terms of $\frac{du}{dx}$.

$$\frac{du}{dx} - 2 = -\frac{1 + 2u}{1 + u}$$

$$\therefore \quad \frac{du}{dx} = \frac{-(1 + 2u) + 2(1 + u)}{1 + u}$$

$$\therefore \quad \frac{du}{dx} = \frac{1}{1 + u}$$

Separate the variables

$$\int (1 + u)\, du = \int 1 \times dx$$

$$\therefore \quad u + \frac{u^2}{2} = x + c, \text{ where } c \text{ is constant}$$

And $\quad (y + 2x) + \frac{(y + 2x)^2}{2} = x + c$

$$2y + 4x + y^2 + 4xy + 4x^2 = 2x + 2c$$

i.e. $4x^2 + 4xy + y^2 + 2y + 2x = k$, where $k = 2c$

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Exercise E, Question 1

Question:

Solve the equation $\frac{dy}{dx} = \frac{x}{\sqrt{x^2 + 16}}$ and sketch three solution curves.

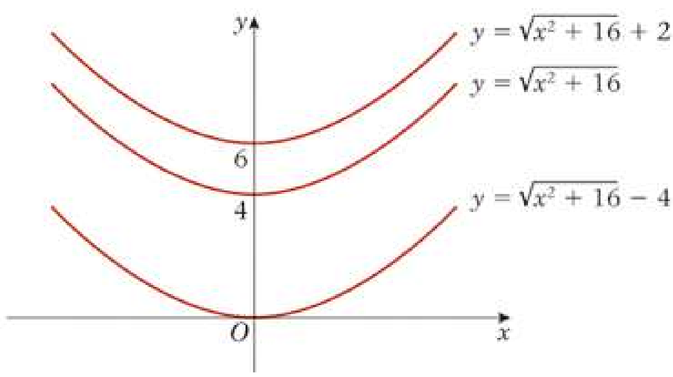
Solution:

$$\frac{dy}{dx} = \frac{x}{\sqrt{x^2 + 16}}$$

$$\therefore y = \int \frac{x}{\sqrt{x^2 + 16}}\, dx$$

$$= (x^2 + 16)^{\frac{1}{2}} + c$$

The integral is of the type $\int [f(x)]^n f'(x)\, dx$, which integrates to give $[f(x)]^{n+1} + c$.



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Exercise E, Question 2

Question:

Solve the equation $\frac{dy}{dx} = xy$ and sketch the solution curves which pass through

a (0, 1) **b** (0, 2) **c** (0, 3)

Solution:

$\frac{dy}{dx} = xy$ — Separate the variables and integrate.

$\therefore \int \frac{1}{y}\,dy = \int x\,dx$

$\therefore \quad \ln y = \frac{1}{2}x^2 + c$, where c is constant

$\therefore \quad y = e^{\frac{1}{2}x^2 + c}$

$= e^c\, e^{\frac{1}{2}x^2} \quad = Ae^{\frac{1}{2}x^2}$, where A is e^c

a The solution which satisfies $x = 0$ when $y = 1$

is $y = Ae^{\frac{1}{2}x^2}$ where $1 = Ae^0$ i.e. $A = 1$

$\therefore \quad y = e^{\frac{1}{2}x^2}$

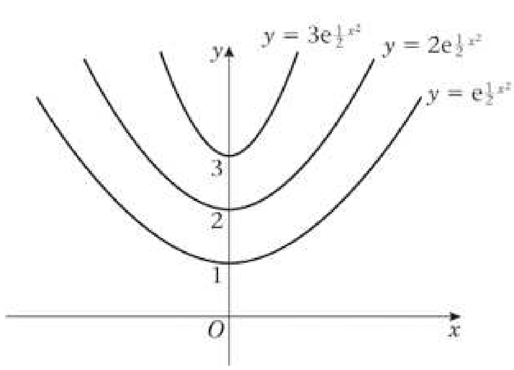
b The solution for which $y = 2$ when $x = 0$ is $y = Ae^{\frac{1}{2}x^2}$

with $2 = Ae^0$ i.e. $A = 2$

$\therefore \quad y = 2e^{\frac{1}{2}x^2}$

c The solution for which $y = 3$ when $x = 0$ is $y = 3e^{\frac{1}{2}x^2}$

The solution curves are shown in the sketch.



