Solutionbank FP2
Edexcel AS and A Level Modular Mathematics

Exercise A, Question 1

Question:

Solve the following inequality

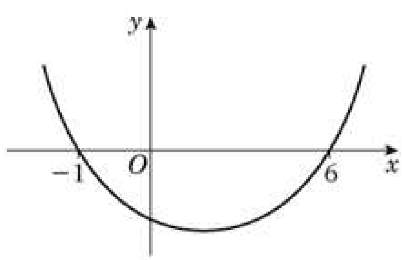
$x^2 < 5x + 6$

Solution:

$$x^2 - 5x - 6 < 0$$
$$(x - 6)(x + 1) < 0$$

critical values $x = -1$ or 6

sketch



solution is $-1 < x < 6$

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Edexcel AS and A Level Modular Mathematics

Exercise A, Question 2

Question:

Solve the following inequality

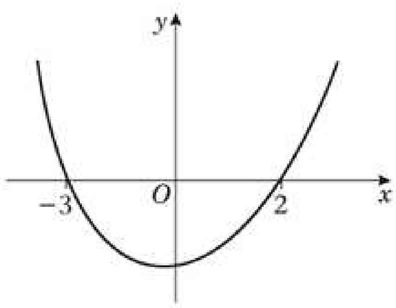
$x(x + 1) \geqslant 6$

Solution:

$$x^2 + x \geqslant 6$$
$$(x + 3)(x - 2) \geqslant 0$$

critical values $x = 2$ or -3

sketch



solution is $x \geqslant 2$ or $x \leqslant -3$

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Exercise A, Question 3

Question:

Solve the following inequality

$$\frac{2}{x^2+1} > 1$$

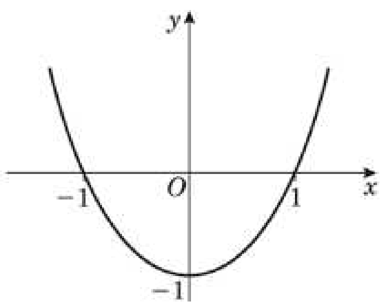
Solution:

$$2 > x^2 + 1$$
$$0 > x^2 - 1$$

You can multiply by $x^2 + 1$ because it is always positive.

critical values $x = \pm 1$

sketch



solution is $-1 < x < 1$

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Exercise A, Question 4

Question:

Solve the following inequality

$$\frac{2}{x^2-1} > 1$$

Solution:

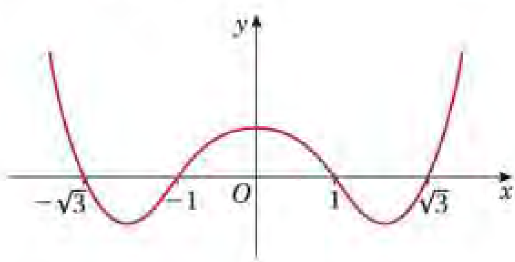
$$\frac{2}{\cancel{(x^2-1)}} \times (x^2-1)^{\cancel{2}} > (x^2-1)^2$$

$$0 > (x^2-1)[x^2-1-2]$$

$$0 > (x-1)(x+1)(x-\sqrt{3})(x+\sqrt{3})$$

critical values $x = \pm 1, \pm\sqrt{3}$

sketch



solution is $-\sqrt{3} < x < -1$ or $1 < x < \sqrt{3}$

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Exercise A, Question 5

Question:

Solve the following inequality

$\frac{x}{x-1} \leqslant 2x \quad x \neq 1$

Solution:

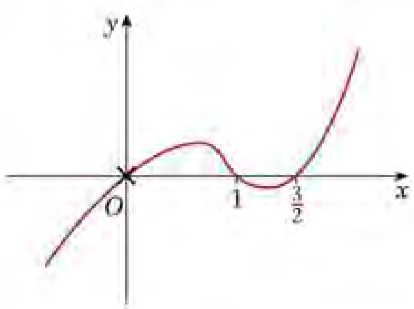
$$\frac{x}{\cancel{(x-1)}} \times (x-1)^{\cancel{2}} \leqslant 2x(x-1)^2$$

$$0 \leqslant x(x-1)[2x-2-1]$$

$$0 \leqslant x(x-1)(2x-3)$$

critical values $x = 0, 1, \frac{3}{2}$

sketch



solution is $x > \frac{3}{2}$ or $0 < x < 1$

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Exercise A, Question 6

Question:

Solve the following inequality

$$\frac{3}{x+1} < \frac{2}{x}$$

Solution:

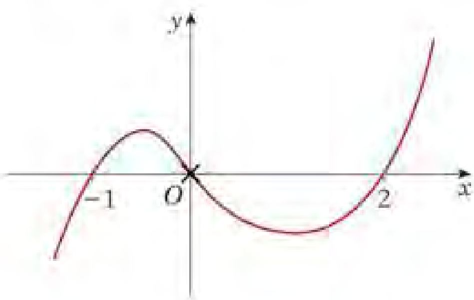
$$\frac{3}{\cancel{(x+1)}} \times (x+1)^{\cancel{2}} x^2 < \frac{2}{\cancel{x}} \times (x+1)^2 x^{\cancel{2}}$$

$$x(x+1)[3x - 2(x+1)] < 0$$

$$x(x+1)(x-2) < 0$$

critical values $x = 0, -1, 2$

sketch



solution is $x < -1$ or $0 < x < 2$

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Exercise A, Question 7

Question:

Solve the following inequality

$$\frac{3}{(x+1)(x-1)} < 1$$

Solution:

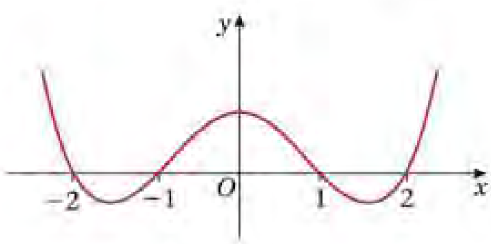
$$\frac{3}{\cancel{(x+1)(x-1)}} \times (x+1)^{\cancel{2}}(x-1)^{\cancel{2}} < (x+1)^2(x-1)^2$$

$$0 < (x+1)(x-1)[x^2 - 1 - 3]$$

$$0 < (x+1)(x-1)(x-2)(x+2)$$

critical values $x = \pm 1, \pm 2$

sketch



solution is $x < -2$ or $-1 < x < 1$ or $x > 2$

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Exercise A, Question 8

Question:

Solve the following inequality

$$\frac{2}{x^2} \geqslant \frac{3}{(x+1)(x-2)}$$

Solution:

$$\frac{2}{x^2} \times (x+1)^2(x-2)^2 \geqslant \frac{3(x+1)^{\cancel{2}}(x-2)^{\cancel{2}}}{\cancel{(x+1)(x-2)}}$$

$$(x+1)(x-2)[2x^2 - 2x - 4 - 3x^2] \geqslant 0$$

You can multiply across x^2 since it is positive.

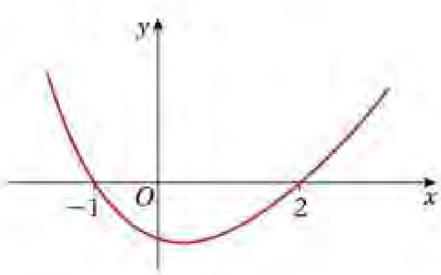
$$(x+1)(x-2)(-4 - 2x - x^2) \geqslant 0$$

or $\quad 0 \geqslant (x+1)(x-2)(x^2 + 2x + 4)$

$x^2 + 2x + 4$ has no real roots

$\therefore$ critical values $x = 2$ or -1

sketch



solution is $-1 < x < 2 \quad x \neq 0$

or $-1 < x < 0$ or $0 < x < 2$

NB $x = 2$ and $x = -1$, $x = 0$ are invalid in the original expression.

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Exercise A, Question 9

Question:

Solve the following inequality

$$\frac{2}{x-4} < 3$$

Solution:

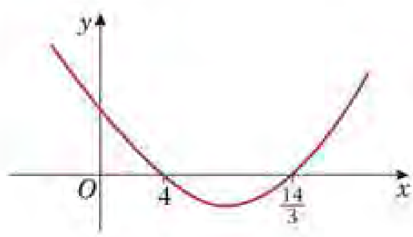
$$\frac{2}{\cancel{x-4}} \times (x-4)^{\cancel{2}} < 3(x-4)^2$$

$$0 < (x-4)[3x-12-2]$$

$$0 < (x-4)(3x-14)$$

critical values $x = 4, \frac{14}{3}$

sketch



solution is $x < 4$ or $x > \frac{14}{3}$

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Exercise A, Question 10

Question:

Solve the following inequality

$$\frac{3}{x+2} > \frac{1}{x-5}$$

Solution:

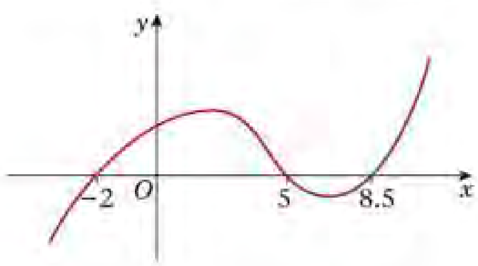
$$\frac{3}{\cancel{(x+2)}} \times (x+2)^{\cancel{2}}(x-5)^2 > \frac{1}{\cancel{(x-5)}} \times (x+2)^2(x-5)^{\cancel{2}}$$

$$(x+2)(x-5)[3x - 15 - (x+2)] > 0$$

$$(x+2)(x-5)(2x-17) > 0$$

critical values $x = -2, 5, 8.5$

sketch



solution is $-2 < x < 5$ or $x > 8.5$

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Exercise A, Question 11

Question:

Solve the following inequality

$$\frac{3x^2+5}{x+5} > 1$$

Solution:

$$\frac{3x^2+5}{\cancel{(x+5)}} \times (x+5)^{\cancel{2}} > (x+5)^2$$

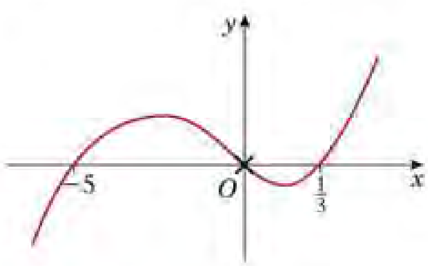
$$(x+5)[3x^2+5-(x+5)] > 0$$

$$(x+5)(3x^2-x) > 0$$

$$(x+5)x(3x-1) > 0$$

critical values $\quad x = 0, \frac{1}{3}, -5$

sketch



solution is $-5 < x < 0$ or $x > \frac{1}{3}$

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Exercise A, Question 12

Question:

Solve the following inequality

$$\frac{3x}{x-2} > x$$

Solution:

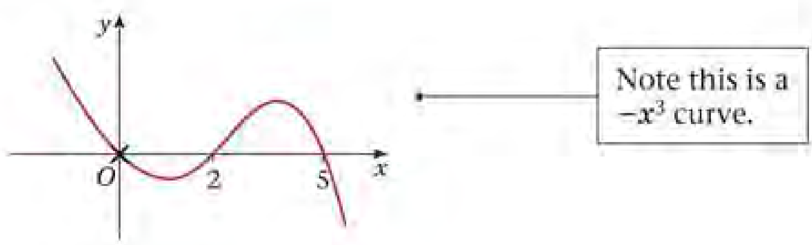
$$\frac{3x}{\cancel{x-2}} \times (x-2)^{\cancel{2}} > x(x-2)^2$$

$$x(x-2)[3-(x-2)] > 0$$

$$x(x-2)(5-x) > 0$$

critical values $\quad x = 0, 2, 5$

sketch



solution is $x < 0$ or $2 < x < 5$

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Exercise A, Question 13

Question:

Solve the following inequality

$\frac{1+x}{1-x} > \frac{2-x}{2+x}$

Solution:

$$\frac{1+x}{\cancel{1-x}} \times (1-x)^{\cancel{2}}(2+x)^2 > \frac{2-x}{\cancel{2+x}} \times (1-x)^2(2+x)^{\cancel{2}}$$

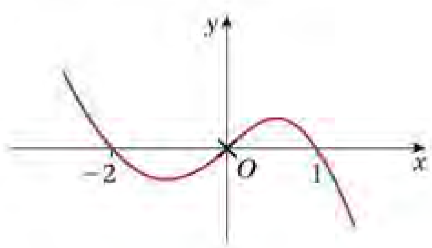
$$(1-x)(2+x)[(1+x)(2+x)-(2-x)(1-x)] > 0$$

$$(1-x)(2+x)(x^2+3x+2-(x^2-3x+2)) > 0$$

$$(1-x)(2+x)6x > 0$$

critical values $x = 1, -2, 0$

sketch



Curve is $y = -x^3$ shape.

solution is $x < -2$ or $0 < x < 1$

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Exercise A, Question 14

Question:

Solve the following inequality

$$\frac{x^2 + 7x + 10}{x + 1} > 2x + 7$$

Solution:

$$\frac{x^2 + 7x + 10}{\cancel{x+1}} \times (x + 1)^{\cancel{2}} > (2x + 7) \times (x + 1)^2$$

$$(x + 1)[x^2 + 7x + 10 - (2x + 7)(x + 1)] > 0$$

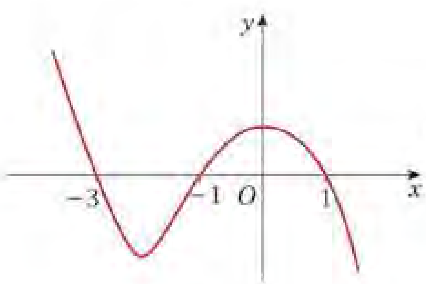
$$(x + 1)[x^2 + 7x + 10 - 2x^2 - 9^x - 7] > 0$$

$$(x + 1)(3 - 2x - x^2) > 0$$

$$(x + 1)(1 - x)(x + 3) > 0$$

critical values $x = -1, 1, -3$

sketch



This is $y = -x^3$ shape.

solution is $x < -3$ or $-1 < x < 1$

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Exercise A, Question 15

Question:

Solve the following inequalities

a $\dfrac{x+1}{x^2} > 6$

b $\dfrac{x^2}{x+1} > \dfrac{1}{6}$

Solution:

a $\dfrac{x+1}{x^2} > 6$

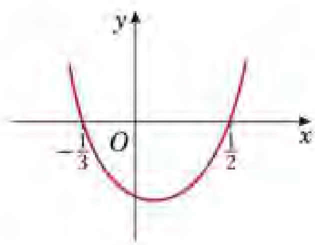
$0 > 6x^2 - x - 1$

You can multiply by x^2 since it is > 0.

$0 > (3x+1)(2x-1)$

critical values $x = -\frac{1}{3}, \frac{1}{2}$

sketch



solution is $-\frac{1}{3} < x < \frac{1}{2}$ But $x \neq 0$

or $-\frac{1}{3} < x < 0$ or $0 < x < \frac{1}{2}$

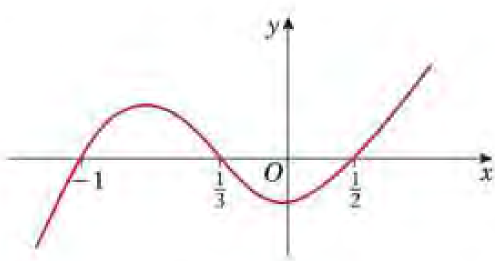
b $\dfrac{x^2}{\cancel{x+1}} \times (x+1)^{\cancel{2}} > \frac{1}{6}(x+1)^2$

$(x+1)[6x^2 - (x+1)] > 0$

$(x+1)(3x+1)(2x-1) > 0$

critical values $x = -1, \frac{1}{2}, -\frac{1}{3}$

sketch



solution is $-1 < x < -\frac{1}{3}$ or $x > \frac{1}{2}$

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Exercise B, Question 1

Question:

Solve the following inequality

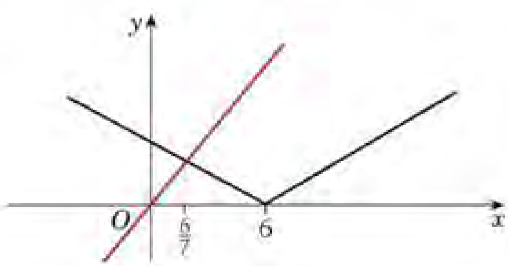
$|x - 6| > 6x$

Solution:

$|x - 6| > 6x$

$$\begin{array}{llll} & x - 6 = 6x & \text{or} & -(x-6) = 6x \\ \Rightarrow & -6 = 5x & \Rightarrow & 6 = 7x \\ & -1.2 = x & \Rightarrow & \frac{6}{7} = x \end{array}$$

sketch



only $x = \frac{6}{7}$ is valid

solution is $x < \frac{6}{7}$

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Exercise B, Question 2

Question:

Solve the following inequality

$|t-3| > t^2$

Solution:

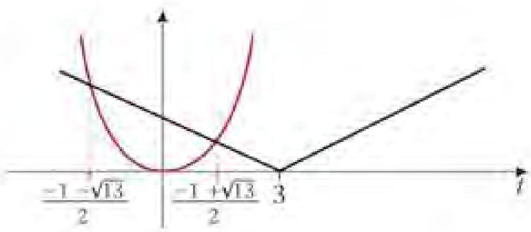
$|t-3| > t^2$

$t - 3 = t^2$ or $-(t-3) = t^2$

$\Rightarrow \quad 0 = t^2 - t + 3 \quad \Rightarrow \quad 0 = t^2 + t - 3$

$t = \text{no solution} \qquad t = \dfrac{-1 \pm \sqrt{1+12}}{2}$

sketch



$|t-3|$ is above t^2 for $\dfrac{-1-\sqrt{13}}{2} < t < \dfrac{-1+\sqrt{13}}{2}$

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Exercise B, Question 3

Question:

Solve the following inequality

$|(x-2)(x+6)| < 9$

Solution:

$|(x-2)(x+6)| < 9$

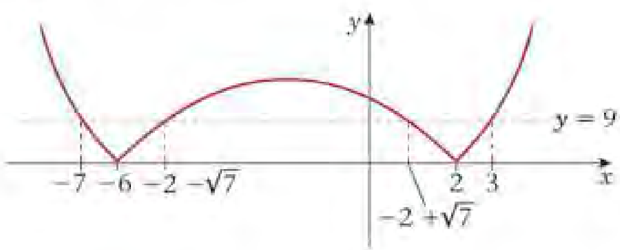
$x^2 + 4x - 12 = 9$ or $-(x^2 + 4x - 12) = 9$

$\Rightarrow x^2 + 4x - 21 = 0$ $\quad 0 = x^2 + 4x - 3$

$(x-3)(x+7) = 0$ $\quad x = \dfrac{-4 \pm \sqrt{16+12}}{2}$

$x = 3$ or -7 $\quad x = -2 \pm \sqrt{7}$

sketch



Line $y = 9$ is above curve for $-7 < x < -2 - \sqrt{7}$ or $-2 + \sqrt{7} < x < 3$

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Exercise B, Question 4

Question:

Solve the following inequality

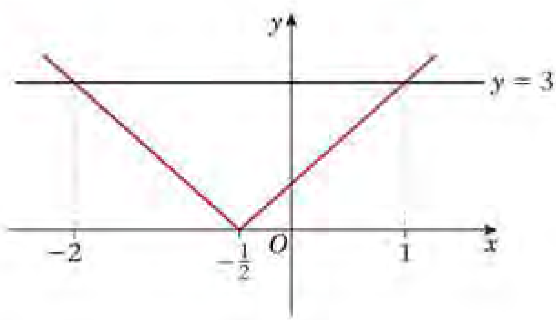
$|2x + 1| \geqslant 3$

Solution:

$|2x + 1| \geqslant 3$

$$2x + 1 = 3 \quad \text{or} \quad -(2x + 1) = 3$$
$$\Rightarrow \quad 2x = 2 \qquad -4 = 2x$$
$$x = 1 \qquad -2 = x$$

sketch



solution is $y = 3$ is below the V when

$x \leqslant -2$ or $x \geqslant 1$

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Exercise B, Question 5

Question:

Solve the following inequality

$|2x| + x > 3$

Solution:

$|2x| + x > 3$

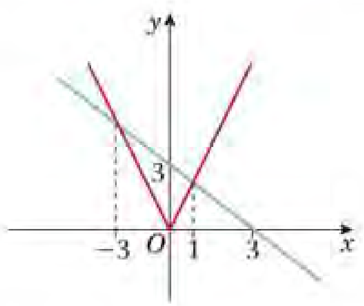
Rearrange: $|2x| > 3 - x$

$2x = 3 - x$ or $-2x = 3 - x$

$\Rightarrow 3x = 3$ $-x = 3$

$\Rightarrow x = 1$ or $x = -3$

sketch



$y = 3 - x$ is below V for

$x < -3$ or $x > 1$

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Exercise B, Question 6

Question:

Solve the following inequality

$$\frac{x+3}{|x|+1} < 2$$

Solution:

$$\frac{x+3}{|x|+1} < 2$$

Rearrange: $x + 3 < 2|x| + 2$

Because $|x| + 1$ is positive you can multiply across.

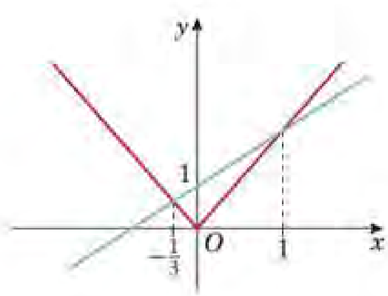
i.e. $x + 1 < 2|x|$

$x + 1 = 2x$ or $x + 1 = -2x$

$\Rightarrow$ $1 = x$ $\Rightarrow$ $3x = -1$

$x = -\frac{1}{3}$

sketch



Line $y = x + 1$ is below V when $x < -\frac{1}{3}$ or $x > 1$

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Exercise B, Question 7

Question:

Solve the following inequality

$$\frac{3-x}{|x|+1} > 2$$

Solution:

$$\frac{3-x}{|x|+1} > 2$$

Rearrange: $3 - x > 2|x| + 2$

You can multiply by $|x| + 1$ since it is > 0.

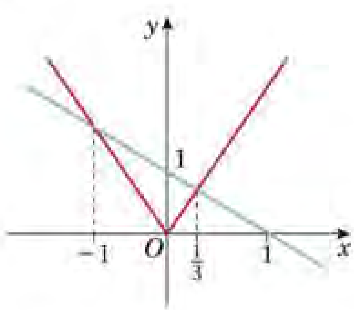
$1 - x > 2|x|$

$1 - x = 2x$ or $1 - x = -2x$

$\Rightarrow$ $1 = 3x$ $\quad$ $x = -1$

$\frac{1}{3} = x$

sketch



The line $y = 1 - x$ is above the V for $-1 < x < \frac{1}{3}$

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Exercise B, Question 8

Question:

Solve the following inequality

$$\left|\frac{x}{x+2}\right| < 1 - x$$

Solution:

$$\left|\frac{x}{x+2}\right| < 1 - x$$

$\frac{x}{x+2} = 1 - x$ or $-\frac{x}{x+2} = 1 - x$

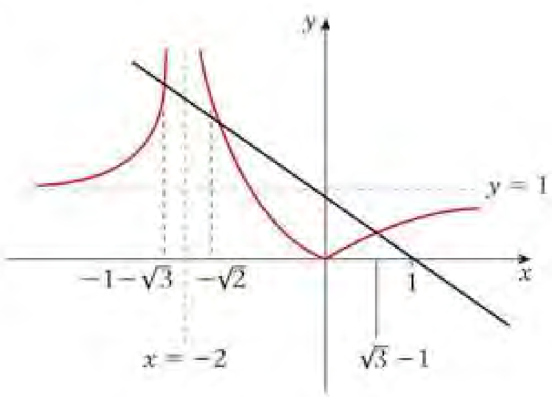
$\Rightarrow$ $x = (1-x)(x+2)$ $\quad$ $-x = (1-x)(x+2)$

$x^2 + 2x - 2 = 0$ $\quad$ $x^2 - 2 = 0$

$x = \frac{-2 \pm \sqrt{12}}{2}$ $\quad$ $x = \pm\sqrt{2}$

$x = -1 \pm \sqrt{3}$

sketch



NB $x = +\sqrt{2}$ is invalid.

The line $y = 1 - x$ is above the curve for $x < -1 - \sqrt{3}$

or $-\sqrt{2} < x < -1 + \sqrt{3}$

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Exercise B, Question 9

Question:

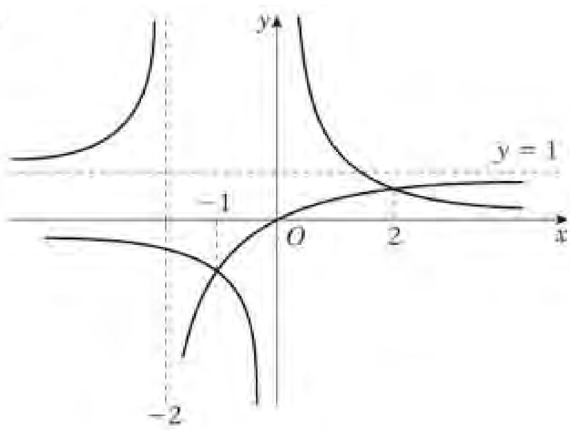
Solve the following inequalities

a On the same axes sketch the graphs of $y = \frac{1}{x}$ and $y = \frac{x}{x+2}$.

b Solve $\frac{1}{x} > \frac{x}{x+2}$.

Solution:

a



b $\frac{1}{x} = \frac{x}{x+2} \quad \Rightarrow \quad x + 2 = x^2$

i.e. $0 = x^2 - x - 2$

$0 = (x-2)(x+1)$

$x = 2$ or -1

$\frac{1}{x}$ is above $\frac{x}{x+2}$ for $-2 < x < -1$ or $0 < x < 2$

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Exercise B, Question 10

Question:

Solve the following inequalities

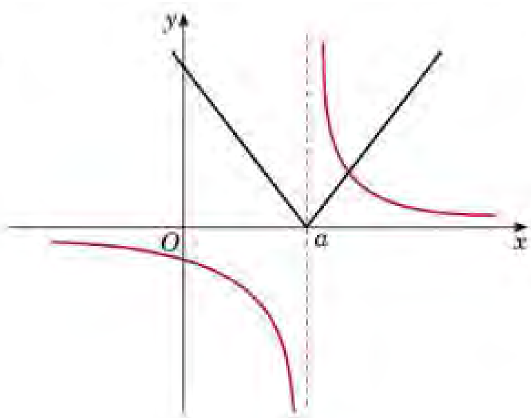
a On the same axes sketch the graphs of $y = \frac{1}{x-a}$ and $y = 4|x - a|$.

b Solve, giving your answers in terms of the constant a, $\frac{1}{x-a} < 4|x - a|$.

Solution:

a

(Assume $a > 0$)



b $\frac{1}{x-a} = 4(x-a)$

Only this case needs to be considered because the right hand branch of V has the intersection.

$\frac{1}{4} = (x-a)^2$

$\pm\frac{1}{2} = x - a$

$x = a \pm \frac{1}{2}$

From sketch $x = a + \frac{1}{2}$.

V is above when $x < a$ or $x > a + \frac{1}{2}$

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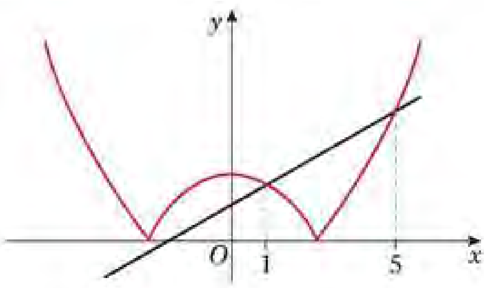
Exercise C, Question 1

Question:

Solve the inequality $|x^2 - 7| < 3(x + 1)$

Solution:

sketch:



$|x^2 - 7| < 3(x + 1)$

$$x^2 - 7 = 3x + 3 \quad \text{or} \quad -(x^2 - 7) = 3x + 3$$

$$\Rightarrow \quad x^2 - 3x - 10 = 0 \qquad \Rightarrow \quad 0 = x^2 + 3x - 4$$

$$(x - 5)(x + 2) = 0 \qquad 0 = (x + 4)(x - 1)$$

$$x = -2 \text{ or } 5 \qquad x = -4 \text{ or } 1$$

From the sketch, only $x = 1$ and $x = 5$ are valid.

Line is above the curve for $1 < x < 5$

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Exercise C, Question 2

Question:

Solve the inequality $\frac{x^2}{|x| + 6} < 1$

Solution:

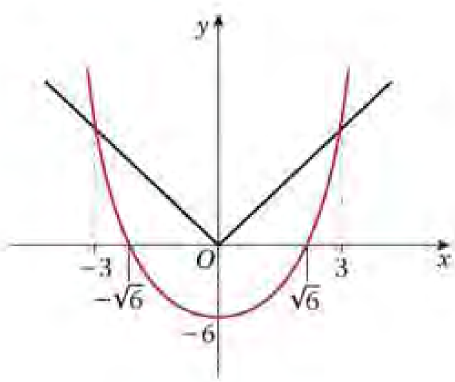
$$\frac{x^2}{|x| + 6} < 1$$

Rearrange: $x^2 < |x| + 6$

or $x^2 - 6 < |x|$

Multiply by $|x| + 6$ since it is positive.

sketch:



$x^2 - 6 = x$ or $x^2 - 6 = -x$

$\Rightarrow$ $x^2 - x - 6 = 0$ $\quad$ $x^2 + x - 6 = 0$

$(x - 3)(x + 2) = 0$ $\quad$ $(x + 3)(x - 2) = 0$

$x = -2$ or 3 $\quad$ $x = 2$ or -3

From the sketch the intersections are $> \sqrt{6}$ $\quad \therefore x = \pm 3$

Curve is below V for $-3 < x < 3$

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Exercise C, Question 3

Question:

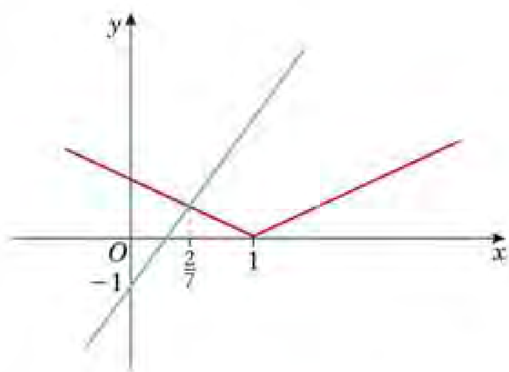
Find the set of values of x for which $|x-1| > 6x-1$

Solution:

$|x-1| > 6x-1$

$$x-1 = 6x-1 \quad \text{or} \quad -(x-1) = 6x-1$$
$$\Rightarrow \quad 0 = 5x \qquad\qquad 2 = 7x$$
$$\Rightarrow \quad x = 0 \qquad\qquad \tfrac{2}{7} = x$$

sketch:



$x = 0$ is not valid so only critical value is $x = \frac{2}{7}$

V is above the line for $x < \frac{2}{7}$

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Exercise C, Question 4

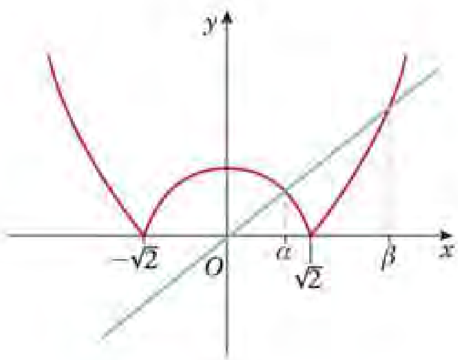
Question:

Find the complete set of values of x for which $|x^2 - 2| > 2x$

Solution:

$|x^2 - 2| > 2x$

sketch:



$x^2 - 2 = 2x$ or $-(x^2 - 2) = 2x$

$\Rightarrow x^2 - 2x - 2 = 0$ $\Rightarrow 0 = x^2 + 2x - 2$

$x = \dfrac{2 \pm \sqrt{12}}{2}$ $x = \dfrac{-2 \pm \sqrt{12}}{2}$

$x = 1 \pm \sqrt{3}$ $x = -1 \pm \sqrt{3}$

β is a solution of this equation so must be $1 + \sqrt{3}$

α is a solution of this equation so must be $\sqrt{3} - 1$

The line is below the curve for $x > 1 + \sqrt{3}$ or $x < \sqrt{3} - 1$

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Exercise C, Question 5

Question:

Find the set of values of x for which $\frac{x+1}{2x-3} < \frac{1}{x-3}$

Solution:

$$\frac{x+1}{\cancel{2x-3}} \times (2x-3)^{\cancel{2}}(x-3)^2 < \frac{1}{\cancel{x-3}} \times (2x-3)^2(x-3)^{\cancel{2}}$$

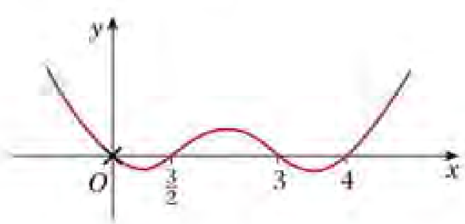
$(2x-3)(x-3)[(x+1)(x-3)-(2x-3)] < 0$

$(2x-3)(x-3)(x^2-2x-\cancel{3}-2x+\cancel{3}) < 0$

$(2x-3)(x-3)x(x-4) < 0$

critical values $x = \frac{3}{2}, 3, 4, 0$

sketch



$0 < x < \frac{3}{2}$ or $3 < x < 4$

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Exercise C, Question 6

Question:

Solve $\dfrac{(x+3)(x+9)}{x-1} > 3x - 5$

Solution:

$$\frac{(x+3)(x+9)}{\cancel{x-1}} \times (x-1)^{\cancel{2}} > (3x-5) \times (x-1)^2$$

$$(x-1)[x^2 + 12x + 27 - (3x^2 - 8x + 5)] > 0$$

$$(x-1)(22 + 20x - 2x^2) > 0$$

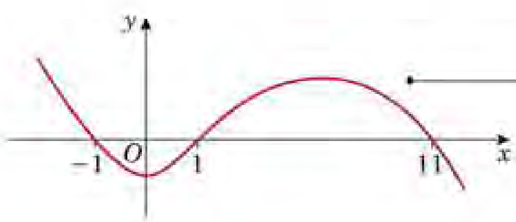
$$(x-1)(11 + 10x - x^2) > 0$$

Divide by 2.

$$(x-1)(11-x)(1+x) > 0$$

critical values $x = 1, -1, 11$

sketch:



This is a $y = -x^3$ shape.

$x < -1$ or $1 < x < 11$

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Exercise C, Question 7

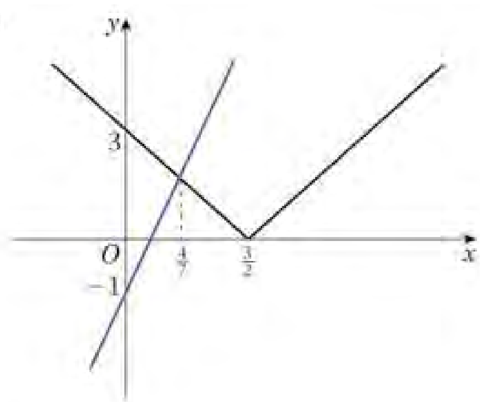
Question:

a Sketch, on the same axes, the graph with equation $y = |2x - 3|$, and the line with equation $y = 5x - 1$

b Solve the inequality $|2x - 3| < 5x - 1$

Solution:

a



b $|2x - 3| < 5x - 1$

$$2x - 3 = 5x - 1 \quad \text{or} \quad -(2x - 3) = 5x - 1$$

$$\Rightarrow \quad -2 = 3x \qquad\qquad 4 = 7x$$

$$-\tfrac{2}{3} = x \qquad\qquad \tfrac{4}{7} = x$$

From sketch this is not valid.

Line is above V for $x > \frac{4}{7}$

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Exercise C, Question 8

Question:

a Use algebra to find the exact solution of $|2x^2 + x - 6| = 6 - 3x$

b On the same diagram, sketch the curve with equation $y = |2x^2 + x - 6|$ and the line with equation $y = 6 - 3x$

c Find the set of values of x for which $|2x^2 + x - 6| > 6 - 3x$

Solution:

a $2x^2 + x - 6 = 6 - 3x$ or $-(2x^2 + x - 6) = 6 - 3x$

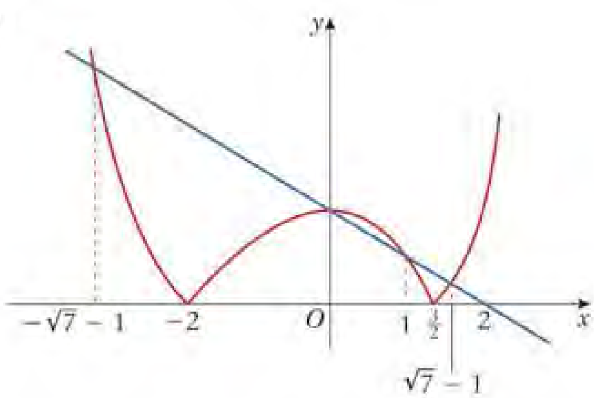
$$2x^2 + 4x - 12 = 0 \qquad 0 = 2x^2 - 2x$$

$$2(x^2 + 2x - 6) = 0 \qquad 0 = 2x(x - 1)$$

$$x = \frac{-2 \pm \sqrt{28}}{2} \qquad x = 0 \text{ or } 1$$

$$= -1 \pm \sqrt{7}$$

b



$$2x^2 + x - 6 = 0$$

$$(2x - 3)(x + 2) = 0$$

$$x = -2 \text{ or } \tfrac{3}{2}$$

c The line is below the curve for $x > \sqrt{7} - 1$ or $0 < x < 1$ or $x < -\sqrt{7} - 1$

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Exercise C, Question 9

Question:

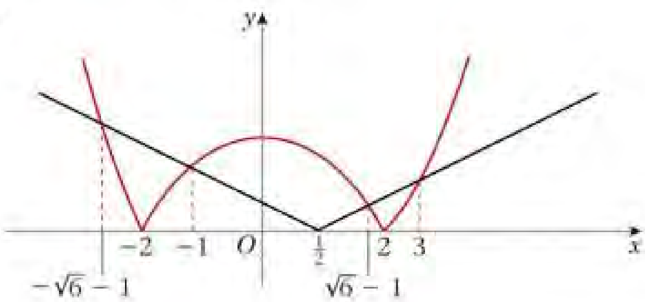
a On the same diagram, sketch the graphs of $y = |x^2 - 4|$ and $y = |2x - 1|$, showing the coordinates of the points where the graphs meet the x-axis.

b Solve $|x^2 - 4| = |2x - 1|$, giving your answers in surd form where appropriate.

c Hence, or otherwise, find the set of values of x for which $|x^2 - 4| > |2x - 1|$

Solution:

a



b $x^2 - 4 = 2x - 1$ or $x^2 - 4 = -(2x - 1)$

$\Rightarrow x^2 - 2x - 3 = 0$ $\Rightarrow$ $x^2 + 2x - 5 = 0$

$(x - 3)(x + 1) = 0$ $x = \dfrac{-2 \pm \sqrt{24}}{2}$

$x = -1 \text{ or } 3$ $x = -1 \pm \sqrt{6}$

c V is below the curve for

$|x^2 - 4| > |2x - 1|$

when $x > 3$ or $-1 < x < \sqrt{6} - 1$ or $x < -\sqrt{6} - 1$

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Exercise A, Question 1

Question:

a Show that $r = \frac{1}{2}(r(r+1) - r(r-1))$.

b Hence show that $\sum_{r=1}^{n} r = \frac{n}{2}(n+1)$ using the method of differences.

Solution:

a $\frac{1}{2}(r(r+1) - r(r-1))$ — Consider RHS.

$= \frac{1}{2}(r^2 + r - r^2 + r)$ — Expand and simplify.

$= \frac{1}{2}(2r)$

$= r$

$=$ LHS

b $\sum_{r=1}^{n} r = \frac{1}{2}\sum_{r=1}^{n} r(r+1) - \frac{1}{2}\sum_{r=1}^{n} r(r-1)$ — Use above.

Use method of differences.

$r = 1 \quad \cancel{\frac{1}{2} \times 1 \times 2} \quad -\frac{1}{2} \times 1 \times 0$

$r = 2 \quad \cancel{\frac{1}{2} \times 2 \times 3} \quad \cancel{-\frac{1}{2} \times 2 \times 1}$

$r = 3 \quad \frac{1}{2} \times 3 \times 4 \quad \cancel{-\frac{1}{2} \times 3 \times 2}$

$\ldots \quad \ldots$

$r = n-1 \quad \cancel{\frac{1}{2}(n-1)(n)} \quad -\frac{1}{2}(n-1)(n-2)$

$r = n \quad \frac{1}{2}n(n+1) \quad \cancel{-\frac{1}{2}n(n-1)}$

When you add, all terms cancel except $\frac{1}{2}n(n+1)$.

Hence $\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$

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Exercise A, Question 2

Question:

Given $\frac{1}{r(r+1)(r+2)} \equiv \frac{1}{2r(r+1)} - \frac{1}{2(r+1)(r+2)}$

find $\sum_{r=1}^{n} \frac{1}{r(r+1)(r+2)}$ using the method of differences.

Solution:

$$\sum_{r=1}^{n} \frac{1}{r(r+1)(r+2)} = \sum_{r=1}^{n} \frac{1}{2r(r+1)} - \sum_{r=1}^{n} \frac{1}{2(r+1)(r+2)}$$

Use the information given and equate the summations.

Put $r = 1$ $\quad \frac{1}{2\times1\times2} - \cancel{\frac{1}{2\times2\times3}}$

Use method of differences.

$r = 2$ $\quad \cancel{\frac{1}{2\times2\times3}} - \cancel{\frac{1}{2\times3\times4}}$

All terms cancel except first and last.