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Exercise E, Question 3

Question:

Solve the equation $\frac{dv}{dx} = -g - kv$ given that $v = u$ when $t = 0$, and that u, g and k are positive constants. Sketch the solution curve indicating the velocity which v approaches as t becomes large.

Solution:

$$\frac{dv}{dt} = -g - kv$$

You can separate the variables by dividing both sides by $(g + kv)$, or you could rearrange the equation as $\frac{dv}{dt} + kv = g$ and use the integrating factor e^{kt}.

$$\therefore \quad \int \frac{dv}{g + kv} = -\int 1 \, dt$$

$\therefore \frac{1}{k} \ln |g + kv| = -t + c$ where c is a constant *

When $t = 0$, $v = u$

$$\therefore \frac{1}{k} \ln |g + ku| = c$$

$\therefore$ Substituting c back into the equation *

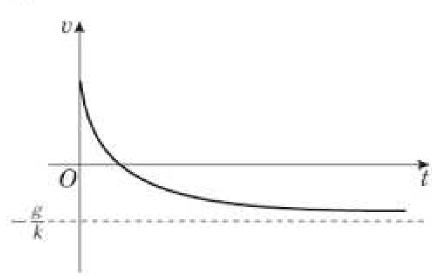
$$\frac{1}{k} \ln |g + kv| = -t + \frac{1}{k} \ln |g + ku|$$

$$\therefore \frac{1}{k} [\ln |g + kv| - \ln |g + ku|] = -t$$

$$\therefore \quad \ln \frac{g + kv}{g + ku} = -kt$$

$$\therefore \quad g + kv = (g + ku) e^{-kt}$$

$$\therefore \quad v = \frac{1}{k} [(g + ku) e^{-kt} - g]$$



The required velocity is $-\frac{g}{k}$ m s^{-1}

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Exercise E, Question 4

Question:

Solve the equation $\frac{dy}{dx} + y \tan x = 2 \sec x$

Solution:

$$\frac{dy}{dx} + y\tan x = 2\sec x$$

Use an integrating factor to solve this equation.

Use the integrating factor $e^{\int \tan x \, dx} = e^{\ln \sec x} = \sec x$

$\therefore \quad \sec x \frac{dy}{dx} + y \sec x \tan x = 2\sec^2 x$

$\therefore \quad \frac{d}{dx}(y \sec x) = 2\sec^2 x$

$\therefore \quad y \sec x = \int 2\sec^2 x \, dx$

$= 2\tan x + c$

$\therefore \quad y = 2\sin x + c\cos x$

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Exercise E, Question 5

Question:

Solve the equation $(1-x^2)\frac{dy}{dx}+xy=5x \qquad -1<x<1$

Solution:

$$(1-x^2)\frac{dy}{dx}+xy=5x$$

Divide through by $(1-x^2)$, then find the integrating factor.

Divide through by $(1-x^2)$

$$\therefore \quad \frac{dy}{dx}+\frac{x}{1-x^2}y=\frac{5x}{1-x^2}$$

Use the integrating factor $e^{\int \frac{x}{1-x^2}dx}=e^{-\frac{1}{2}\ln(1-x^2)}$

$$=e^{\ln(1-x^2)^{-\frac{1}{2}}}=\frac{1}{\sqrt{1-x^2}}$$

$$\therefore \quad \frac{1}{\sqrt{1-x^2}}\frac{dy}{dx}+\frac{x}{(1-x^2)^{\frac{3}{2}}}y=\frac{5x}{(1-x^2)^{\frac{3}{2}}}$$

$$\therefore \quad \frac{d}{dx}\left[(1-x^2)^{-\frac{1}{2}}y\right]=\frac{5x}{(1-x^2)^{\frac{3}{2}}}$$

$$\therefore \quad (1-x^2)^{-\frac{1}{2}}y=\int\frac{5x}{(1-x^2)^{\frac{3}{2}}}dx$$

$$=5(1-x^2)^{-\frac{1}{2}}+c$$

$$\therefore \quad y=5+c(1-x^2)^{\frac{1}{2}}$$

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Exercise E, Question 6

Question:

Solve the equation $x\frac{dy}{dx} + x + y = 0$

Solution:

$$x\frac{dy}{dx} + x + y = 0$$

Take the 'x' term to the other side of the equation.

$$\therefore \quad x\frac{dy}{dx} + y = -x$$

This is an exact equation.