$r = 3$ $\quad \cancel{\frac{1}{2\times3\times4}} - \frac{1}{2\times4\times5}$

$\vdots$

$r = n$ $\quad \cancel{\frac{1}{2n(n+1)}} - \frac{1}{2(n+1)(n+2)}$

Add $\sum_{r=1}^{n} \frac{1}{r(r+1)(r+2)} = \frac{1}{4} - \frac{1}{2(n+1)(n+2)}$

First and last from above.

$$= \frac{(n+1)(n+2)-2}{4(n+1)(n+2)}$$

Simplify.

$$= \frac{n^2+3n+2-2}{4(n+1)(n+2)}$$

$$= \frac{n(n+3)}{4(n+1)(n+2)}$$

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Exercise A, Question 3

Question:

a Express $\frac{1}{r(r+2)}$ in partial fractions.

b Hence find the sum of the series $\sum_{r=1}^{n} \frac{1}{r(r+2)}$ using the method of differences.

Solution:

a $\frac{1}{r(r+2)} \equiv \frac{A}{r} + \frac{B}{r+2}$

Set $\frac{1}{r(r+2)}$ identical to $\frac{A}{r} + \frac{B}{r+2}$.

$\equiv \frac{A(r+2)+Br}{r(r+2)}$

Add the two fractions.

$1 \equiv A(r+2)+Br$

Put $r = 0$

$1 = 2A$

$A = \frac{1}{2}$

$\cancel{\frac{1}{2} = A}$

Put $r = 1$

$1 = \frac{1}{2}(3) + B$

$B = -\frac{1}{2}$

$\therefore \frac{1}{r(r+2)} = \frac{1}{2r} - \frac{1}{2(r+2)}$

b $\sum_{r=1}^{n} \frac{1}{r(r+2)} = \sum_{r=1}^{n} \frac{1}{2r} - \sum_{r=1}^{n} \frac{1}{2(r+2)}$

Use method of differences.

$r = 1 \quad \frac{1}{2\times 1} - \cancel{\frac{1}{2\times 3}}$

$r = 2 \quad \frac{1}{2\times 2} - \cancel{\frac{1}{2\times 4}}$

$r = 3 \quad \cancel{\frac{1}{2\times 3}} - \cancel{\frac{1}{2\times 5}}$

$\vdots$

$r = n-1 \quad \cancel{\frac{1}{2(n-1)}} - \frac{1}{2(n+1)}$

$r = n \quad \cancel{\frac{1}{2n}} - \frac{1}{2(n+2)}$

All terms cancel except $\frac{1}{2}, \frac{1}{4}$, $\frac{1}{2(n+1)}$ and $\frac{1}{2(n+2)}$

Add

$$\begin{aligned}
\sum_{r=1}^{n} \frac{1}{r(r+2)} &= \frac{1}{2} + \frac{1}{4} - \frac{1}{2(n+1)} - \frac{1}{2(n+2)} \\
&= \frac{2(n+1)(n+2) + (n+1)(n+2) - 2(n+2) - 2(n+1)}{4(n+1)(n+2)} \\
&= \frac{2n^2 + 6n + 4 + n^2 + 3n + 2 - 2n - 4 - 2n - 2}{4(n+1)(n+2)} \\
&= \frac{3n^2 + 5n}{4(n+1)(n+2)} \\
&= \frac{n(3n+5)}{4(n+1)(n+2)}
\end{aligned}$$

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Exercise A, Question 4

Question:

a Express $\frac{1}{(r+2)(r+3)}$ in partial fractions.

b Hence find the sum of the series $\sum_{r=1}^{n} \frac{1}{(r+2)(r+3)}$ using the method of differences.

Solution:

a
$$\frac{1}{(r+2)(r+3)} \equiv \frac{A}{r+2} + \frac{B}{r+3}$$

Set $\frac{1}{(r+2)(r+3)}$ identical to $\frac{A}{r+2} + \frac{B}{r+3}$.

$$\equiv \frac{A(r+3) + B(r+2)}{(r+2)(r+3)}$$

Add the two fractions.

$$1 \equiv A(r+3) + B(r+2)$$

Compare numerators as they are equivalent.

$$r = -3 \Rightarrow B = -1$$
$$r = -2 \Rightarrow A = 1$$

Solve for A and B.

$$\therefore \frac{1}{(r+2)(r+3)} = \frac{1}{r+2} - \frac{1}{r+3}$$

b
$$\sum_{r=1}^{n} \frac{1}{(r+2)(r+3)} \equiv \sum_{r=1}^{n} \frac{1}{(r+2)} - \sum_{r=1}^{n} \frac{1}{(r+3)}$$

Use the method of differences.

$r = 1$ $\quad \frac{1}{3} - \cancel{\frac{1}{4}}$

$r = 2$ $\quad \cancel{\frac{1}{4}} - \cancel{\frac{1}{5}}$

$r = 3$ $\quad \cancel{\frac{1}{5}} - \cancel{\frac{1}{6}}$

$\vdots$

$r = n$ $\quad \cancel{\frac{1}{n+2}} - \frac{1}{n+3}$

All cancel except first and last.

$$\text{Add } \sum_{r=1}^{n} \frac{1}{(r+2)(r+3)} = \frac{1}{3} - \frac{1}{n+3}$$
$$= \frac{n+3-3}{3(n+3)}$$
$$= \frac{n}{3(n+3)}$$

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Exercise A, Question 5

Question:

a Express $\dfrac{5r+4}{r(r+1)(r+2)}$ in partial fractions.

b Hence or otherwise, show that $\displaystyle\sum_{r=1}^{n} \frac{5r+4}{r(r+1)(r+2)} = \frac{7n^2+11n}{2(n+1)(n+2)}$

Solution:

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Exercise A, Question 6

Question:

Given that $\frac{r}{(r+1)!} \equiv \frac{1}{r!} - \frac{1}{(r+1)!}$

find $\sum_{r=1}^{n} \frac{r}{(r+1)!}$

Solution:

$$\sum_{r=1}^{n} \frac{r}{(r+1)!} \equiv \sum_{r=1}^{n} \frac{1}{r!} - \sum_{r=1}^{n} \frac{1}{(r+1)!}$$

Use given.

$r = 1 \quad \frac{1}{1!} - \cancel{\frac{1}{2!}}$

$r = 2 \quad \cancel{\frac{1}{2!}} - \cancel{\frac{1}{3!}}$

$r = 3 \quad \cancel{\frac{1}{3!}} - \cancel{\frac{1}{4!}}$

$\vdots$

$r = n \quad \cancel{\frac{1}{n!}} - \frac{1}{(n+1)!}$

Use method of differences.

All cancel except first and last term.

$\therefore$

Add $\quad \sum_{r=1}^{n} \frac{r}{(r+1)!} = 1 - \frac{1}{(n+1)!}$

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Exercise A, Question 7

Question:

Given that $\frac{2r+1}{r^2(r+1)^2} \equiv \frac{1}{r^2} - \frac{1}{(r+1)^2}$

find $\sum_{r=1}^{n} \frac{2r+1}{r^2(r+1)^2}$.

Solution:

$$\sum_{r=1}^{n} \frac{2r+1}{r^2(r+1)^2} = \sum_{r=1}^{n} \frac{1}{r^2} - \sum_{r=1}^{n} \frac{1}{(r+1)^2}$$

Use given.

Use method of differences.

$r = 1 \quad \frac{1}{1} - \cancel{\frac{1}{2^2}}$

$r = 2 \quad \cancel{\frac{1}{2^2}} - \cancel{\frac{1}{3^2}}$

$r = 3 \quad \cancel{\frac{1}{3^2}} - \cancel{\frac{1}{4^2}}$

$\vdots$

$r = n \quad \cancel{\frac{1}{n^2}} - \frac{1}{(n+1)^2}$

All terms cancel except first and last.

So adding $\sum_{r=1}^{n} \frac{2r+1}{r^2(r+1)^2} = 1 - \frac{1}{(n+1)^2}$

$$= \frac{(n+1)^2 - 1}{(n+1)^2}$$

Simplify.

$$= \frac{n^2 + 2n}{(n+1)^2}$$

$$= \frac{n(n+2)}{(n+1)^2}$$

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Exercise A, Question 1

Question:

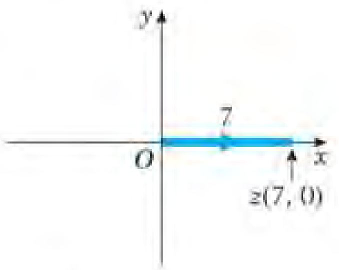
Express the following in the form $r(\cos\theta + i\sin\theta)$, where $-\pi < \theta \leqslant \pi$. Give the exact values of r and θ where possible, or values to 2 d.p. otherwise.

a 7 **b** $-5i$ **c** $\sqrt{3} + i$ **d** $2 + 2i$ **e** $1 - i$

f -8 **g** $3 - 4i$ **h** $-8 + 6i$ **i** $2 - \sqrt{3}i$

Solution:

a 7

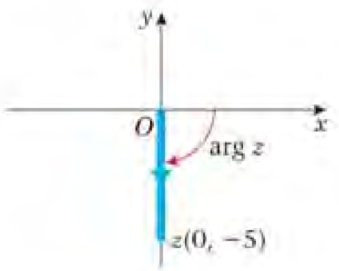


$r = 7$

$\theta = \arg z = 0$

$\therefore\ 7 = 7\ (\cos\theta + \mathrm{i}\sin\theta)$

b $-5\mathrm{i}$

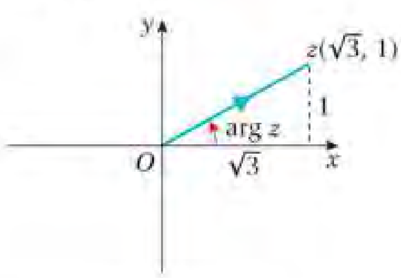


$r = 5$

$\theta = \arg z = -\frac{\pi}{2}$

$\therefore\ -5\mathrm{i} = 5\left(\cos\left(-\frac{\pi}{2}\right) + \mathrm{i}\sin\left(-\frac{\pi}{2}\right)\right)$

c $\sqrt{3} + \mathrm{i}$

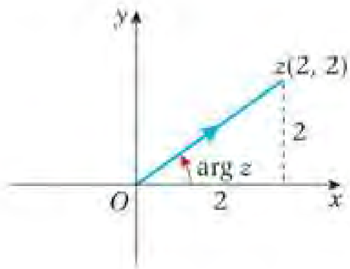


$r = \sqrt{(\sqrt{3})^2 + 1^2} = \sqrt{4} = 2$

$\theta = \arg z = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$

$\therefore\ \sqrt{3} + \mathrm{i} = 2\left(\cos\frac{\pi}{6} + \mathrm{i}\sin\frac{\pi}{6}\right)$

d $2 + 2i$

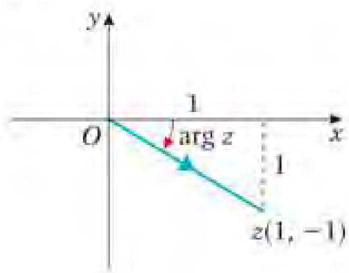


$r = \sqrt{2^2 + 2^2} = \sqrt{8} = 2\sqrt{2}$

$\theta = \arg z = \tan^{-1}\left(\frac{2}{2}\right) = \frac{\pi}{4}$

$\therefore 2 + 2i = 2\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

e $1 - i$

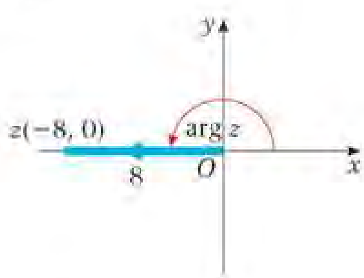


$r = \sqrt{1^2 + (-1)^2} = \sqrt{2}$

$\theta = \arg z = \tan^{-1}\left(\frac{1}{1}\right) = -\frac{\pi}{4}$

$\therefore 1 - i = \sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right)$

f -8

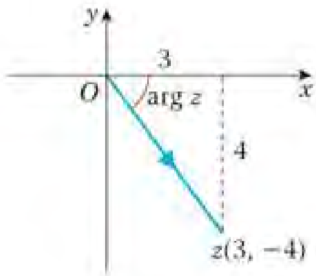


$r = 8$

$\theta = \arg z = \pi$

$\therefore -8 = 8(\cos\pi + i\sin\pi)$

g $3 - 4i$

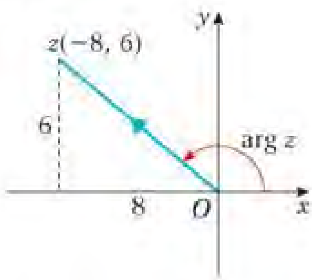


$r = \sqrt{3^2 + (-4)^2} = \sqrt{25} = 5$

$\theta = \arg z = -\tan^{-1}\left(\frac{4}{3}\right) = -0.93^c$ (2 d.p.)

$\therefore 3 - 4i = 5(\cos(-0.93^c) + i\sin(-0.93^c))$

h $-8 + 6i$

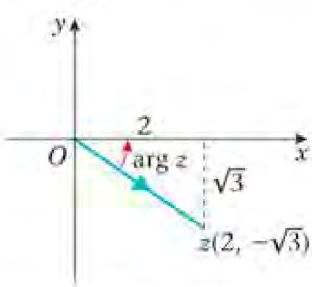


$r = \sqrt{(-8)^2 + 6^2} = \sqrt{100} = 10$

$\theta = \arg z = \pi - \tan^{-1}\left(\frac{6}{8}\right) = 2.50^c$ (2 d.p.)

$\therefore -8 + 6i = 10(\cos(2.50^c) + i\sin(2.50^c))$

i $2 - \sqrt{3}i$



$r = \sqrt{2^2 + (-\sqrt{3})^2} = \sqrt{7}$

$\theta = \arg z = -\tan^{-1}\left(\frac{\sqrt{3}}{2}\right) = -0.71^c$ (2 d.p.)

$\therefore 2 - \sqrt{3}i = \sqrt{7}\,(\cos(-0.71^c) + i\sin(-0.71^c))$

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Exercise A, Question 2

Question:

Express the following in the form $x + iy$, where $x \in \mathbb{R}$ and $y \in \mathbb{R}$.

a $5\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$

b $\frac{1}{2}\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$

c $6\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$

d $3\left(\cos\left(-\frac{2\pi}{3}\right) + i\sin\left(-\frac{2\pi}{3}\right)\right)$

e $2\sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right)$

f $-4\left(\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}\right)$

Solution:

a $5\left(\cos\frac{\pi}{2} + \text{i}\sin\frac{\pi}{2}\right)$

$= 5(0 + \text{i})$

$= 5\text{i}$

b $\frac{1}{2}\left(\cos\frac{\pi}{6} + \text{i}\sin\frac{\pi}{6}\right)$

$= \frac{1}{2}\left(\frac{\sqrt{3}}{2} + \frac{1}{2}\text{i}\right)$

$= \frac{\sqrt{3}}{4} + \frac{1}{4}\text{i}$

c $6\left(\cos\frac{5\pi}{6} + \text{i}\sin\frac{5\pi}{6}\right)$

$= 6\left(-\frac{\sqrt{3}}{2} + \frac{1}{2}\text{i}\right)$

$= -3\sqrt{3} + 3\text{i}$

d $3\left(\cos\left(-\frac{2\pi}{3}\right) + \text{i}\sin\left(-\frac{2\pi}{3}\right)\right)$

$= 3\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}\text{i}\right)$

$= -\frac{3}{2} - \frac{3\sqrt{3}}{2}\text{i}$

e $2\sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + \text{i}\sin\left(-\frac{\pi}{4}\right)\right)$

$= 2\sqrt{2}\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\text{i}\right)$

$= 2 - 2\text{i}$

f $-4\left(\cos\frac{7\pi}{6} + \text{i}\sin\frac{7\pi}{6}\right)$

$= -4\left(-\frac{\sqrt{3}}{2} - \frac{1}{2}\text{i}\right)$

$= 2\sqrt{3} + 2\text{i}$

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Exercise A, Question 3

Question:

Express the following in the form $re^{i\theta}$, where $-\pi < \theta \leq \pi$. Give the exact values of r and θ where possible, or values to 2 d.p. otherwise.

a -3

b $6i$

c $-2\sqrt{3} - 2i$

d $-8 + i$

e $2 - 5i$

f $-2\sqrt{3} + 2\sqrt{3}i$

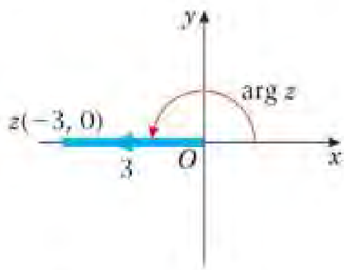
g $\sqrt{8}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

h $8\left(\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right)$

i $2\left(\cos\frac{\pi}{5} - i\sin\frac{\pi}{5}\right)$

Solution:

a -3

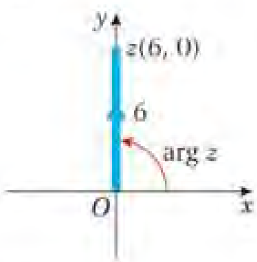


$r = 3$

$\theta = \arg z = \pi$

$\therefore\ -3 = 3e^{\pi i}$

b 6i

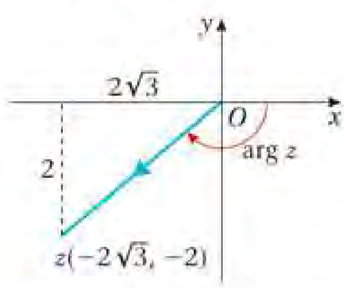


$r = 6$

$\theta = \arg z = \dfrac{\pi}{2}$

$\therefore\ 6i = 6e^{\frac{\pi i}{2}}$

c $-2\sqrt{3} - 2i$

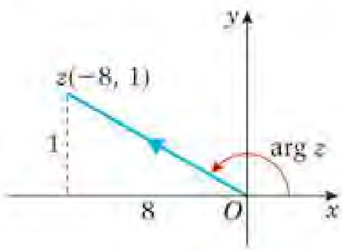


$r = \sqrt{(-2\sqrt{3})^2 + (-2)^2} = \sqrt{12 + 4} = \sqrt{16} = 4$

$\theta = \arg z = -\pi + \tan^{-1}\left(\dfrac{2}{2\sqrt{3}}\right) = -\pi + \dfrac{\pi}{6} = -\dfrac{5\pi}{6}$

$\therefore\ -2\sqrt{3} - 2i = 4e^{\frac{-5\pi i}{6}}$

d $-8 + i$

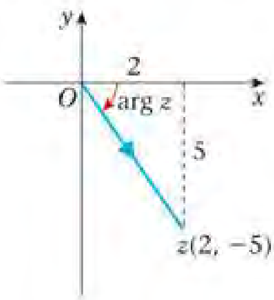


$r = \sqrt{(-8)^2 + 1^2} = \sqrt{65}$

$\theta = \pi - \tan^{-1}\left(\frac{1}{8}\right) = 3.02^c$ (2 d.p.)

$\therefore -8 + i = \sqrt{65}\, e^{3.02i}$

e $2 - 5i$

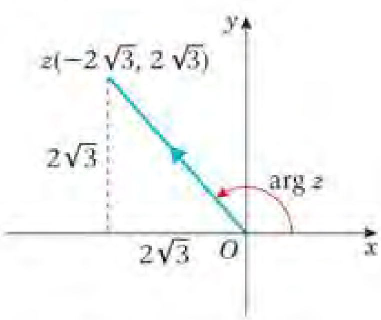


$r = \sqrt{2^2 + (-5)^2} = \sqrt{29}$

$\theta = -\tan^{-1}\left(\frac{5}{2}\right) = -1.19^c$ (2 d.p.)

$\therefore 2 - 5i = \sqrt{29}\, e^{-1.19i}$

f $-2\sqrt{3} + 2\sqrt{3}i$



$r = \sqrt{(-2\sqrt{3})^2 + (2\sqrt{3})^2} = \sqrt{12 + 12} = \sqrt{24}$

$= \sqrt{4}\sqrt{6} = 2\sqrt{6}$

$$\theta = \pi - \tan^{-1}\left(\frac{2\sqrt{3}}{2\sqrt{3}}\right) = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$$

$$\therefore\ -2\sqrt{3} + 2\sqrt{3}\,\mathrm{i} = 2\sqrt{6}\,\mathrm{e}^{\frac{3\pi\mathrm{i}}{4}}$$

g $\sqrt{8}\left(\cos\frac{\pi}{4} + \mathrm{i}\sin\frac{\pi}{4}\right)$

$= 2\sqrt{2}\,\mathrm{e}^{\frac{\pi\mathrm{i}}{4}}$

$$\boxed{\begin{array}{l} r = \sqrt{8} = \sqrt{4}\sqrt{2} = 2\sqrt{2} \\ \theta = \frac{\pi}{4} \end{array}}$$

h $8\left(\cos\frac{\pi}{6} - \mathrm{i}\sin\frac{\pi}{6}\right)$

$= 8\left(\cos\left(-\frac{\pi}{6}\right) + \mathrm{i}\sin\left(-\frac{\pi}{6}\right)\right)$

$= 8\mathrm{e}^{-\frac{\pi\mathrm{i}}{6}}$

$$\boxed{r = 8,\ \theta = -\frac{\pi}{6}}$$

i $2\left(\cos\frac{\pi}{5} - \mathrm{i}\sin\frac{\pi}{5}\right)$

$= 2\left(\cos\left(-\frac{\pi}{5}\right) + \mathrm{i}\sin\left(-\frac{\pi}{5}\right)\right)$

$= 2\mathrm{e}^{-\frac{\pi\mathrm{i}}{5}}$

$$\boxed{r = 2,\ \theta = -\frac{\pi}{5}}$$

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Exercise A, Question 4

Question:

Express the following in the form $x + \text{i}y$ where $x \in \mathbb{R}$ and $y \in \mathbb{R}$.

a $e^{\frac{\pi}{3}\text{i}}$ **b** $4e^{\pi\text{i}}$ **c** $3\sqrt{2}\,e^{\frac{\pi\text{i}}{4}}$

d $8e^{\frac{\pi\text{i}}{6}}$ **e** $3e^{-\frac{\pi\text{i}}{2}}$ **f** $e^{\frac{5\pi\text{i}}{6}}$

g $e^{-\pi\text{i}}$ **h** $3\sqrt{2}e^{\frac{-3\pi}{4}\text{i}}$ **i** $8e^{-\frac{4\pi\text{i}}{3}}$

Solution:

a $e^{\frac{\pi\text{i}}{3}} = \cos\frac{\pi}{3} + \text{i}\sin\frac{\pi}{3}$

$= \frac{1}{2} + \frac{\sqrt{3}}{2}\text{i}$

b $4e^{\pi\text{i}} = 4(\cos\pi + \text{i}\sin\pi)$

$= 4(-1 + \text{i}(0))$

$= -4$

c $3\sqrt{2}\,e^{\frac{\pi\text{i}}{4}} = 3\sqrt{2}\left(\cos\frac{\pi}{4} + \text{i}\sin\frac{\pi}{4}\right)$

$= 3\sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\text{i}\right)$

$= 3 + 3\text{i}$

d $8e^{\frac{\pi\text{i}}{6}} = 8\left(\cos\frac{\pi}{6} + \text{i}\sin\frac{\pi}{6}\right)$

$= 8\left(\frac{\sqrt{3}}{2} + \frac{1}{2}\text{i}\right)$

$= 4\sqrt{3} + 4\text{i}$

e $3e^{-\frac{\pi\text{i}}{2}} = 8\left(\cos\left(-\frac{\pi}{2}\right) + \text{i}\sin\left(-\frac{\pi}{2}\right)\right)$

$= 3(0 - \text{i})$

$= -3\text{i}$

f $e^{\frac{5\pi\text{i}}{6}} = \cos\frac{5\pi}{6} + \text{i}\sin\frac{5\pi}{6}$

$= -\frac{\sqrt{3}}{2} + \frac{1}{2}\text{i}$

g $e^{-\pi i} = \cos(-\pi) + i\sin(-\pi)$

$= -1 + i(0)$

$= -1$

h $3\sqrt{2}e^{-\frac{3\pi}{4}i} = 3\sqrt{2}\left(\cos\left(-\frac{3\pi}{4}\right) + i\sin\left(-\frac{3\pi}{4}\right)\right)$

$= 3\sqrt{2}\left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right)$

$= -3 - 3i$

i $8e^{-\frac{4\pi i}{3}} = 8\left(\cos\left(-\frac{4\pi}{3}\right) + i\sin\left(-\frac{4\pi}{3}\right)\right)$

$= 8\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)$

$= -4 + 4\sqrt{3}\,i$

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Exercise A, Question 5

Question:

Express the following in the form $r(\cos\theta + \mathrm{i}\sin\theta)$, where $-\pi < \theta \leqslant \pi$.

a $e^{\frac{16\pi}{13}\mathrm{i}}$ **b** $4e^{\frac{17\pi}{5}\mathrm{i}}$ **c** $5e^{-\frac{9\pi}{8}\mathrm{i}}$

Solution:

a $$e^{\frac{16\pi i}{13}} = \cos\left(\frac{16\pi}{13}\right) + i\sin\left(\frac{16\pi}{13}\right)$$

$$= \cos\left(-\frac{10\pi}{13}\right) + i\sin\left(-\frac{10\pi}{13}\right)$$

-2π from the argument.

b $$4e^{\frac{17\pi i}{5}} = 4\left(\cos\left(\frac{17\pi}{5}\right) + i\sin\left(\frac{17\pi}{5}\right)\right)$$

$$= 4\left(\cos\left(\frac{7\pi}{5}\right) + i\sin\left(\frac{7\pi}{5}\right)\right)$$

$$= 4\left(\cos\left(-\frac{3\pi}{5}\right) + i\sin\left(-\frac{3\pi}{5}\right)\right)$$

c $$5e^{\frac{-9\pi i}{8}} = 5\left(\cos\left(-\frac{9\pi}{8}\right) + i\sin\left(-\frac{9\pi}{8}\right)\right)$$

$$= 5\left(\cos\left(\frac{7\pi}{8}\right) + i\sin\left(\frac{7\pi}{8}\right)\right)$$

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Exercise A, Question 6

Question:

Use $e^{i\theta} = \cos\theta + i\sin\theta$ to show that $\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$.

Solution:

$e^{i\theta} = \cos\theta + i\sin\theta$ ①

$e^{-i\theta} = \cos(-\theta) + i\sin(-\theta) = \cos\theta - i\sin\theta$ ②

① − ② $\Rightarrow e^{i\theta} - e^{-i\theta} = 2i\sin\theta$

$\frac{1}{2i}(e^{i\theta} - e^{-i\theta}) = \sin\theta$

$\therefore \sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$ (as required)

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Exercise B, Question 1

Question:

Express the following in the form $x + iy$.

a $(\cos 2\theta + i \sin 2\theta)(\cos 3\theta + i \sin 3\theta)$

b $\left(\cos \frac{3\pi}{11} + i \sin \frac{3\pi}{11}\right)\left(\cos \frac{8\pi}{11} + i \sin \frac{8\pi}{11}\right)$

c $3\left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4}\right) \times 2\left(\cos \frac{\pi}{12} + i \sin \frac{\pi}{12}\right)$

d $\sqrt{6}\left(\cos \left(\frac{-\pi}{12}\right) + i \sin \left(\frac{-\pi}{12}\right)\right) \times \sqrt{3}\left(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3}\right)$

e $4\left(\cos \left(\frac{-5\pi}{9}\right) + i \sin \left(\frac{-5\pi}{9}\right)\right) \times \frac{1}{2}\left(\cos \left(\frac{-5\pi}{18}\right) + i \sin \left(\frac{-5\pi}{18}\right)\right)$

f $6\left(\cos \frac{\pi}{10} + i \sin \frac{\pi}{10}\right) \times 5\left(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3}\right) \times \frac{1}{3}\left(\cos \frac{2\pi}{5} + i \sin \frac{2\pi}{5}\right)$

g $(\cos 4\theta + i \sin 4\theta)(\cos \theta - i \sin \theta)$

h $3\left(\cos \frac{\pi}{12} + i \sin \frac{\pi}{12}\right) \times \sqrt{2}\left(\cos \frac{\pi}{3} - i \sin \frac{\pi}{3}\right)$

Solution:

a $(\cos 2\theta + i \sin 2\theta)(\cos 3\theta + i \sin 3\theta)$

$= \cos(2\theta + 3\theta) + i\sin(2\theta + 3\theta)$

$= \cos 5\theta + i\sin 5\theta$

b $\left(\cos \frac{3\pi}{11} + i \sin \frac{3\pi}{11}\right)\left(\cos \frac{8\pi}{11} + i \sin \frac{8\pi}{11}\right)$

$= \cos\left(\frac{3\pi}{11} + \frac{8\pi}{11}\right) + i\sin\left(\frac{3\pi}{11} + \frac{8\pi}{11}\right)$

$= \cos \pi + i\sin \pi$

$= -1 + i(0)$

$= -1$

c $3\left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4}\right) \times 2\left(\cos \frac{\pi}{12} + i \sin \frac{\pi}{12}\right)$

$= 3(2)\left(\cos\left(\frac{\pi}{4} + \frac{\pi}{12}\right) + i\sin\left(\frac{\pi}{4} + \frac{\pi}{12}\right)\right)$

$= 6\left(\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right)$

$= 6\left(\frac{1}{2} 1 \frac{\sqrt{3}}{2}i\right)$

$= 3 + 3\sqrt{3}\,i$

d $\sqrt{6}\left(\cos\left(-\frac{\pi}{12}\right) + i \sin\left(-\frac{\pi}{12}\right)\right) \times \sqrt{3}\left(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3}\right)$

$= (\sqrt{6})(\sqrt{3})\left(\cos\left(-\frac{\pi}{12} + \frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{12} + \frac{\pi}{3}\right)\right)$

$= \sqrt{18}\left(\cos \frac{\pi}{4} + i\sin \frac{\pi}{4}\right)$

$$\boxed{\begin{aligned}\sqrt{18} &= \sqrt{9}\sqrt{2} \\ &= 3\sqrt{2}\end{aligned}}$$

$= 3\left(\sqrt{2}\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right)$

$= 3 + 3i$

e $4\left(\cos\left(-\frac{5\pi}{9}\right) + \mathrm{i}\sin\left(-\frac{5\pi}{9}\right)\right) \times \frac{1}{2}\left(\cos\left(-\frac{5\pi}{18}\right) + \mathrm{i}\sin\left(-\frac{5\pi}{18}\right)\right)$

$= 4\left(\frac{1}{2}\right)\left(\cos\left(-\frac{5\pi}{9} + -\frac{5\pi}{18}\right) + \mathrm{i}\sin\left(-\frac{5\pi}{9} + -\frac{5\pi}{18}\right)\right)$

$= 2\left(\cos\left(-\frac{15\pi}{18}\right) + \mathrm{i}\sin\left(-\frac{15\pi}{18}\right)\right)$

$= 2\left(\cos\left(-\frac{5\pi}{6}\right) 1 \mathrm{i}\sin\left(-\frac{5\pi}{6}\right)\right)$

$= 2\left(-\frac{\sqrt{3}}{2} - \frac{1}{2}\mathrm{i}\right)$

$= -\sqrt{3} - \mathrm{i}$

f $6\left(\cos\frac{\pi}{10} + \mathrm{i}\sin\frac{\pi}{10}\right) \times 5\left(\cos\frac{\pi}{3} + \mathrm{i}\sin\frac{\pi}{3}\right) \times \frac{1}{3}\left(\cos\frac{2\pi}{5} + \mathrm{i}\sin\frac{2\pi}{5}\right)$

$= 6(5)\left(\frac{1}{3}\right)\left(\cos\left(\frac{\pi}{10} + \frac{\pi}{3} + \frac{2\pi}{5}\right) + \mathrm{i}\sin\left(\frac{\pi}{10} + \frac{\pi}{3} + \frac{2\pi}{5}\right)\right)$

$= 10\left(\cos\frac{5\pi}{6} + \mathrm{i}\sin\frac{5\pi}{6}\right)$

$= 10\left(-\frac{\sqrt{3}}{2} + \frac{1}{2}\mathrm{i}\right)$

$= -5\sqrt{3} + 5\mathrm{i}$

g $(\cos 4\theta + \mathrm{i}\sin 4\theta)(\cos\theta - \mathrm{i}\sin\theta)$

$= (\cos 4\theta + \mathrm{i}\sin 4\theta)(\cos(-\theta) + \mathrm{i}\sin(-\theta))$

$= \cos(4\theta + -\theta) + \mathrm{i}\sin(4\theta + -\theta)$

$= \cos 3\theta + \mathrm{i}\sin 3\theta$

h $3\left(\cos\frac{\pi}{12} + \mathrm{i}\sin\frac{\pi}{12}\right) \times \sqrt{2}\left(\cos\frac{\pi}{3} - \mathrm{i}\sin\frac{\pi}{3}\right)$

$= 3\left(\cos\frac{\pi}{12} + \mathrm{i}\sin\frac{\pi}{12}\right) \times \sqrt{2}\left(\cos\left(-\frac{\pi}{3}\right) + \mathrm{i}\sin\left(-\frac{\pi}{3}\right)\right)$

$= 3(\sqrt{2})\left(\cos\left(\frac{\pi}{12} - \frac{\pi}{3}\right) + \mathrm{i}\sin\left(\frac{\pi}{12} - \frac{\pi}{3}\right)\right)$

$= 3\sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + \mathrm{i}\sin\left(-\frac{\pi}{4}\right)\right)$

$= 3\sqrt{2}\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\mathrm{i}\right)$

$= 3 - 3\mathrm{i}$

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Exercise B, Question 2

Question:

Express the following in the form $x + \text{i}y$.

a $\dfrac{\cos 5\theta + \text{i} \sin 5\theta}{\cos 2\theta + \text{i} \sin 2\theta}$

b $\dfrac{\sqrt{2}\left(\cos \frac{\pi}{2} + \text{i} \sin \frac{\pi}{2}\right)}{\frac{1}{2}\left(\cos \frac{\pi}{4} + \text{i} \sin \frac{\pi}{4}\right)}$

c $\dfrac{3\left(\cos \frac{\pi}{3} + \text{i} \sin \frac{\pi}{3}\right)}{4\left(\cos \frac{5\pi}{6} + \text{i} \sin \frac{5\pi}{6}\right)}$

d $\dfrac{\cos 2\theta - \text{i} \sin 2\theta}{\cos 3\theta + \text{i} \sin 3\theta}$

Solution:

a $\dfrac{\cos 5\theta + i\sin 5\theta}{\cos 2\theta + i\sin 2\theta}$

$= \cos(5\theta - 2\theta) + i\sin(5\theta - 2\theta)$

$= \cos 3\theta + i\sin 3\theta$

b $\dfrac{\sqrt{2}\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)}{\frac{1}{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)}$

$= \dfrac{\sqrt{2}}{\left(\frac{1}{2}\right)}\left(\cos\left(\frac{\pi}{2} - \frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{2} - \frac{\pi}{4}\right)\right)$

$= 2\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

$= 2\sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right)$

$= 2 + 2i$

c $\dfrac{3\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)}{4\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)}$

$= \frac{3}{4}\left(\cos\left(\frac{\pi}{3} - \frac{5\pi}{6}\right) + i\sin\left(\frac{\pi}{3} - \frac{5\pi}{6}\right)\right)$

$= \frac{3}{4}\left(\cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)\right)$

$= \frac{3}{4}(0 - i)$

$= -\frac{3}{4}i$

d $\dfrac{\cos 2\theta - i\sin 2\theta}{\cos 3\theta + i\sin 3\theta}$

$= \dfrac{\cos(-2\theta) + i\sin(-2\theta)}{\cos 3\theta + i\sin 3\theta}$

$= \cos(-2\theta - 3\theta) + i\sin(-2\theta - 3\theta)$

$= \cos(-5\theta) + i\sin(-5\theta)$ *or* $\cos 5\theta - i\sin 5\theta$

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Exercise B, Question 3

Question:

z and w are two complex numbers where

$$z = -9 + 3\sqrt{3}\mathrm{i},\ |w| = \sqrt{3} \text{ and } \arg w = \frac{7\pi}{12}.$$

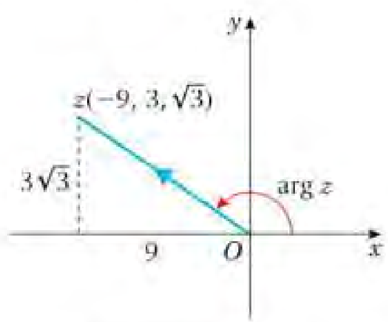
Express the following in the form $r(\cos\theta + \mathrm{i}\sin\theta)$,

a z, **b** w, **c** zw, **d** $\frac{z}{w}$,

where $-\pi < \theta \leqslant \pi$.

Solution:

a $z = -9 + 3\sqrt{3}\mathrm{i}$



$$r = \sqrt{(-9)^2 + (3\sqrt{3})^2} = \sqrt{81 + 27} = \sqrt{108}$$
$$= \sqrt{36}\sqrt{3} = 6\sqrt{3}$$

$$\theta = \arg z = \pi - \tan^{-1}\left(\frac{3\sqrt{3}}{9}\right) = \frac{5\pi}{6}$$

$$z = 6\sqrt{3}\left(\cos\frac{5\pi}{6} + \mathrm{i}\sin\frac{5\pi}{6}\right)$$

b $r = |w| = \sqrt{3}$

$$\theta = \arg w = \frac{7\pi}{12}$$

$$w = \sqrt{3}\left(\cos\frac{7\pi}{12} + \mathrm{i}\sin\frac{7\pi}{12}\right)$$

c $zw = 6\sqrt{3}\left(\cos\frac{5\pi}{6} + \mathrm{i}\sin\frac{5\pi}{6}\right) \times \sqrt{3}\left(\cos\frac{7\pi}{12} + \mathrm{i}\sin\frac{7\pi}{12}\right)$

$= (6\sqrt{3})(\sqrt{3})\left(\cos\left(\frac{5\pi}{6} + \frac{7\pi}{12}\right) + \mathrm{i}\sin\left(\frac{5\pi}{6} + \frac{7\pi}{12}\right)\right)$

$= 18\left(\cos\left(\frac{17\pi}{12}\right) + \mathrm{i}\sin\left(\frac{17\pi}{12}\right)\right)$

$= 18\cos\left(-\frac{7\pi}{12}\right) + \mathrm{i}\sin\left(-\frac{7\pi}{12}\right)$

d $\frac{z}{w} = \frac{6\sqrt{3}\left(\cos\frac{5\pi}{6} + \mathrm{i}\sin\frac{5\pi}{6}\right)}{\sqrt{3}\left(\cos\frac{7\pi}{12} + \mathrm{i}\sin\frac{7\pi}{12}\right)}$

$= \frac{6\sqrt{3}}{\sqrt{3}}\left(\cos\left(\frac{5\pi}{6} - \frac{7\pi}{12}\right) + \mathrm{i}\sin\left(\frac{5\pi}{6} - \frac{7\pi}{12}\right)\right)$

$= 6\left(\cos\left(\frac{3\pi}{12}\right) + \mathrm{i}\sin\left(\frac{3\pi}{12}\right)\right)$

$= 6\left(\cos\frac{\pi}{4} + \mathrm{i}\sin\frac{\pi}{4}\right)$

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Exercise C, Question 1

Question:

Use de Moivre's theorem to simplify each of the following:

a $(\cos\theta + i\sin\theta)^6$

b $(\cos 3\theta + i\sin 3\theta)^4$

c $\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)^5$

d $\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)^8$

e $\left(\cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}\right)^5$

f $\left(\cos\frac{\pi}{10} - i\sin\frac{\pi}{10}\right)^{15}$

g $\dfrac{\cos 5\theta + i\sin 5\theta}{(\cos 2\theta + i\sin 2\theta)^2}$

h $\dfrac{(\cos 2\theta + i\sin 2\theta)^7}{(\cos 4\theta + i\sin 4\theta)^3}$

i $\dfrac{1}{(\cos 2\theta + i\sin 2\theta)^3}$

j $\dfrac{(\cos 2\theta + i\sin 2\theta)^4}{(\cos 3\theta + i\sin 3\theta)^3}$

k $\dfrac{\cos 5\theta + i\sin 5\theta}{(\cos 3\theta - i\sin 3\theta)^2}$

l $\dfrac{\cos\theta - i\sin\theta}{(\cos 2\theta - i\sin 2\theta)^3}$

Solution:

a $(\cos\theta + i\sin\theta)^6$
$= \cos 6\theta + i\sin 6\theta$

b $(\cos 3\theta + i\sin 3\theta)^4$
$= \cos(4(3\theta)) + i\sin(4(3\theta))$
$= \cos 12\theta + i\sin 12\theta$

c $\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)^5$

$= \cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}$

$= -\frac{\sqrt{3}}{2} + \frac{1}{2}i$

d $\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)^8$

$= \cos\frac{8\pi}{3} + i\sin\frac{8\pi}{3}$

$= \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$

$= -\frac{1}{2} + \frac{\sqrt{3}}{2}i$

e $\left(\cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}\right)^5$

$= \cos\frac{10\pi}{5} + i\sin\frac{10\pi}{5}$

$= \cos 2\pi + i\sin 2\pi$
$= \cos 0 + i\sin 0$
$= 1 + 1\,(0)$
$= 1$

f $\left(\cos\frac{\pi}{10} - i\sin\frac{\pi}{10}\right)^{15}$

$= \left(\cos\left(-\frac{\pi}{10}\right) + i\sin\left(-\frac{\pi}{10}\right)\right)^{15}$

$= \cos\left(-\frac{15\pi}{10}\right) + i\sin\left(-\frac{15\pi}{10}\right)$

$= \cos\left(\frac{5\pi}{10}\right) + i\sin\left(\frac{5\pi}{10}\right)$

$= \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$

$= 0 + i$
$= i$

g $\dfrac{\cos 5\theta + i \sin 5\theta}{(\cos 2\theta + i \sin 2\theta)^2}$

$= \dfrac{\cos 5\theta + i \sin 5\theta}{\cos 4\theta + i \sin 4\theta}$

$= \cos(5\theta - 4\theta) + i \sin(5\theta - 4\theta)$

$= \cos\theta + i \sin\theta$

h $\dfrac{(\cos 2\theta + i \sin 2\theta)^7}{(\cos 4\theta + i \sin 4\theta)^3}$

$= \dfrac{\cos 14\theta + i \sin 14\theta}{\cos 12\theta + i \sin 12\theta}$

$= \cos(14\theta - 12\theta) + i \sin(14\theta - 12\theta)$

$= \cos 2\theta + i \sin 2\theta$

i $\dfrac{1}{(\cos 2\theta + i \sin 2\theta)^3}$

$= (\cos 2\theta + i \sin 2\theta)^{-3}$

$= \cos(-6\theta) + i \sin(-6\theta)$

$= \cos 6\theta - i \sin 6\theta$

j $\dfrac{(\cos 2\theta + i \sin 2\theta)^4}{(\cos 3\theta + i \sin 3\theta)^3}$

$= \dfrac{\cos 8\theta + i \sin 8\theta}{\cos 9\theta + i \sin 9\theta}$

$= \cos(8\theta - 9\theta) + i \sin(8\theta - 9\theta)$

$= \cos(-\theta) + i \sin(-\theta)$

$= \cos\theta - i \sin\theta$

k $\dfrac{\cos 5\theta + i \sin 5\theta}{(\cos 3\theta + i \sin 3\theta)^2}$

$= \dfrac{\cos 5\theta + i \sin 5\theta}{(\cos(-3\theta) + i \sin(-3\theta))^2}$

$= \dfrac{\cos 5\theta + i \sin 5\theta}{\cos(-6\theta) + i \sin(-6\theta)}$

$= \cos(5\theta - -6\theta) + i \sin(5\theta - -6\theta)$

$= \cos 11\theta - i \sin 11\theta$

l $\dfrac{\cos\theta - i \sin\theta}{(\cos 2\theta - i \sin 2\theta)^3}$

$= \dfrac{\cos(-\theta) - i \sin(-\theta)}{(\cos(-2\theta) - i \sin(-2\theta))^3}$

$= \dfrac{\cos(-\theta) - i \sin(-\theta)}{\cos(-6\theta) - i \sin(-6\theta)}$

$= \cos(-\theta - -6\theta) + i \sin(-\theta - -6\theta)$

$= \cos 5\theta - i \sin 5\theta$

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Edexcel AS and A Level Modular Mathematics

Exercise C, Question 2

Question:

Evaluate $\dfrac{\left(\cos\frac{7\pi}{13} + \mathrm{i}\sin\frac{7\pi}{13}\right)^4}{\left(\cos\frac{4\pi}{13} - \mathrm{i}\sin\frac{4\pi}{13}\right)^6}$.

Solution:

$$\frac{\left(\cos\frac{7\pi}{13} + \mathrm{i}\sin\frac{7\pi}{13}\right)^4}{\left(\cos\frac{4\pi}{13} - \mathrm{i}\sin\frac{4\pi}{13}\right)^6}$$

$$= \frac{\left(\cos\frac{7\pi}{13} + \mathrm{i}\sin\frac{7\pi}{13}\right)^4}{\left(\cos\left(-\frac{4\pi}{13}\right) - \mathrm{i}\sin\left(-\frac{4\pi}{13}\right)\right)^6}$$

$$= \frac{\cos\left(\frac{28\pi}{13}\right) + \mathrm{i}\sin\left(\frac{28\pi}{13}\right)}{\cos\left(-\frac{24\pi}{13}\right) - \mathrm{i}\sin\left(-\frac{24\pi}{13}\right)}$$

$$= \cos\left(\frac{28\pi}{13} - -\frac{24\pi}{13}\right) + \mathrm{i}\sin\left(\frac{28\pi}{13} - -\frac{24\pi}{13}\right)$$

$$= \cos\left(\frac{52\pi}{13}\right) + \mathrm{i}\sin\left(\frac{52\pi}{13}\right)$$

$$= \cos 4\pi + \mathrm{i}\sin 4\pi$$

$$= \cos 0 + \mathrm{i}\sin 0$$

$$= 1 + \mathrm{i}(0)$$

$$= 1$$

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Exercise C, Question 3

Question:

Express the following in the form $x + \mathrm{i}y$ where $x \in \mathbb{R}$ and $y \in \mathbb{R}$.

a $(1 + \mathrm{i})^5$ **b** $(-2 + 2\mathrm{i})^8$ **c** $(1 - \mathrm{i})^6$

d $(1 - \sqrt{3}\mathrm{i})^6$ **e** $\left(\frac{3}{2} - \frac{1}{2}\sqrt{3}\mathrm{i}\right)^9$ **f** $\left(-2\sqrt{3} - 2\mathrm{i}\right)^5$

Solution:

a $(1+\mathrm{i})^5$

If $z = 1 + \mathrm{i}$, then

$r = \sqrt{1^2 + 1^2} = \sqrt{2}$

$\theta = \arg z = \tan^{-1}\left(\frac{1}{1}\right) = \frac{\pi}{4}$

So, $1 + \mathrm{i} = \sqrt{2}\left(\cos\frac{\pi}{4} + \mathrm{i}\sin\frac{\pi}{4}\right)$

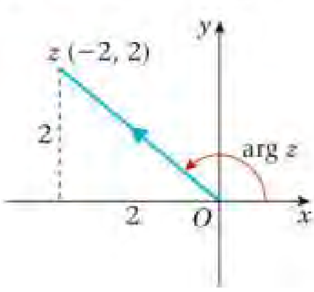
$$\begin{aligned}\therefore (1+\mathrm{i})^5 &= \left[\sqrt{2}\left(\cos\frac{\pi}{4} + \mathrm{i}\sin\frac{\pi}{4}\right)\right]^5 \\ &= (\sqrt{2})^5\left(\cos\frac{5\pi}{4} + \mathrm{i}\sin\frac{5\pi}{4}\right) \\ &= 4\sqrt{2}\left(\cos\frac{5\pi}{4} + \mathrm{i}\sin\frac{5\pi}{4}\right) \\ &= 4\sqrt{2}\left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\mathrm{i}\right) \\ &= -4 - 4\mathrm{i}\end{aligned}$$

$$\boxed{\begin{aligned}(\sqrt{2}\,) &= \sqrt{2}\sqrt{2}\sqrt{2}\sqrt{2}\sqrt{2} \\ &= 4\sqrt{2}\end{aligned}}$$

Therefore, $(1+\mathrm{i})^5 = -4 - 4\mathrm{i}$

b $(-2+2\mathrm{i})^8$

If $z = -2 + \mathrm{i}$, then



$r = \sqrt{(-2)^2 + 2^2} = \sqrt{4+4} = \sqrt{8} = \sqrt{4}\sqrt{2} = 2\sqrt{2}$

$\theta = \arg z = \pi - \tan^{-1}\left(\frac{2}{2}\right) = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$

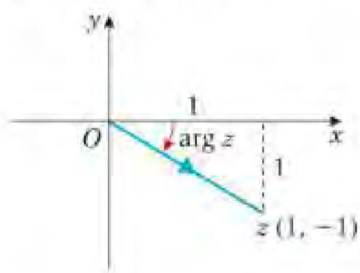
So, $-2 + 2\mathrm{i} = 2\sqrt{2}\left(\cos\frac{3\pi}{4} + \mathrm{i}\sin\frac{3\pi}{4}\right)$

$$\begin{aligned}
\therefore (-2+2\mathrm{i})^8 &= \left[2\sqrt{2}\left(\cos\left(\frac{3\pi}{4}\right) + \mathrm{i}\sin\left(\frac{3\pi}{4}\right)\right)\right]^8 \\
&= (2\sqrt{2})^8\left(\cos\left(\frac{24\pi}{4}\right) + \mathrm{i}\sin\left(\frac{24\pi}{4}\right)\right) \\
&= (256)(16)\,(\cos 6\pi + \mathrm{i}\sin 6\pi) \\
&= 4096\,(1 + \mathrm{i}(0)) \\
&= 4096
\end{aligned}$$

Therefore, $(-2+2\mathrm{i})^8 = 4096$

c $(1-\mathrm{i})^6$

If $z = 1 - \mathrm{i}$, then



$r = \sqrt{1^2 + 1^2} = \sqrt{2}$

$\theta = \arg z = \pi - \tan^{-1}\left(\frac{1}{1}\right) = \pi - \frac{\pi}{4}$

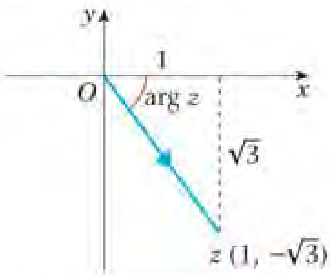
So, $1 - \mathrm{i} = \sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + \mathrm{i}\sin\left(-\frac{\pi}{4}\right)\right)$

$$\begin{aligned}
\therefore (1-\mathrm{i})^6 &= \left[\sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + \mathrm{i}\sin\left(-\frac{\pi}{4}\right)\right)\right]^6 \\
&= (\sqrt{2})^6\left(\cos\left(-\frac{6\pi}{4}\right) + \mathrm{i}\sin\left(-\frac{6\pi}{4}\right)\right) \\
&= 8\left(\cos\left(-\frac{3\pi}{2}\right) + \mathrm{i}\sin\left(-\frac{3\pi}{2}\right)\right) \\
&= 8\,(0 + \mathrm{i}) \\
&= 8\mathrm{i}
\end{aligned}$$

Therefore, $(1-i)^6 = 8i$

d $(1-\sqrt{3}i)^6$

If $z = 1 - \sqrt{3}\,i$, then



$r = \sqrt{1^2 + (-\sqrt{3}\,)^2} = \sqrt{1+3} = \sqrt{4} = 2$

$\theta = \arg z = -\tan^{-1}\left(\frac{\sqrt{3}}{1}\right) = -\frac{\pi}{3}$

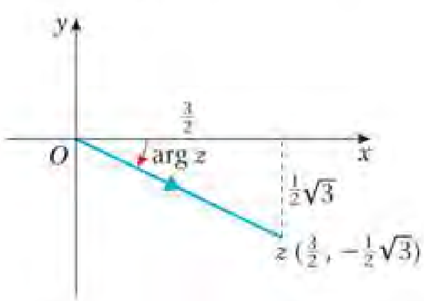
So, $1 - \sqrt{3}\,i = 2\left(\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right)$

$$\begin{aligned}\therefore\ (1-\sqrt{3}\,i)^6 &= \left[2\left(\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right)\right]^6\\ &= (2)^6\left(\cos\left(-\frac{6\pi}{3}\right) + i\sin\left(-\frac{6\pi}{3}\right)\right)\\ &= 64(\cos(-2\pi) + i\sin(-2\pi))\\ &= 64(1 + i(0))\\ &= 64\end{aligned}$$

Therefore, $(1-\sqrt{3}\,i)^6 = 64$

e $\left(\frac{3}{2} - \frac{1}{2}\sqrt{3}i\right)^9$

If $z = \frac{3}{2} - \frac{1}{2}\sqrt{3}\,i$, then



$$r = \sqrt{\left(\frac{3}{2}\right)^2 + \left(-\frac{1}{2}\sqrt{3}\right)^2} = \sqrt{\frac{9}{4} + \frac{3}{4}} = \sqrt{\frac{12}{4}} = \sqrt{3}$$

$$\theta = \arg z = -\tan^{-1}\left(\frac{\frac{1}{2}\sqrt{3}}{\frac{3}{2}}\right) = -\tan^{-1}\frac{\sqrt{3}}{3} = -\frac{\pi}{6}$$

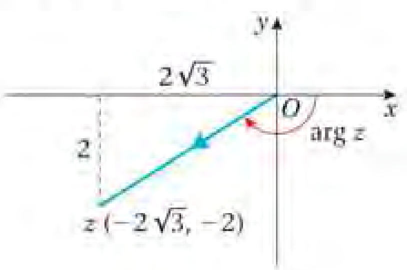
So, $\frac{3}{2} - \frac{1}{2}\sqrt{3}\,\mathrm{i} = \sqrt{3}\left(\cos\left(-\frac{\pi}{6}\right) + \mathrm{i}\sin\left(-\frac{\pi}{6}\right)\right)$

$$\begin{aligned}\therefore\ \left(\frac{3}{2} - \frac{1}{2}\sqrt{3}\,\mathrm{i}\right)^9 &= \left[\sqrt{3}\left(\cos\left(-\frac{\pi}{6}\right) + \mathrm{i}\sin\left(-\frac{\pi}{6}\right)\right)\right]^9 \\ &= (\sqrt{3})^9\left(\cos\left(-\frac{9\pi}{6}\right) + \mathrm{i}\sin\left(-\frac{9\pi}{6}\right)\right) \\ &= 81\sqrt{3}\left(\cos\left(-\frac{3\pi}{2}\right) + \mathrm{i}\sin\left(-\frac{3\pi}{2}\right)\right) \\ &= 81\sqrt{3}\,(0 + \mathrm{i}) \\ &= 81\sqrt{3}\,\mathrm{i}\end{aligned}$$

Therefore, $\left(\frac{3}{2} - \frac{1}{2}\sqrt{3}\,\mathrm{i}\right)^9 = 81\sqrt{3}\,\mathrm{i}$

f $(-2\sqrt{3} - 2\mathrm{i})^5$

If $z = -2\sqrt{3} - 2\mathrm{i}$, then



$$r = \sqrt{(-2\sqrt{3})^2 + (-2)^2} = \sqrt{12 + 4} = \sqrt{16} = 4$$

$$\theta = \arg z = -\pi - \tan^{-1}\left(\frac{2}{2\sqrt{3}}\right) = -\pi + \frac{\pi}{6} = -\frac{5\pi}{6}$$

So, $-2\sqrt{3} - 2\mathrm{i} = 4\left(\cos\left(-\frac{5\pi}{6}\right) + \mathrm{i}\sin\left(-\frac{5\pi}{6}\right)\right)$

$$\begin{aligned}\therefore\ (-2\sqrt{3} - 2\mathrm{i})^5 &= \left[4\left(\cos\left(-\frac{5\pi}{6}\right) + \mathrm{i}\sin\left(-\frac{5\pi}{6}\right)\right)\right]^5 \\ &= 4^5\left(\cos\left(-\frac{25\pi}{6}\right) + \mathrm{i}\sin\left(-\frac{25\pi}{6}\right)\right) \\ &= 1024\left(\cos\left(-\frac{\pi}{6}\right) + \mathrm{i}\sin\left(-\frac{\pi}{6}\right)\right) \\ &= 1024\left(\frac{\sqrt{3}}{2} - \frac{1}{2}\mathrm{i}\right) \\ &= 512\sqrt{3} - 512\mathrm{i}\end{aligned}$$

Therefore, $(-2\sqrt{3} - 2\mathrm{i})^5 = 512\sqrt{3} - 512\mathrm{i}$

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Exercise C, Question 4

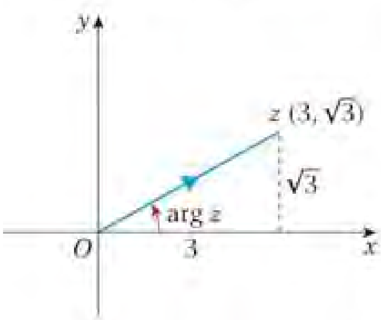
Question:

Express $(3+\sqrt{3}\mathrm{i})^5$ in the form $a+b\sqrt{3}\mathrm{i}$ where a and b are integers.

Solution:

$(3+\sqrt{3}\mathrm{i})^5$

If $z=3+\sqrt{3}\,\mathrm{i}$, then



$$r=\sqrt{3^2+(\sqrt{3})^2}=\sqrt{9+3}=\sqrt{12}=\sqrt{4}\sqrt{3}=2\sqrt{3}$$

$$\theta=\arg z=\tan^{-1}\left(\frac{\sqrt{3}}{3}\right)=\frac{\pi}{6}$$

So, $3+\sqrt{3}\,\mathrm{i}=2\sqrt{3}\left(\cos\frac{\pi}{6}+\mathrm{i}\sin\frac{\pi}{6}\right)$

$$\begin{aligned}
\therefore (3+\sqrt{3}\,\mathrm{i})^5 &= \left[2\sqrt{3}\left(\cos\frac{\pi}{6}+\mathrm{i}\sin\frac{\pi}{6}\right)\right]^5 \\
&= (2\sqrt{3})^5\left(\cos\frac{5\pi}{6}+\mathrm{i}\sin\frac{5\pi}{6}\right) \\
&= 32(9\sqrt{3})\left(-\frac{\sqrt{3}}{2}+\frac{1}{2}\mathrm{i}\right) \\
&= 288\sqrt{3}\left(-\frac{\sqrt{3}}{2}+\frac{1}{2}\mathrm{i}\right) \\
&= -144\sqrt{3}\sqrt{3}+144\sqrt{3}\,\mathrm{i} \\
&= -432+144\sqrt{3}\,\mathrm{i}
\end{aligned}$$

Therefore, $(3+\sqrt{3}\,\mathrm{i})^5=-432+144\sqrt{3}\,\mathrm{i}$

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Exercise D, Question 1

Question:

$\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

Solution:

$(\cos\theta + i\sin\theta)^3 = \cos^3\theta + i\sin^3\theta$

de Moivre's Theorem.

$= \cos^3\theta + {}^3C_1\cos^2\theta(i\sin\theta)$

$\quad + {}^3C_2\cos\theta(i\sin\theta)^2 + (i\sin\theta)^3$

Binomial expansion.

$= \cos^3\theta + 3i\cos^2\theta\sin\theta + 3i^2\cos\theta\sin^2\theta + i^3\sin^3\theta$

$= \cos^3\theta + 3i\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta$

Hence,

$\cos 3\theta + i\sin 3\theta = \cos^3\theta + 3i\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta$

Equating the imaginary parts gives,

$\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$

Applying $\cos^2\theta = 1 - \sin^2\theta$.

$= 3(1 - \sin^2\theta)\sin\theta - \sin^3\theta$

$= 3\sin\theta(1 - \sin^2\theta) - \sin^3\theta$

$= 3\sin\theta - 3\sin^3\theta - \sin^3\theta$

$= 3\sin\theta - 4\sin^3\theta$

Hence, $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$ (as required)

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Exercise D, Question 2

Question:

$\sin 5\theta = 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$

Solution:

$(\cos\theta + i\sin\theta)^5 = \cos 5\theta + i\sin 5\theta$

de Moivre's Theorem.

$= \cos^5\theta + {}^5C_1\cos^4\theta(i\sin\theta) + {}^5C_2\cos^3\theta(i\sin\theta)^2$
$\quad + {}^5C_3\cos^2\theta(i\sin\theta)^3 + {}^5C_4\cos\theta(i\sin\theta)^4 + (i\sin\theta)^5$

Binomial expansion.

$= \cos^5\theta + 5i\cos^4\theta\sin\theta + 10i^2\cos^3\theta\sin^2\theta + 10i^3\cos^2\theta\sin^3\theta$
$\quad + 5i^4\cos\theta\sin^4\theta + i^5\sin^5\theta$

Hence,

$\cos 5\theta + i\sin 5\theta = \cos^5\theta + 5i\cos^4\theta\sin\theta - 10\cos^3\theta\sin^2\theta - 10i\cos^2\theta\sin^3\theta$
$\quad + 5\cos\theta\sin^4\theta + i\sin^5\theta$

Equating the imaginary parts gives,

$\sin 5\theta = 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta$

$= 5(\cos^2\theta)^2\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta$

$= 5(1 - \sin^2\theta)^2\sin\theta - 10(1 - \sin^2\theta)\sin^3\theta + \sin^5\theta$

Applying $\cos^2\theta = 1 - \sin^2\theta$.

$= 5\sin\theta(1 - 2\sin^2\theta + \sin^4\theta) - 10\sin^3\theta(1 - \sin^2\theta) + \sin^5\theta$

$= 5\sin\theta - 10\sin^3\theta + 5\sin^5\theta - 10\sin^3\theta + 10\sin^5\theta + \sin^5\theta$

$= 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$

Hence, $\sin 5\theta = 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$ (as required)

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Exercise D, Question 3

Question:

$\cos 7\theta = 64\cos^7\theta - 112\cos^5\theta + 56\cos^3\theta - 7\cos\theta$

Solution:

$$(\cos\theta + \mathrm{i}\sin\theta)^7 = \cos 7\theta + \mathrm{i}\sin 7\theta$$

de Moivre's Theorem.

$$= \cos^7\theta + {}^7C_1\cos^6\theta(\mathrm{i}\sin\theta) + {}^7C_2\cos^5\theta(\mathrm{i}\sin\theta)^2$$
$$+ {}^7C_3\cos^4\theta(\mathrm{i}\sin\theta)^3 + {}^7C_4\cos^3\theta(\mathrm{i}\sin\theta)^4 + {}^7C_5\cos^2\theta(\mathrm{i}\sin\theta)^5$$
$$+ {}^7C_6\cos\theta(\mathrm{i}\sin\theta)^6 + (\mathrm{i}\sin\theta)^7$$

Binomial expansion.

$$= \cos^7\theta + 7\mathrm{i}\cos^6\theta\sin\theta + 21\mathrm{i}^2\cos^5\theta\sin^2\theta$$
$$+ 35\mathrm{i}^3\cos^4\theta\sin^3\theta + 35\mathrm{i}^4\cos^3\theta\sin^4\theta + 21\mathrm{i}^5\cos^2\theta\sin^5\theta$$
$$+ 7\mathrm{i}^6\cos\theta\sin^6\theta + \mathrm{i}^7\sin^7\theta$$

Hence,

$$\cos 7\theta + \mathrm{i}\sin 7\theta = \cos^7\theta + 7\mathrm{i}\cos^6\theta\sin\theta - 21\cos^5\theta\sin^2\theta$$
$$- 35\mathrm{i}^3\cos^4\theta\sin^3\theta + 35\mathrm{i}^4\cos^3\theta\sin^4\theta + 21\mathrm{i}^5\cos^2\theta\sin^5\theta$$
$$- 7\cos\theta\sin^6\theta - \mathrm{i}\sin^7\theta$$

Equating the imaginary parts gives,

$$\cos 7\theta = \cos^7\theta - 21\cos^5\theta\sin^2\theta + 35\cos^3\theta\sin^4\theta - 7\cos\theta\sin^6\theta$$
$$= \cos^7\theta - 21\cos^5\theta(1-\cos^2\theta) + 35\cos^3\theta(1-\cos^2\theta)^2$$
$$- 7\cos\theta(1-\cos^2\theta)^3$$

Applying $\cos^2\theta = 1 - \sin^2\theta$.

$$= \cos^7\theta - 21\cos^5\theta + 21\cos^7\theta + 35\cos^3\theta(1 - 2\cos^2\theta + \cos^4\theta)$$
$$- 7\cos\theta(1 - 3\cos^2\theta + 3\cos^4\theta - \cos^6\theta)$$
$$= \cos^7\theta - 21\cos^5\theta + 21\cos^7\theta + 35\cos^3\theta - 70\cos^5\theta + 35\cos^7\theta$$
$$- 7\cos\theta + 21\cos^3\theta - 21\cos^5\theta + 7\cos^7\theta$$
$$= 64\cos^7\theta - 112\cos^5\theta + 56\cos^3\theta - 7\cos\theta$$

Hence, $\cos 7\theta = 64\cos^7\theta - 112\cos^5\theta + 56\cos^3\theta - 7\cos^5\theta$ (as required)

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Exercise D, Question 4

Question:

$\cos^4\theta = \frac{1}{8}(\cos 4\theta + 4\cos 2\theta + 3)$

Solution:

Let $z = \cos\theta + i\sin\theta$

$$\left(z+\frac{1}{z}\right)^4 = (2\cos\theta)^4 = 16\cos^4\theta$$

$$z + \frac{1}{z} = 2\cos\theta$$

$$= z^4 + {}^4C_1 z^3\left(\frac{1}{z}\right) + {}^4C_2 z^2\left(\frac{1}{z}\right)^2 + {}^4C_3 z\left(\frac{1}{z}\right)^3 + \left(\frac{1}{z}\right)^4$$

$$= z^4 + 4z^3\left(\frac{1}{z}\right) + 6z^2\left(\frac{1}{z^2}\right) + 4z^2\left(\frac{1}{z^3}\right) + \frac{1}{z^4}$$

$$= z^4 + 4z^2 + 6 + \frac{4}{z^2} + \frac{1}{z^4}$$

$$= \left(z^4 + \frac{1}{z^4}\right) + 4\left(z^2 + \frac{1}{z^2}\right) + 6$$

$$= 2\cos 4\theta + 4(2\cos 2\theta) + 6$$

$$z^n + \frac{1}{z^n} = 2\cos n\theta$$

So, $16\cos^4\theta = 2\cos 4\theta + 8\cos 2\theta + 6$

$16\cos^4\theta = 2(\cos 4\theta + 4\cos 2\theta + 3)$

$\cos^4\theta = \frac{2}{16}(\cos 4\theta + 4\cos 2\theta + 3)$

Therefore, $\cos^4\theta = \frac{1}{8}(\cos 4\theta + 4\cos 2\theta + 3)$ (as required)

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Exercise D, Question 5

Question:

$\sin^5\theta = \frac{1}{16}(\sin 5\theta - 5\sin 3\theta + 10\sin\theta)$

Solution:

Let $z = \cos\theta + i\sin\theta$

$$\left(z + \frac{1}{z}\right)^5 = (2i\sin\theta)^5 = 32i^5\sin^5\theta = 32i\sin^5\theta \qquad \boxed{z - \frac{1}{z} = 2i\sin\theta}$$

$$= z^5 + {}^5C_1 z^4\left(-\frac{1}{z}\right) + {}^5C_2 z^3\left(-\frac{1}{z}\right)^2 + {}^5C_3 z^2\left(-\frac{1}{z}\right)^3 + {}^5C_4 z\left(-\frac{1}{z}\right)^4 + \left(-\frac{1}{z}\right)^5$$

$$= z^5 + 5z^4\left(-\frac{1}{z}\right) + 10z^3\left(-\frac{1}{z}\right)^2 + 10z^2\left(-\frac{1}{z}\right)^3 + 5z\left(-\frac{1}{z}\right)^4 + \left(-\frac{1}{z}\right)^5$$

$$= z^5 + 5z^4\left(\frac{1}{z}\right) + 10z^3\left(\frac{1}{z^2}\right) - 10z^2\left(\frac{1}{z^3}\right) + 5z\left(\frac{1}{z^4}\right) - \frac{1}{z^5}$$

$$= z^5 - 5z^3 + 10z - \frac{10}{z} + \frac{5}{z^3} - \frac{1}{z^5}$$

$$= \left(z^5 - \frac{1}{z^5}\right) - 5\left(z^3 - \frac{1}{z^3}\right) + 10\left(z - \frac{1}{z}\right)$$

$$= 2i\sin 5\theta - 5(2i\sin 3\theta) + 10(2i\sin\theta) \qquad \boxed{z^n + \frac{1}{z^n} = 2i\sin n\theta}$$

So, $32i\sin^5\theta = 2i\sin 5\theta - 10i\sin 3\theta + 20i\sin\theta \quad (\div 2i)$

$16\sin^5\theta = \sin 5\theta - 5\sin 3\theta + 10\sin\theta$

$\sin^5\theta = \frac{1}{16}(\sin 5\theta - 5\sin 3\theta + 10\sin\theta)$

Therefore, $\sin^5\theta = \frac{1}{16}(\sin 5\theta - 5\sin 3\theta + 10\sin\theta)$

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Exercise D, Question 6

Question:

a Show that $32\cos^6\theta = \cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10$.

b Hence find $\int_0^{\frac{\pi}{6}} \cos^6\theta \, d\theta$ in the form $a\pi + b\sqrt{3}$ where a and b are constants.

Solution:

Let $z = \cos\theta + \text{i}\sin\theta$

a $\left(z + \frac{1}{z}\right)^6 = (2\cos\theta)^6 = 64\cos^6\theta$

$\boxed{z - \frac{1}{z} = 2\cos\theta}$

$$= z^6 + {}^6\text{C}_1 z^5\left(\frac{1}{z}\right) + {}^6\text{C}_2 z^4\left(\frac{1}{z}\right)^2 + {}^6\text{C}_3 z^3\left(\frac{1}{z}\right)^3 + {}^6\text{C}_4 z^2\left(\frac{1}{z}\right)^4 + {}^6\text{C}_5\, {}^2\left(\frac{1}{z}\right)^5 + \left(\frac{1}{z}\right)^6$$

$$= z^6 + 6z^5\left(\frac{1}{z}\right) + 15z^4\left(\frac{1}{z^2}\right) + 20z^3\left(\frac{1}{z^3}\right) + 15z^2\left(\frac{1}{z^4}\right) + 6z\left(\frac{1}{z^5}\right) + \frac{1}{z^6}$$

$$= z^6 - 6z^4 + 15z^2 + 20 + \frac{15}{z^2} + \frac{6}{z^4} + \frac{1}{z^6}$$

$$= \left(z^6 - \frac{1}{z^6}\right) + 6\left(z^4 + \frac{1}{z^4}\right) + 15\left(z^2 + \frac{1}{z^2}\right) + 20$$

$$= 2\cos 6\theta + 6(2\cos 4\theta) + 15(2\sin 2\theta) + 20$$

$\boxed{z^n + \frac{1}{z^n} = 2\cos n\theta}$

So, $64\cos^6\theta = 2\cos 6\theta + 12\cos 4\theta + 30\cos 2\theta + 20$

$32\cos^6\theta = \cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10$ (as required)

b $\int_0^{\frac{\pi}{6}} \cos^6\theta = \frac{1}{32}\int_0^{\frac{\pi}{6}} \cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10\theta$

$$= \frac{1}{32}\left[\frac{\sin^6\theta}{6} + \frac{6\sin^4\theta}{4} + \frac{15\sin^2\theta}{2} + 10\theta\right]_0^{\frac{\pi}{6}}$$

$$= \frac{1}{32}\left[\left(\frac{\sin(\pi)}{6} + \frac{6\sin\left(\frac{2\pi}{3}\right)}{4} + \frac{15\sin\left(\frac{\pi}{3}\right)}{2} + \frac{10\pi}{6}\right) - (0)\right]$$

$$= \frac{1}{32}\left[0 + \frac{3}{2}\frac{\sqrt{3}}{2} + \frac{15}{2}\frac{\sqrt{3}}{2} + \frac{5\pi}{3}\right]$$

$$= \frac{1}{32}\left[\frac{3}{4}\sqrt{3} + \frac{15}{4}\sqrt{3} + \frac{5\pi}{3}\right]$$

$$= \frac{1}{32}\left[\frac{9}{2}\sqrt{3} + \frac{5\pi}{3}\right]$$

$$= \frac{5\pi}{96} + \frac{9}{64}\sqrt{3}$$

$\therefore \int_0^{\frac{\pi}{6}} \cos^6\theta = \frac{5\pi}{96} + \frac{9}{64}\sqrt{3}$ $\boxed{a = \frac{5}{96},\ b = \frac{9}{64}}$

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Exercise D, Question 7

Question:

a Use de Moivre's theorem to show that $\sin 4\theta = 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta$.

b Hence, or otherwise, show that $\tan 4\theta = \dfrac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}$.

c Use your answer to part **b** to find, to 2 d.p., the four solutions of the equation $x^4 + 4x^3 - 6x^2 - 4x + 1 = 0$.

Solution:

a $(\cos\theta + \mathrm{i}\sin\theta)^4 = \cos 4\theta + \mathrm{i}\sin 4\theta$

de Moivre's Theorem.

$$= \cos^4\theta + {}^4C_1\cos^3\theta(\mathrm{i}\sin\theta) + {}^4C_2\cos^2\theta(\mathrm{i}\sin\theta)^2 + {}^4C_3\cos\theta(\mathrm{i}\sin\theta)^3 + (\mathrm{i}\sin\theta)^4$$

Binomial expansion.

$$= \cos^4\theta + 4\mathrm{i}\cos^3\theta\sin\theta + 6\mathrm{i}^2\cos^2\theta\sin^2\theta + 4\mathrm{i}^3\cos\theta\sin^3\theta + \mathrm{i}^4\sin^4\theta$$

$$= \cos^4\theta + 4\mathrm{i}\cos^3\theta\sin\theta - 6\cos^2\theta\sin^2\theta - 4\mathrm{i}\cos\theta\sin^3\theta + \sin^4\theta$$

Hence,

$$\cos 4\theta + \mathrm{i}\sin 4\theta = \cos^4\theta + 4\mathrm{i}\cos^3\theta\sin\theta - 6\cos^2\theta\sin^2\theta - 4\mathrm{i}\cos\theta\sin^3\theta + \sin^4\theta \quad ①$$

Equating the imaginary parts of ① gives:

$\sin^4\theta = 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta$ (as required)

b Equating the real parts of ① gives:

$$\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$$

$$\tan 4\theta = \frac{\sin 4\theta}{\cos 4\theta} = \frac{4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta}{\cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta} \qquad \frac{(\cos 4\theta \div \cos^4\theta)}{(\cos 4\theta \div \cos^4\theta)}$$

$$= \frac{\dfrac{4\cos^3\theta\sin\theta}{\cos^4\theta} - \dfrac{4\cos\theta\sin^3\theta}{\cos^4\theta}}{\dfrac{\cos^4\theta}{\cos^4\theta} - \dfrac{6\cos^2\theta\sin^2\theta}{\cos^4\theta} + \dfrac{\sin^4\theta}{\cos^4\theta}}$$

$$= \frac{\dfrac{4\cancel{\cos^3\theta}}{\cancel{\cos^3\theta}}\dfrac{\sin\theta}{\cos\theta} - \dfrac{4\cancel{\cos\theta}\sin^3\theta}{\cancel{\cos\theta}\cos^3\theta}}{\dfrac{\cancel{\cos^4\theta}}{\cancel{\cos^4\theta}} - \dfrac{6\cancel{\cos^2\theta}\sin^2\theta}{\cancel{\cos^2\theta}\cos^2\theta} + \dfrac{\sin^4\theta}{\cos^4\theta}}$$

$$= \frac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}$$

Therefore, $\tan^4\theta = \dfrac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}$ (as required)

c $x^4 + 4x^3 - 6x^2 - 4x + 1 = 0$

$x^4 - 6x^2 + 1 = 4x - 4x^3$

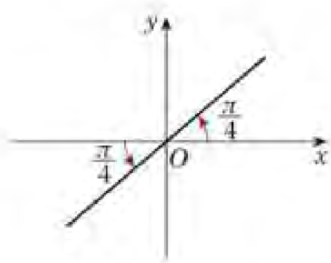
$1 = \dfrac{4x - 4x^3}{x^4 - 6x^2 + 1}$ ②

Let $x = \tan\theta$; then

② $\Rightarrow \dfrac{4\tan\theta - 4\tan^3\theta}{\tan^4\theta - 6\tan^2\theta + 1} = 1$

$\tan 4\theta = 1$ From part **b**.

$\alpha = \dfrac{\pi}{4}$



$4\theta = \left\{\dfrac{\pi}{4}, \dfrac{5\pi}{4}, \dfrac{9\pi}{4}, \dfrac{13\pi}{4}, \ldots\right\}$

$\theta = \left\{\dfrac{\pi}{16}, \dfrac{5\pi}{16}, \dfrac{9\pi}{16}, \dfrac{13\pi}{16}, \ldots\right\}$

$\therefore\ x = \tan\theta = \tan\dfrac{\pi}{16}, \tan\dfrac{5\pi}{16}, \tan\dfrac{9\pi}{16}, \tan\dfrac{13\pi}{16}$

$x = 0.19891\ldots, 1.49660\ldots, -5.02733\ldots, -0.66817\ldots,$

$x = 0.20, 1.50, -5.03, -0.67$ (2 d.p.)

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Exercise E, Question 1

Question:

Solve the following equations, expressing your answers for z in the form $x + iy$, where $x \in \mathbb{R}$ and $y \in \mathbb{R}$.

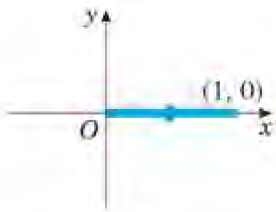
a $z^4 - 1 = 0$ **b** $z^3 - i = 0$ **c** $z^3 = 27$

d $z^4 + 64 = 0$ **e** $z^4 + 4 = 0$ **f** $z^3 + 8i = 0$

Solution:

a $z^4 - 1 = 0$

$z^4 = 1$



for 1, $r = 1$ and $\theta = 0$

So $z^4 = 1\,(\cos 0 + \mathrm{i}\sin 0)$

$z^4 = \cos(0 + 2k\pi) + \mathrm{i}\sin(0 + 2k\pi) \quad k \in \mathbb{Z}$

Hence, $z = [\cos(2k\pi) + \mathrm{i}\sin(2k\pi)]^{\frac{1}{4}}$

$$z = \cos\left(\frac{2k\pi}{4}\right) + \mathrm{i}\sin\left(\frac{2k\pi}{4}\right)$$

de Moivre's Theorem.

$$z = \cos\left(\frac{k\pi}{2}\right) + \mathrm{i}\sin\left(\frac{k\pi}{2}\right)$$

$k = 0$, $z = \cos 0 + \mathrm{i}\sin 0 = 1$

$k = 1$, $z = \cos\frac{\pi}{2} + \mathrm{i}\sin\frac{\pi}{2} = \mathrm{i}$

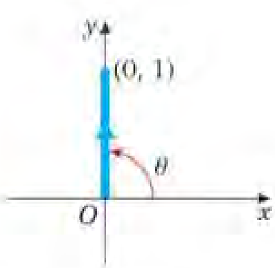
$k = 2$, $z = \cos\pi + \mathrm{i}\sin\pi = -1$

$k = -1$, $z = \cos\left(-\frac{\pi}{2}\right) + \mathrm{i}\sin\left(-\frac{\pi}{2}\right) = -\mathrm{i}$

Therefore, $z = 1, \mathrm{i}, -1, -\mathrm{i}$

b $z^3 - \mathrm{i} = 0$

$z^3 = \mathrm{i}$



for i, $r = 1$ and $\theta = \frac{\pi}{2}$

So $z^3 = 1\left(\cos\frac{\pi}{2} + \mathrm{i}\sin\frac{\pi}{2}\right)$

$$z^3 = \cos\left(\frac{\pi}{2} + 2k\pi\right) + \mathrm{i}\sin\left(\frac{\pi}{2} + 2k\pi\right), \quad k \in \mathbb{Z}$$

Hence, $z = \left[\cos\left(\frac{\pi}{2} + 2k\pi\right) + \mathrm{i}\sin\left(\frac{\pi}{2} + 2k\pi\right)\right]^{\frac{1}{3}}$

$$z = \cos\left(\frac{\frac{\pi}{2} + 2k\pi}{3}\right) + \mathrm{i}\sin\left(\frac{\frac{\pi}{2} + 2k\pi}{3}\right)$$

de Moivre's Theorem.

$$z = \cos\left(\frac{\pi}{6} + \frac{2k\pi}{3}\right) + \mathrm{i}\sin\left(\frac{\pi}{6} + \frac{2k\pi}{3}\right)$$

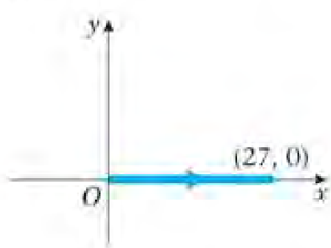
$\therefore k = 0, z = \cos\frac{\pi}{6} + \mathrm{i}\sin\frac{\pi}{6} = \frac{\sqrt{3}}{2} + \frac{1}{2}\mathrm{i}$

$k = 1, z = \cos\frac{5\pi}{6} + \mathrm{i}\sin\frac{5\pi}{6} = -\frac{\sqrt{3}}{2} + \frac{1}{2}\mathrm{i}$

$k = -1, z = \cos\left(-\frac{\pi}{2}\right) + \mathrm{i}\sin\left(-\frac{\pi}{2}\right) = 0 - \mathrm{i}$

Therefore, $z = \frac{\sqrt{3}}{2} + \frac{1}{2}\mathrm{i}, -\frac{\sqrt{3}}{2} + \frac{1}{2}\mathrm{i}, -\mathrm{i}$

c $z^3 = 27$



for 27, $r = 27$ and $\theta = 0$

So $z^3 = 27(\cos 0 + \mathrm{i}\sin 0)$

$z^3 = 27[\cos(0 + 2k\pi) + \mathrm{i}\sin(0 + 2k\pi)] \quad k \in \mathbb{Z}$

Hence, $z = [27(\cos(2k\pi) + \mathrm{i}\sin(2k\pi))]^{\frac{1}{3}}$

de Moivre's Theorem.

$$z = 3\left[\cos\left(\frac{2k\pi}{3}\right) + \mathrm{i}\sin\left(\frac{2k\pi}{3}\right)\right]$$

$k = 0$; $z = 3(\cos 0 + \mathrm{i}\sin 0) = 3$

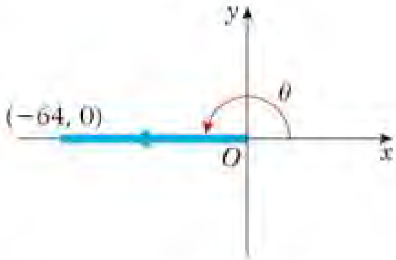
$k = 1$; $z = 3\left(\cos\frac{2\pi}{3} + \mathrm{i}\sin\frac{2\pi}{3}\right) = 3\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}\mathrm{i}\right) = -\frac{3}{2} + \frac{3\sqrt{3}}{2}\mathrm{i}$

$k = -1$; $z = 3\left(\cos\left(\frac{-2\pi}{3}\right) + \mathrm{i}\sin\left(\frac{-2\pi}{3}\right)\right) = 3\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}\mathrm{i}\right) = -\frac{3}{2} - \frac{3\sqrt{3}}{2}\mathrm{i}$

Therefore, $z = 3, -\frac{3}{2} + \frac{3\sqrt{3}}{2}\mathrm{i}, -\frac{3}{2} - \frac{3\sqrt{3}}{2}\mathrm{i}$

d $z^4 + 64 = 0$

$z^4 = -64$



for -64, $r = 64$ and $\theta = \pi$

So $z^4 = 64(\cos\pi + i\sin\pi)$

$z^4 = 64(\cos(\pi + 2k\pi) + i\sin(\pi + 2k\pi)) \quad k \in \mathbb{Z}$

Hence, $z = [64(\pi + 2k\pi) + i\sin(\pi + 2k\pi))]^{\frac{1}{4}}$

$z = 64^{\frac{1}{4}}\left(\cos\left(\frac{\pi + 2k\pi}{4}\right) + i\sin\left(\frac{\pi + 2k\pi}{4}\right)\right)$ de Moivre's Theorem.