So $\frac{d}{dx}(xy) = -x$

$$\therefore \quad xy = -\int x \, dx$$

$$= -\tfrac{1}{2}x^2 + c$$

$$\therefore \quad y = -\tfrac{1}{2}x + \frac{c}{x}$$

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Exercise E, Question 7

Question:

Solve the equation $\frac{dy}{dx} + \frac{y}{x} = \sqrt{x}$

Solution:

$$\frac{dy}{dx} + \frac{y}{x} = \sqrt{x}$$

The integrating factor is $e^{\int \frac{1}{x} dx} = e^{\ln x} = x$

Multiply the differential equation by the integrating factor:

$$x\frac{dy}{dx} + y = x\sqrt{x}$$

$$\therefore \quad \frac{d}{dx}(xy) = x^{\frac{3}{2}}$$

$$\therefore \quad xy = \int x^{\frac{3}{2}} dx$$

$$= \frac{2}{5}x^{\frac{5}{2}} + c$$

$$\therefore \quad y = \frac{2}{5}x^{\frac{3}{2}} + \frac{c}{x}$$

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Exercise E, Question 8

Question:

Solve the equation $\frac{dy}{dx} + 2xy = x$

Solution:

$$\frac{dy}{dx} + 2xy = x$$

The integrating factor is $e^{\int 2x\,dx} = e^{x^2}$

Multiply the differential equation by e^{x^2}

$$\therefore \quad e^{x^2}\frac{dy}{dx} + 2xe^{x^2}y = xe^{x^2}$$

$$\therefore \quad \frac{d}{dx}(e^{x^2}y) = xe^{x^2}$$

$$\therefore \quad ye^{x^2} = \int xe^{x^2}\,dx$$

$$= \tfrac{1}{2}e^{x^2} + c$$

$$\therefore \quad y = \tfrac{1}{2} + ce^{-x^2}$$

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Exercise E, Question 9

Question:

Solve the equation $x(1-x^2)\frac{dy}{dx}+(2x^2-1)y=2x^3 \qquad 0<x<1$

Solution:

$$x(1-x^2)\frac{dy}{dx}+(2x^2-1)y=2x^3$$

Divide through by $x(1-x^2)$

$$\therefore \quad \frac{dy}{dx}+\frac{2x^2-1}{x(1-x^2)}y=\frac{2x^3}{x(1-x^2)} \quad *$$

You will need to use partial fractions to integrate $\frac{2x^2-1}{x(1-x^2)}$ and to find the integrating factor.

The integrating factor is $e^{\int\frac{2x^2-1}{x(1-x^2)}dx}$

$$\int\frac{2x^2-1}{x(1-x)(1+x)}dx=\int\left(-\frac{1}{x}+\frac{1}{2(1-x)}-\frac{1}{2(1+x)}\right)dx$$

$$=-\ln x-\tfrac{1}{2}\ln(1-x)-\tfrac{1}{2}\ln(1+x)$$

$$=-\ln x\sqrt{1-x^2}$$

So the integrating factor is $e^{-\ln x\sqrt{1-x^2}}=e^{\ln\frac{1}{x\sqrt{1-x^2}}}=\frac{1}{x\sqrt{1-x^2}}$