$z = 2\sqrt{2}\left(\cos\left(\frac{\pi}{4} + \frac{k\pi}{2}\right) + i\sin\left(\frac{\pi}{4} + \frac{k\pi}{2}\right)\right)$

$k = 0;\ z = 2\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = 2\sqrt{2}\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) = 2 + 2i$

$k = 1;\ z = 2\sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right) = 2\sqrt{2}\left(-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right) = -2 + 2i$

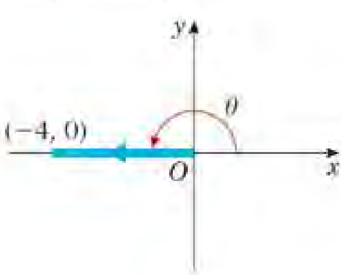
$k = -1;\ z = 2\sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right) = 2\sqrt{2}\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right) = 2 - 2i$

$k = -2;\ z = 2\sqrt{2}\left(\cos\left(-\frac{3\pi}{4}\right) + i\sin\left(-\frac{3\pi}{4}\right)\right) = 2\sqrt{2}\left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right) = -2 - 2i$

Therefore, $z = 2 + 2i,\ -2 + 2i,\ 2 - 2i,\ -2 - 2i$

e $z^4 + 4 = 0$

$z^4 = -4$



for -4, $r = 4$ and $\theta = \pi$

So $z^4 = 4(\cos\pi + i\sin\pi)$

$z^4 = 4(\cos(\pi + 2k\pi) + i\sin(\pi + 2k\pi)) \quad k \in \mathbb{Z}$

Hence, $z = [4(\cos(\pi + 2k\pi) + i\sin(\pi + 2k\pi))]^{\frac{1}{4}}$

$$z = 4^{\frac{1}{4}}\left(\cos\left(\frac{\pi + 2k\pi}{4}\right) + i\sin\left(\frac{\pi + 2k\pi}{4}\right)\right)$$

de Moivre's Theorem.

$$z = \sqrt{2}\left(\cos\left(\frac{\pi}{4} + \frac{k\pi}{2}\right) + i\sin\left(\frac{\pi}{4} + \frac{k\pi}{2}\right)\right)$$

$k = 0$; $z = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = \sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right) = 1 + i$

$k = 1$; $z = \sqrt{2}\left(\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right) = \sqrt{2}\left(-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right) = -1 + i$

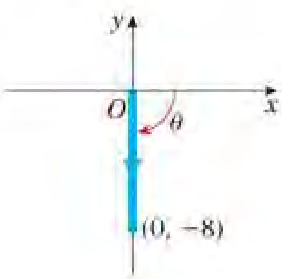
$k = -1$; $z = \sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right) = \sqrt{2}\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right) = 1 - i$

$k = -2$; $z = \sqrt{2}\left(\cos\left(-\frac{3\pi}{4}\right) + i\sin\left(-\frac{3\pi}{4}\right)\right) = \sqrt{2}\left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right) = -1 - i$

Therefore, $z = 1 + i, -1 + i, 1 - i, -1 - i$

f $z^3 + 8i = 0$

$z^3 = -8i$



for -8i, $r = 8$, $\theta = -\frac{\pi}{2}$

So $z^3 = 8\left(\cos\left(-\frac{\pi}{2}\right) + \text{i}\sin\left(-\frac{\pi}{2}\right)\right)$

$$z^4 = 8\left(\cos\left(-\frac{\pi}{2} + 2k\pi\right) + \text{i}\sin\left(-\frac{\pi}{2} + 2k\pi\right)\right) \quad k \in \mathbb{Z}$$

Hence, $z = \left[8\left(\cos\left(-\frac{\pi}{2} + 2k\pi\right) + \text{i}\sin\left(-\frac{\pi}{2} + 2k\pi\right)\right)\right]^{\frac{1}{3}}$

$$z = 8^{\frac{1}{3}}\left(\cos\left(\frac{-\frac{\pi}{2} + 2k\pi}{3}\right) + \text{i}\sin\left(\frac{-\frac{\pi}{2} + 2k\pi}{3}\right)\right)$$

de Moivre's Theorem.

$$z = 2\left(\cos\left(-\frac{\pi}{6} + \frac{2k\pi}{3}\right) + \text{i}\sin\left(-\frac{\pi}{6} + \frac{2k\pi}{3}\right)\right)$$

$k = 0$; $z = 2\left(\cos\left(-\frac{\pi}{6}\right) + \text{i}\sin\left(-\frac{\pi}{6}\right)\right) = 2\left(\frac{\sqrt{3}}{2} - \frac{1}{2}\text{i}\right) = \sqrt{3} - \text{i}$

$k = 1$; $z = 2\left(\cos\frac{\pi}{2} + \text{i}\sin\frac{\pi}{2}\right) = 2(0 + \text{i}) = 2\text{i}$

$k = -1$; $z = 2\left(\cos\left(-\frac{5\pi}{6}\right) + \text{i}\sin\left(-\frac{5\pi}{6}\right)\right) = 2\left(-\frac{\sqrt{3}}{2} - \frac{1}{2}\text{i}\right) = -\sqrt{3} - \text{i}$

Therefore, $z = \sqrt{3} - \text{i}$, 2i, $-\sqrt{3} - \text{i}$

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Exercise E, Question 2

Question:

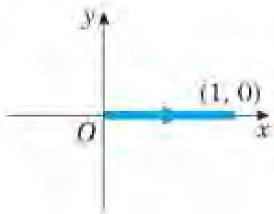
Solve the following equations, expressing your answers for z in the form $r(\cos\theta + i\sin\theta)$, where $-\pi < \theta \leqslant \pi$.

a $z^7 = 1$ **b** $z^4 + 16i = 0$ **c** $z^5 + 32 = 0$

d $z^3 = 2 + 2i$ **e** $z^4 + 2\sqrt{3}i = 2$ **f** $z^3 + 32\sqrt{3} + 32i = 0$

Solution:

a $z^7 = 1$



for 1, $r = 1$ and $\theta = 0$

So $z^7 = 1\,(\cos 0 + \mathrm{i}\sin 0)$

$z^7 = \cos(0 + 2k\pi) + \mathrm{i}\sin(0 + 2k\pi) \quad k \in \mathbb{Z}$

Hence, $z = (\cos(2k\pi) + \mathrm{i}\sin(2k\pi))^{\frac{1}{7}}$

$$z = \cos\left(\frac{2k\pi}{7}\right) + \mathrm{i}\sin\left(\frac{2k\pi}{7}\right)$$

de Moivre's Theorem.

$k = 0,\ z = \cos 0 + \mathrm{i}\sin 0$

$k = 1,\ z = \cos\left(\frac{2\pi}{7}\right) + \mathrm{i}\sin\left(\frac{2\pi}{7}\right)$

$k = 2,\ z = \cos\left(\frac{4\pi}{7}\right) + \mathrm{i}\sin\left(\frac{4\pi}{7}\right)$

$k = 3,\ z = \cos\left(\frac{6\pi}{7}\right) + \mathrm{i}\sin\left(\frac{6\pi}{7}\right)$

$k = -1,\ z = \cos\left(-\frac{2\pi}{7}\right) + \mathrm{i}\sin\left(-\frac{2\pi}{7}\right)$

$k = -2,\ z = \cos\left(-\frac{4\pi}{7}\right) + \mathrm{i}\sin\left(-\frac{4\pi}{7}\right)$

$k = -3,\ z = \cos\left(-\frac{6\pi}{7}\right) + \mathrm{i}\sin\left(-\frac{6\pi}{7}\right)$

Therefore, $z = \cos 0 + \mathrm{i}\sin 0,\ \cos\frac{2\pi}{7} + \mathrm{i}\sin\frac{2\pi}{7}$

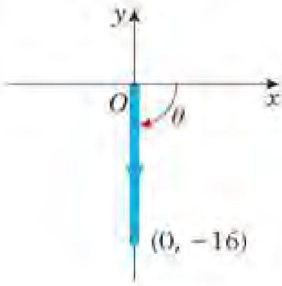
$\cos\frac{4\pi}{7} + \mathrm{i}\sin\frac{4\pi}{7},\ \cos\frac{6\pi}{7} + \mathrm{i}\sin\frac{6\pi}{7}$

$\cos\left(-\frac{2\pi}{7}\right) + \mathrm{i}\sin\left(-\frac{2\pi}{7}\right),\ \cos\left(-\frac{4\pi}{7}\right) + \mathrm{i}\sin\left(-\frac{4\pi}{7}\right)$

$\cos\left(-\frac{6\pi}{7}\right) + \mathrm{i}\sin\left(-\frac{6\pi}{7}\right)$

b $z^4 + 16\text{i} = 0$

$$z^4 = -16\text{i}$$



for -16i, $r = 16$ and $\theta = -\frac{\pi}{2}$

So $z^4 = 16\left(\cos\left(-\frac{\pi}{2}\right) + \text{i}\sin\left(-\frac{\pi}{2}\right)\right)$

$$z^4 = 16\left(\cos\left(-\frac{\pi}{2} + 2k\pi\right) + \text{i}\sin\left(-\frac{\pi}{2} + 2k\pi\right)\right) \quad k \in \mathbb{Z}$$

Hence, $z = \left[16\left(\cos\left(-\frac{\pi}{2} + 2k\pi\right) + \text{i}\sin\left(-\frac{\pi}{2} + 2k\pi\right)\right)\right]^{\frac{1}{4}}$

$$z = 16^{\frac{1}{4}}\left(\cos\left(\frac{-\frac{\pi}{2} + 2k\pi}{4}\right) + \text{i}\sin\left(\frac{-\frac{\pi}{2} + 2k\pi}{4}\right)\right)$$ de Moivre's Theorem.

$$z = \left(\cos\left(-\frac{\pi}{8} + \frac{k\pi}{2}\right) + \text{i}\sin\left(-\frac{\pi}{8} + \frac{k\pi}{2}\right)\right)$$

$k = 0$, $z = 2\left(\cos\left(-\frac{\pi}{8}\right) + \text{i}\sin\left(-\frac{\pi}{8}\right)\right)$

$k = 1$, $z = 2\left(\cos\frac{3\pi}{8} + \text{i}\sin\frac{3\pi}{8}\right)$

$k = 2$, $z = 2\left(\cos\frac{7\pi}{8} + \text{i}\sin\frac{7\pi}{8}\right)$

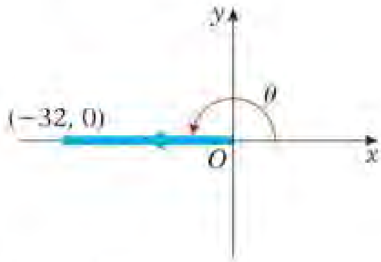
$k = -1$, $z = 2\left(\cos\left(-\frac{5\pi}{8}\right) + \text{i}\sin\left(-\frac{5\pi}{8}\right)\right)$

Therefore, $z = 2\left(\cos\left(-\frac{\pi}{8}\right) + \text{i}\sin\left(-\frac{\pi}{8}\right)\right), 2\left(\cos\left(\frac{3\pi}{8}\right) + \text{i}\sin\left(\frac{3\pi}{8}\right)\right)$

$$2\left(\cos\left(\frac{7\pi}{8}\right) + \text{i}\sin\left(\frac{7\pi}{8}\right)\right), 2\left(\cos\left(-\frac{5\pi}{8}\right) + \text{i}\sin\left(-\frac{5\pi}{8}\right)\right)$$

c $z^5 + 32 = 0$

$z^5 = -32$



for -32, $r = 32$ and $\theta = \pi$

So $z^5 = 32(\cos \pi + i \sin \pi)$

$z^5 = 32(\cos(\pi + 2k\pi) + i \sin(\pi + 2k\pi)) \quad k \in \mathbb{Z}$

Hence, $z = [32(\cos(\pi + 2k\pi) + i \sin(\pi + 2k\pi))]^{\frac{1}{5}}$

$$z = 32^{\frac{1}{5}}\left(\cos\left(\frac{\pi + 2k\pi}{5}\right) + i \sin\left(\frac{\pi + 2k\pi}{5}\right)\right)$$

$$z = 2\left(\cos\left(\frac{\pi}{5} + \frac{2k\pi}{5}\right) + i \sin\left(\frac{\pi}{5} + \frac{2k\pi}{5}\right)\right)$$

de Moivre's Theorem.

$k = 0$, $z = 2\left(\cos\frac{\pi}{5} + i \sin\frac{\pi}{5}\right)$

$k = 1$, $z = 2\left(\cos\frac{3\pi}{5} + i \sin\frac{3\pi}{5}\right)$

$k = 1$, $z = 2(\cos \pi + i \sin \pi)$

$k = 2$, $z = 2\left(\cos\left(-\frac{\pi}{5}\right) + i \sin\left(-\frac{\pi}{5}\right)\right)$

$k = -1$, $z = 2\left(\cos\left(-\frac{5\pi}{8}\right) + i \sin\left(-\frac{5\pi}{8}\right)\right)$

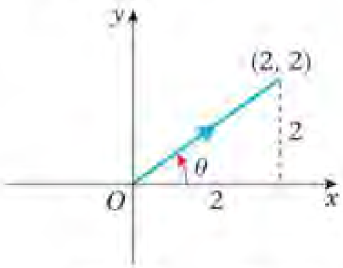
$k = -2$, $z = 2\left(\cos\left(-\frac{3\pi}{5}\right) + i \sin\left(-\frac{3\pi}{5}\right)\right)$

Therefore, $z = 2\left(\cos\frac{\pi}{5} + i \sin\frac{\pi}{5}\right), 2\left(\cos\frac{3\pi}{5} + i \sin\frac{3\pi}{5}\right),$

$2(\cos \pi + i \sin \pi), 2\left(\cos\left(-\frac{\pi}{5}\right) + i \sin\left(-\frac{\pi}{5}\right)\right),$

$2\left(\cos\left(-\frac{3\pi}{5}\right) + i \sin\left(-\frac{3\pi}{5}\right)\right)$

d $z^3 = 2 + 2i$



$r = \sqrt{2^2 + 2^2} = \sqrt{4 + 4} = \sqrt{8} = 2\sqrt{2}$

$\theta = \tan^{-1}\left(\frac{2}{2}\right) = \frac{\pi}{4}$

So $z^3 = 2\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

$z^3 = 2\sqrt{2}\left(\cos\left(\frac{\pi}{4} + 2k\pi\right) + i\sin\left(\frac{\pi}{4} + 2k\pi\right)\right) \quad k \in \mathbb{Z}$

Hence, $z = \left[2\sqrt{2}\left(\cos\left(\frac{\pi}{4} + 2k\pi\right) + i\sin\left(\frac{\pi}{4} + 2k\pi\right)\right)\right]^{\frac{1}{3}}$

$z = (2\sqrt{2})^{\frac{1}{3}}\left(\cos\left(\frac{\frac{\pi}{4} + 2k\pi}{3}\right) + i\sin\left(\frac{\frac{\pi}{4} + 2k\pi}{3}\right)\right)$

de Moivre's Theorem.

$z = \sqrt{2}\left(\cos\left(\frac{\pi}{12} + \frac{2k\pi}{3}\right) + i\sin\left(\frac{\pi}{12} + \frac{2k\pi}{3}\right)\right)$

$k = 0,\ z = \sqrt{2}\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right)$

$k = 1,\ z = \sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$

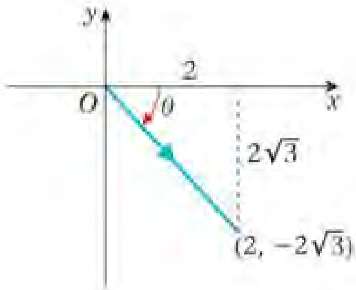
$k = -1,\ z = \sqrt{2}\left(\cos\left(-\frac{7\pi}{12}\right) + i\sin\left(\frac{-7\pi}{12}\right)\right)$

Therefore, $z = \sqrt{2}\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right),\ \sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right),$

$\sqrt{2}\left(\cos\left(-\frac{7\pi}{12}\right) + i\sin\left(\frac{-7\pi}{12}\right)\right)$

e $z^4 + 2\sqrt{3}\,\mathrm{i} = 2$

$z^4 = 2 - 2\sqrt{3}\,\mathrm{i}$



$$r = \sqrt{2^2 + (-2\sqrt{3})^2} = \sqrt{4 + 12} = \sqrt{16} = 4$$

$$\theta = -\tan^{-1}\left(\frac{2\sqrt{3}}{2}\right) = -\frac{\pi}{3}$$

$$\text{So } z^4 = 4\left(\cos\left(-\frac{\pi}{3}\right) + \mathrm{i}\sin\left(-\frac{\pi}{3}\right)\right)$$

$$z^4 = 4\left(\cos\left(-\frac{\pi}{3} + 2k\pi\right) + \mathrm{i}\sin\left(-\frac{\pi}{3} + 2k\pi\right)\right) \quad k \in \mathbb{Z}$$

$$\text{Hence, } z = \left[4\left[\cos\left(-\frac{\pi}{3} + 2k\pi\right) + \mathrm{i}\sin\left(-\frac{\pi}{3} + 2k\pi\right)\right]\right]^{\frac{1}{4}}$$

$$z = 4^{\frac{1}{4}}\left(\cos\left(\frac{-\frac{\pi}{3} + 2k\pi}{4}\right) + \mathrm{i}\sin\left(\frac{-\frac{\pi}{3} + 2k\pi}{4}\right)\right) \qquad \boxed{\text{de Moivre's Theorem.}}$$

$$z = \sqrt{2}\left(\cos\left(-\frac{\pi}{12} + \frac{k\pi}{2}\right) + \mathrm{i}\sin\left(-\frac{\pi}{12} + \frac{k\pi}{2}\right)\right)$$

$$k = 0,\ z = \sqrt{2}\left(\cos\left(-\frac{\pi}{12}\right) + \mathrm{i}\sin\left(-\frac{\pi}{12}\right)\right)$$

$$k = 1,\ z = \sqrt{2}\left(\cos\left(\frac{5\pi}{12}\right) + \mathrm{i}\sin\left(\frac{5\pi}{12}\right)\right)$$

$$k = 1,\ z = \sqrt{2}\left(\cos\left(\frac{11\pi}{12}\right) + \mathrm{i}\sin\left(\frac{11\pi}{12}\right)\right)$$

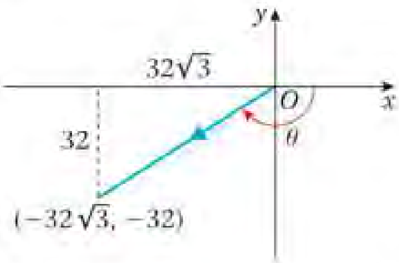
$$k = -1,\ z = \sqrt{2}\left(\cos\left(-\frac{7\pi}{12}\right) + \mathrm{i}\sin\left(-\frac{7\pi}{12}\right)\right)$$

$$\text{Therefore, } z = \sqrt{2}\left(\cos\left(-\frac{\pi}{12}\right) + \mathrm{i}\sin\left(-\frac{\pi}{12}\right)\right), \sqrt{2}\left(\cos\left(\frac{5\pi}{12}\right) + \mathrm{i}\sin\left(\frac{5\pi}{12}\right)\right),$$

$$\sqrt{2}\left(\cos\left(\frac{11\pi}{12}\right) + \mathrm{i}\sin\left(\frac{11\pi}{12}\right)\right), \sqrt{2}\left(\cos\left(-\frac{7\pi}{12}\right) + \mathrm{i}\sin\left(-\frac{7\pi}{12}\right)\right)$$

f $z^3 + 32\sqrt{3} + 32\text{i} = 0$

$z^3 = -32\sqrt{3} - 32\text{i}$



$$r = \sqrt{(-32\sqrt{3})^2 + (-32)^2} = \sqrt{3072 + 1024} = \sqrt{4096} = 64$$

$$\theta = -\pi + \tan^{-1}\left(\frac{32}{32\sqrt{3}}\right) = -\pi + \frac{\pi}{6} = -\frac{5\pi}{6}$$

So $z^4 = 64\left(\cos\left(-\frac{5\pi}{6}\right) + \text{i}\sin\left(-\frac{5\pi}{6}\right)\right)$

$$z^3 = 64\left(\cos\left(-\frac{5\pi}{6} + 2k\pi\right) + \text{i}\sin\left(-\frac{5\pi}{6} + 2k\pi\right)\right) \quad k \in \mathbb{Z}$$

Hence, $z = \left[64\left(\cos\left(-\frac{5\pi}{6} + 2k\pi\right) + \text{i}\sin\left(-\frac{5\pi}{6} + 2k\pi\right)\right)\right]^{\frac{1}{3}}$

$$z = 64^{\frac{1}{3}}\left(\cos\left(\frac{-\frac{5\pi}{6} + 2k\pi}{3}\right) + \text{i}\sin\left(\frac{-\frac{5\pi}{6} + 2k\pi}{3}\right)\right)$$

de Moivre's Theorem.

$$z = 4\left(\cos\left(-\frac{5\pi}{18} + \frac{2k\pi}{3}\right) + \text{i}\sin\left(-\frac{5\pi}{18} + \frac{2k\pi}{3}\right)\right)$$

$k = 0$, $z = 4\left(\cos\left(-\frac{5\pi}{18}\right) + \text{i}\sin\left(-\frac{5\pi}{18}\right)\right)$

$k = 1$, $z = 4\left(\cos\left(\frac{7\pi}{18}\right) + \text{i}\sin\left(\frac{7\pi}{18}\right)\right)$

$k = -1$, $z = 4\left(\cos\left(-\frac{17\pi}{18}\right) + \text{i}\sin\left(-\frac{17\pi}{18}\right)\right)$

Therefore, $z = 4\left(\cos\left(-\frac{5\pi}{18}\right) + \text{i}\sin\left(-\frac{5\pi}{18}\right)\right), 4\left(\cos\left(\frac{7\pi}{18}\right) + \text{i}\sin\left(\frac{7\pi}{18}\right)\right),$

$$4\left(\cos\left(-\frac{17\pi}{18}\right) + \text{i}\sin\left(-\frac{17\pi}{18}\right)\right)$$

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Exercise E, Question 3

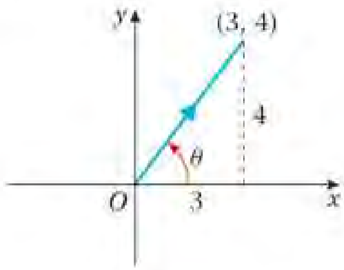
Question:

Solve the following equations, expressing your answers for z in the form $re^{i\theta}$, where $r > 0$ and $-\pi < \theta \leqslant \pi$. Give θ to 2 d.p.

a $z^4 = 3 + 4i$ **b** $z^3 = \sqrt{11} - 4i$ **c** $z^4 = -\sqrt{7} + 3i$

Solution:

a $z^4 = 3 + 4i$



$r = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$

$\theta = \tan^{-1}\left(\frac{4}{3}\right) = 0.927295...$

So, $z^4 = 5e^{i(0.927295...)}$

$z^4 = 5e^{i(0.927295... + 2k\pi)}, \quad k \in \mathbb{Z}$

Hence, $z = [5e^{i(0.927295... + 2k\pi)}]^{\frac{1}{4}}$

$= 5^{\frac{1}{4}}e^{i\left(\frac{0.927295... + 2k\pi}{4}\right)}$ de Moivre's Theorem.

$= 5^{\frac{1}{4}}e^{i\left(\frac{0.927295...}{4} + \frac{k\pi}{2}\right)}$

$k = 0, z = 5^{\frac{1}{4}}e^{i(0.2318...)}$

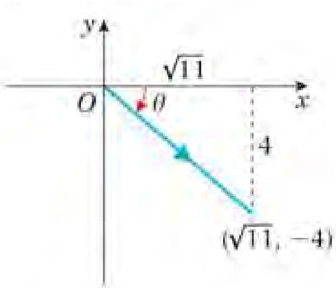
$k = 1, z = 5^{\frac{1}{4}}e^{i(1.8026...)}$

$k = -1, z = 5^{\frac{1}{4}}e^{i(-1.3389...)}$

$k = -2, z = 5^{\frac{1}{4}}e^{i(-2.9097...)}$

Therefore, $z = 5^{\frac{1}{4}}e^{0.23i}, 5^{\frac{1}{4}}e^{1.80i}, 5^{\frac{1}{4}}e^{-1.34i}, 5^{\frac{1}{4}}e^{-2.91i}$

b $z^3 = \sqrt{11} + 4i$



$r = \sqrt{(\sqrt{11})^2 + (-4^2)} = \sqrt{11 + 16} = \sqrt{27}$

$\theta = -\tan^{-1}\left(\frac{4}{\sqrt{11}}\right) = 0.878528...$

So, $z^3 = \sqrt{27}\, e^{i(-0.878528...)}$

$z^3 = \sqrt{27}\, e^{i(-0.878528... + 2k\pi)}, \quad k \in \mathbb{Z}$

Hence, $z = [\sqrt{27}\, e^{i(-0.878528... + 2k\pi)}]^{\frac{1}{3}}$

$= (\sqrt{27})^{\frac{1}{3}} e^{i\left(\frac{-0.878528... + 2k\pi}{3}\right)}$

de Moivre's Theorem.

$= \sqrt{3}\, e^{i\frac{-0.878528...}{3} + \frac{2k\pi}{3}}$

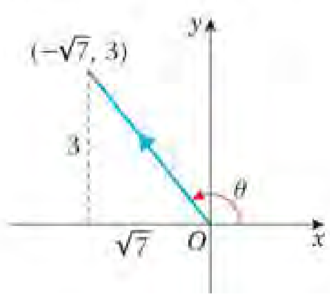
$k = 0,\ z = \sqrt{3}\, e^{i(-0.2928...)}$

$k = 1,\ z = \sqrt{3}\, e^{i(1.8015...)}$

$k = -1,\ z = \sqrt{3}\, e^{i(-2.3872...)}$

Therefore, $z = \sqrt{3}\, e^{-0.29i}, \sqrt{3}\, e^{1.80i}, \sqrt{3}\, e^{-2.39i}$

c $z^4 = -\sqrt{7} + 3i$



$r = \sqrt{(-\sqrt{7})^2 + 3^2} = \sqrt{7 + 9} = \sqrt{16} = 4$

$\theta = \pi - \tan^{-1}\left(\frac{3}{\sqrt{7}}\right) = 2.293530\ldots$

So, $z^4 = 4e^{i(2.293530\ldots)}$

$z^4 = 4e^{i(2.293530\ldots + 2k\pi)}, \quad k \in \mathbb{Z}$

Hence, $z = [4e^{i(2.293530\ldots + 2k\pi)}]^{\frac{1}{4}}$

$= 4^{\frac{1}{4}} e^{i\left(\frac{2.293530\ldots + 2k\pi}{4}\right)}$

de Moivre's Theorem.

$= \sqrt{2}\, e^{i\left(\frac{2.293530\ldots}{4} + \frac{k\pi}{2}\right)}$

$k = 0,\ z = \sqrt{2}\, e^{i(0.5733\ldots)}$

$k = 1,\ z = \sqrt{2}\, e^{i(2.1441\ldots)}$

$k = -1,\ z = \sqrt{2}\, e^{i(-0.9974\ldots)}$

$k = -2,\ z = \sqrt{2}\, e^{i(-2.5682\ldots)}$

Therefore, $z = \sqrt{2}\, e^{0.57i}$, $z = \sqrt{2}\, e^{2.14i}$, $z = \sqrt{2}\, e^{-1.00i}$, $z = \sqrt{2}\, e^{-2.57i}$

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Exercise E, Question 4

Question:

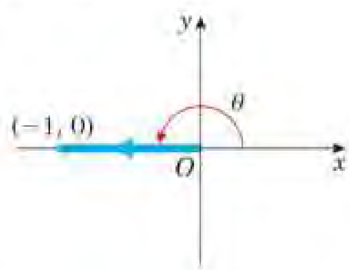
a Find the three roots of the equation $(z+1)^3 = -1$.
Give your answers in the form $x + \mathrm{i}y$, where $x \in \mathbb{R}$ and $y \in \mathbb{R}$.

b Plot the points representing these three roots on an Argand diagram.

c Given that these three points lie on a circle, find its centre and radius.

Solution:

a $(z+1)^3 = -1$



For -1, $r = 1$ and $\theta = \pi$

So, $(z+1)^3 = 1(\cos\pi + \mathrm{i}\sin\pi)$

$(z+1)^3 = (\pi + 24\,\pi) + \mathrm{i}\sin(\pi + 2k\,\pi) \quad k \in \mathbb{Z}$

Hence, $z + 1 = [\cos(\pi + 2k\,\pi) + \mathrm{i}\sin(\pi + 2k\,\pi)]$

$$z + 1 = \cos\left(\frac{\pi + 2k\pi}{3}\right) + \mathrm{i}\sin\left(\frac{\pi + 2k\pi}{3}\right)$$ de Moivre's Theorem.

$$z + 1 = \cos\left(\frac{\pi}{3} + \frac{2k\pi}{3}\right) + \mathrm{i}\sin\left(\frac{\pi}{3} + \frac{2k\pi}{3}\right)$$

$k = 0$, $z + 1 = \cos\frac{\pi}{3} + \mathrm{i}\sin\frac{\pi}{3} = \frac{1}{2} + \frac{\sqrt{3}}{2}\mathrm{i}$

$\Rightarrow z = -\frac{1}{2} + \sqrt{\frac{3}{2}}\mathrm{i}$

$k = 1$, $z + 1 = \cos\pi + \mathrm{i}\sin\pi = -1 + 0\mathrm{i}$

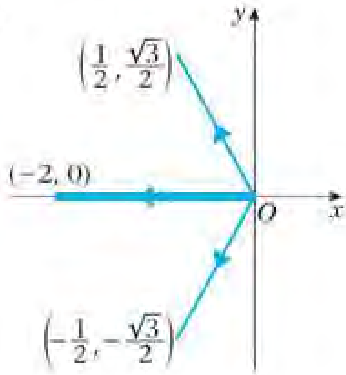
$\Rightarrow z = -2$

$k = -1$, $z + 1 = \cos\left(-\frac{\pi}{3}\right) + \mathrm{i}\sin\left(-\frac{\pi}{3}\right) = \frac{1}{2} - \frac{\sqrt{3}}{2}\mathrm{i}$

$\Rightarrow z = -\frac{1}{2} - \frac{\sqrt{3}}{2}\mathrm{i}$

Therefore, $z = -\frac{1}{2} + \frac{\sqrt{3}}{2}\mathrm{i}, -2, \frac{1}{2} - \frac{\sqrt{3}}{2}\mathrm{i}$

b



c The solutions to $w^3 = -1$, lie on a circle centre $(0, 0)$, radius 1.

As $w = z + 1$, then the three solutions for z are the three solutions for w translated by $\begin{pmatrix} -1 \\ 0 \end{pmatrix}$.

Hence the three points (the solutions for z), lie on a circle centre $(-1, 0)$, radius 1.

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Exercise E, Question 5

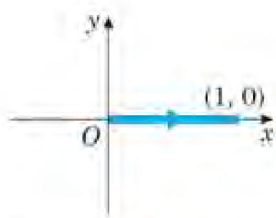
Question:

a Find the five roots of the equation $z^5 - 1 = 0$.
Give your answers in the form $r(\cos\theta + i\sin\theta)$, where $-\pi < \theta \leqslant \pi$.

b Given that the sum of all five roots of $z^5 - 1 = 0$ is zero, show that
$\cos\left(\frac{2\pi}{5}\right) + \cos\left(\frac{4\pi}{5}\right) = -\frac{1}{2}$.

Solution:

a $z^5 - 1 = 0$
$z^5 = 1$



For 1, $r = 1$ and $\theta = 0$

So, $z^5 = 1(\cos 0 + i\sin 0)$

$z^5 = \cos(0 + 2k\pi) + i\sin(0 + 2k\pi) \quad k \in \mathbb{Z}$

Hence, $z = [\cos(2k\pi) + i\sin(2k\pi)]^{\frac{1}{5}}$

$z = \cos\left(\frac{2k\pi}{5}\right) + i\sin\left(\frac{2k\pi}{5}\right)$ de Moivre's Theorem.

$k = 0,\ z_1 = \cos 0 + i\sin 0 = 1 + i(0) = 1$

$k = 1,\ z_2 = \cos\left(\frac{2\pi}{5}\right) + i\sin\left(\frac{2\pi}{5}\right)$

$k = 2,\ z_3 = \cos\left(\frac{4\pi}{5}\right) + i\sin\left(\frac{4\pi}{5}\right)$

$k = -1,\ z_4 = \cos\left(-\frac{2\pi}{5}\right) + i\sin\left(-\frac{2\pi}{5}\right)$

$k = -2,\ z_5 = \cos\left(-\frac{4\pi}{5}\right) + i\sin\left(-\frac{4\pi}{5}\right)$

Therefore $z = 1,\ \cos\left(\frac{2\pi}{5}\right) + i\sin\left(\frac{2\pi}{5}\right),\ \cos\left(\frac{4\pi}{5}\right) + i\sin\left(\frac{4\pi}{5}\right),$

$\cos\left(-\frac{2\pi}{5}\right) + i\sin\left(-\frac{2\pi}{5}\right),\ \cos\left(-\frac{4\pi}{5}\right) + i\sin\left(-\frac{4\pi}{5}\right)$

b So, $z_1 + z_2 + z_3 + z_4 + z_5 = 0$

$$1 + \cos\left(\frac{2\pi}{5}\right) + i\sin\left(\frac{2\pi}{5}\right) + \cos\left(\frac{4\pi}{5}\right) + i\sin\left(\frac{4\pi}{5}\right)$$

$$+ \cos\left(-\frac{2\pi}{5}\right) + i\sin\left(-\frac{2\pi}{5}\right) + \cos\left(-\frac{4\pi}{5}\right) + i\sin\left(-\frac{4\pi}{5}\right) = 0$$

$$\Rightarrow 1 + \cos\left(\frac{2\pi}{5}\right) + i\sin\left(\frac{2\pi}{5}\right) + \cos\left(\frac{4\pi}{5}\right) + i\sin\left(\frac{4\pi}{5}\right)$$

$$+ \cos\left(\frac{2\pi}{5}\right) - i\sin\left(\frac{2\pi}{5}\right) + \cos\left(\frac{4\pi}{5}\right) - i\sin\left(\frac{4\pi}{5}\right) = 0$$

$$1 + 2\cos\left(\frac{2\pi}{5}\right) + 2\cos\left(\frac{4\pi}{5}\right) = 0$$

$$2\cos\left(\frac{2\pi}{5}\right) + 2\cos\left(\frac{4\pi}{5}\right) = -1$$

$$2\left(\cos\left(\frac{2\pi}{5}\right) + 2\cos\left(\frac{4\pi}{5}\right)\right) = -1$$

$$\cos\left(\frac{2\pi}{5}\right) + \cos\left(\frac{4\pi}{5}\right) = -\frac{1}{2} \text{ (as required)}$$

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Edexcel AS and A Level Modular Mathematics

Exercise E, Question 6

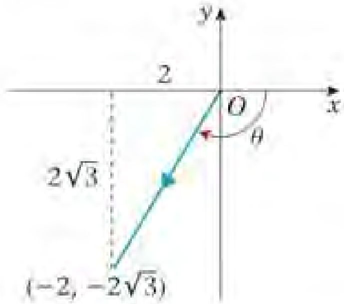
Question:

a Find the modulus and argument of $-2 - 2\sqrt{3}\text{i}$.

b Hence find all the solutions of the equation $z^4 + 2 + 2\sqrt{3}\text{i} = 0$.
Give your answers in the form $re^{\text{i}\theta}$, where $r > 0$ and $-\pi < \theta \leqslant \pi$.

Solution:

a $-2 - 2\sqrt{3}\text{i}.$



modulus $= r\sqrt{(-2)^2 + (-2\sqrt{3})^2} = \sqrt{4 + 12} = \sqrt{16} = 4$

argument $= \theta = 2\pi + \tan^{-1}\left(\frac{2\sqrt{3}}{2}\right) = -\pi + \frac{\pi}{3} = -\frac{2\pi}{3}$

Therefore, $r = 4,\ \theta = -\frac{2\pi}{3}$

b $z^4 + 2 + 2\sqrt{3}\,\text{i} = 0$

$z^4 = -2 - 2\sqrt{3}\,\text{i}$

and $r = 4,\ \theta = -\frac{2\pi}{3}$ for $-2 - 2\sqrt{3}\,\text{i}$

So $z^4 = 4e^{\text{i}\left(-\frac{2\pi}{3}\right)}$

$z^4 = 4e^{\text{i}\left(-\frac{2\pi}{3} + 2k\pi\right)},\quad k \in \mathbb{Z}$

Hence, $z = \left[4e^{\text{i}\left(-\frac{2\pi}{3} + 2k\pi\right)}\right]^{\frac{1}{4}}$

$= 4^{\frac{1}{4}}e^{\text{i}\left(\frac{-\frac{2\pi}{3} + 2k\pi}{4}\right)}$ de Moivre's Theorem.

$= \sqrt{2}\,e^{\text{i}\left(-\frac{\pi}{6} + \frac{k\pi}{2}\right)}$

$k = 0,\ z = \sqrt{2}\,e^{\text{i}\left(-\frac{\pi}{6}\right)}$

$k = 1,\ z = \sqrt{2}\,e^{\text{i}\left(\frac{\pi}{3}\right)}$

$k = 2,\ z = \sqrt{2}\,e^{\text{i}\left(\frac{5\pi}{6}\right)}$

$k = -1,\ z = \sqrt{2}\,e^{\text{i}\left(-\frac{2\pi}{3}\right)}$

Therefore, $z = \sqrt{2}\,e^{-\frac{\pi\text{i}}{6}},\ \sqrt{2}\,e^{\frac{\pi\text{i}}{3}},\ \sqrt{2}\,e^{\frac{5\pi\text{i}}{6}},\ \sqrt{2}\,e^{-\frac{2\pi\text{i}}{3}}$

Solutionbank FP2
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Exercise E, Question 7

Question:

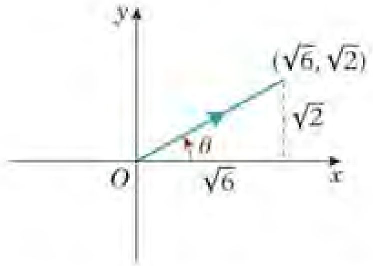
a Find the modulus and argument of $\sqrt{6} + \sqrt{2}i$.

b Solve the equation $z^{\frac{1}{4}} = \sqrt{6} + \sqrt{2}i$.

Give your answers in the form $re^{i\theta}$, where $r > 0$ and $-\pi < \theta \leq \pi$.