Multiply the differential equation $*$ by $\frac{1}{x\sqrt{1-x^2}}$

$$\therefore \quad \frac{1}{x\sqrt{1-x^2}}\frac{dy}{dx}+\frac{2x^2-1}{x^2(1-x^2)^{\frac{3}{2}}}y=\frac{2x}{(1-x^2)^{\frac{3}{2}}}$$

$$\therefore \quad \frac{d}{dx}\left[\frac{1}{x\sqrt{1-x^2}}y\right]=\frac{2x}{(1-x^2)^{\frac{3}{2}}}$$

$$\therefore \quad \frac{y}{x\sqrt{1-x^2}}=\int\frac{2x}{(1-x^2)^{\frac{3}{2}}}dx$$

$$=2(1-x^2)^{-\frac{1}{2}}+c$$

$$\therefore \quad y=2x+cx\sqrt{1-x^2}$$

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Exercise E, Question 10

Question:

Solve the equation $R\frac{dq}{dt}+\frac{q}{c}=E$ when

a $E=0$ **b** $E=$ constant **c** $E=\cos pt$

(R, c and p are constants)

Solution:

This is a difficult question – particularly part **c**. You may decide to omit this question, unless you want a challenge.

$$R\frac{dq}{dt} + \frac{q}{c} = E$$

$$\therefore \quad \frac{dq}{dt} + \frac{1}{Rc}q = \frac{E}{R}$$

The integrating factor is $e^{\int \frac{1}{Rc}dt} = e^{\frac{t}{Rc}}$

$$\therefore \quad e^{\frac{t}{Rc}}\frac{dq}{dt} + \frac{1}{Rc}e^{\frac{t}{Rc}}q = \frac{E}{R}e^{\frac{t}{Rc}}$$

$$\therefore \quad \frac{d}{dt}\left(qe^{\frac{t}{Rc}}\right) = \frac{E}{R}e^{\frac{t}{Rc}}$$

$$\therefore \quad qe^{\frac{t}{Rc}} = \int \frac{E}{R}e^{\frac{t}{Rc}}dt$$

a When $E = 0$

$\therefore \quad qe^{\frac{t}{Rc}} = k$, where k is constant.

$$\therefore \quad q = ke^{-\frac{t}{Rc}}$$

b When $E =$ constant

$$qe^{\frac{t}{Rc}} = \int \frac{E}{R}e^{\frac{t}{Rc}}dt$$

$= Ece^{\frac{t}{Rc}} + k$, where k is constant

$$\therefore \quad q = Ec + ke^{-\frac{t}{Rc}}$$

c When $E = \cos pt$

$$qe^{\frac{t}{Rc}} = \int \frac{1}{R}\cos pt\, e^{\frac{t}{Rc}}dt \quad *$$

i.e. $\int \frac{1}{R}\cos pt\, e^{\frac{t}{Rc}} = ce^{\frac{t}{Rc}}\cos pt + \int cpe^{\frac{t}{Rc}}\sin pt\, dt$ — Use integration by parts.

$\int \frac{1}{R}\cos pt e^{\frac{t}{Rc}}dt = ce^{\frac{t}{Rc}}\cos pt + Rpc^2 e^{\frac{t}{Rc}}\sin pt - \int Rp^2c^2e^{\frac{t}{Rc}}\cos pt\, dt$ — Use 'parts' again.

$\therefore \quad \int\left(\frac{1}{R} + Rp^2c^2\right)e^{\frac{t}{Rc}}\cos pt\, dt = ce^{\frac{t}{Rc}}(\cos pt + Rpc\sin pt) + k$, where k is a constant

$$\therefore \quad \frac{1}{R}\int e^{\frac{t}{Rc}}\cos pt\, dt = \frac{c}{(1 + R^2p^2c^2)}e^{\frac{t}{Rc}}(\cos pt + Rpc\sin pt) + \frac{k}{(1 + R^2p^2c^2)}$$

From $*$

$$qe^{\frac{t}{Rc}} = \frac{c}{(1 + R^2p^2c^2)}e^{\frac{t}{Rc}}(\cos pt + Rpc\sin pt) + \frac{k}{(1 + R^2p^2c^2)}$$

$\therefore \quad q = \frac{c}{(1 + R^2p^2c^2)}(\cos pt + Rpc\sin pt) + k'e^{-\frac{t}{Rc}}$, where $k' = \frac{k}{1 + R^2p^2c^2}$ is constant

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Exercise E, Question 11

Question:

Find the general solution of the equation $\frac{dy}{dx} - ay = Q$, where a is a constant, giving your answer in terms of a, when

a $Q = ke^{\lambda x}$ **b** $Q = ke^{ax}$ **c** $Q = kx^n e^{ax}$.