Solution:

a $\sqrt{6} + \sqrt{2}\text{i}$.



$$\text{modulus} = r\sqrt{(\sqrt{6})^2 + (\sqrt{2})^2} = \sqrt{6+2} = \sqrt{8}$$

$$\text{argument} = \theta = \tan^{-1}\left(\frac{\sqrt{2}}{\sqrt{6}}\right) = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$$

Therefore, $r = \sqrt{8},\ \theta = \frac{\pi}{6}$

b $z^{\frac{3}{4}} = \sqrt{6} + \sqrt{2}\,\text{i}$

For $\sqrt{6} + \sqrt{2}\,\text{i}$, $r = \sqrt{8}$, $\theta = \frac{\pi}{6}$

So, $z^{\frac{3}{4}} = \sqrt{8}\,\text{e}^{\text{i}\left(\frac{\pi}{6}\right)}$

$$z^3 = \left[\sqrt{8}\,\text{e}^{\text{i}\left(\frac{\pi}{6}\right)}\right]^4$$

$$z^3 = (\sqrt{8})^4\text{e}^{\text{i}\left(\frac{4\pi}{6}\right)}$$

de Moivre's Theorem.

$$(\sqrt{8})^4 = \left(8^{\frac{1}{2}}\right)^4 = 8^2 = 64$$

$$z^3 = 64\text{e}^{\text{i}\left(\frac{2\pi}{3}\right)}$$

$$z^3 = 64\text{e}^{\text{i}\left(\frac{2\pi}{3} + 2k\pi\right)}, \quad k \in \mathbb{Z}$$

Hence, $z = \left[64\text{e}^{\text{i}\left(\frac{2\pi}{3} + 2k\pi\right)}\right]^{\frac{1}{3}}$

$$= (64)^{\frac{1}{3}}\text{e}^{\text{i}\left(\frac{\frac{2\pi}{3} + 2k\pi}{3}\right)}$$

$$= 4\text{e}^{\text{i}\left(\frac{2\pi}{9} + \frac{2k\pi}{3}\right)}$$

$k = 0$, $z = 4\text{e}^{\text{i}\left(\frac{2\pi}{9}\right)}$

$k = 1$, $z = 4\text{e}^{\text{i}\left(\frac{8\pi}{9}\right)}$

$k = -1$, $z = 4\text{e}^{\text{i}\left(-\frac{4\pi}{9}\right)}$

Therefore, $z = 4\text{e}^{\frac{2\pi\text{i}}{9}}$, $z = 4\text{e}^{\frac{8\pi\text{i}}{9}}$, $z = 4\text{e}^{-\frac{4\pi\text{i}}{9}}$

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Exercise F, Question 1

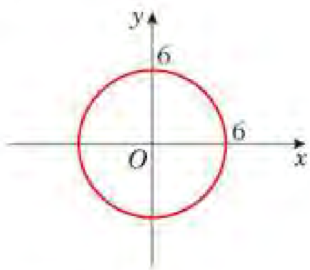
Question:

Sketch the locus of z and give the Cartesian equation of the locus of z when:

a $|z| = 6$
b $|z| = 10$
c $|z - 3| = 2$
d $|z + 3i| = 3$
e $|z - 4i| = 5$
f $|z + 1| = 1$
g $|z - 1 - i| = 5$
h $|z + 3 + 4i| = 4$
i $|z - 5 + 6i| = 5$
j $|2z + 6 - 4i| = 6$
k $|3z - 9 - 6i| = 12$

Solution:

a $|z| = 6$

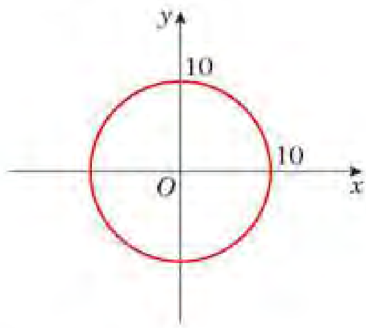


circle centre (0, 0), radius 6

equation: $x^2 + y^2 = 6^2$

$x^2 + y^2 = 36$

b $|z| = 10$

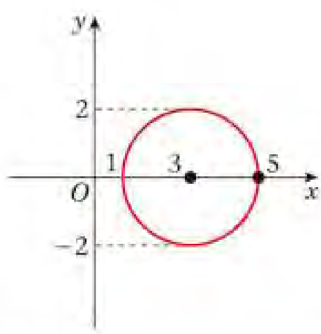


circle centre (0, 0), radius 10

equation: $x^2 + y^2 = 10^2$

$x^2 + y^2 = 100$

c $|z - 3| = 2$

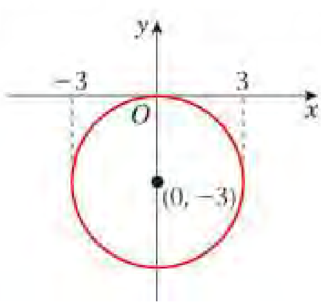


circle centre (3, 0), radius 2

equation: $(x + 3)^2 + y^2 = 2^2$

$(x + 3)^2 + y^2 = 4$

d $|z + 3\text{i}| = 3 \Rightarrow |z - (-3\text{i})| = 3$

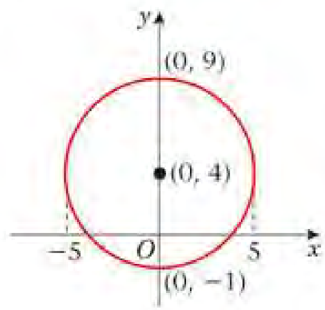


circle centre (0, −3), radius 3

equation: $x^2 + (y - 3)^2 = 3^2$

$x^2 + (y - 3)^2 = 9$

e $|z - 4\mathrm{i}| = 5$

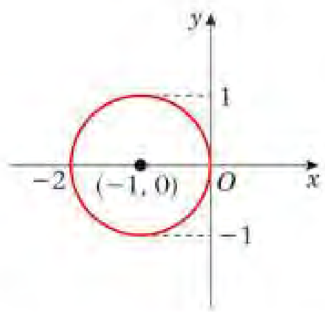


circle centre (0, 4), radius 5

equation: $x^2 + (y - 4)^2 = 5^2$

$x^2 + (y - 4)^2 = 25$

f $|z + 1| = 1 \Rightarrow |z - (-1)| = 1$

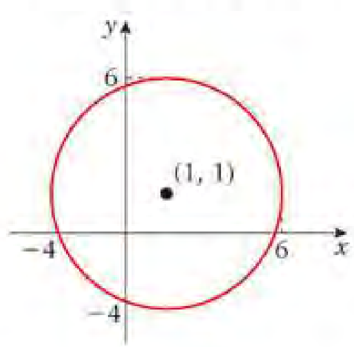


circle centre (−1, 0), radius 1

equation: $(x + 1)^2 + y^2 = 1^2$

$(x + 1)^2 + y^2 = 1$

g $|z - 1 - \mathrm{i}| = 5 \Rightarrow |z - (1 + \mathrm{i})| = 5$

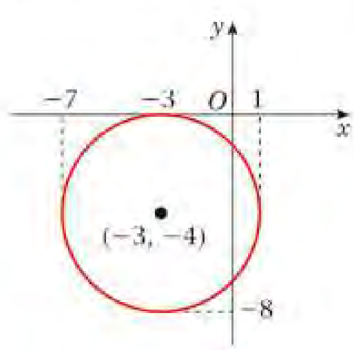


circle centre (1, 1), radius 5

equation: $(x - 1)^2 + (y - 1)^2 = 5^2$

$(x - 1)^2 + (y - 1)^2 = 25$

h $|z + 3 + 4\mathrm{i}| = 4 \Rightarrow |z - (-3 - 4\mathrm{i})| = 4$

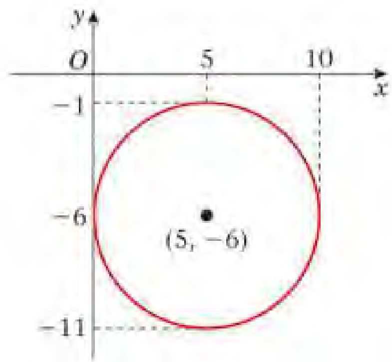


circle centre (−3, −4), radius 4

equation: $(x + 3)^2 + (y + 4)^2 = 4^2$

$(x + 3)^2 + (y + 4)^2 = 16$

i $|z - 5 + 6i| = 5 \Rightarrow |z - (5 - 6i)| = 4$



circle centre $(5, -6)$, radius 5

equation: $(x - 5)^2 + (y + 6)^2 = 5^2$

$(x - 5)^2 + (y + 6)^2 = 25$

j $|2z + 6 - 4i| = 6$

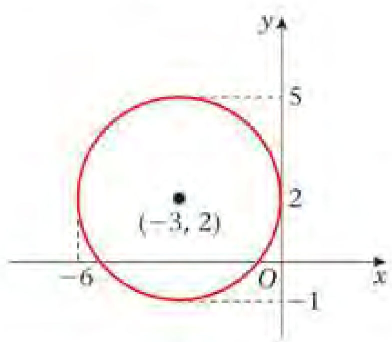
$\Rightarrow |2(z + 3 - 2i)| = 6$

$\Rightarrow |2|\,|z + 3 - 2i| = 6$

$\Rightarrow 2|z + 3 - 2i| = 6$

$\Rightarrow |z + 3 - 2i| = 3$

$\Rightarrow |z - (-3 + 2i)| = 3$



circle centre $(-3, 2)$, radius 3

equation: $(x + 3)^2 + (y - 2)^2 = 3^2$

$(x + 3)^2 + (y - 2)^2 = 9$

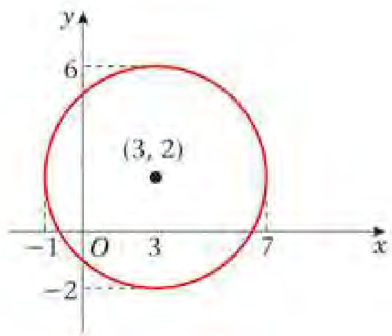
k $|3z - 9 - 6i| = 12$

$\Rightarrow |3(z - 3 - 2i)| = 12$

$\Rightarrow |3|\,|z - 3 - 2i| = 12$

$\Rightarrow 3|z - (3 + 2i)| = 12$

$\Rightarrow |z - (3 + 2i)| = 4$



circle centre (3, 2), radius 4

equation: $(x-3)^2+(y-2)^2=4^2$

$(x-3)^2+(y-2)^2=16$

l $|3z-9-6\text{i}|=9$

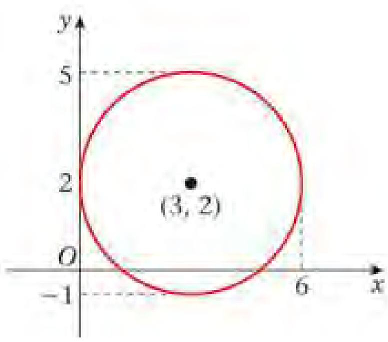
$\Rightarrow |3(z-3-2\text{i})|=9$

$\Rightarrow |3|\,|z-3-2\text{i}|=9$

$\Rightarrow 3|z-3-2\text{i}|=9$

$\Rightarrow |z-3-2\text{i}|=3$

$\Rightarrow |z-(3+2\text{i})|=3$



circle centre (3, 2), radius 3

equation: $(x-3)^2+(y-2)^2=3^2$

$(x-3)^2+(y-2)^2=9$

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Exercise F, Question 2

Question:

Sketch the locus of z when:

a $\arg z = \frac{\pi}{3}$

b $\arg(z + 3) = \frac{\pi}{4}$

c $\arg(z - 2) = \frac{\pi}{2}$

d $\arg(z + 2 + 2i) = -\frac{\pi}{4}$

e $\arg(z - 1 - i) = \frac{3\pi}{4}$

f $\arg(z + 3i) = \pi$

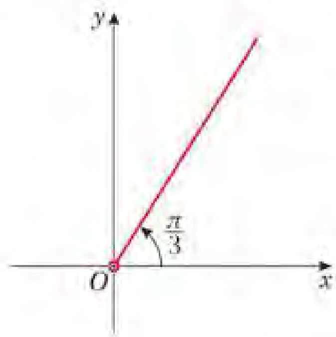
g $\arg(z - 1 + 3i) = \frac{2\pi}{3}$

h $\arg(z - 3 + 4i) = -\frac{\pi}{2}$

i $\arg(z - 4i) = -\frac{3\pi}{4}$

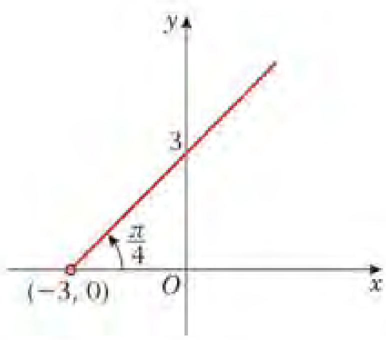
Solution:

a $\arg z = \frac{\pi}{3}$

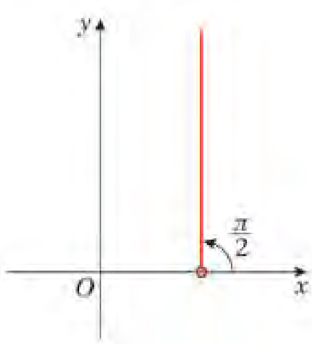


b $\arg(z + 3) = \frac{\pi}{4}$

$\Rightarrow \arg(z - (-3)) = \frac{\pi}{4}$

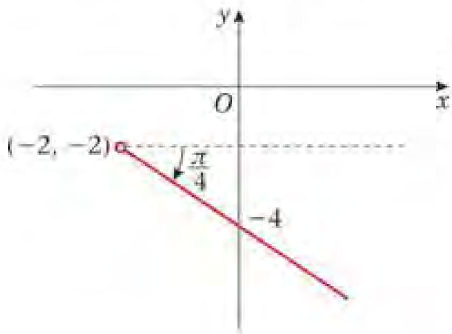


c $\arg(z - 2) = \frac{\pi}{2}$



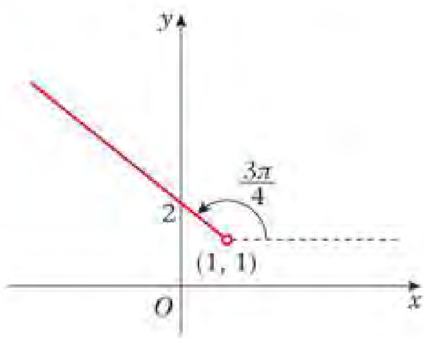
d $\arg(z + 2 + 2i) = -\frac{\pi}{4}$

$\Rightarrow \arg(z - (-2 - 2i)) = -\frac{\pi}{4}$



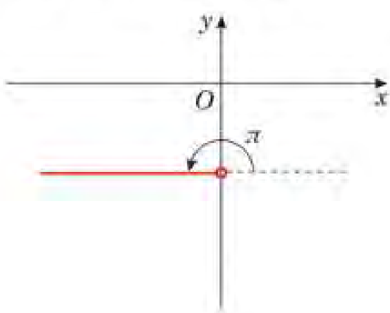
e $\arg(z - 1 - i) = \frac{3\pi}{4}$

$\Rightarrow \arg(z - (1 + i)) = \frac{3\pi}{4}$



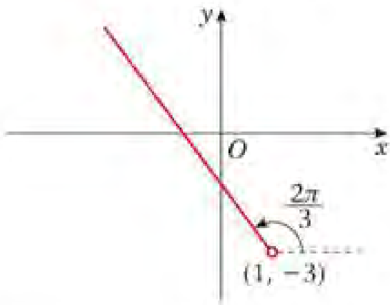
f $\arg(z + 3i) = \pi$

$\Rightarrow \arg(z - (-3i)) = \pi$



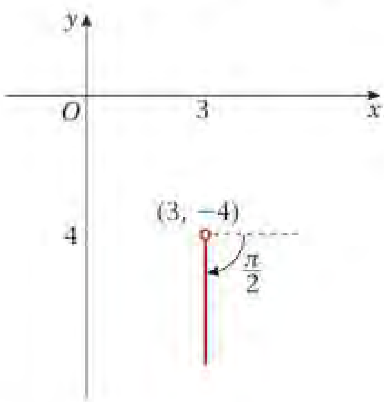
g $\arg(z - 1 + 3i) = \frac{2\pi}{3}$

$\Rightarrow \arg(z - (1 - 3i)) = \frac{2\pi}{3}$

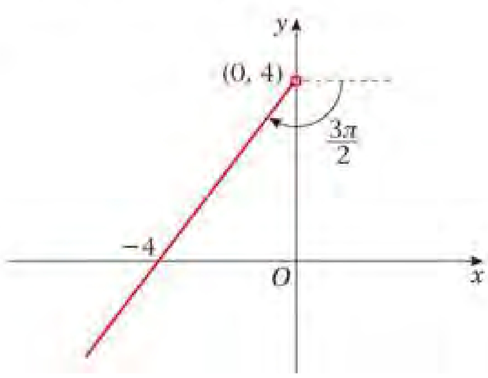


h $\arg(z - 3 + 4i) = -\frac{\pi}{2}$

$\Rightarrow \arg(z - (3 - 4i)) = -\frac{\pi}{2}$



i $\arg(z - 4i) = -\frac{3\pi}{4}$



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Exercise F, Question 3

Question:

Sketch the locus of z and give the Cartesian equation of the locus of z when:

a $|z-6| = |z-2|$

b $|z+8| = |z-4|$

c $|z| = |z+6\mathrm{i}|$

d $|z+3\mathrm{i}| = |z-8\mathrm{i}|$

e $|z-2-2\mathrm{i}| = |z+2+2\mathrm{i}|$

f $|z+4+\mathrm{i}| = |z+4+6\mathrm{i}|$

g $|z+3-5\mathrm{i}| = |z-7-5\mathrm{i}|$

h $|z+4-2\mathrm{i}| = |z-8+2\mathrm{i}|$

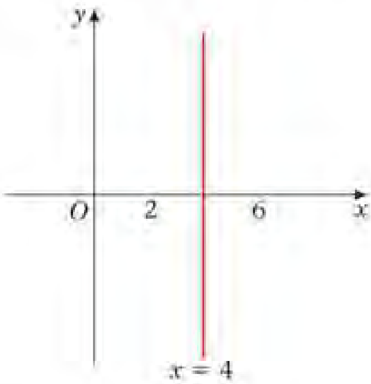
i $\dfrac{|z+3\mathrm{i}|}{|z-6\mathrm{i}|} = 1$

j $|z+7+2\mathrm{i}| = |z-4-3\mathrm{i}|$

k $|z+1-6\mathrm{i}| = |2+3\mathrm{i}-z|$

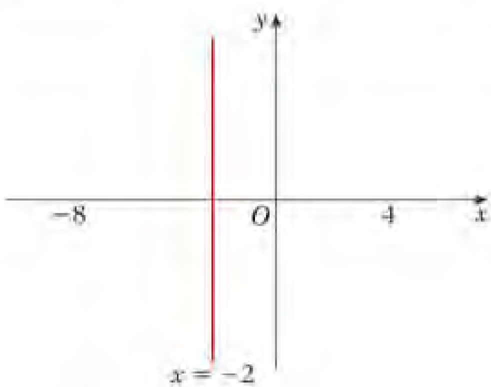
Solution:

a $|z-6| = |z-2|$
perpendicular bisector of the line joining (6, 0) and (2, 0).



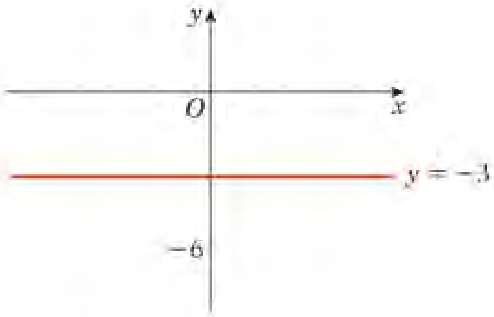
Equation: $x = 4$

b $|z+8| = |z-4|$
$\Rightarrow |z-(-8)| = |z-4|$
perpendicular bisector of the line joining (−8, 0) and (4, 0).



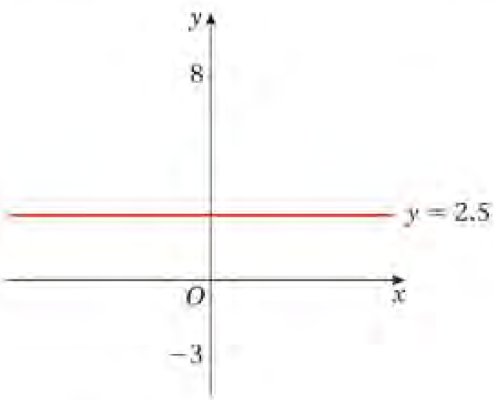
Equation: $x = -2$

c $|z| = |z+6\text{i}|$
$\Rightarrow |z| = |z-(-6\text{i})|$
perpendicular bisector of the line joining (0, 0) to (0, −6).



Equation: $y = -3$

d $|z + 3i| = |z - 8i|$
$\Rightarrow |z - (-3i)| = |z - 8i|$
perpendicular bisector of the line joining $(0, -3)$ to $(0, 8)$.



Equation: $y = 2.5$

Equation: $y = 2.5$

e $|z - 2 - 2\mathrm{i}| = |z + 2 + 2\mathrm{i}|$

$\Rightarrow |z - (2 + 2\mathrm{i})| = |z - (-2 - 2\mathrm{i})|$

perpendicular bisector of the line joining $(2, 2)$ to $(-2, -2)$.

So, $|x + \mathrm{i}y - 2 - 2\mathrm{i}| = |x + \mathrm{i}y + 2 + 2\mathrm{i}|$

$\Rightarrow |(x - 2) + \mathrm{i}(y - 2)| = |(x + 2) + \mathrm{i}(y + 2)|$

$\Rightarrow (x - 2)^2 + (y - 2)^2 = (x + 2)^2 + (y + 2)^2$

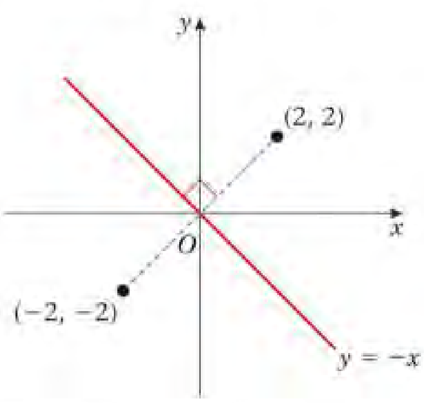
$\Rightarrow \cancel{x^2} - 4x + 4 + \cancel{y^2} - 4y + 4 = \cancel{x^2} + 4x + 4 + \cancel{y^2} + 4y + 4$

$\Rightarrow -4x - 4y^2 + 8 = 4x + 4y + 8$

$\Rightarrow 0 = 8x + 8y$

$\Rightarrow -8x = 8y$

$\Rightarrow y = -x$

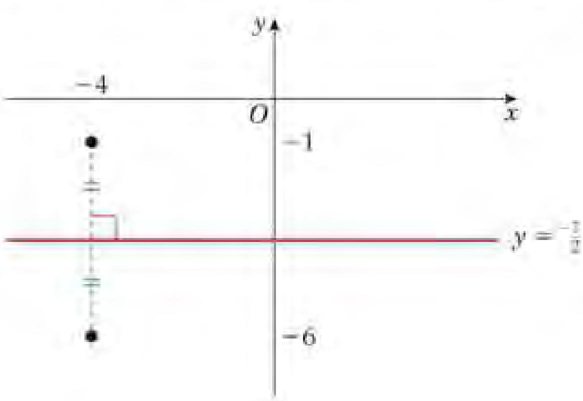


Equation: $y = -x$

f $|z + 4 + \mathrm{i}| = |z + 4 + 6\mathrm{i}|$

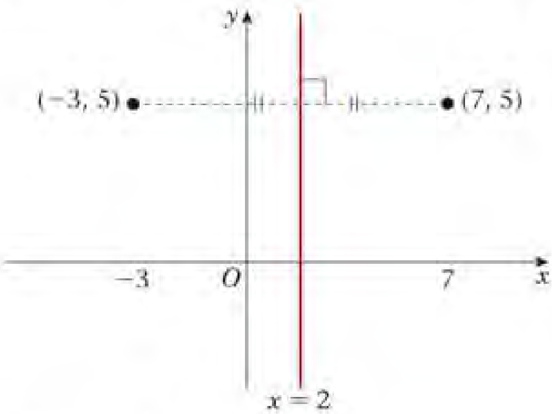
$\Rightarrow |z - (-4 - \mathrm{i})| = |z + (-4 - 6\mathrm{i})|$

perpendicular bisector of the line joining $(-4, -1)$ to $(-4, -6)$.



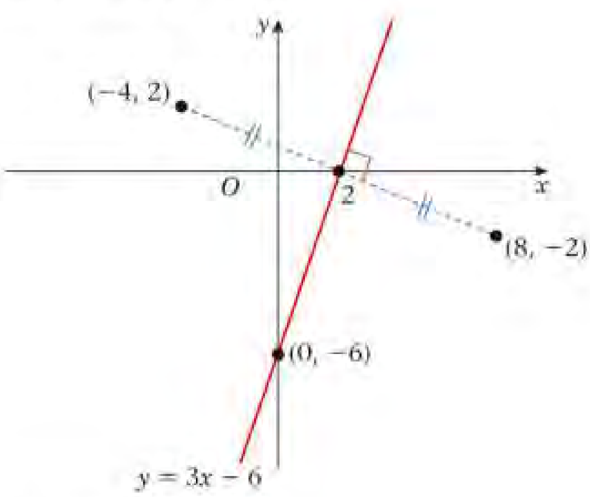
Equation: $y = -\frac{7}{2}$

g $|z + 3 - 5i| = |z - 7 - 5i|$
$\Rightarrow |z - (-3 + 5i)| = |z - (7 + 5i)|$
perpendicular bisector of the line joining $(-3, 5)$ to $(7, 5)$.



Equation: $x = 2$

h $|z + 4 - 2i| = |z - 8 + 2i|$
$\Rightarrow |z - (-4 + 2i)| = |z - (8 - 2i)|$
perpendicular bisector of the line joining $(-4, 2)$ to $(8, -2)$.
So, $|x + iy + 4 - 2i| = |x + iy - 8 + 2i|$
$\Rightarrow |(x + 4) + i(y - 2)| = |(x - 8) + i(y + 2)|$
$\Rightarrow (x + 4)^2 + (y - 2)^2 = (x - 8)^2 + (y + 2)^2$
$\Rightarrow \cancel{x^2} + 8x + 16 + \cancel{y^2} - 4y + 4 = \cancel{x^2} - 16x + 64 + \cancel{y^2} + 4y + 4$
$\Rightarrow 8x - 4y + 20 = -16x + 4y + 68$
$\Rightarrow 0 = -24x + 8y + 48$
$\Rightarrow 0 = -3x + y + 6$
$\Rightarrow 3x - 6 = y$



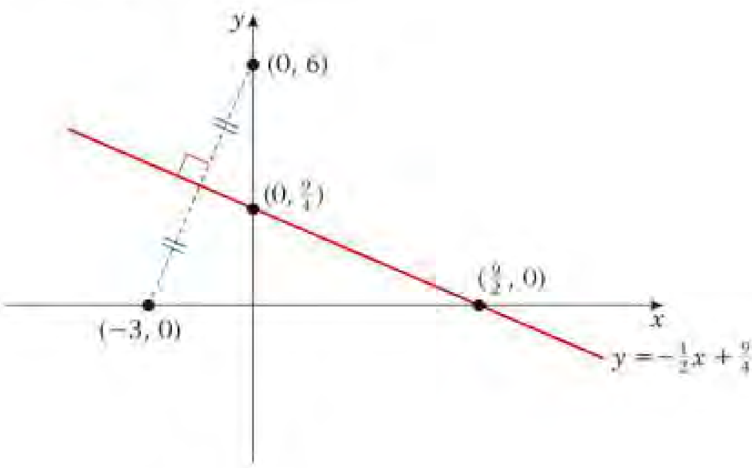
$y = 0$
$\Rightarrow 3x - 6 = 0$
$\Rightarrow 3x = 6$
$\Rightarrow x = 2$

$x = 0, y = -6$

Equation: $y = 3x - 6$

i $\dfrac{|z+3|}{|z-6i|} = 1$

$\Rightarrow |z+3| = |z-6i|$
$\Rightarrow |z-(-3)| = |z-6i|$
perpendicular bisector of the line joining $(-3, 0)$ to $(0, 6)$.
So, $|x+iy+3| = |x+iy-6i|$
$\Rightarrow |(x+3)+iy| = |x+i(y-6)|$
$\Rightarrow (x+3)^2 + y^2 = x^2 + (y-6)^2$
$\Rightarrow \cancel{x^2} + 6x + 9 + \cancel{y^2} = \cancel{x^2} + \cancel{y^2} - 12y + 36$
$\Rightarrow 6x + 12y = 36 - 9$
$\Rightarrow 6x + 12y = 27$
$\Rightarrow 2x + 4y = 9$
$\Rightarrow 4y = 9 - 2x$
$\Rightarrow y = -\frac{1}{2}x + \frac{9}{4}$



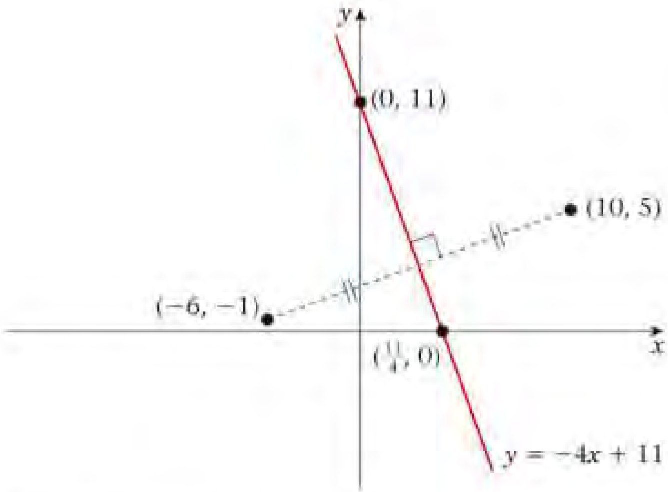
$y = 0$
$\Rightarrow 0 = 9 - 2x$
$\Rightarrow 2x = 9$
$\Rightarrow x = \frac{9}{2}$

$x = 0, y = \frac{9}{4}$

Equation: $y = -\frac{1}{2}x + \frac{9}{4}$

j $\dfrac{|z+6-i|}{|z-10-5i|} = 1$

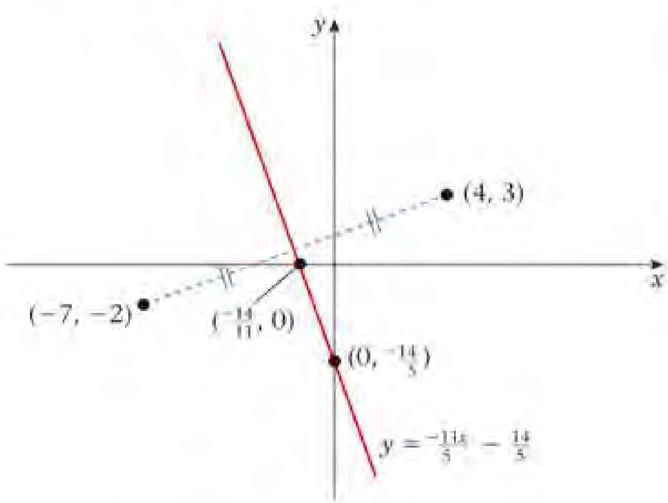
$\Rightarrow |z+6-i| = |z-10-5i|$
$\Rightarrow |z-(-6+i)| = |z-(10+5i)|$
perpendicular bisector of the line joining $(-6, 1)$ to $(10, 5)$.
So, $|x+iy+6-i| = |x+iy-10-5i|$
$\Rightarrow |(x+6)+i(y-1)| = |(x-10)+i(y-5)|$
$\Rightarrow (x+6)^2 + (y-1)^2 = (x-10)^2 + (y-5)^2$
$\Rightarrow \cancel{x^2} + 12x + 36 + \cancel{y^2} - 2y + 1 = \cancel{x^2} - 20x + 100 + \cancel{y^2} - 10y + 25$
$\Rightarrow 12x - 2y + 37 = -20x - 10y + 125$
$\Rightarrow 32x + 8y + 37 - 125 = 0$
$\Rightarrow 32x + 8y - 88 = 0$
$\Rightarrow 4x + y - 11 = 0$
$\Rightarrow y = -4x + 11$



$y = 0$
$\Rightarrow 0 = -4x + 11$
$\Rightarrow 4x = 11$
$\Rightarrow x = \frac{11}{4}$

Equation: $y = -4x + 11$

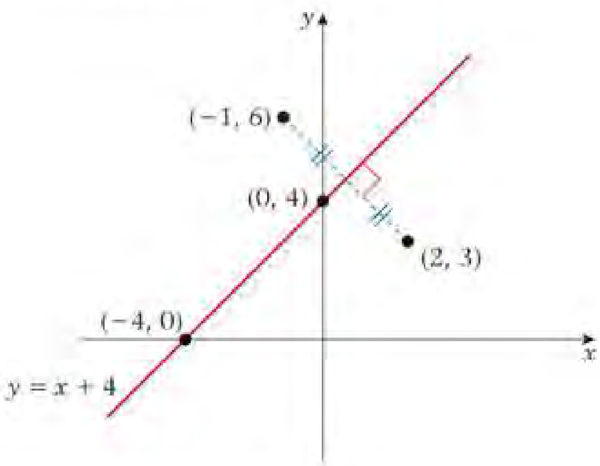
k $|z + 7 + 2\text{i}| = |z - 4 - 3\text{i}|$
$\Rightarrow |z - (-7 - 2\text{i})| = |z - (4 + 3\text{i})|$
perpendicular bisector of the line joining $(-7, -2)$ to $(4, 3)$.
So, $|x + \text{i}y + 7 + 2\text{i}| = |x + \text{i}y - 4 - 3\text{i}|$
$\Rightarrow |(x + 7) + \text{i}(y + 2)| = |(x - 4) + \text{i}(y - 3)|$
$\Rightarrow (x + 7)^2 + (y + 2)^2 = (x - 4)^2 + (y - 3)^2$
$\Rightarrow \cancel{x^2} + 14x + 49 + \cancel{y^2} + 4y + 4 = \cancel{x^2} - 8x + 16 + \cancel{y^2} - 6y + 9$
$\Rightarrow 14x + 4y + 53 = -8x - 6y + 25$
$\Rightarrow 22x + 10y + 28 = 0$
$\Rightarrow 11x + 5y + 14 = 0$
$\Rightarrow 5x = -11x - 14$
$\Rightarrow y = -\frac{11x}{5} - \frac{14}{5}$



when $x = 0$, $y = -\frac{14}{5}$
when $y = 0$; $0 = -11x - 14$
$14 = -11x$
$-\frac{14}{11} = x$

Equation: $y = -\frac{11x}{5} - \frac{14}{5}$

I $|z + 1 - 6i| = |2 + 3i - z|$
$\Rightarrow |z + 1 - 6i| = |(-1)(z - 2 - 3i)|$
$\Rightarrow |z + 1 - 6i| = |(-1)||z - 2 - 3i|$
$\Rightarrow |z - (-1 + 6i)| = |z - (2 + 3i)|$
perpendicular bisector of the line joining $(-1, 6)$ to $(2, 3)$.
So, $|x + iy + 1 - 6i| = |x + iy - 2 - 3i|$
$\Rightarrow |(x + 1) + i(y - 6)| = |(x - 2) + i(y - 3)|$
$\Rightarrow (x + 1)^2 + (y - 6)^2 = (x - 2)^2 + (y - 3)^2$
$\Rightarrow \cancel{x^2} + 2x + 1 + \cancel{y^2} - 12y + 36 = \cancel{x^2} - 4x + 4 + \cancel{y^2} - 6y + 9$
$\Rightarrow 2x - 12y + 37 = -4x - 6y + 13$
$\Rightarrow 6x - 6y + 24 = 0$
$\Rightarrow x - y + 4 = 0$
$\Rightarrow y = x + 4$



$y = 0$
$x = -4$

$x = 0, y = 4$

Equation: $y = x + 4$

Solutionbank FP2
Edexcel AS and A Level Modular Mathematics

Exercise F, Question 4

Question:

Find the Cartesian equation of the locus of z when:

a $z - z^* = 0$

b $z + z^* = 0$

Solution:

a $z - z^* = 0$

$\Rightarrow (x + \mathrm{i}y) - (x - \mathrm{i}y) = 0$

$\Rightarrow 2\mathrm{i}y = 0 \quad (\times \mathrm{i})$

$\Rightarrow -2y = 0$

$\Rightarrow y = 0$

$\boxed{\begin{array}{l} z = x + \mathrm{i}y \\ z^* = x - \mathrm{i}y \end{array}}$

The Cartesian equation of the locus of $z - z^* = 0$ is $y = 0$.

b $z + z^* = 0$

$\Rightarrow (x + \mathrm{i}y) + (x - \mathrm{i}y) = 0$

$\Rightarrow 2x = 0$

$x = 0$

$\boxed{\begin{array}{l} z = x + \mathrm{i}y \\ z^* = x - \mathrm{i}y \end{array}}$

The Cartesian equation of the locus of $z + z^* = 0$ is $x = 0$.

Solutionbank FP2
Edexcel AS and A Level Modular Mathematics

Exercise F, Question 5

Question:

Sketch the locus of z and give the Cartesian equation of the locus of z when:

a $|2 - z| = 3$ **b** $|5\text{i} - z| = 4$ **c** $|3 - 2\text{i} - z| = 3$

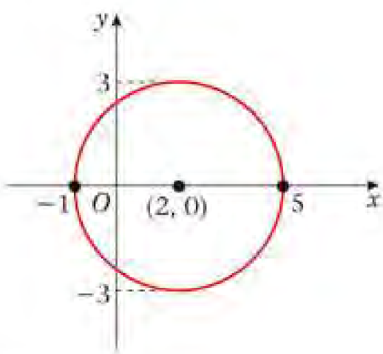
Solution:

a $|2 - z| = 3$

$\Rightarrow |(-1)(z - 2)| = 3$

$\Rightarrow |(-1)||(z - 2)| = 3$ — $\boxed{|-1| = 1}$

$\Rightarrow |(z - 2)| = 3$



circle centre $(2, 0)$, radius 3

equation: $(x - 2)^2 + y^2 = 3^2$

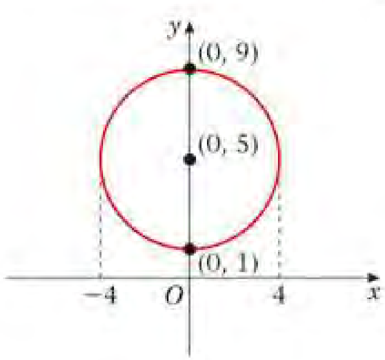
$(x - 2)^2 + y^2 = 9$

b $|5\text{i} - z| = 4$

$\Rightarrow |(-1)(z - 5\text{i})| = 4$

$\Rightarrow |(-1)||(z - 5\text{i})| = 4$

$\Rightarrow |(z - 5\text{i})| = 4$ — $\boxed{|-1| = 1}$



circle centre $(0, 5)$, radius 4

equation: $x^2 + (y - 5)^2 = 4^2$

$x^2 + (y - 5)^2 = 16$

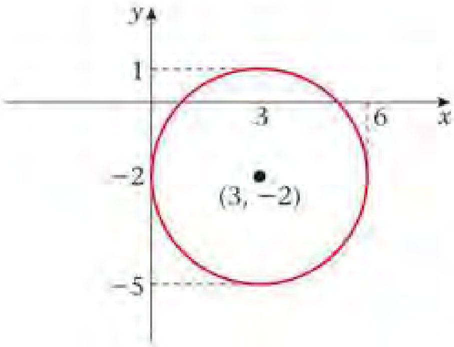
c $|3 - 2\mathrm{i} - z| = 3$

$\Rightarrow |(-1)(z - 3 + 2\mathrm{i})| = 3$

$\Rightarrow |(-1)||(z - 3 + 2\mathrm{i})| = 3$

$\Rightarrow |z - 3 + 2\mathrm{i}| = 3$ — $\boxed{|-1| = 1}$

$\Rightarrow |z - (3 - 2\mathrm{i})| = 3$



circle centre $(3, -2)$, radius 3

equation: $(x - 3)^2 + (y + 2)^2 = 3^2$

$(x - 3)^2 + (y + 2)^2 = 9$

Solutionbank FP2
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Exercise F, Question 6

Question:

Sketch the locus of z and give the Cartesian equation of the locus of z when:

a $|z+3| = 3|z-5|$

b $|z-3| = 4|z+1|$

c $|z-\mathrm{i}| = 2|z+\mathrm{i}|$

d $|z+2-7\mathrm{i}| = 2|z-10+2\mathrm{i}|$

e $|z+4-2\mathrm{i}| = 2|z-2-5\mathrm{i}|$

f $|z| = 2|2-z|$

Solution:

a $|z+3| = 3|z-5|$

$\Rightarrow |x + \mathrm{i}y + 3| = 3|x + \mathrm{i}y - 3|$

$\Rightarrow |(x+3) + \mathrm{i}y| = 3|(x-5) + \mathrm{i}y|$

$\Rightarrow |(x+3) + \mathrm{i}y|^2 = 3^2|(x-5) + \mathrm{i}y|^2$

$\Rightarrow (x+3)^2 + y^2 = 9[(x-5)^2 + y^2]$

$\Rightarrow x^2 + 6x + 9 + y^2 = 9[(x^2 - 10x + 25 + y^2]$

$\Rightarrow x^2 + 6x + 9 + y^2 = 9x^2 - 90x + 225 + 9y^2$

$\Rightarrow 0 = 8x^2 - 96x + 8y^2 + 216 \quad (\div 8)$

$\Rightarrow x^2 - 12x + y^2 + 27 = 0$

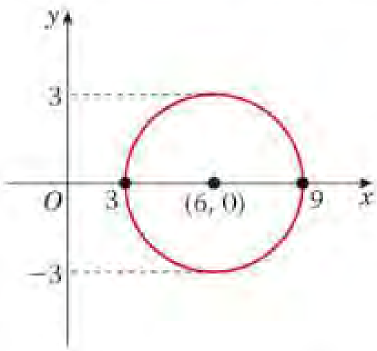
$\Rightarrow (x-6)^2 - 36 + y^2 + 27 = 0$

$\Rightarrow (x-6)^2 + y^2 - 9 = 0$

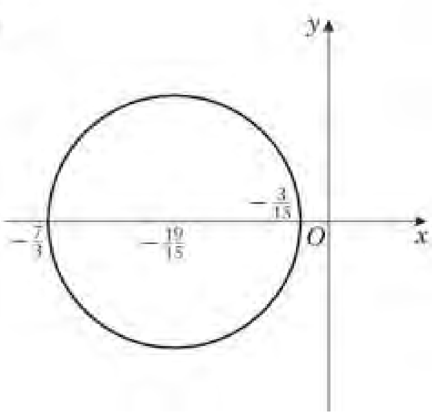
$\Rightarrow (x-6)^2 + y^2 = 9$

The Cartesian equation of the locus of z is $(x-6)^2 + y^2 = 9$

This is a circle centre $(6, 0)$, radius $= 3$



b



$|z-3| = 4|z+1|$

$|x + \mathrm{i}y - 3| = 4|x + \mathrm{i}y + 1|$

$|x - 3 + \mathrm{i}y|^2 = 16|x + 1 + \mathrm{i}y|^2$

$(x-3)^2 + y^2 = 16((x+1)^2 + y)^2$

$x^2 - 6x + 9 + y^2 = 16(x^2 + 2x + 1 + y^2)$

$= 16x^2 + 32x + 16 + 16y^2$

$15x^2 + 38x + 15y^2 + 7 = 0$

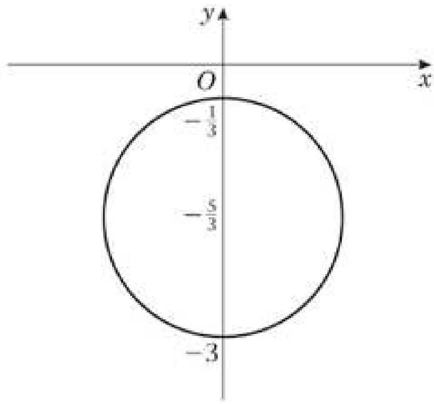
$x^2 + \frac{38}{15}x + y^2 + \frac{7}{15} = 0$