(k, λ and n are constants).

Solution:

Given that $\frac{dy}{dx} - ay = Q$

The integrating factor is $e^{\int -a\,dx} = e^{-ax}$

Then $e^{-ax}\frac{dy}{dx} - ae^{-ax}y = Qe^{-ax}$

$\therefore \quad \frac{d}{dx}(ye^{-ax}) = Qe^{-ax}$

$\therefore \quad ye^{-ax} = \int Qe^{-ax}\,dx$

a When $Q = ke^{\lambda x}$

$ye^{-ax} = \int ke^{(\lambda - a)x}\,dx$

$= \frac{k}{\lambda - a}e^{(\lambda - a)x} + c$, where c is constant

$\therefore \quad y = \frac{k}{\lambda - a}e^{\lambda x} + ce^{ax}$

When $\lambda \neq a$.
For $\lambda = a$, see part **b**.

b When $Q = ke^{ax}$

$ye^{-ax} = \int k\,dx$

$= kx + c$, where c is constant

$\therefore \quad y = (kx + c)e^{ax}$

c When $Q = kx^n e^{ax}$

$ye^{-ax} = \int kx^n\,dx$

$= \frac{kx^{n+1}}{n+1} + c$, where c is constant

$\therefore \quad y = \frac{kx^{n+1}}{n+1}e^{ax} + ce^{ax}$

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Exercise E, Question 12

Question:

Use the substitution $z = y^{-1}$ to transform the differential equation $x\frac{dy}{dx} + y = y^2 \ln x$, into a linear equation. Hence obtain the general solution of the original equation.

Solution:

Given that $z = y^{-1}$, then $y = z^{-1}$ so $\frac{dy}{dx} = -z^{-2}\frac{dz}{dx}$.

Use the substitution to express y in terms of z and $\frac{dy}{dx}$ in terms of z and $\frac{dz}{dx}$.

The equation $x\frac{dy}{dx} + y = y^2 \ln x$ becomes

$$-xz^{-2}\frac{dz}{dx} + z^{-1} = z^{-2}\ln x$$

Divide through by $-xz^{-2}$

$$\therefore \quad \frac{dz}{dy} - \frac{z}{x} = -\frac{\ln x}{x}$$

The integrating factor is $e^{-\int \frac{1}{x}dx} = e^{-\ln x} = e^{\ln \frac{1}{x}} = \frac{1}{x}$

$$\therefore \quad \frac{1}{x}\frac{dz}{dx} - \frac{z}{x^2} = -\frac{\ln x}{x^2}$$

$$\therefore \quad \frac{d}{dx}\left(\frac{1}{x}z\right) = -\frac{\ln x}{x^2}$$

$$\therefore \quad \frac{1}{x}z = -\int \frac{1}{x^2}\ln x\, dx$$

$$= -\left[-\frac{1}{x}\ln x + \int \frac{1}{x^2}dx\right]$$

$$= \frac{1}{x}\ln x + \frac{1}{x} + c$$

$$\therefore \quad z = \ln x + 1 + cx.$$

As $y = z^{-1}$ $\therefore$ $y = \frac{1}{1 + cx + \ln x}$

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Exercise E, Question 13

Question:

Use the substitution $z = y^2$ to transform the differential equation $2\cos x \frac{dy}{dx} - y\sin x + y^{-1} = 0$, into a linear equation. Hence obtain the general solution of the original equation.