$\left(x + \frac{19}{15}\right)^2 - \frac{19^2}{15^2} + y^2 + \frac{7}{15} = 0$

$\left(x + \frac{19}{15}\right)^2 + y^2 = \frac{256}{225}$

Circle centre $\left(-\frac{19}{15}, 0\right)$ radius $\frac{16}{15}$

c



$|z - i| = 2|z + i|$

$|x + iy - i| = 2|x + iy + i|$

$|x + i(y - 1)|^2 = 4|x + i(y + 1)|^2$

$x^2 + (y - 1)^2 = 4[x^2 + (y + 1)^2]$

$x^2 + y^2 - 2y + 1 = 4(x^2 + y^2 + 2y)$

$= 4x^2 + 4y^2 + 8y$

$3x^2 + 3y^2 + 10y + 3 = 0$

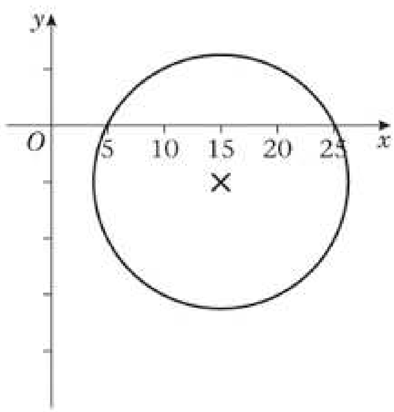
$x^2 + y^2 + \frac{10}{3}y + 1 = 0$

$x^2 + \left(y + \frac{5}{3}\right)^2 - \frac{25}{9} + 1 = 0$

$x^2 + \left(y + \frac{5}{3}\right)^2 = \frac{16}{9}$

Circle centre $\left(0, -\frac{5}{3}\right)$ radius $\frac{4}{3}$

d



$|z + 2 - 7i| = 2|z - 10 + 2i|$

$|x + iy + 2 - 7i| = 2|x + iy - 10 + 2i|$

$|(x + 2) + i(y - 7)|^2 = 4|(x - 10) + i(y + 2)|^2$

$(x + 2)^2 + (y - 7)^2 = 4[(x - 10)^2 + (y + 2)^2]$

$x^2 + 4x^2 + 4 + y^2 - 14y + 49 = [x^2 - 20x + 100$

$+ y^2 + 4y + 4]$

$3x^2 - 84x + 3y^2 + 30y + 363 = 0$

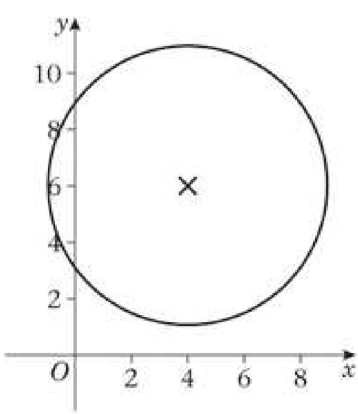
$x^2 - 28x + y^2 + 10y + 121 = 0$

$(x - 14)^2 - 14^2 + (y + 5)^2 - 5^2 + 121 = 0$

$(x - 14)^2 + (y + 5)^2 = 100$

Circle centre $(14, -5)$ radius 10

e



$|z + 4 - 2i| = 2|z - 2 - 5i|$

$|x + iy + 4 - 2i| = 2|x + iy - 2 - 5i|$

$|(x + 4) + i(y - 2)|^2 = 4|(x - 2) + i(y - 5)|^2$

$(x + 4)^2 + (y - 2)^2 = 4[(x - 2)^2 + (y - 5)^2]$

$x^2 + 8x^2 + 16 + y^2 - 4y + 4 = [x^2 - 4x + 4$

$+ y^2 + 10y + 25]$

$3x^2 - 24x + 3y^2 - 36y + 96 = 0$

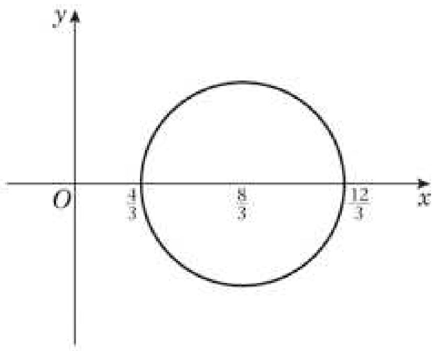
$x^2 - 8x + y^2 - 12y + 32 = 0$

$(x - 4)^2 - 16 + (y - 6)^2 - 36 + 32 = 0$

$(x - 4)^2 + (y - 6)^2 = 20$

Circle centre $(4, 6)$ radius $\sqrt{20} = 2\sqrt{5}$

f



$$|z| = 2|2 - z|$$
$$= 2|-1||z - 2|$$
$$|x + \mathrm{i}y| = 2 \times 1 \times |x + \mathrm{i}y - 2|$$
$$x^2 + y^2 = 4((x - 2)^2 + y^2)$$
$$x^2 + y^2 = 4(x^2 - 4x + 4 + y^2)$$
$$3x^2 - 16x + 3y^2 + 16 = 0$$
$$x^2 - \frac{16}{3}x + y^2 + \frac{16}{3} = 0$$
$$\left(x - \frac{8}{3}\right)^2 - \frac{64}{9} + y^2 + \frac{16}{3} = 0$$
$$\left(x - \frac{8}{3}\right)^2 + y^2 = \frac{16}{9}$$

Circle centre $\left(\frac{8}{3}, 0\right)$ radius $\frac{4}{3}$

Solutionbank FP2
Edexcel AS and A Level Modular Mathematics

Exercise F, Question 7

Question:

Sketch the locus of z when:

a $\arg\left(\frac{z}{z+3}\right) = \frac{\pi}{4}$

b $\arg\left(\frac{z-3i}{z+4}\right) = \frac{\pi}{6}$

c $\arg\left(\frac{z}{z-2}\right) = \frac{\pi}{3}$

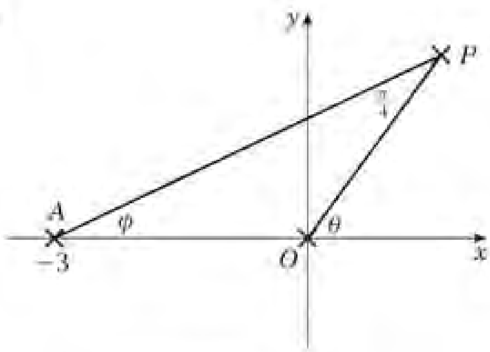
d $\arg\left(\frac{z-3i}{z-5}\right) = \frac{\pi}{4}$

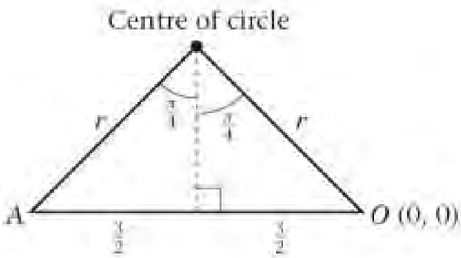
e $\arg z - \arg(z-2+3i) = \frac{\pi}{3}$

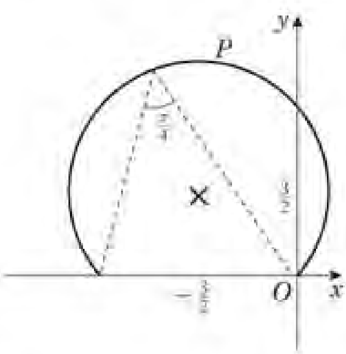
f $\arg\left(\frac{z-4i}{z+4}\right) = \frac{\pi}{2}$

Solution:

a







$$\arg\left(\frac{z}{z+3}\right) = \frac{\pi}{4}$$

$$\arg z - \arg(z+3) = \frac{\pi}{4}$$

$$\arg z - \arg(z-(-3)) = \frac{\pi}{4}$$

$$\arg z = \theta$$

$$\arg(z-(-3)) = \phi$$

$$\theta - \phi = \frac{\pi}{4}$$

$$\theta = \phi + \frac{\pi}{4}$$

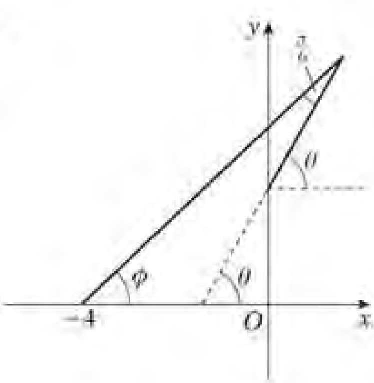
P lies on an arc of a circle cut off at $A(-3, 0)$ and $O(0, 0)$

Angle at the centre is twice the angle at the circumference $\therefore \frac{\pi}{2}$

It follows that the centre is at $\left(-\frac{3}{2}, \frac{3}{2}\right)$

and the radius is $\frac{3}{2}\sqrt{2}$

b



$$\arg\left(\frac{z-3i}{z+4}\right) = \frac{\pi}{6}$$

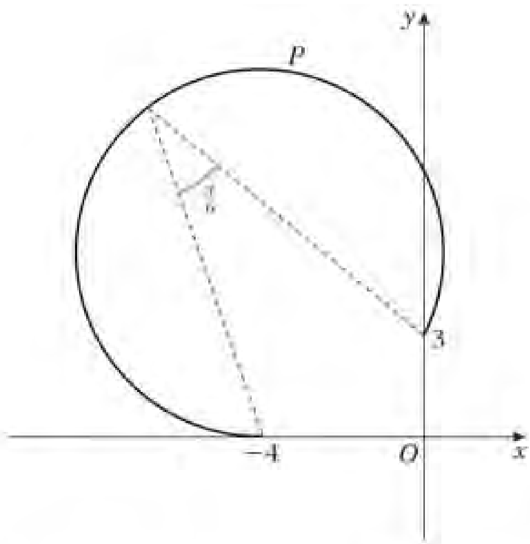
$$\arg(z-3i) - \arg(z-(-4)) = \frac{\pi}{6}$$

$$\arg(z-3i) = \theta.$$

$$\arg(z-(-4)) = \phi$$

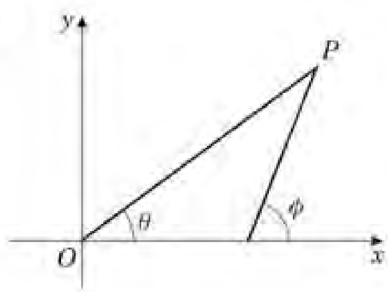
$$\theta - \phi = \frac{\pi}{6}$$

Arc of a circle from $(-4, 0)$ to $(0, 3)$



(The centre is at $\left(-\frac{4+3\sqrt{3}}{2}, \frac{3+4\sqrt{3}}{2}\right)$ you do not need to calculate this for a sketch!)

c



$\arg\left(\frac{z}{z-2}\right) = \frac{\pi}{3}$

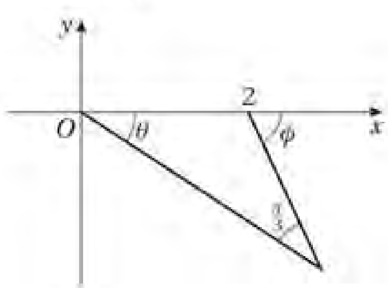
$\arg z = \theta$

$\arg(z-2) = \phi$

$\theta - \phi = \frac{\pi}{3}$

As our diagram has $\phi > \theta$, we have P on the wrong side of the line joining O or ϕ.
We want the arc below the x-axis.

Redrawing:



$\arg z = -\theta$

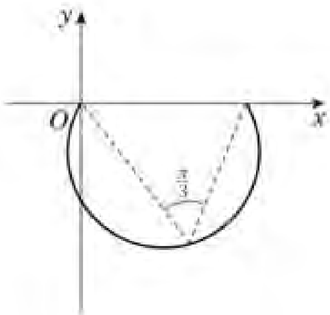
$\arg(z-2) = -\phi$

Hence $\arg z - \arg(z-2) = \frac{\pi}{3}$

becomes $-\theta - (-\phi) = \frac{\pi}{3}$

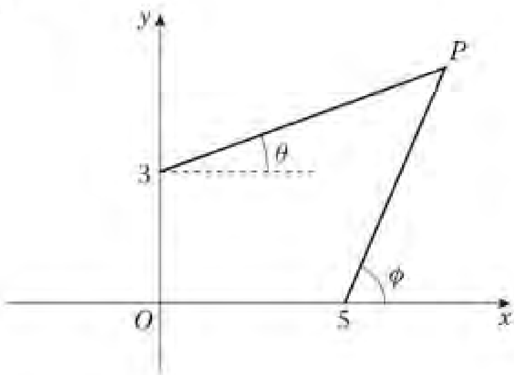
$\phi = \theta + \frac{\pi}{3}$

Arc of a circle, ends 0 and 2, subtending angle $\frac{\pi}{3}$



[The centre is at $\left(1, -\frac{1}{\sqrt{3}}\right)$ radius $\frac{2\sqrt{3}}{3}$ not needed to be calculated for a sketch]

d



$\arg\left(\frac{z-3i}{z-5}\right) = \frac{\pi}{4}$

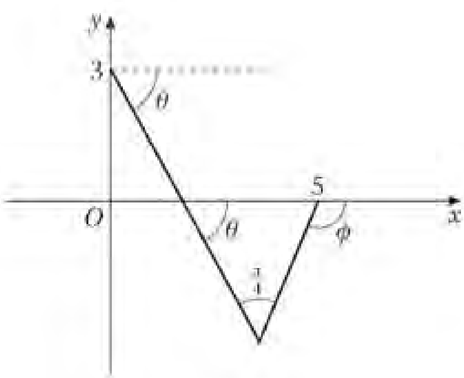
$\arg(z-3i) - \arg(z-5) = \frac{\pi}{4}$

$\arg(z-3i) = \theta$

$\arg(z-5) = \phi$

$\theta - \phi = \frac{\pi}{4}$

But $\phi > \theta$, we have P on the wrong side of the line joining 3i and 5.

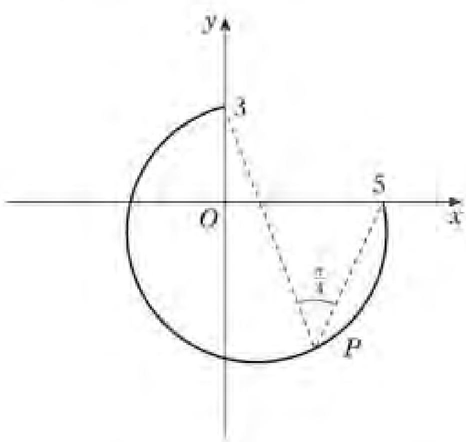


$\arg(z-3i) = -\theta$

$\arg(z-5) = -\phi$

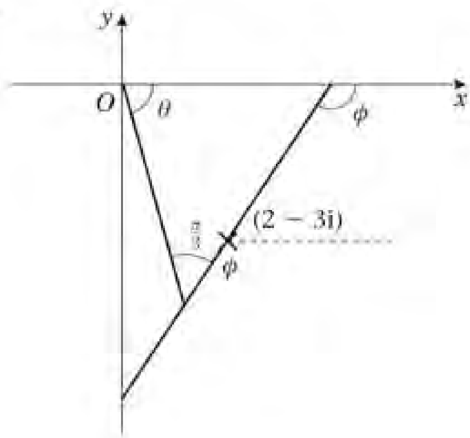
$-\theta - (-\phi) = \frac{\pi}{4}$

$\phi = \theta + \frac{\pi}{4}$



(Arc of Circle centre $(1, -1)$ radius $\sqrt{17}$ not needed for sketch)

e



$\arg z - \arg(z - 2 + 3\mathrm{i}) = \frac{\pi}{3}$

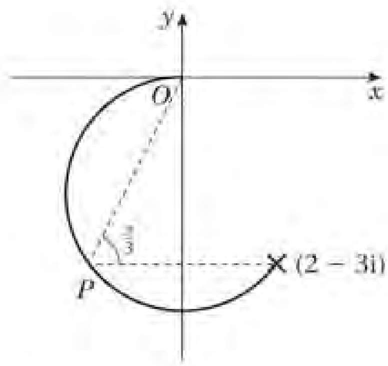
$\arg z - \arg(z - (2 - 3\mathrm{i})) = \frac{\pi}{3}$

$\arg z = -\theta$

$\arg(z - (2 - 3\mathrm{i})) = -\phi$

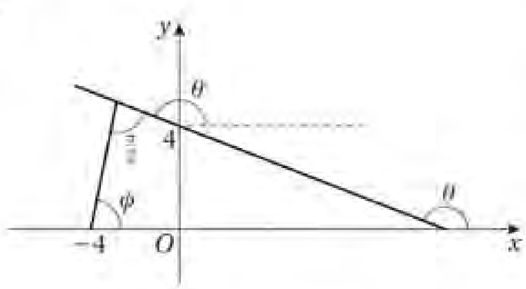
$-\theta - (-\phi) = \frac{\pi}{3}$

$\phi = \theta + \frac{\pi}{3}$



Arc of circle, centre at $\left(\frac{2-\sqrt{3}}{2}, -\frac{9+2\sqrt{3}}{6}\right)$, this need not be calculated for your sketch.

f



$\arg\left(\frac{z - 4\mathrm{i}}{z + 4}\right) = \frac{\pi}{2}$

$\arg(z - 4\mathrm{i}) - \arg(z + 4) = \frac{\pi}{2}$

$\arg(z - 4\mathrm{i}) = \theta$

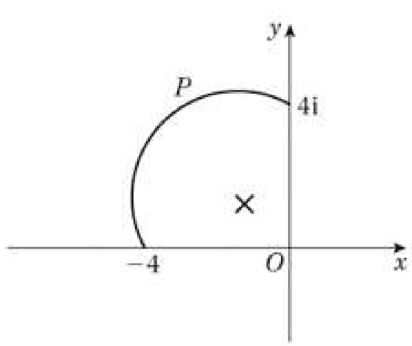
$\arg(z + 4) = \phi = \arg(z - (-4\mathrm{i}))$

$\theta - \phi = \frac{\pi}{2}$

$\theta = \phi + \frac{\pi}{2}$

The locus is an arc of a circle, ends at -4 and $4\mathrm{i}$, angle subtended being $\frac{\pi}{2}$.

$\therefore$ It is a semi-circle.



(Circle arc has centre $(-2, 2)$, radius $2\sqrt{2}$)

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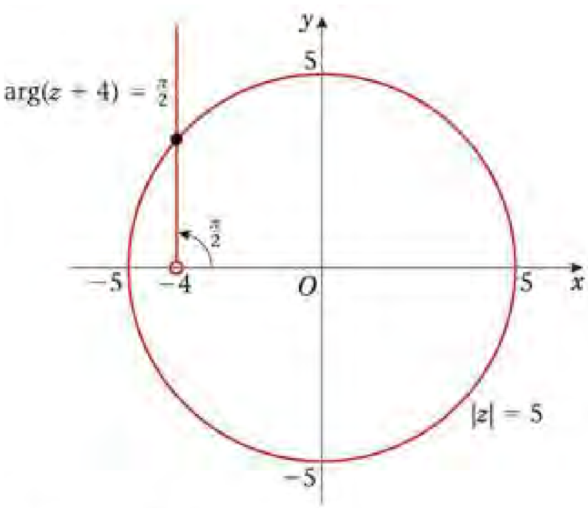
Edexcel AS and A Level Modular Mathematics

Exercise F, Question 8

Question:

Use the Argand diagram to find the value of z that satisfies the equations $|z| = 5$ and $\arg(z + 4) = \frac{\pi}{2}$.

Solution:



$|z| = 5$
is a circle centre (0, 0), radius 5

$\arg(z + 4) = \frac{\pi}{2}$ is a half-line from $(-4, 0)$ making an angle of $\frac{\pi}{2}$ with the positive x-axis.

$|z| = 5 \Rightarrow x^2 + y^2 = 25$ ①

$\arg(z + 4) = \frac{\pi}{2} \Rightarrow x = -4$ and $y > 0$ ②

Substituting ② into ① gives
$$(-4)^2 + y^2 = 25$$
$$16 + y^2 = 25$$
$$y^2 = 9$$
$$y = 3$$

Therefore, $z = -4 + 3i$

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Exercise F, Question 9

Question:

Given that the complex number z satisfies $|z - 2 - 2i| = 2$,

a sketch, on an Argand diagram, the locus of z.

Given further that $\arg(z - 2 - 2i) = \frac{\pi}{6}$,

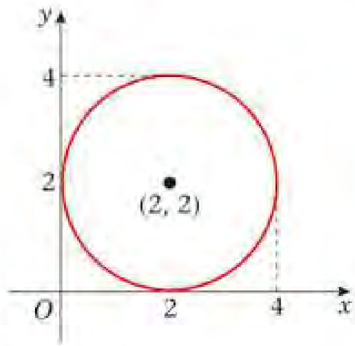
b find the value of z in the form $a + ib$, where $a \in \mathbb{R}$ and $b \in \mathbb{R}$.

Solution:

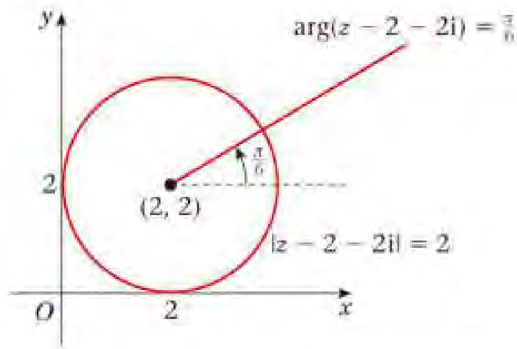
a $|z - 2 - 2i| = 2$

$\Rightarrow |z - (2 + 2i)| = 2$

The locus of z is a circle centre (2, 2), radius 2.



b $\arg(z - 2 - 2i) = \frac{\pi}{6}$, is a half-line from (2, 2), as shown below.



$|z - 2 - 2i| = 2 \Rightarrow (x - 2)^2 + (y - 2)^2 = 4$ ①

$$\arg(z - 2 - 2i) = \frac{\pi}{6} \Rightarrow \arg(x + iy - 2 - 2i) = \frac{\pi}{6}$$

$$\Rightarrow \arg((x - 2) + i(y - 2)) = \frac{\pi}{6}$$

$$\Rightarrow \frac{y - 2}{x - 2} = \tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}}$$

$$\Rightarrow y - 2 = \frac{1}{\sqrt{3}}(x - 2)$$

$$\Rightarrow (y - 2)^2 = \left[\frac{1}{\sqrt{3}}(x - 2)\right]^2$$

$\Rightarrow (y - 2)^2 = \frac{1}{3}(x - 2)^2$ ②

Substituting ② into ① gives $(x-2)^2 + \frac{1}{3}(x-2)^2 = 4$

$\Rightarrow \frac{4}{3}(x-2)^2 = 4$

$\Rightarrow 4(x-2)^2 = 12$

$\Rightarrow (x-2)^2 = 3$

$\Rightarrow x - 2 = \pm\sqrt{3}$

$\Rightarrow x = 2 \pm \sqrt{3}$

From the Argand diagram, $x > 2$.

So $x = 2 + \sqrt{3}$ ③

As $y - 2 = \frac{1}{\sqrt{3}}(x - 20$ ④

Substituting ③ into ④ gives $y - 2 = \frac{1}{\sqrt{3}}(2 + \sqrt{3} - 2)$

$\Rightarrow y - 2 = \frac{1}{\sqrt{3}}(\sqrt{3})$

$\Rightarrow y - 2 = 1$

$\Rightarrow y = 3$

Therefore, $z = (2 + \sqrt{3}) + 3\text{i}$

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Exercise F, Question 10

Question:

Sketch on the same Argand diagram the locus of points satisfying

a $|z - 2i| = |z - 8i|$, **b** $\arg(z - 2 - i) = \frac{\pi}{4}$.

The complex number z satisfies both $|z - 2i| = |z - 8i|$ and $\arg(z - 2 - i) = \frac{\pi}{4}$.

c Use your answers to parts **a** and **b** to find the value of z.

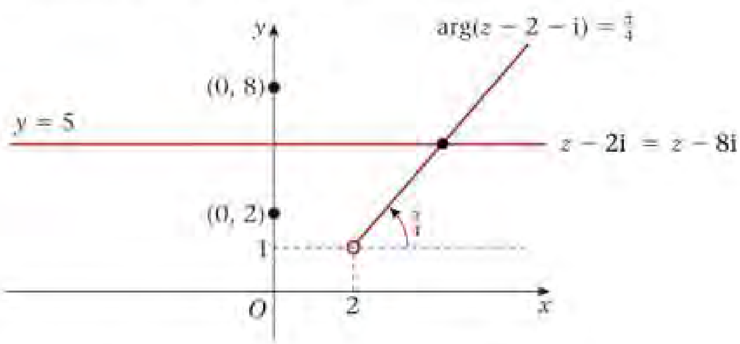
Solution:

a $|z - 2i| = |z - 8i|$

perpendicular bisector of the line joining $(0, 2)$ to $(0, 8)$, having equation $y = 5$.

b $\arg(z - 2 - i) = \frac{\pi}{4}$

is a half-line from $(1, 1)$, as shown below.



c $|z - 2i| = |z - 8i| \Rightarrow y = 5$ ①

$$\arg(z - 2 - i) = \frac{\pi}{4} \Rightarrow \arg(x + iy - 2 - i) = \frac{\pi}{4}$$

$$\Rightarrow \arg((x - 2) + i(y - 1)) = \frac{\pi}{4}$$

$$\Rightarrow \frac{y - 1}{x - 2} = \tan\frac{\pi}{4}$$

$$\Rightarrow \frac{y - 1}{x - 2} = 1$$

$$\Rightarrow y - 1 = x - 2$$

$$\Rightarrow y - x - 1 \quad ②$$

Substituting ① into ② gives $5 = x - 1$

$$\Rightarrow 6 = x$$

Therefore, $z = 6 + 5i$

Solutionbank FP2

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Exercise F, Question 11

Question:

Sketch on the same Argand diagram the locus of points satisfying

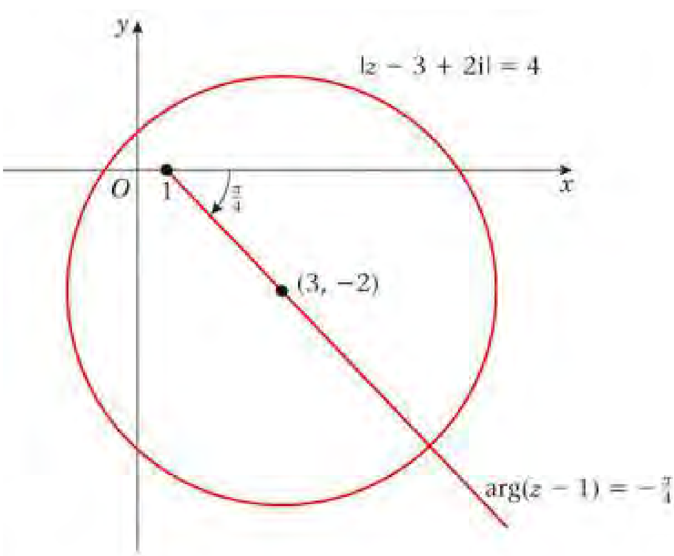
a $|z - 3 + 2i| = 4$ **b** $\arg(z - 1) = -\frac{\pi}{4}$.

The complex number z satisfies both $|z - 3 + 2i| = 4$ and $\arg(z - 1) = -\frac{\pi}{4}$.

Given that $z = a + ib$ where $a \in \mathbb{R}$ and $b \in \mathbb{R}$,

c find the exact value of a and the exact value of b.

Solution:



a $|z - 3 + 2i| = 4$
is a circle centre $(3, -2)$ radius 4.

b $\arg(z - 1) = -\frac{\pi}{4}$
is a half-line from $(1, 0)$ making an angle of $-\frac{\pi}{4}$ with the positive x-axis.

c $|z - 3 + 2i| = 4 \Rightarrow (x - 3)^2 + (y + 2)^2 = 16$ ①

$$\arg(z - 1) = -\frac{\pi}{4} \Rightarrow \arg(x + iy - 1) = -\frac{\pi}{4}$$
$$\Rightarrow \arg((x - 1) + iy) = -\frac{\pi}{4}$$
$$\Rightarrow \frac{y}{x - 1} = \tan\left(-\frac{\pi}{4}\right)$$
$$\Rightarrow \frac{y}{x - 1} = -1$$
$$\Rightarrow y = -1(x - 1)$$
$$\Rightarrow y = -x + 1$$ ② for $x > 1, y < 0$

Substituting ② into ① gives $(x-3)^2 + (-x+1+2)^2 = 16$

$\Rightarrow (x-3)^2 + (-x+3)^2 = 16$

$\Rightarrow x^2 - 6x + 9 + x^2 - 6x + 9 = 16$

$\Rightarrow 2x^2 - 12x + 18 = 16$

$\Rightarrow 2x^2 - 12x + 2 = 0$

$\Rightarrow x^2 - 6x + 1 = 0$

$$\Rightarrow x = \frac{6 \pm \sqrt{36 - 4(1)(1)}}{2}$$

$$\Rightarrow x = \frac{6 \pm \sqrt{32}}{2}$$

$$\Rightarrow x = \frac{6 \pm \sqrt{16}\sqrt{2}}{2}$$

$$\Rightarrow x = \frac{6 \pm 4\sqrt{2}}{2}$$

$\Rightarrow x = 3 \pm 2\sqrt{2}$

as $x > 1$ then $x = 3 \pm 2\sqrt{2}$

② $\Rightarrow y = -(3 + 2\sqrt{2}) + 1$

$\Rightarrow y = -3 - 2\sqrt{2} + 1$

$\Rightarrow y = -2 - 2\sqrt{2}$

Therefore, $z = (3 + 2\sqrt{2}) + (-2 - 2\sqrt{2})\mathrm{i}$

$\boxed{\text{Note: } z = a + \mathrm{i}b}$

So $a = 3 + 2\sqrt{2}$, $b = -2 - 2\sqrt{2}$

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Exercise F, Question 12

Question:

On an Argand diagram the point P represents the complex number z.
Given that $|z - 4 - 3\text{i}| = 8$,

a find the Cartesian equation for the locus of P,

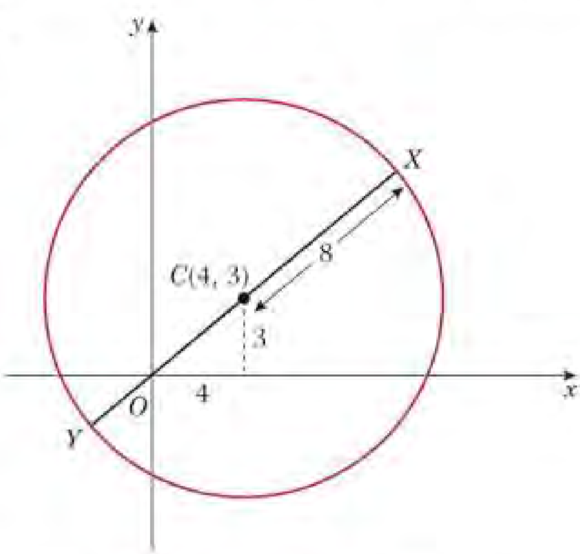
b sketch the locus of P,

c find the maximum and minimum values of $|z|$ for points on this locus.

Solution:

a $|z - 4 - 3\text{i}| = 8 \Rightarrow |z - (4 + 3\text{i})| = 8$
circle centre $(4, 3)$, radius 8
Hence the Cartesian equation of the locus of P is $(x - 4)^2 + (y - 3)^2 = 64$

b



c $|z|$ is the distance from $(0, 0)$ to the locus of points.
$|z|_{\max}$ is the distance OX.
$|z|_{\min}$ is the distance OY.
Note radius $= CY = CX = 8$
and $OC = \sqrt{4^2 + 3^2} = \sqrt{25} = 5$
$|z|_{\max} = OC + CX = 5 + 8 = 13$
$|z|_{\min} = CY - OC = 8 - 5 = 3$

The maximum value of $|z|$ is 13 and the minimum value of $|z|$ is 3.

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Exercise F, Question 13

Question:

On an Argand diagram the point P represents the complex number z.
Given that $|z - 4 - 3i| = 8$,

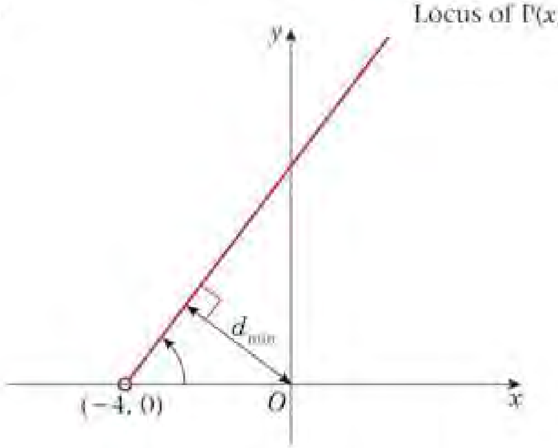
a find the Cartesian equation for the locus of P,

b sketch the locus of P,

c find the maximum and minimum values of $|z|$ for points on this locus.

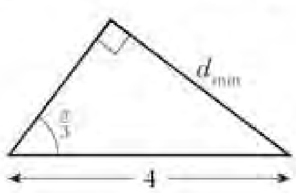
Solution:

a



b $|z|$ is the distance from (0, 0) to the locus of points.
Marked as d_{min} on the Argand diagram is the minimum value of $|z|$.

Hence,



$$\frac{d_{min}}{4} = \sin\left(\frac{\pi}{3}\right)$$

$$d_{min} = 4\sin\left(\frac{\pi}{3}\right)$$

$$d_{min} = \frac{4\sqrt{3}}{2} = 2\sqrt{3}$$

Hence the minimum value of $|z|$ is $|z|_{min} = 2\sqrt{3}$.

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Exercise F, Question 14

Question:

The complex number $z = x + \mathrm{i}y$ satisfies the equation $|z + 1 + \mathrm{i}| = 2|z + 4 - 2\mathrm{i}|$.

The complex number z is represented by the point P on the Argand diagram.

a Show that the locus of P is a circle with centre $(-5, 3)$.

b Find the exact radius of this circle.

Solution:

a $|z + 1 + \mathrm{i}| = 2|z + 4 - 2\mathrm{i}|$

$\Rightarrow |x + \mathrm{i}y + 1 + \mathrm{i}| = 2|x + \mathrm{i}y + 4 - 2\mathrm{i}|$

$\Rightarrow |(x + 1) + \mathrm{i}(y + 1)| = 2|(x + 4) + \mathrm{i}(y - 2)|$

$\Rightarrow |(x + 1) + \mathrm{i}(y + 1)|^2 = 2^2|(x + 4) + \mathrm{i}(y - 2)|^2$

$\Rightarrow (x + 1)^2 + (y + 1)^2 = 4[(x + 4)^2 + (y - 2)^2]$

$\Rightarrow x^2 + 2x + 1 + y^2 + 2y + 1 = 4[x^2 + 8x + 16 + y^2 - 4y + 4]$

$\Rightarrow x^2 + 2x + 1 + y^2 + 2y + 1 = 4x^2 + 32x + 64 + 4y^2 - 16y + 16$

$\Rightarrow 0 = 3x^2 + 30x + 3y^2 - 18y + 64 + 16 - 1 - 1$

$\Rightarrow 3x^2 + 30x + 3y^2 - 18y + 78 = 0$

$\Rightarrow x^2 + 10x + y^2 - 6y + 26 = 0$

$\Rightarrow (x + 5)^2 - 25 + (y - 3)^2 - 9 + 26 = 0$

$\Rightarrow (x + 5)^2 + (y - 3)^2 = 25 + 9 - 26$

$\Rightarrow (x + 5)^2 + (y - 3)^2 = 8$

Therefore the locus of P is a circle centre $(-5, 3)$. (as required)

b radius $= \sqrt{8} = \sqrt{4}\sqrt{2} = 2\sqrt{2}$

The exact radius is $2\sqrt{2}$.

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Exercise F, Question 15

Question:

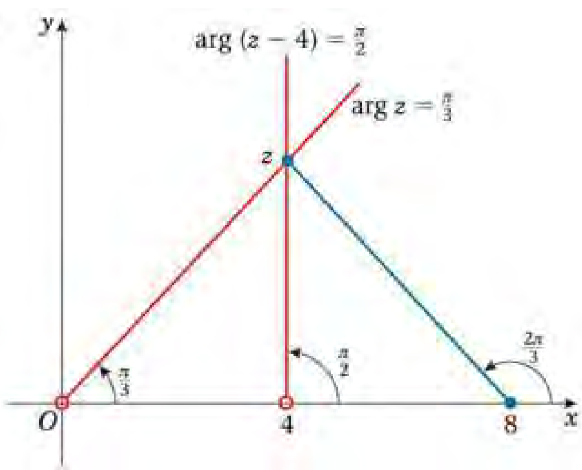
If the complex number z satisfies both $\arg z = \frac{\pi}{3}$ and $\arg(z-4) = \frac{\pi}{2}$,

a find the value of z in the form $a + \mathrm{i}b$, where $a \in \mathbb{R}$ and $b \in \mathbb{R}$.

b Hence, find $\arg(z-8)$.

Solution:

a



From part **b** $\arg(z-8) = \frac{2\pi}{3}$.

$$\arg z = \frac{\pi}{3} \Rightarrow \arg(x + \mathrm{i}y) = \frac{\pi}{3}$$

$$\Rightarrow \frac{y}{x} = \tan\frac{\pi}{3}$$

$$\Rightarrow \frac{y}{x} = \sqrt{3}$$

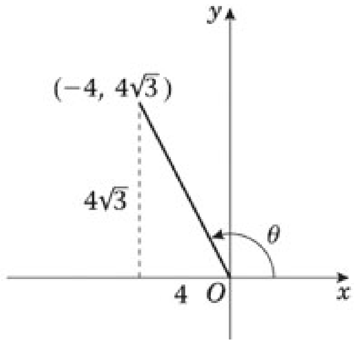
$$\Rightarrow y = \sqrt{3}x \text{ (for } x > 0, y > 0) \qquad ①$$

$$\arg(z-4) = \frac{\pi}{2} \Rightarrow x = 4 \text{ (for } y > 0) \qquad ②$$

Substituting ② and ① gives $y = \sqrt{3}(4) = 4\sqrt{3}$

The value of z satisfying both equations is $z = 4 + 4\sqrt{3}\,\mathrm{i}$.

b $\arg(z - 8\text{i}) = \arg(4 + 4\sqrt{3}\,\text{i} - 8)$
$= \arg(-4 + 4\sqrt{3}\,\text{i}) = \theta$



$\therefore \theta = \pi - \tan^{-1}\left(\frac{4\sqrt{3}}{4}\right) = \pi - \frac{\pi}{3}$

$\theta = \frac{2\pi}{3}$

Therefore, $\arg(z - 8) = \frac{2\pi}{3}$.

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Exercise F, Question 16

Question:

The point P represents a complex number z in an Argand diagram.

Given that $|z + 2 - 2\sqrt{3}i| = 2$,

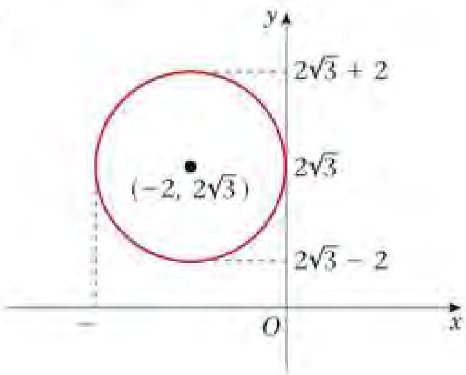
a sketch the locus of P on an Argand diagram.

b Write down the minimum value of arg z.

c Find the maximum value of arg z.

Solution:

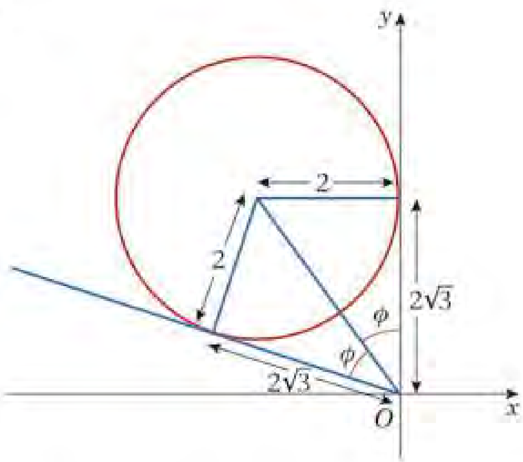
a



$|z + 2 - 2\sqrt{3}\,i| = 2$ is a circle centre $(-2, 2\sqrt{3})$, radius 2.

b From the diagram, the minimum value of $\arg(z)$ is $\frac{\pi}{2}$.

c



The maximum value of $\arg z$ is $\frac{\pi}{2} + \phi + \phi = \frac{\pi}{2} + 2\phi$.

$\tan\phi = \frac{2}{2\sqrt{3}}$

$\Rightarrow \tan\phi = \frac{1}{\sqrt{3}}$

$\Rightarrow \phi = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$

$\arg(z)_{\max} = \frac{\pi}{2} + 2\left(\frac{\pi}{6}\right) = \frac{5\pi}{6}$.

The maximum value of $\arg(z)$ is $\frac{5\pi}{6}$.

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Exercise F, Question 17

Question:

The point P represents a complex number z in an Argand diagram.

Given that $\arg z - \arg(z + 4) = \frac{\pi}{4}$ is a locus of points P lying on an arc of a circle C,

a sketch the locus of points P,

b find the coordinates of the centre of C,

c find the radius of C,

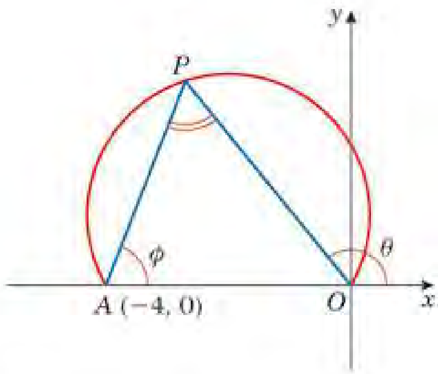
d find a Cartesian equation for the circle C,

e find the finite area bounded by the locus of P and the x-axis.