Solution:

Given that $z = y^2$, $y = z^{\frac{1}{2}}$ and $\frac{dy}{dx} = \frac{1}{2}z^{-\frac{1}{2}}\frac{dz}{dx}$

The differential equation

$$2\cos x \frac{dy}{dx} - y\sin x + y^{-1} = 0 \text{ becomes}$$

$$\cos x\, z^{-\frac{1}{2}}\frac{dz}{dx} - z^{\frac{1}{2}}\sin x + z^{-\frac{1}{2}} = 0$$

Divide through by $z^{-\frac{1}{2}}$

then $\cos x \frac{dz}{dx} - z\sin x = -1$

This becomes an exact equation which can be solved directly.

$$\therefore \quad \frac{d}{dx}(z\cos x) = -1$$

$$\therefore \quad z\cos x = -\int 1\,dx$$

$$= -x + c$$

$$\therefore \quad z = \frac{c - x}{\cos x}$$

$$\therefore \quad y = \sqrt{\frac{c - x}{\cos x}}$$

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Exercise E, Question 14

Question:

Use the substitution $z = \frac{y}{x}$ to transform the differential equation $(x^2 - y^2)\frac{dy}{dx} - xy = 0$, into a linear equation. Hence obtain the general solution of the original equation.

Solution:

Given that $z = \frac{y}{x}$, $y = zx$ so $\frac{dy}{dx} = z + x\frac{dz}{dx}$

The equation $(x^2 - y^2)\frac{dy}{dx} - xy = 0$ becomes

$$(x^2 - z^2x^2)\left(z + x\frac{dz}{dx}\right) - xzx = 0$$

$\therefore \quad (1 - z^2)z + (1 - z^2)x\frac{dz}{dx} - z = 0$

$\therefore \quad x\frac{dz}{dx} = \frac{z}{1 - z^2} - z$

i.e. $\quad x\frac{dz}{dx} = \frac{z^3}{1 - z^2}$

Separate the variables to give

$$\int \frac{1 - z^2}{z^3}\,dz = \int \frac{1}{x}\,dx$$

$\therefore \quad \int (z^{-3} - z^{-1})\,dz = \int x^{-1}\,dx$

$\therefore \quad \frac{z^{-2}}{-2} - \ln z = \ln x + c$

$\therefore \quad -\frac{1}{2z^2} = \ln x + \ln z + c$

$\quad = \ln xz + c$

But $\quad y = zx$

$\therefore \quad (c + \ln y) = -\frac{x^2}{2y^2}$

$\therefore \quad 2y^2(\ln y + c) + x^2 = 0$

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Exercise E, Question 15

Question:

Use the substitution $z = \frac{y}{x}$ to transform the differential equation $\frac{dy}{dx} = \frac{y(x+y)}{x(y-x)}$, into a linear equation. Hence obtain the general solution of the original equation.

Solution:

$z = \frac{y}{x}, \quad \therefore \quad y = xz \text{ and } \frac{dy}{dx} = z + x\frac{dz}{dx}$

$\therefore \quad \frac{dy}{dx} = \frac{y(x+y)}{x(y-x)} \text{ becomes } z + x\frac{dz}{dx} = \frac{xz(x+xz)}{x(xz-x)}$

$\therefore \quad z + x\frac{dz}{dx} = \frac{z(1+z)}{(z-1)}$

So $\quad x\frac{dz}{dx} = \frac{z(1+z)}{z-1} - z$

$= \frac{2z}{z-1}$

Separating the variables

$$\int \frac{(z-1)}{2z} dz = \int \frac{1}{x} dx$$

$$\therefore \quad \int \left(\frac{1}{2} - \frac{1}{2z}\right) dz = \int \frac{1}{x} dx$$

$$\therefore \quad \tfrac{1}{2}z - \tfrac{1}{2}\ln z = \ln x + c$$

As $z = \frac{y}{x} \quad \therefore \quad \frac{y}{2x} - \frac{1}{2}\ln\frac{y}{x} = \ln x + c$

$$\therefore \quad \frac{y}{2x} - \tfrac{1}{2}\ln y + \tfrac{1}{2}\ln x = \ln x + c$$

$$\therefore \quad \frac{y}{2x} - \tfrac{1}{2}\ln y = \tfrac{1}{2}\ln x + c$$

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Exercise E, Question 16

Question:

Use the substitution $z = \frac{y}{x}$ to transform the differential equation $\frac{dy}{dx} = \frac{-3xy}{(y^2 - 3x^2)}$, into a linear equation. Hence obtain the general solution of the original equation.