Solution:

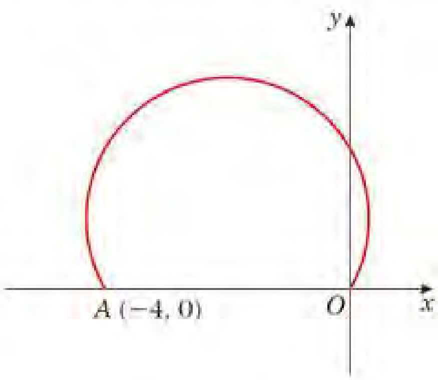
a $\arg(z) - \arg(z+4) = \frac{\pi}{4}$

$\Rightarrow \theta - \phi = \frac{\pi}{4}$, where $\arg(z) = \theta$ and $\arg(z+4) = \phi$

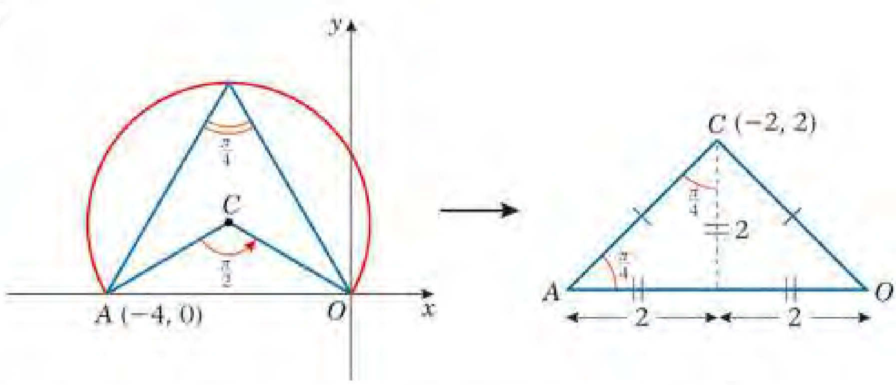


from $\triangle AOP$,
$A\hat{P}O + \phi = \theta$
$\Rightarrow A\hat{P}O = \theta - \phi$
$\Rightarrow A\hat{P}O = \frac{\pi}{4}$

The locus of points P is an arc of a circle cut off at $(-4, 0)$ and $(0, 0)$, as shown below.



b



Therefore the centre of the circle has coordinates $(-2, 2)$.

c $r = \sqrt{2^2 + 2^2} = \sqrt{8} = \sqrt{4}\sqrt{2} = 2\sqrt{2}$

Therefore, the radius of C is $2\sqrt{2}$.

d The Cartesian equation of C is $(x+2)^2 + (y-2)^2 = 8$.

e Finite area = Area of major sector ACO + Area $\triangle ACO$

$$= \frac{1}{2}(\sqrt{8})^2\left(2\pi - \frac{\pi}{2}\right) + \frac{1}{2}(4)(2)$$

$$= \frac{1}{2}(8)\left(2\pi - \frac{\pi}{2}\right) + 4$$

$$= 4\left(\frac{3\pi}{2}\right) + 4$$

$$= 6\pi + 4$$

Finite area bounded by the locus of P and the x-axis is $6\pi + 4$.

b, **c**, **d** Method ②:

$$\arg z - \arg(z + 4) = \arg\left(\frac{z}{z+4}\right)$$

$$= \arg\left(\frac{x + iy}{x + iy + 4}\right)$$

$$= \arg\left[\frac{x + iy}{(x + 4) + iy}\right]$$

$$= \arg\left[\frac{(x + iy)}{(x + 4) + iy} \times \frac{(x + 4) - iy}{(x + 4) - iy}\right]$$

$$= \arg\left[\frac{x(x + 4) - iyx + iy(x + 4) + y^2}{(x + 4)^2 + y^2}\right]$$

$$= \arg\left[\left(\frac{x(x + 4) + y^2}{(x + 4)^2 + y^2}\right) + i\left(\frac{y(x + 4) - yx}{(x + 4)^2 + y^2}\right)\right]$$

$$= \arg\left[\left(\frac{x^2 + 4x + y^2}{(x + 4)^2 + y^2}\right) + i\left(\frac{xy + 4y - xy}{(x + 4)^2 + y^2}\right)\right]$$

$$= \arg\left[\left(\frac{x^2 + 4x + y^2}{(x + 4)^2 + y^2}\right) + i\left(\frac{4y}{(x + 4)^2 + y^2}\right)\right]$$

Applying $\arg\left(\frac{z}{z+4}\right) = \frac{\pi}{4} \Rightarrow \dfrac{\left(\frac{4y}{(x + 4)^2 + y^2}\right)}{\left(\frac{x^2 + 4x + y^2}{(x + 4)^2 + y^2}\right)} = \tan\left(\frac{\pi}{4}\right) = 1$

$$\Rightarrow \frac{4y}{x^2 + 4x + y^2} = 1$$

$$\Rightarrow 4y = x^2 + 4x + y^2$$

$$\Rightarrow 0 = x^2 + 4x + y^2 - 4y$$

$$\Rightarrow (x + 2)^2 - 4 + (y - 2)^2 - 4 = 0$$

$$\Rightarrow (x + 2)^2 + (y - 2)^2 = 8$$

$$\Rightarrow (x + 2)^2 + (y - 2)^2 = (2\sqrt{2})^2$$

C is a circle with centre $(-2, 2)$, radius $2\sqrt{2}$ and has Cartesian equation $(x + 2)^2 + (y - 2)^2 = 8$.

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Exercise G, Question 1

Question:

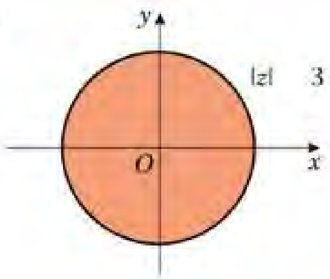
On an Argand diagram shade in the regions represented by the following inequalities:

a $|z| < 3$ **b** $|z - 2i| > 2$ **c** $|z + 7| \geqslant |z - 1|$ **d** $|z + 6| > |z + 2 + 8i|$

e $2 \leqslant |z| \leqslant 3$??? **f** $1 \leqslant |z + 4i| \leqslant 4$ **g** $3 \leqslant |z - 3 + 5i| \leqslant 5$ **h** $2|z|\ .\ |z - 3|$

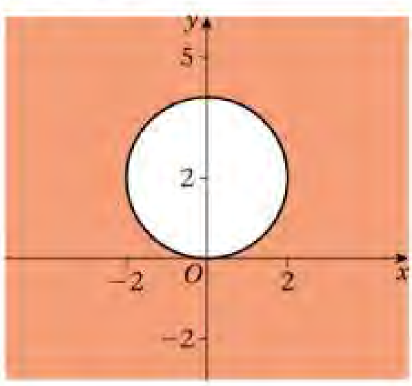
Solution:

a

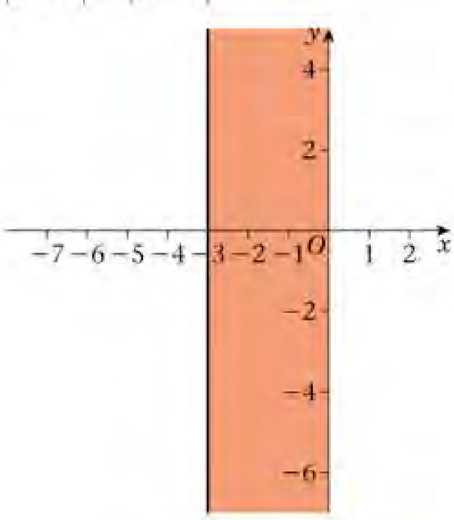


$|z| = 3$ represents a circle centre $(0, 0)$, radius 3

b $|z - (2i)| > 2$

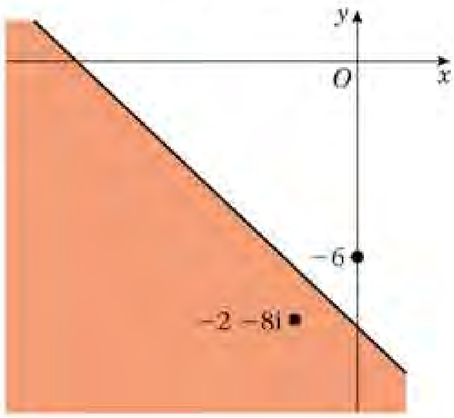


c $|z + 7| \leqslant |z - 1|$



$|z + 7| = |z - 1|$ represents a perpendicular bisector of the line joining $(-7, 0)$ to $(1, 0)$ which has equation $x = -3$.

d $|z + 6| > |z + 2 + 8\text{i}|$



$|z + 6| = |z + 2 + 8\text{i}|$ represents a perpendicular bisector of the line joining $(-6, 0)$ to $(-2, -8)$.

$|x + \text{i}y + 6| = |x + \text{i}y + 2 + 8\text{i}|$

$\Rightarrow |x + 6 + \text{i}y| = |(x + 2) + \text{i}(y + 8)|$

$\Rightarrow |(x + 6) + \text{i}y|^2 = |(x + 2) + \text{i}(y + 8)|^2$

$\Rightarrow (x + 6)^2 + y^2 = (x + 2)^2 + (y + 8)^2$

$\Rightarrow x^2 + 12x + 36 + y^2 = x^2 + 4x + 4 + y^2 + 16y + 64$

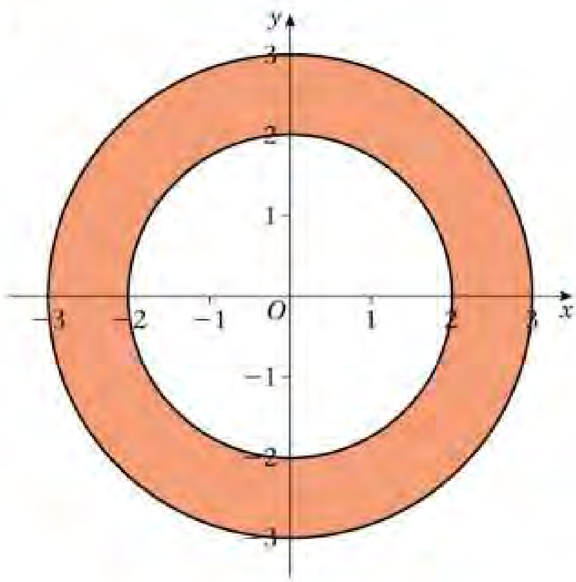
$\Rightarrow (2x + 36 = 4x + 16y + 68$

$\Rightarrow 8x + 36 - 68 = 16y$

$\Rightarrow 8x - 32 = 16y$

$\Rightarrow y = \frac{1}{2}x - 2$

e

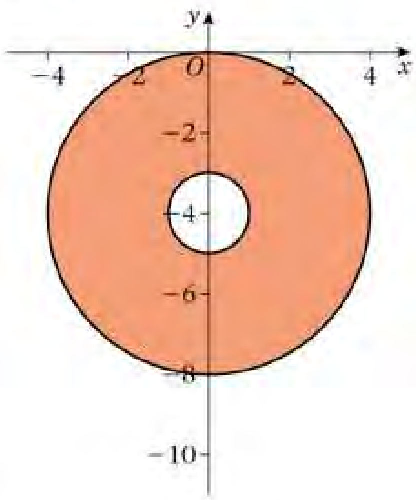


$2 \leqslant |z| \leqslant 3$

$|z| = 2$ represents a circle centre $(0, 0)$, radius 2

$|z| = 3$ represents a circle centre $(0, 0)$, radius 3

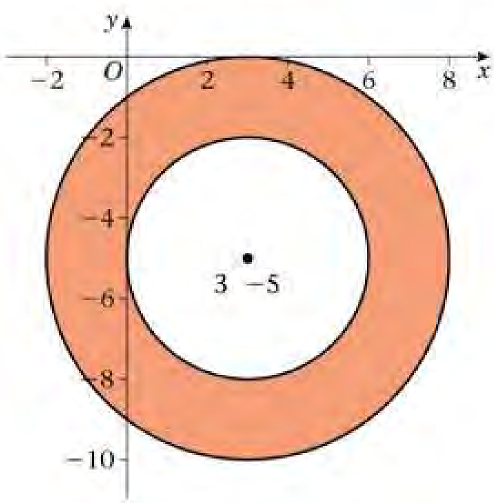
f $1 \leqslant |z + 4\text{i}|$



$|z + 4\text{i}| = 1$ represents a circle centre (0, 4), radius 1.

$|z + 4\text{i}| = 4$ represents a circle centre (0, −4), radius 4.

g $3 \leqslant |z - 3 + 5\text{i}| \leqslant 5$



h $2|z| \geqslant |z - 3|$

Consider $2|z| = |z - 3|$ let $z = x + \text{i}y$

$2|x + \text{i}y| = |x + \text{i}y - 3|$

$4(x^2 + y^2) = (x - 3)^2 + y^2$

$4x^2 + 4y^2 = x^2 - 6x + 9 + y^2$

$3x^2 + 6x + 3y^2 - 9 = 0$

$(x + 1)^2 - 1 + y^2 - 3 = 0$

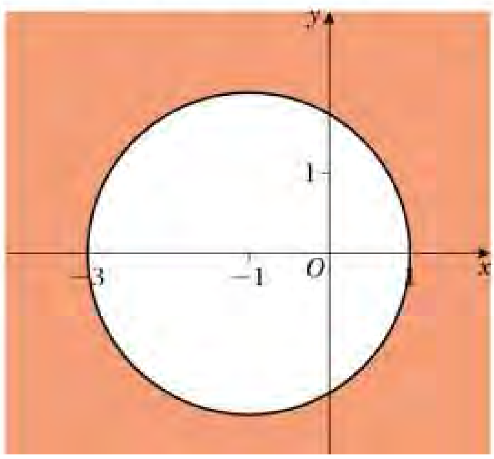
$(x + 1)^2 + y^2 = 4$

Circle centre $(-1, 0)$ radius 2.

Consider $z = 0$ in $2|z| \geqslant |z - 3|$

$2 \times 0 \geqslant 3$

So $z = 0$ is not in the region.



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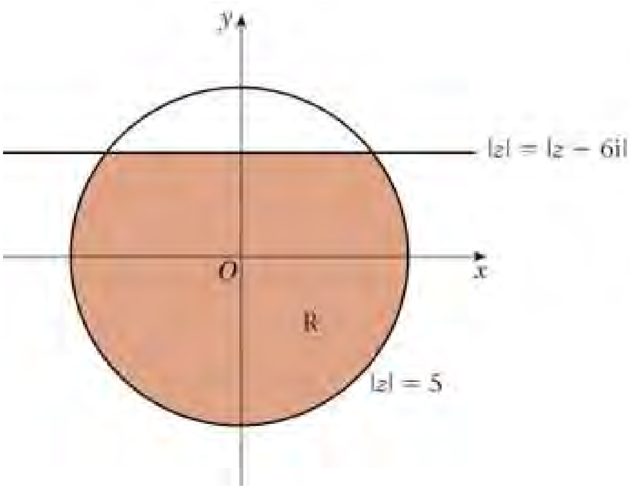
Exercise G, Question 2

Question:

The region R in an Argand diagram is satisfied by the inequalities $|z| \leqslant 5$ and $|z| \leqslant |z - 6\text{i}|$. Draw an Argand diagram and shade in the region R.

Solution:



$|z| \leqslant 5$
$|z| \leqslant |z - 6\text{i}|$

$|z| = 5$ represents a circle centre $(0, 0)$, radius 5

$|z| = |z - 6\text{i}|$ represents a perpendicular bisector of the line joining $(0, 0)$, to $(0, 6)$ and has the equation $y = 3$.

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Exercise G, Question 3

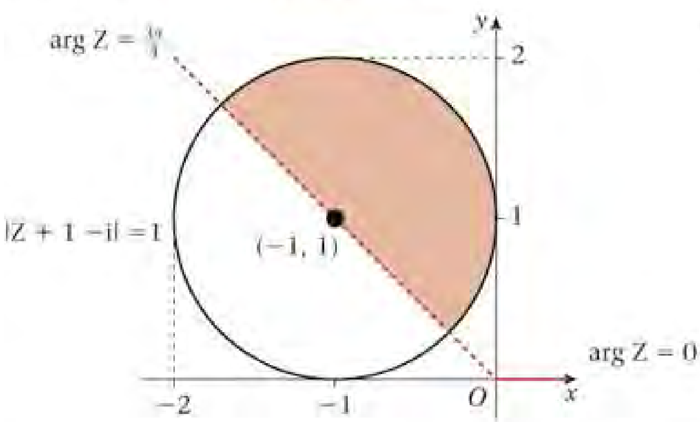
Question:

Shade in on an Argand diagram the region satisfied by the set of points $P(x, y)$, where $|z + 1 - i| \leqslant 1$ and $0 \leqslant \arg z < \frac{3\pi}{4}$.

Solution:

$|z + 1 - i| \leqslant 1$ $\qquad |z - (-1 - i)| \leqslant 1$

Inside of a circle centre $(-1, 1)$ radius 1



$\arg z = \frac{3\pi}{4}$ is a half-line with equation $y = -x$, which goes through the centre of the circle, $(-1, 1)$.

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Exercise G, Question 4

Question:

Shade in on an Argand diagram the region satisfied by the set of points $P(x, y)$, where $|z| \leqslant 3$ and $\frac{\pi}{4} \leqslant \arg(z + 3) \leqslant \pi$.

Solution:

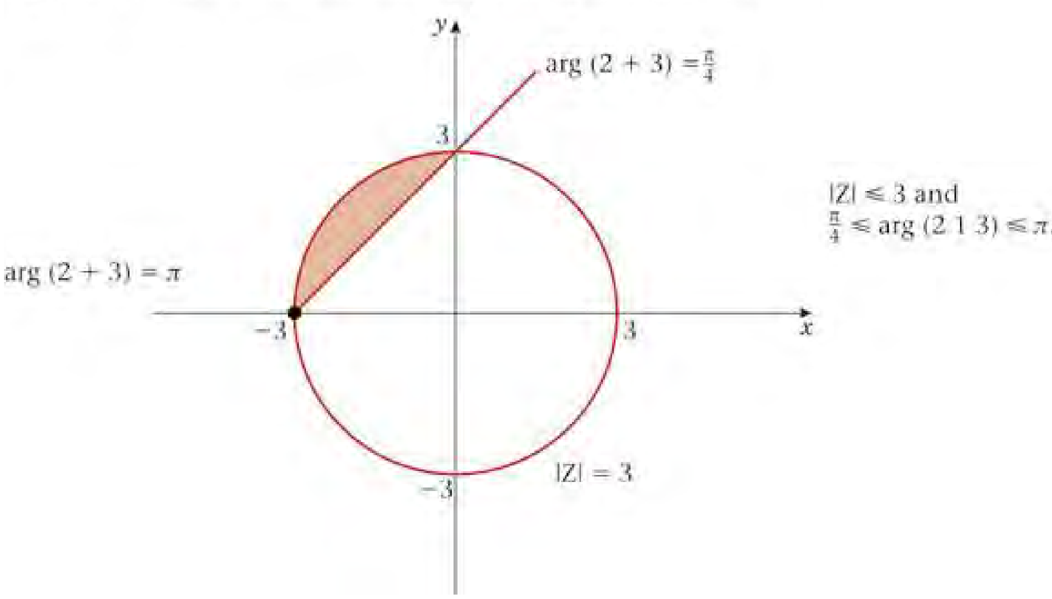
$|z| \leqslant 3$ and $\frac{\pi}{4} \leqslant \arg(z + 3) \leqslant \pi$

$|z| = 3$ represents a circle centre $(0, 0)$ radius 3.

$\arg(z + 3) = \frac{\pi}{4}$ is a half-line with equation $y - 0 = 1\,(x + 3) \Rightarrow y = x + 3, x > 0$.

Note it passes through the points $(-3, 0)$ and $(0, 3)$.

$\arg(z + 3) = \pi$ is a half-line with equation $y = 0, x < -3$.



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Exercise G, Question 5

Question:

a Sketch on the same Argand diagram:

i the locus of points representing $|z-2| = |z-6-8\mathrm{i}|$,

ii the locus of points representing $\arg(z-4-2\mathrm{i}) = 0$,

iii the locus of points representing $\arg(z-4-2\mathrm{i}) = \frac{\pi}{2}$.

The region R is defined by the inequalities $|z-2| \leqslant |z-6-8\mathrm{i}|$ and $0 \leqslant \arg(z-4-2\mathrm{i}) \leqslant \frac{\pi}{2}$.

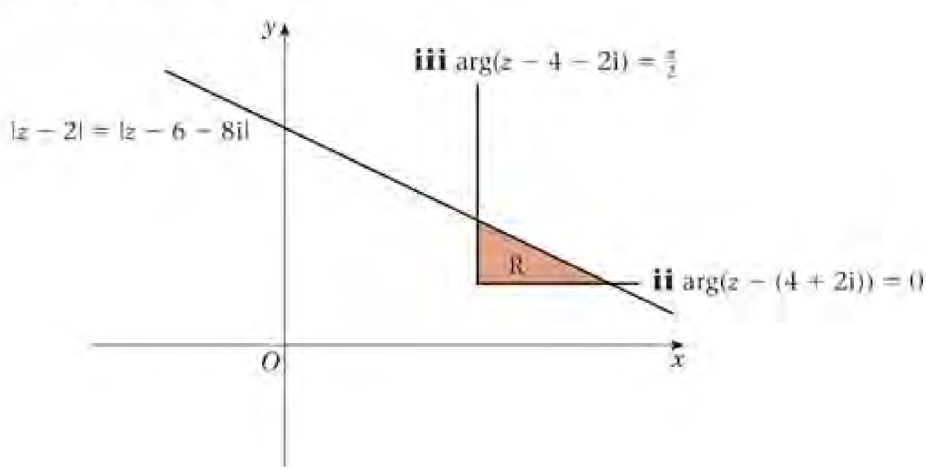
b On your sketch in part **a**, identify, by shading, the region R.

Solution:

a $|z-2| = |z-6-8\mathrm{i}|$ represents a perpendicular bisector of the line joining (2, 0) to (6, 8).

$$|x+\mathrm{i}y-2| = |x+\mathrm{i}y-6-8\mathrm{i}|$$
$$\Rightarrow |(x-2)+\mathrm{i}y| = |(x-6)+\mathrm{i}(y-8)|$$
$$\Rightarrow |(x-2)+\mathrm{i}y|^2 = |(x-6)+\mathrm{i}(y-8)|^2$$
$$\Rightarrow (x-2)^2+y^2 = (x-6)^2+(y-8)^2$$
$$\Rightarrow x^2-4x+4+y^2 = x^2-12x+36+y^2-16y+64$$
$$\Rightarrow -4x+4 = -12x-16y+100$$
$$\Rightarrow 8x+16y-96 = 0 \qquad (\div 8)$$
$$\Rightarrow x+2y-12 = 0$$
$$\Rightarrow 2y = -x+12$$
$$\Rightarrow y = \frac{-1}{2}x+6$$

i $|z-2| = |z-(6-8\mathrm{i})|$



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Exercise G, Question 6

Question:

a Find the Cartesian equations of:

i the locus of points representing $|z + 10| = |z - 6 - 4\sqrt{2}i|$,

ii the locus of points representing $|z + 1| = 3$.

b Find the two values of z that satisfy both $|z + 10| = |z - 6 - 4\sqrt{2}i|$ and $|z + 1| = 3$.

c Hence shade in the region R on an Argand diagram which satisfies both $|z + 10| \leqslant |z - 6 - 4\sqrt{2}i|$ and $|z + 1| \leqslant 3$.

Solution:

a **i** $|x + \mathrm{i}y + 10| = |x + \mathrm{i}y - 6 - 4\sqrt{2}\mathrm{i}|$

so $(x + 10)^2 + y^2 = (x - 6)^2 + (y - 4\sqrt{2})^2$

$\cancel{x^2} + 20x + 100 + \cancel{y^2} = \cancel{x^2} + 12x + 36 + \cancel{y^2} - 8\sqrt{2}y + 32$

$32x = -8\sqrt{2}y - 32$

$8\sqrt{2}y + (x + 1)32 = 0$

$y + (x + 1)2\sqrt{2} = 0$

$y = -2\sqrt{2}(x + 1)$

ii $(x + 1)^2 + y^2 = 9$

$(x^2 + 2x + y^2 = 8)$

b Substitute $y = -2\sqrt{2}(x + 1)$ into $(x + 1)^2 + y^2 = 9$

$(x + 1)^2 + 8(x + 1)^2 = 9$

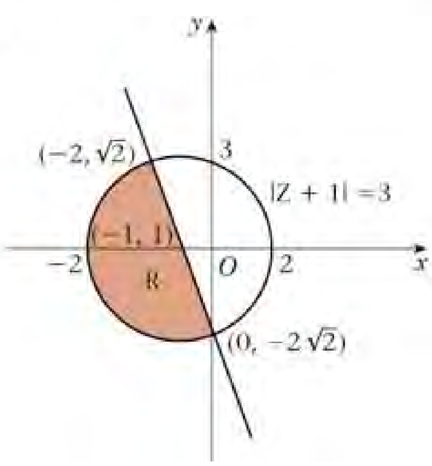
$9(x + 1)^2 = 9$

$x + 1 = \pm 1$

$x = 0, -2$ $\quad (0, -2\sqrt{2})$ and $(-2, 2\sqrt{2})$

$z = -2\sqrt{2}\mathrm{i}$ and $z = -2 + 2\sqrt{2}\mathrm{i}$

c



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Exercise H, Question 1

Question:

For the transformation $w = z + 4 + 3\text{i}$, sketch on separate Argand diagrams the locus of w when

a z lies on the circle $|z| = 1$,

b z lies on the half-line $\arg z = \frac{\pi}{2}$,

c z lies on the line $y = x$.

Solution:

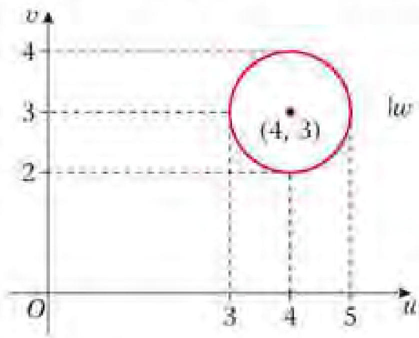
$w = z + 4 + 3\text{i}$

a $|z| = 1$ is a circle, centre (0, 0), radius 1

METHOD ① $|z|$ is translated by a translation vector $\begin{pmatrix}4\\3\end{pmatrix}$ to give a circle, centre (4, 3), radius 1, in the w plane.

METHOD ② $w = z + 4 + 3\text{i}$
$\Rightarrow w - 4 - 3\text{i} = z$
$\Rightarrow |w - 4 - 3\text{i}| = |z|$
$\Rightarrow |w - 4 - 3\text{i}| = 1$

The locus of w is a circle centre (4, 3), radius 1.

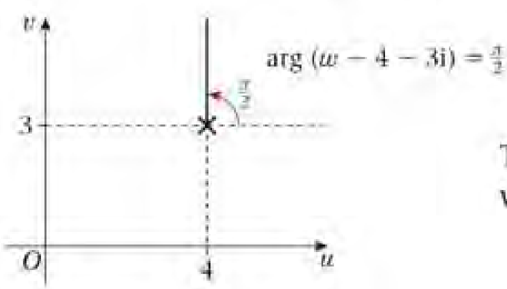


b $\arg z = \frac{\pi}{2}$

METHOD ① $\arg z = \frac{\pi}{2}$ is translated by a translation vector $\begin{pmatrix}4\\3\end{pmatrix}$ to give a half-line from (4, 3) at $\frac{\pi}{2}^c$ with the positive real axis.

METHOD ② $w = z + 4 + 3\text{i}$
$\Rightarrow w - 4 - 3\text{i} = z$

So $\arg z = \frac{\pi}{2} \Rightarrow \arg(w - 4 - 3\text{i}) = \frac{\pi}{2}$



The locus of w is the half-line with equation $u = 4$, $v > 3$.

c $y = x$

$w = z + 4 + 3\text{i}$

$\Rightarrow z = w - 4 - 3\text{i}$

$\Rightarrow x + \text{i}y = u + \text{i}v - 4 - 4\text{i}$

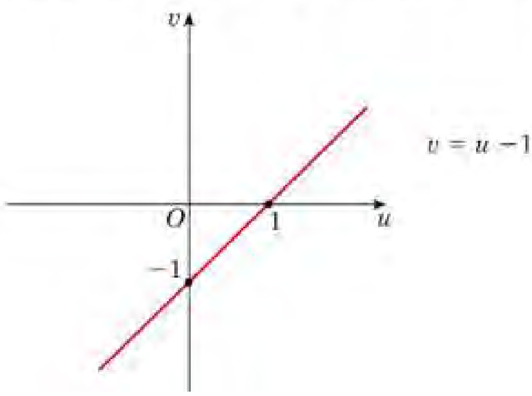
$\Rightarrow x + \text{i}y = (u - 4) + \text{i}(v - 3)$

$y = x \Rightarrow v - 3 = u - 4$

$\Rightarrow v = u - 4 + 3$

$\Rightarrow v = u - 1$

The locus of w is a line with equation $v = u - 1$.



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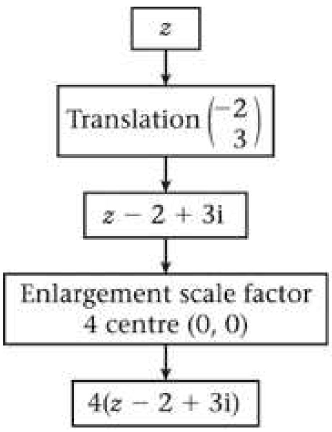
Exercise H, Question 2

Question:

A transformation T from the z-plane to the w-plane is a translation with translation vector $\begin{pmatrix} -2 \\ 3 \end{pmatrix}$ followed by an enlargement scale factor 4, centre O. Write down the transformation T in the form $w = a z + b$, where $a, b \in \mathbb{C}$.

Solution:



Hence T: $w = 4(z - 2 + 3i)$
$= 4z - 8 + 12i$

The transformation T is $w = 4z - 8 + 12i$

$\boxed{\text{Note: } a = 4,\ b = -8 + 12i.}$

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Exercise H, Question 3

Question:

For the transformation $w = 3z + 2 - 5\mathrm{i}$, find the equation of the locus of w when z lies on a circle centre O, radius 2.

Solution:

$w = 3z + 2 - 5\mathrm{i}$

METHOD ① z lies on a circle, centre 0, radius 2.

$$\begin{aligned}
&\Rightarrow |z| = 2\\
&w = 3z + 2 - 5\mathrm{i}\\
&\Rightarrow w - 2 + 5\mathrm{i} = 3z\\
&\Rightarrow |w - 2 + 5\mathrm{i}| = |3z|\\
&\Rightarrow |w - 2 - 5\mathrm{i}| = |3|\,|z|\\
&\Rightarrow |w - 2 - 5\mathrm{i}| = 3|z|\\
&\Rightarrow |w - 2 - 5\mathrm{i}| = 3(2)\\
&\Rightarrow |w - 2 - 5\mathrm{i}| = 6\\
&\Rightarrow |w - (2 - 5\mathrm{i})| = 6
\end{aligned}$$

So the locus of w is a circle centre $(2, -5)$, radius 6 with equation $(u - 2)^2 + (v + 5)^2 = 36$.

METHOD ② z lies on a circle, centre 0, radius 2.

$\downarrow$ $\boxed{\text{enlargement scale factor 3, centre 0.}}$

$3z$ lies on a circle, centre 0, radius 6.

$\downarrow$ $\boxed{\text{translation by a translation vector } \begin{pmatrix} 2 \\ -5 \end{pmatrix}.}$

$3z + 2 - 5\mathrm{i}$ lies on a circle centre $(2, -5)$, radius 6.

So the locus of w is a circle, centre $(2, -5)$, radius 6 with equation $(u - 2)^2 + (v + 5)^2 = 36$.

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Exercise H, Question 4

Question:

For the transformation $w = 2z - 5 + 3\text{i}$, find the equation of the locus of w as z moves on the circle $|z - 2| = 4$.

Solution:

z moves on a circle $|z - 2| = 4$

METHOD ① $w = 2z - 5 + 3\text{i}$

$$\Rightarrow w + 5 - 3\text{i} = 2z$$

$$\Rightarrow \frac{w + 5 - 3\text{i}}{2} = z$$

$$\Rightarrow \frac{w + 5 - 3\text{i}}{2} - 2 = z - 2$$

$$\Rightarrow \frac{w + 5 - 3\text{i} - 4}{2} = z - 2$$

$$\Rightarrow \frac{w + 1 - 3\text{i}}{2} = z - 2$$

$$\Rightarrow \left|\frac{w + 1 - 3\text{i}}{2}\right| = |z - 2|$$

$$\Rightarrow \frac{|w + 1 - 3\text{i}|}{|2|} = |z - 2|$$

$$\Rightarrow |w + 1 - 3\text{i}| = 2|z - 2|$$

$$\Rightarrow |w + 1 - 3\text{i}| = 2(4)$$

$$\Rightarrow |w + 1 - 3\text{i}| = 8$$

$$\Rightarrow |w - (-1 + 3\text{i})| = 8$$

So the locus of w is a circle centre $(-1, 3)$, radius 8 with equation $(u + 1)^2 + (v - 3)^2 = 8$.

METHOD ② $|z - 2| = 4$

z lies on a circle, centre $(2, 0)$, radius 4

$\downarrow$ $\boxed{\text{enlargement scale factor 2, centre 0.}}$

$2z$ lies on a circle, centre $(4, 0)$, radius 8.

$\downarrow$ $\boxed{\text{translation by a translation vector } \begin{pmatrix} 2 \\ -5 \end{pmatrix}.}$

$w = 2z - 5 + 3\text{i}$ lies on a circle centre $(-1, 3)$, radius 8.

So the locus of w is a circle, centre $(-1, 3)$, radius 8 with equation $(u - 1)^2 + (v - 3)^2 = 8$.

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Exercise H, Question 5

Question:

For the transformation $w = z - 1 + 2\text{i}$ sketch on separate Argand diagrams the locus of w when:

a z lies on the circle $|z - 1| = 3$,

b z lies on the half-line $\arg(z - 1 + \text{i}) = \frac{\pi}{4}$,

c z lies on the line $y = 2x$.

Solution:

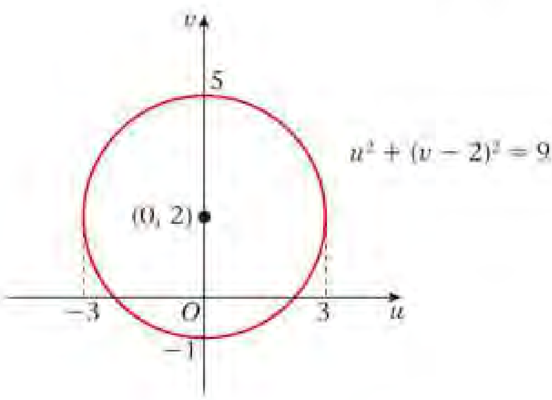
$w = z - 1 + 2i$

a $|z - 1| = 3$ circle centre (1, 0) radius 3.

METHOD ① $|z - 1| = 3$ is translated by a translation vector $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ to give a circle, centre (0, 2), radius 3, in the w-plane.

METHOD ② $w = z - 1 + 2i$
$\Rightarrow w - 2i = z - 1$
$\Rightarrow |w - 2i| = |z - 1|$
$\Rightarrow |w - 2i| = 3$

The locus of w is a circle, centre (0, 2), radius 3.



b $\arg(z - 1 + i) = \frac{\pi}{4}$ half-line from $(1, -1)$ at $\frac{\pi}{4}^c$ with the positive real axis.

METHOD ① $\arg(z - 1 + i) = \frac{\pi}{4}$ is translated by a translation vector $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ to give a half-line from (0, 1) at $\frac{\pi}{4}^c$ with the positive real axis.

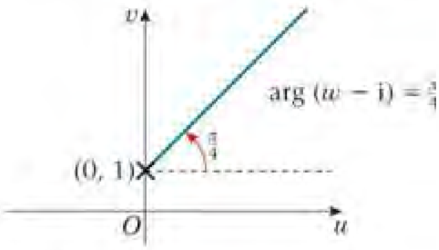
METHOD ② $w = z - 1 + 2i$
$\Rightarrow w + 1 - 2i = z$

So $\arg(z - 1 + i) = \frac{\pi}{4}$

becomes $\arg(w + 1 - 2i - 1 + i) = \frac{\pi}{4}$

$\Rightarrow \arg(w - i) = \frac{\pi}{4}$

Therefore, the locus of w is a half-line from (0, 1) at $\frac{\pi}{4}^c$ with the positive real axis.

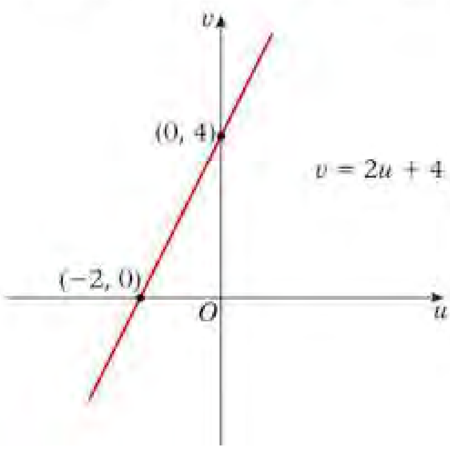


c $y = 2x$

$w = z - 1 + 2i$
$\Rightarrow z = w + 1 - 2i$
$\Rightarrow x + iy = u + iv + 1 - 2i$
$\Rightarrow x + iy = u + 1 + i(v - 2)$

So $y = 2x \Rightarrow v - 2 = 2(u + 1)$
$\Rightarrow v - 2 = 2u + 2$
$\Rightarrow v = 2u + 4$

The locus of w is a line with equation $v = 2u + 4$.



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Exercise H, Question 6

Question:

For the transformation $w = \frac{1}{z}$, $z \neq 0$, find the locus of w when:

a z lies on the circle $|z| = 2$,

b z lies on the half-line with equation $\arg z = \frac{\pi}{4}$,

c z lies on the line with equation $y = 2x + 1$.

Solution:

$w = \frac{1}{z}$, $z \neq 0$

a z lies on a circle, $|z| = 2$

$w = \frac{1}{z}$

$\Rightarrow |w| = \left|\frac{1}{z}\right|$

$\Rightarrow |w| = \frac{|1|}{|z|}$

$\Rightarrow |w| = \frac{1}{2}$ — $\boxed{\text{apply } |z| = 2}$

Therefore the locus of w is a circle, centre $(0, 0)$, radius $\frac{1}{2}$, with equation $u^2 + v^2 = \frac{1}{4}$.

b z lies on the half-line, $\arg z = \frac{\pi}{4}$

$w = \frac{1}{z} \Rightarrow wz = 1 \Rightarrow z = \frac{1}{w}$

So $\arg z = \frac{\pi}{4}$, becomes $\arg\left(\frac{1}{w}\right) = \frac{\pi}{4}$

$\Rightarrow \arg(1) - \arg(w) = \frac{\pi}{4}$

$\Rightarrow -\arg w = \frac{\pi}{4}$ — $\boxed{\arg 1 = 0}$

$\Rightarrow \arg w = -\frac{\pi}{4}$

Therefore the locus of w is a half-line from $(0, 0)$ at $-\frac{\pi}{4}^{c}$ with the positive x-axis.
The locus of w has equation, $v = -u$, $u > 0$, $v < 0$.

c z lies on the line $y = 2x + 1$

$w = \frac{1}{z} \Rightarrow wz = 1 \Rightarrow z = \frac{1}{w}$.

$\Rightarrow x + \mathrm{i}y = \frac{1}{u + \mathrm{i}v}$

$\Rightarrow x + \mathrm{i}y = \frac{1}{(u + \mathrm{i}v)}\frac{(u - \mathrm{i}v)}{(u - \mathrm{i}v)}$

$\Rightarrow x + \mathrm{i}y = \frac{u - \mathrm{i}v}{u^2 + v^2}$

$\Rightarrow x + \mathrm{i}y = \frac{u}{u^2 + v^2} + \mathrm{i}\left(\frac{-v}{u^2 + v^2}\right)$

So $x = \frac{u}{u^2 + v^2}$ and $y = \frac{-v}{u^2 + v^2}$

Hence $y = 2x + 1$ becomes $\frac{-v}{u^2 + v^2} = \frac{2u}{u^2 + v^2} + 1 \qquad \times (u^2 + v^2)$

$$\Rightarrow -v = 2u + u^2 + v^2$$
$$\Rightarrow 0 = u^2 + 2u + v^2 + v$$
$$\Rightarrow (u + 1)^2 - 1 + \left(v + \frac{1}{2}\right)^2 - \frac{1}{4} = 0$$
$$\Rightarrow (u + 1)^2 + \left(v + \frac{1}{2}\right)^2 = \frac{5}{4}$$
$$\Rightarrow (u + 1)^2 + \left(v + \frac{1}{2}\right)^2 = \left(\frac{\sqrt{5}}{4}\right)^2$$

Therefore, the locus of w is a circle, centre $\left(-1, -\frac{1}{2}\right)$, radius $\frac{\sqrt{5}}{2}$, with equation $(u + 1)^2 + \left(v + \frac{1}{2}\right)^2 = \frac{5}{4}$.

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Exercise H, Question 7

Question:

For the transformation $w = z^2$,

a show that as z moves once round a circle centre (0, 0), radius 3, w moves twice round a circle centre (0, 0), radius 9,

b find the locus of w when z lies on the real axis, with equation $y = 0$,

c find the locus of w when z lies on the imaginary axis.