Solution:

Given that $z = \frac{y}{x}$, so $y = zx$ and $\frac{dy}{dx} = z + x\frac{dz}{dx}$

The equation $\frac{dy}{dx} = \frac{-3xy}{y^2 - 3x^2}$ becomes

$$z + x\frac{dz}{dx} = \frac{-3x^2z}{z^2x^2 - 3x^2}$$

i.e. $$x\frac{dz}{dx} = \frac{-3z}{z^2 - 3} - z$$

$$= \frac{-z^3}{z^2 - 3}$$

Separate the variables:

Then $\int \left(\frac{z^2 - 3}{z^3}\right) dz = -\int \frac{1}{x} dx.$

$\therefore \quad \int \left(\frac{1}{z} - 3z^{-3}\right) dz = -\ln x + c$

$\therefore \quad \ln z + \frac{3}{2} z^{-2} = -\ln x + c$

$\therefore \quad \ln zx + \frac{3}{2z^2} = c$

But $zx = y$ and $z = \frac{y}{x}$

$\therefore \quad \ln y + \frac{3x^2}{2y^2} = c$

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Exercise E, Question 17

Question:

Use the substitution $u = x + y$ to transform the differential equation $\frac{dy}{dx} = (x + y + 1)(x + y - 1)$ into a differential equation in u and x. By first solving this new equation, find the general solution of the original equation, giving y in terms of x.

Solution:

Let $u = x + y$, then $\frac{du}{dx} = 1 + \frac{dy}{dx}$ and so $\frac{dy}{dx} = (x + y + 1)(x + y - 1)$ becomes

$$\frac{du}{dx} - 1 = (u + 1)(u - 1)$$
$$= u^2 - 1$$
$$\therefore \quad \frac{du}{dx} = u^2$$

Separate the variables.

Then $$\int \frac{1}{u^2} du = \int 1 \, dx$$

$$\therefore \quad -\frac{1}{u} = x + c$$

But $u = x + y$ $\therefore$ $-\frac{1}{x + y} = x + c$

$$\therefore \quad y + x = \frac{-1}{x + c}$$

$$y = \frac{-1}{x + c} - x$$

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Exercise E, Question 18

Question:

Use the substitution $u = y - x - 2$ to transform the differential equation $\frac{dy}{dx} = (y - x - 2)^2$ into a differential equation in u and x. By first solving this new equation, find the general solution of the original equation, giving y in terms of x.

Solution:

Given that $u = y - x - 2$, and so $\frac{du}{dx} = \frac{dy}{dx} - 1$

$\therefore \quad \frac{dy}{dx} = (y - x - 2)^2$ becomes $\frac{du}{dx} + 1 = u^2$

i.e. $$\frac{du}{dx} = u^2 - 1$$

$$\therefore \quad \int \frac{1}{u^2 - 1}\,du = \int 1\,dx$$

Factorise $\frac{1}{u^2 - 1}$ into $\frac{1}{(u-1)(u+1)}$ and use partial fractions.

$$\therefore \quad \int \left(\frac{1}{2(u-1)} - \frac{1}{2(u+1)} \right) du = x + c \text{ where } c \text{ is constant}$$

$$\therefore \quad \tfrac{1}{2}\ln(u-1) - \tfrac{1}{2}\ln(u+1) = x + c$$

$$\therefore \quad \tfrac{1}{2}\ln\frac{u-1}{u+1} = x + c$$

$$\therefore \quad \frac{u-1}{u+1} = e^{2c+2x} = Ae^{2x} \text{ where } A = e^{2c} \text{ is a constant}$$

$$\therefore \quad u - 1 = Aue^{2x} + Ae^{2x}$$

$$\therefore \quad u(1 - Ae^{2x}) = (1 + Ae^{2x})$$

$$\therefore \quad u = \frac{1 + Ae^{2x}}{1 - Ae^{2x}}$$

But $\quad u = y - x - 2$

$$\therefore \quad y = x + 2 + \frac{1 + Ae^{2x}}{1 - Ae^{2x}}$$