Solution:

$w = z^2$

a z moves once round a circle, centre (0, 0), radius 3.
The equation of the circle, $|z| = 3$ is also $r = 3$.
The equation of the circle can be written as $z = 3e^{i\theta}$
or $z = 3(\cos\theta + i\sin\theta)$

$$\Rightarrow w = z^2 = (3(\cos\theta + i\sin\theta))^2$$
$$= 3^2(\cos 2\theta + i\sin 2\theta)$$
$$= 9(\cos 2\theta + i\sin 2\theta)$$

de Moivre's Theorem.

So, $w = 9(\cos 2\theta + i\sin 2\theta)$ can be written as $|w| = 9$
Hence, as $|w| = 9$ and $\arg w = 2\theta$ then w moves twice round a circle, centre (0, 0), radius 9.

b z lies on the real-axis $\Rightarrow y = 0$
So $z = x + iy$ becomes $z = x$ (as $y = 0$)
$\Rightarrow w = z^2 = x^2$
$\Rightarrow u + iv = x^2 + i(0)$
$\Rightarrow u = x^2$ and $v = 0$
As $v = 0$ and $u = x^2 \geqslant 0$ then w lies on the positive real-axis including the origin, 0.

c z lies on the imaginary axis $\Rightarrow x = 0$
So $z = x + iy$ becomes $z = iy$ (as $x = 0$)
$\Rightarrow w = z^2 = (iy)^2 = -y^2$
$\Rightarrow u + iv = -y^2 + i(0)$
$\Rightarrow u = -y^2$ and $v = 0$
As $v = 0$ and $u = -y^2 \leqslant 0$ then w lies on the negative real-axis including the origin, 0.

Solutionbank FP2
Edexcel AS and A Level Modular Mathematics

Exercise H, Question 8

Question:

If z is any point in the region R for which $|z + 2\text{i}| < 2$,

a shade in on an Argand diagram the region R.

Sketch on separate Argand diagrams the corresponding regions for $\mathbb{R}$ where:

b $w = z - 2 + 5\text{i}$,

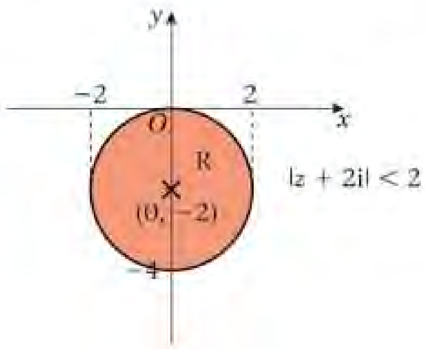
c $w = 4z + 2 + 4\text{i}$,

d $|zw + 2\text{i}w| = 1$.

Solution:

$|z + 2i| < 2$

a $|z + 2i| = 2$ is a circle, centre $(0, -2)$, radius 2.



b $w = z - 2 + 5i$

$\Rightarrow w + 2 - 5i = z$

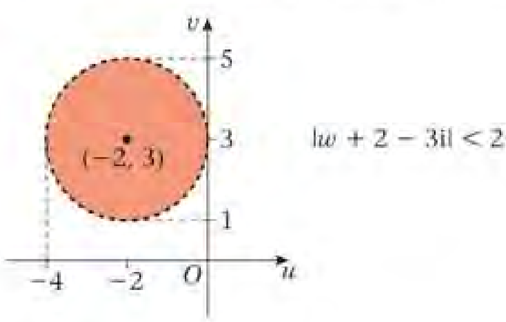
$\Rightarrow z + 2i = w + 2 - 5i + 2i$

$\Rightarrow z + 2i = w + 2 - 3i$

$\Rightarrow |z + 2i| = |w + 2 - 3i|$

As $|z + 2i| < 2$, then $|z + 2i| = |w + 2 - 3i| < 2$

Note that $|w + 2 - 3i| = 2$ is a circle, centre $(-2, 3)$, radius 2.



c $w = 4z + 2 + 4i$

$\Rightarrow w - 2 - 4i = 4z$

$\Rightarrow \dfrac{w - 2 - 4i}{4} = z$

$\Rightarrow z + 2i = \dfrac{w - 2 - 4i}{4} + 2i$

$\Rightarrow z + 2i = \dfrac{w - 2 - 4i + 8i}{4}$

$\Rightarrow z + 2i = \dfrac{w - 2 + 4i}{4}$

$\Rightarrow |z + 2i| = \left|\dfrac{w - 2 + 4i}{4}\right|$

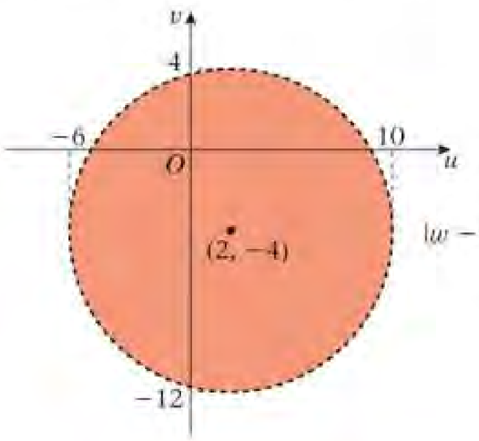
$$\Rightarrow |z + 2i| = \frac{|w - 2 + 4i|}{|4|}$$

$$\Rightarrow |z + 2i| = \frac{|w - 2 + 4i|}{4}$$

As $|z + 2i| < 2$, then $|z + 2i| = \dfrac{|w - 2 + 4i|}{4} < 2$

$\Rightarrow |w - 2 + 4i| < 8$

Note that $|w - 2 + 4i| = 8$ is a circle, centre $(2, -4)$, radius 8.



d $|zw + 2iw| = 1$

$\Rightarrow |w(z + 2i)| = 1$

$\Rightarrow |w|\ |z + 2i| = 1$

$$\Rightarrow |z + 2i| = \frac{1}{|w|}$$

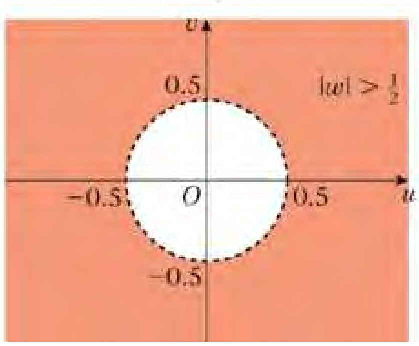
As $|z + 2i| < 2$, then $|z + 2i| = \dfrac{1}{|w|} < 2$

$\Rightarrow 1 < 2|w|$

$\Rightarrow \frac{1}{2} < |w|$

$\Rightarrow |w| > \frac{1}{2}$

Note that $|w| = \frac{1}{2}$ is a circle, centre $(0, 0)$ radius $\frac{1}{2}$.



Solutionbank FP2

Edexcel AS and A Level Modular Mathematics

Exercise H, Question 9

Question:

For the transformation $w = \frac{1}{2 - z}$, $z \neq 2$, show that the image, under T, of the circle centre O, radius 2 in the z-plane is a line l in the w-plane. Sketch l on an Argand diagram.

Solution:

Circle, centre 0, radius 2 in the z-plane $\Rightarrow |z| = 2$

T: $w = \dfrac{1}{2 - z}$

$\Rightarrow w(2 - z) = 1$

$\Rightarrow 2w - wz = 1$

$\Rightarrow 2w - 1 = wz$

$\Rightarrow \dfrac{2w - 1}{w} = z$

$\Rightarrow \left|\dfrac{2w - 1}{w}\right| = |z|$

$\Rightarrow \dfrac{|2w - 1|}{|w|} = |z|$

Applying $|z| = 2$ gives $\dfrac{|2w - 1|}{|w|} = 2$

$\Rightarrow |2w - 1| = 2|w|$

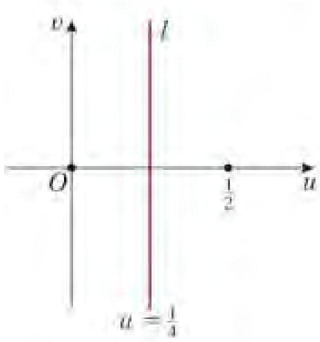
$\Rightarrow |2(w - \frac{1}{2})| = 2|w|$

$\Rightarrow |2|\,|(w - \frac{1}{2})| = 2|w|$

$\Rightarrow 2|w - \frac{1}{2}| = 2|w|$

$\Rightarrow |w - \frac{1}{2}| = |w|$

The image under T of $|z| = 2$ is the perpendicular bisector of the line segment joining $(0, 0)$ and $(\frac{1}{2}, 0)$. Therefore the line l has equation $u = \frac{1}{4}$.

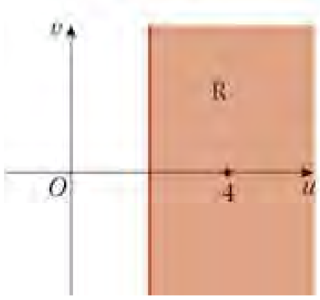


$|z - 4| < 4$ gives $\dfrac{|16 - 4w|}{|w|} < 4$

$\Rightarrow |16 - 4w| < 4|w|$

which leads to $|w - 4| < |w|$

$\Rightarrow |w| > |w - 4|$



Solutionbank FP2
Edexcel AS and A Level Modular Mathematics

Exercise H, Question 10

Question:

The transformation T from the z-plane, where $z = x + \mathrm{i}y$, to the w-plane where $w = u + \mathrm{i}v$, is given by $w = \frac{16}{z}$, $z \neq 0$.

a The transformation T maps the points on the circle $|z - 4| = 4$, in the z-plane, to points on a line l in the w-plane. Find the equation of l.

b Hence, or otherwise, shade and label on an Argand diagram the region R which is the image of $|z - 4| < 4$ under T.

Solution:

T: $w = \frac{16}{z}$

$|z - 4| = 4$

$w = \frac{16}{z}$

$\Rightarrow wz = 16$

$\Rightarrow z = \frac{16}{w}$

$\Rightarrow z - 4 = \frac{16}{w} - 4$

$\Rightarrow z - 4 = \frac{16 - 4w}{w}$

$\Rightarrow |z - 4| = \left|\frac{16 - 4w}{w}\right|$

$\Rightarrow |z - 4| = \frac{|16 - 4w|}{|w|}$

Applying $|z - 4| = 4$ gives $\frac{|16 - 4w|}{|w|} = 4$

$\Rightarrow \frac{|-4(w - 4)|}{|w|} = 4|w|$

$\Rightarrow |-4||w - 4| = 4|w|$

$\Rightarrow 4|w - 4| = 4|w|$

$\Rightarrow |w - 4| = |w|$

The image under T of $|z - 4| = 4$ is the perpendicular bisector of the line segment joining (0, 0) to (4, 0). Therefore the line l has equation $u = 2$.

Solutionbank FP2
Edexcel AS and A Level Modular Mathematics

Exercise H, Question 11

Question:

The transformation T from the z-plane, where $z = x + \mathrm{i}y$, to the w-plane where $w = u + \mathrm{i}v$, is given by $w = \frac{3}{2 - z}$, $z \neq 2$.

Show that under T the straight line with equation $2y = x$ is transformed to a circle in the w-plane with centre $\left(\frac{3}{4}, \frac{3}{2}\right)$, radius $\frac{3}{4}\sqrt{5}$.

Solution:

T: $w = \frac{3}{2 - z}$, $z \neq 2$

$\Rightarrow w(2 - z) = 3$

$\Rightarrow 2w - wz = 3$

$\Rightarrow 2w = 3 + wz$

$\Rightarrow 2w - 3 = wz$

$\Rightarrow \frac{2w - 3}{w} = z$

$\Rightarrow z = \frac{2w - 3}{w}$

$\Rightarrow z = \frac{2(u + \mathrm{i}v) - 3}{u + \mathrm{i}v}$

$\Rightarrow z = \frac{(2u - 3) + 2\mathrm{i}v}{u + \mathrm{i}v}$

$\Rightarrow z = \frac{[(2u - 3) + 2\mathrm{i}v]}{[u + \mathrm{i}v]} \times \frac{[u - \mathrm{i}v]}{[u - \mathrm{i}v]}$

$\Rightarrow z = \frac{(2u - 3)u - \mathrm{i}v(2u - 3) + 2\mathrm{i}uv + 2v^2}{u^2 + v^2}$

$\Rightarrow z = \frac{2u^2 - 3u - \cancel{2uv\mathrm{i}} + 3\mathrm{i}v + \cancel{2uv\mathrm{i}} + 2v^2}{u^2 + v^2}$

$\Rightarrow z = \frac{2u^2 - 3u + 2v^2}{u^2 + v^2} + \mathrm{i}\left[\frac{3v}{u^2 + v^2}\right]$

So, $x + \mathrm{i}y = \frac{2u^2 - 3u + 2v^2}{u^2 + v^2} + \mathrm{i}\left[\frac{3v}{u^2 + v^2}\right]$

$\Rightarrow x = \frac{2u^2 - 3u + 2v^2}{u^2 + v^2}$

and $y = \frac{3v}{u^2 + v^2}$

As, $2y = x \Rightarrow 2\left(\frac{3v}{u^2 + v^2}\right) = \frac{2u^2 - 3u + 2v^2}{u^2 + v^2}$

$$\Rightarrow \frac{6v}{u^2 + v^2} = \frac{2u^2 - 3u + 2v^2}{u^2 + v^2}$$

$$\Rightarrow 6v = 2u^2 - 3u + 2v^2$$

$$\Rightarrow 0 = 2u^2 - 3u + 2v^2 - 6v$$

$$\Rightarrow 2u^2 - 3u + 2v^2 - 6v = 0 \quad (\div 2)$$

$$\Rightarrow u^2 - \frac{3}{2}u + v^2 - 3v = 0$$

$$\Rightarrow \left(u - \frac{3}{4}\right)^2 - \frac{9}{16} + \left(v - \frac{3}{2}\right)^2 - \frac{9}{4} = 0$$

$$\Rightarrow \left(u - \frac{3}{4}\right)^2 + \left(v - \frac{3}{2}\right)^2 = \frac{9}{16} + \frac{9}{4}$$

$$\Rightarrow \left(u - \frac{3}{4}\right)^2 + \left(v - \frac{3}{2}\right)^2 = \frac{45}{16}$$

$$\Rightarrow \left(u - \frac{3}{4}\right)^2 + \left(v - \frac{3}{2}\right)^2 = \left(\frac{3\sqrt{5}}{4}\right)^2$$

The image under T of $2y = x$ is a circle centre $\left(\frac{3}{4}, \frac{3}{2}\right)$, radius $\frac{3}{4}\sqrt{5}$, as required.

Solutionbank FP2

Edexcel AS and A Level Modular Mathematics

Exercise H, Question 12

Question:

The transformation T from the z-plane, where $z = x + \mathrm{i}y$, to the w-plane where $w = u + \mathrm{i}v$, is given by $w = \dfrac{-\mathrm{i}z + \mathrm{i}}{z + 1}$, $z \neq -1$.

a The transformation T maps the points on the circle with equation $x^2 + y^2 = 1$ in the z-plane, to points on a line l in the w-plane. Find the equation of l.

b Hence, or otherwise, shade and label on an Argand diagram the region R of the w-plane which is the image of $|z| \leqslant 1$ under T.

c Show that the image, under T, of the circle with equation $x^2 + y^2 = 4$ in the z-plane is a circle C in the w-plane. Find the equation of C.

Solution:

$T: w = \dfrac{-\mathrm{i}z + \mathrm{i}}{z + 1}, z \neq -1$

a Circle with equation $x^2 + y^2 = 1 \Rightarrow |z| = 1$

$w = \dfrac{-\mathrm{i}z + \mathrm{i}}{z + 1}$

$\Rightarrow w(z + 1) = -\mathrm{i}z + \mathrm{i}$

$\Rightarrow wz + w = -\mathrm{i}z + \mathrm{i}$

$\Rightarrow wz + \mathrm{i}z = -\mathrm{i} - w$

$\Rightarrow z(w + \mathrm{i}) = \mathrm{i} - w$

$\Rightarrow z = \dfrac{\mathrm{i} - w}{w + \mathrm{i}}$

$\Rightarrow |z| = \left|\dfrac{\mathrm{i} - w}{w + \mathrm{i}}\right|$

$\Rightarrow |z| = \dfrac{|\mathrm{i} - w|}{|w + \mathrm{i}|}$

Applying $|z| = 1 \Rightarrow 1 = \dfrac{|\mathrm{i} - w|}{|w + \mathrm{i}|}$

$\Rightarrow |w + \mathrm{i}| = |\mathrm{i} - w|$

$\Rightarrow |w + \mathrm{i}| = |(-1)(w - \mathrm{i})|$

$\Rightarrow |w + \mathrm{i}| = |(-1)||(w - \mathrm{i})|$

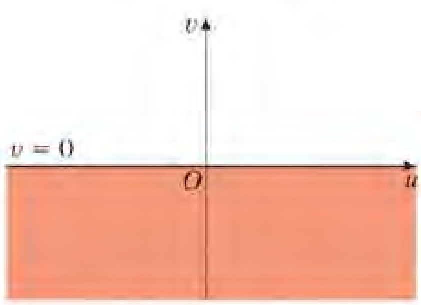
$\Rightarrow |w + \mathrm{i}| = |w - \mathrm{i}|$

The image under T of $x^2 + y^2 = 1$ is the perpendicular bisector of the line segment joining $(0, -1)$ to $(0, 1)$. Therefore the line l, has equation $v = 0$. (i.e. the u-axis.)

b $|z| \leqslant 1 \Rightarrow 1 \leqslant \dfrac{|\mathrm{i} - w|}{|w + \mathrm{i}|}$

$\Rightarrow |w + \mathrm{i}| \leqslant |\mathrm{i} - w|$

$\Rightarrow |w + \mathrm{i}| \leqslant |w - \mathrm{i}|$



c Circle with equation $x^2 + y^2 = 4 \Rightarrow |z| = 2$

from part **a** $\quad w = \dfrac{-\mathrm{i}z + \mathrm{i}}{z + 1}$

$$\Rightarrow z = \frac{\mathrm{i} - w}{w + \mathrm{i}}$$

$$\Rightarrow |z| = \frac{|\mathrm{i} - w|}{|w + \mathrm{i}|}$$

Applying $|z| = 2 \Rightarrow 2 = \dfrac{|\mathrm{i} - w|}{|w + \mathrm{i}|}$

$$\Rightarrow 2|w + \mathrm{i}| = |\mathrm{i} - w|$$

$$\Rightarrow 2|w + \mathrm{i}| = |(-1)(w - \mathrm{i})|$$

$$\Rightarrow 2|w + \mathrm{i}| = |(-1)|\,|(w - \mathrm{i})|$$

$$\Rightarrow 2|w + \mathrm{i}| = |w - \mathrm{i}|$$

$$\Rightarrow 2|u + \mathrm{i}v + \mathrm{i}| = |u + \mathrm{i}v - \mathrm{i}|$$

$$\Rightarrow 2|u + \mathrm{i}(v + 1)| = |u + \mathrm{i}(v - 1)|$$

$$\Rightarrow 2^2|u + \mathrm{i}(v + 1)|^2 = |u + \mathrm{i}(v - 1)|^2$$

$$\Rightarrow 4[u^2 + (v + 1)^2] = u^2 + (v - 1)^2$$

$$\Rightarrow 4[u^2 + v^2 + 2v + 1] = u^2 + v^2 - 2v + 1$$

$$\Rightarrow 4u^2 + 4v^2 + 8v + 4 = u^2 + v^2 - 2v + 1$$

$$\Rightarrow 3u^2 + 3v^2 + 10v + 3 = 0$$

$$\Rightarrow u^2 + v^2 + \frac{10}{3}v + 1 = 0$$

$$\Rightarrow u^2 + \left(v + \frac{5}{3}\right)^2 - \frac{25}{9} + 1 = 0$$

$$\Rightarrow u^2 + \left(v + \frac{5}{3}\right)^2 = \frac{25}{9} - 1$$

$$\Rightarrow u^2 + \left(v + \frac{5}{3}\right)^2 = \frac{16}{9}$$

$$\Rightarrow u^2 + \left(v + \frac{5}{3}\right)^2 = \left(\frac{4}{3}\right)^2$$

The image under T of $x^2 + y^2 = 4$ is a circle C with centre $\left(0, -\frac{5}{3}\right)$, radius $\frac{4}{3}$.

Therefore, the equation of C is $u^2 + \left(v + \frac{5}{3}\right)^2 = \frac{16}{9}$.

Solutionbank FP2
Edexcel AS and A Level Modular Mathematics

Exercise H, Question 13

Question:

The transformation T from the z-plane, where $z = x + \mathrm{i}y$, to the w-plane where $w = u + \mathrm{i}v$, is given by $w = \frac{4z - 3\mathrm{i}}{z - 1}$, $z \neq 1$.

Show that the circle $|z| = 1$ is mapped by T onto a circle C.

Find the centre and radius of C.

Solution:

T: $w = \frac{4z - 3\mathrm{i}}{z - 1}$, $z \neq 1$

Circle with equation $|z| = 3$

$w = \frac{4z - 3\mathrm{i}}{z - 1}$,

$\Rightarrow w(z - 1) = 4z - 3\mathrm{i}$

$\Rightarrow wz - w = 4z - 3\mathrm{i}$

$\Rightarrow wz + 4z = w - 3\mathrm{i}$

$\Rightarrow z(w - 4) = w - 3\mathrm{i}$

$\Rightarrow z = \frac{w - 3\mathrm{i}}{w - 4}$

$\Rightarrow |z| = \left|\frac{w - 3\mathrm{i}}{w - 4}\right|$

$\Rightarrow |z| = \frac{|w - 3\mathrm{i}|}{|w - 4|}$

Applying $|z| = 3 \Rightarrow 3 = \frac{|w - 3\mathrm{i}|}{|w - 4|}$

$$\Rightarrow 3|w - 4| = |w - 3i|$$
$$\Rightarrow 3|u + iv - 4| = |u + iv - 3i|$$
$$\Rightarrow 3|(u - 4) + iv| = |u + i(v - 3)|$$
$$\Rightarrow 3^2|(u - 4) + iv|^2 = |u + i(v - 3)|^2$$
$$\Rightarrow 9[(u - 4)^2 + v^2] = u^2 + (v - 3)^2$$
$$\Rightarrow 9[u^2 - 8u + 16 + v^2] = u^2 + v^2 - 6v + 9$$
$$\Rightarrow 9u^2 - 72u + 144 + 9v^2 = u^2 + v^2 - 6v + 9$$
$$\Rightarrow 8u^2 - 72u + 8v^2 + 6v + 144 - 9 = 0$$
$$\Rightarrow 8u^2 - 72u + 8v^2 + 6v + 135 = 0 \quad (\div 8)$$
$$\Rightarrow u^2 - 9u + v^2 + \frac{3}{4}v + \frac{135}{8} = 0$$
$$\Rightarrow \left(u - \frac{9}{2}\right)^2 - \frac{81}{4} + \left(v + \frac{3}{8}\right)^2 - \frac{9}{64} + \frac{135}{8} = 0$$
$$\Rightarrow \left(u - \frac{9}{2}\right)^2 + \left(v + \frac{3}{8}\right)^2 = \frac{81}{4} + \frac{9}{64} - \frac{135}{8}$$
$$\Rightarrow \left(u - \frac{9}{2}\right)^2 + \left(v + \frac{3}{8}\right)^2 = \frac{225}{64}$$
$$\Rightarrow \left(u - \frac{9}{2}\right)^2 + \left(v + \frac{3}{8}\right)^2 = \left(\frac{15}{8}\right)^2$$

Therefore, the circle with equation $|z| = 1$ is mapped onto a circle C with centre $\left(\frac{9}{2} \ -\frac{3}{8}\right)$, radius $\frac{15}{8}$.

Solutionbank FP2
Edexcel AS and A Level Modular Mathematics

Exercise H, Question 14

Question:

The transformation T from the z-plane, where $z = x + \mathrm{i}y$, to the w-plane where $w = u + \mathrm{i}v$, is given by $w = \frac{1}{z+\mathrm{i}}$, $z \neq -\mathrm{i}$.

a Show that the image, under T, of the real axis in the z-plane is a circle C_1 in the w-plane. Find the equation of C_1.

b Show that the image, under T, of the line $x = 4$ in the z-plane is a circle C_2 in the w-plane. Find the equation of C_2.

Solution:

T: $w = \frac{1}{z+\mathrm{i}}$, $z \neq -\mathrm{i}$

a Real axis in the z-plane $\Rightarrow y = 0$

$$w = \frac{1}{z - \mathrm{i}}$$

$$\Rightarrow w(z + \mathrm{i}) = 1$$

$$\Rightarrow wz + \mathrm{i}w = 1$$

$$\Rightarrow wz = 1 - \mathrm{i}w$$

$$\Rightarrow z = \frac{1 - \mathrm{i}w}{w}$$

$$\Rightarrow z = \frac{1 - \mathrm{i}(u + \mathrm{i}v)}{u + \mathrm{i}v}$$

$$\Rightarrow z = \frac{1 - \mathrm{i}u + v}{u + \mathrm{i}v}$$

$$\Rightarrow z = \frac{((1 + v) - \mathrm{i}u)}{(u + \mathrm{i}v)} \times \frac{(u - \mathrm{i}v)}{(u - \mathrm{i}v)}$$

$$\Rightarrow z = \frac{(1 + v)u - \mathrm{i}v(1 + v) - \mathrm{i}u^2 - uv}{u^2 + v^2}$$

$$\Rightarrow z = \frac{(1 + v)u - uv}{u^2 + v^2} + \frac{\mathrm{i}(-v(1 + v) - u^2)}{u^2 + v^2}$$

$$\Rightarrow z = \frac{u + uv - uv}{u^2 + v^2} + \frac{\mathrm{i}(-v - v^2 - u^2)}{u^2 + v^2}$$

$$\Rightarrow z = \frac{u}{u^2 + v^2} + \frac{\mathrm{i}(-v - v^2 - u^2)}{u^2 + v^2}$$

$$\text{So } x + \mathrm{i}y = \frac{u}{u^2 + v^2} + \frac{\mathrm{i}(-v - v^2 - u^2)}{u^2 + v^2}$$

$$\Rightarrow x = \frac{u}{u^2 + v^2} \quad \textit{and} \quad y = \frac{-v - v^2 - u^2}{u^2 + v^2}$$

As $y = 0$, $\dfrac{-v - v^2 - u^2}{u^2 + v^2} = 0$

$\Rightarrow -v - v^2 - u^2 = 0$

$\Rightarrow u^2 + v^2 + v = 0$

$\Rightarrow u^2 + \left(v + \frac{1}{2}\right)^2 - \frac{1}{4} = 0$

$\Rightarrow u^2 + \left(v + \frac{1}{2}\right)^2 = \frac{1}{4}$

$\Rightarrow u^2 + \left(v + \frac{1}{2}\right)^2 = \left(\frac{1}{2}\right)^2$

Therefore, the image under T of the real axis in the z-plane is a circle C_1 with centre $\left(0, -\frac{1}{2}\right)$, radius $\frac{1}{2}$. The equation of C_1 is $u^2 + \left(v + \frac{1}{2}\right)^2 = \frac{1}{4}$.

b As $x = 4$, $\dfrac{u}{u^2 + v^2} = 2$

$\Rightarrow u = 2(u^2 + v^2)$

$\Rightarrow u = 2u^2 + 2v^2$

$\Rightarrow 0 = 2u^2 - u + 2v^2 \quad (\div 2)$

$\Rightarrow 0 = u^2 - \frac{1}{2}u + v^2$

$\Rightarrow 0 = \left(u - \frac{1}{4}\right)^2 - \frac{1}{16} + v^2$

$\Rightarrow \left(u - \frac{1}{4}\right)^2 + v^2 = \frac{1}{16}$

$\Rightarrow \left(u - \frac{1}{4}\right)^2 + v^2 = \left(\frac{1}{4}\right)^2$

Therefore, the image under T of the line $x = 2$ is a circle C_2 with centre $\left(\frac{1}{4}, 0\right)$, radius $\frac{1}{4}$. The equation of C_2 is $\left(u - \frac{1}{4}\right)^2 + v^2 = \frac{1}{16}$.

Solutionbank FP2
Edexcel AS and A Level Modular Mathematics

Exercise H, Question 15

Question:

The transformation T from the z-plane, where $z = x + \mathrm{i}y$, to the w-plane where $w = u + \mathrm{i}v$, is given by $w = z + \frac{4}{z}$, $z \neq 0$.

Show that the transformation T maps the points on a circle $|z| = 2$ to points in the interval $[-k, k]$ on the real axis. State the value of the constant k.

Solution:

$T: w = z + \frac{4}{z}, z \neq 0$

Circle with equation $|z| = 2 \Rightarrow x^2 + y^2 = 4$

$w = z + \frac{4}{z}$

$\Rightarrow w = \frac{z^2 + 4}{z}$

$\Rightarrow w = \frac{(x + \mathrm{i}y)^2 + 4}{x + \mathrm{i}y}$

$\Rightarrow w = \frac{x^2 + 2xy\mathrm{i} - y^2 + 4}{x + \mathrm{i}y}$

$\Rightarrow w = \frac{[(x^2 - y^2 + 4) + \mathrm{i}(2xy)]}{x + \mathrm{i}y}$

$\Rightarrow w = \frac{[(x^2 - y^2 + 4) + \mathrm{i}(2xy)]}{(x + \mathrm{i}y)} \times \frac{(x - \mathrm{i}y)}{(x - \mathrm{i}y)}$

$\Rightarrow w = \frac{x^3 - xy^2 + 4x + 2xy^2 + \mathrm{i}(2x^2y - x^2y + y^3 - 4y)}{x^2 + y^2}$

$\Rightarrow w = \left(\frac{x^3 - xy^2 + 4x}{x^2 + y^2}\right) + \mathrm{i}\left(\frac{y^3 - x^2y - 4y}{x^2 + y^2}\right)$

$\Rightarrow w = \frac{x(x^2 + y^2 + 4)}{x^2 + y^2} + \frac{\mathrm{i}y(x^2 + y^2 - 4)}{x^2 + y^2}$

Apply $x^2 + y^2 + 4 \Rightarrow w = \frac{x(4 + 4)}{4} + \frac{\mathrm{i}y(4 - 4)}{4}$

$\Rightarrow w = \frac{8x}{4} + \frac{\mathrm{i}y(0)}{4}$

$\Rightarrow w = 2x + 0\mathrm{i}$

$\Rightarrow u + \mathrm{i}v = 2x + 0\mathrm{i}$

$\Rightarrow u = 2x, v = 0$

As $|z| = 2 \Rightarrow -2 \leqslant x \leqslant 2$

So $-4 \leqslant 2x \leqslant 4$

and $-4 \leqslant u \leqslant 4$

Therefore the transformation T maps the points on a circle $|z| = 2$ in the z-plane to points in the interval $[-4, 4]$ on the real axis in the w-plane. Hence $k = 4$.

Solutionbank FP2
Edexcel AS and A Level Modular Mathematics

Exercise H, Question 16

Question:

The transformation T from the z-plane, where $z = x + \mathrm{i}y$, to the w-plane where $w = u + \mathrm{i}v$, is given by $w = \frac{1}{z+3}$, $z \neq -3$.

Show that the line with equation $2x - 2y + 7 = 0$ is mapped by T onto a circle C. State the centre and the exact radius of C.

Solution:

$T: w = \dfrac{1}{z+3}, z \neq -3$

Line with equation $2x - 2y + 7 = 0$ in the z-plane

$$w = \frac{1}{z+3}$$

$$\Rightarrow w(z+3) = 1$$

$$\Rightarrow wz + 3w = 1$$

$$\Rightarrow wz = 1 - 3w$$

$$\Rightarrow z = \frac{1-3w}{w}$$

$$\Rightarrow z = \frac{1 - 3(u + \mathrm{i}v)}{u + \mathrm{i}v}$$

$$\Rightarrow z = \frac{1 - 3u - 3\mathrm{i}v}{u + \mathrm{i}v}$$

$$\Rightarrow z = \frac{[(1-3u) - (3v)\mathrm{i}]}{[(u + \mathrm{i}v)]} \times \frac{(u - \mathrm{i}v)}{(u - \mathrm{i}v)}$$

$$\Rightarrow z = \frac{(1-3u)u - 3v^2 - \mathrm{i}v(1-3u) - \mathrm{i}(3uv)}{u^2 + v^2}$$

$$\Rightarrow z = \frac{u - 3u^2 - 3v^2}{u^2+v^2} + \frac{\mathrm{i}(-v + 3uv - 3uv)}{u^2+v^2}$$

$$\Rightarrow z = \frac{u - 3u^2 - 3v^2}{u^2+v^2} + \frac{\mathrm{i}(-v)}{u^2+v^2}$$

So, $x + \mathrm{i}y = \dfrac{u - 3u^2 - 3v^2}{u^2+v^2} + \dfrac{\mathrm{i}(-v)}{u^2+v^2}$

$$\Rightarrow x = \frac{u - 3u^2 - 3v^2}{u^2+v^2}$$

and $y = \dfrac{-v}{u^2+v^2}$

As $2x - 2y + 7 = 0$, then

$$2\left(\frac{u - 3u^2 - 3v^2}{u^2+v^2}\right) - 2\left(\frac{-v}{u^2+v^2}\right) + 7 = 0$$

$$\Rightarrow \frac{2u - 6u^2 - 6v^2}{u^2+v^2} + \frac{2v}{u^2+v^2} + 7 = 0 \quad (\times\ (u^2+v^2))$$

$$\Rightarrow 2u - 6u^2 - 6v^2 + 2v + 7(u^2+v^2) = 0$$

$$\Rightarrow 2u - 6u^2 - 6v^2 + 2v + 7u^2 + 7v^2 = 0$$

$$\Rightarrow u^2 + 2u + v^2 + 2v = 0$$

$$\Rightarrow (u+1)^2 - 1 + (v+1)^2 - 1 = 0$$

$$\Rightarrow (u+1)^2 + (v+1)^2 = 2$$

$$\Rightarrow (u+1)^2 + (v+1)^2 = (\sqrt{2})^2$$

Therefore the transformation T maps the line $2x - 2y + 7 = 0$ in the z plane to a circle C with

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Exercise I, Question 1

Question:

Express $\frac{(\cos 3x + \mathrm{i} \sin 3x)^2}{\cos x - \mathrm{i} \sin x}$ in the form $\cos nx + \mathrm{i} \sin nx$ where n is an integer to be determined.

Solution:

$$\frac{(\cos 3x + \mathrm{i} \sin 3x)^2}{\cos x - \mathrm{i} \sin x}$$

$$= \frac{(\cos 3x + \mathrm{i} \sin 3x)^2}{\cos(-x) + \mathrm{i} \sin(-x)}$$

$$= \frac{\cos 6x + \mathrm{i} \sin 6x}{\cos(-x) + \mathrm{i} \sin(-x)}$$

$$= \cos(6x - -x) + \mathrm{i} \sin(6x - -x)$$

$$= \cos 7x + \mathrm{i} \sin 7x$$

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Exercise I, Question 2

Question:

Use de Moivre's theorem to evaluate

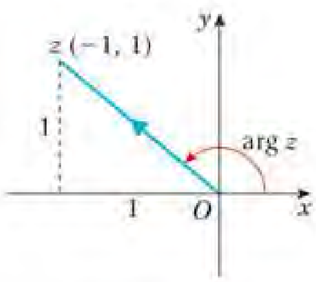
a $(-1 + \mathrm{i})^8$

b $\dfrac{1}{\left(\frac{1}{2} - \frac{1}{2}\mathrm{i}\right)^{16}}$

Solution:

a $(-1 + \mathrm{i})^8$

If $z = -1 + \mathrm{i}$, then



$r = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$

$\theta = \arg z = \pi - \tan^{-1}\left(\frac{1}{1}\right) = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$

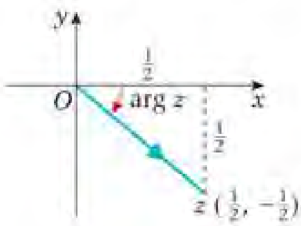
So, $-1 + \mathrm{i} = \sqrt{2}\left(\cos\frac{3\pi}{4} + \mathrm{i}\sin\frac{3\pi}{4}\right)$

$$\therefore (-1 + \mathrm{i})^8 = \left[\sqrt{2}\left(\cos\frac{3\pi}{4} + \mathrm{i}\sin\frac{3\pi}{4}\right)\right]^8$$
$$= (\sqrt{2})^8\left(\cos\frac{24\pi}{4} + \mathrm{i}\sin\frac{24\pi}{4}\right)$$
$$= 16(\cos 6\pi + \mathrm{i}\sin 6\pi)$$
$$= 16(1 + \mathrm{i}(0))$$

Therefore, $(-1 + \mathrm{i})^8 = 16$

b $\frac{1}{\left(\frac{1}{2}-\frac{1}{2}\mathrm{i}\right)^{16}} = \left(\frac{1}{2}-\frac{1}{2}\mathrm{i}\right)^{-16}$

Let $z = \frac{1}{2} - \frac{1}{2}\mathrm{i}$, then



$$r = \sqrt{\left(\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right)^2} = \sqrt{\frac{1}{4}+\frac{1}{4}} = \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}}$$

$$\theta = \tan^{-1}\left(\frac{\frac{1}{2}}{\frac{1}{2}}\right) = -\frac{\pi}{4}$$

So $\frac{1}{2} - \frac{1}{2}\mathrm{i} = \frac{1}{\sqrt{2}}\left[\cos\left(-\frac{\pi}{4}\right) + \mathrm{i}\sin\left(-\frac{\pi}{4}\right)\right]$

$$\begin{aligned}\left(\tfrac{1}{2}-\tfrac{1}{2}\mathrm{i}\right)^{-16} &= \left[\frac{1}{\sqrt{2}}\left(\cos\left(-\frac{\pi}{4}\right) + \mathrm{i}\sin\left(-\frac{\pi}{4}\right)\right)\right]^{-16} \\ &= \left(2^{-\frac{1}{2}}\right)^{-16}\left(\cos\left(\frac{16\pi}{4}\right) + \mathrm{i}\sin\left(\frac{16\pi}{4}\right)\right) \\ &= 2^8(\cos 4\pi + \mathrm{i}\sin 4\pi) \\ &= 256(1 + \mathrm{i}(0)) \\ &= 256\end{aligned}$$

Therefore, $\frac{1}{\left(\frac{1}{2}-\frac{1}{2}\mathrm{i}\right)^{16}} = 256$

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Exercise I, Question 3

Question:

a If $z = \cos\theta + \mathrm{i}\sin\theta$, use de Moivre's theorem to show that $z^n + \frac{1}{z^n} = 2\cos n\theta$.

b Express $\left(z^2 + \frac{1}{z^2}\right)^3$ in terms of $\cos 6\theta$ and $\cos 2\theta$.

c Hence, or otherwise, show that $\cos^3 2\theta = a\cos 6\theta + b\cos 2\theta$, where a and b are constants.

d Hence, or otherwise, show that $\int_0^{\frac{\pi}{6}} \cos^3 2\theta \, \mathrm{d}\theta = k\sqrt{3}$, where k is a constant.

Solution:

a $z = \cos\theta + \mathrm{i}\sin\theta$

$$z^n = (\cos\theta + \mathrm{i}\sin\theta)^n$$
$$= \cos n\theta + \mathrm{i}\sin n\theta$$

de Moivre's Theorem.

$$\frac{1}{z^n} = z^{-n} = (\cos\theta + \mathrm{i}\sin\theta)^n$$
$$= \cos(-n\theta) + \mathrm{i}\sin(-n\theta)$$
$$= \cos n\theta - \mathrm{i}\sin n\theta$$

de Moivre's Theorem.

$\cos(-n\theta) = \cos n\theta$
$\sin(-n\theta) = -\sin n\theta$

Therefore $z^n + \frac{1}{z^n} = \cos n\theta + \mathrm{i}\sin n\theta + \cos n\theta - \mathrm{i}\sin n\theta$

i.e. $z^n + \frac{1}{z^n} = 2\cos n\theta$ (as required)

b $$\left(z^2 + \frac{1}{z^2}\right)^3 = (z^2)^3 + {}^3C_1(z^2)^2\left(\frac{1}{z^2}\right) + {}^3C_2(z^2)\left(\frac{1}{z^2}\right)^2 + \left(\frac{1}{z^2}\right)^3$$
$$= z^6 + 3z^4\left(\frac{1}{z^2}\right) + 3z^2\left(\frac{1}{z^4}\right) + \frac{1}{z^6}$$
$$= z^6 + 3z^2 + \frac{3}{z^2} + \frac{1}{z^6}$$
$$= \left(z^6 + \frac{1}{z^6}\right) = 3\left(z^2 + \frac{1}{z^2}\right)$$
$$= 2\cos 6\theta + 3(2)\cos 2\theta$$
$$= 2\cos 6\theta + 6\cos 2\theta$$

Hence, $\left(z^2 + \frac{1}{z^2}\right)^3 = 2\cos 6\theta + 6\cos 2\theta$

c $\left(z^2 + \frac{1}{z^2}\right)^3 = (2\cos 2\theta)^3 = 8\cos^3 2\theta = 2\cos 6\theta + 6\cos 2\theta$

$\therefore\ \cos^3 2\theta = \frac{2}{8}\cos 6\theta + \frac{6}{8}\cos 2\theta$

Hence, $\cos^3 2\theta = \frac{1}{4}\cos 6\theta + \frac{3}{4}\cos 2\theta$

d $\int_0^{\frac{\pi}{6}} \cos^3 2\theta \, d\theta = \int_0^{\frac{\pi}{6}} \frac{1}{4}\cos 6\theta + \frac{3}{4}\cos 2\theta \, d\theta$

$$= \left[\frac{1}{24}\sin 6\theta + \frac{3}{8}\sin 2\theta\right]_0^{\frac{\pi}{6}}$$

$$= \left(\frac{1}{24}\sin \pi + \frac{3}{8}\sin\left(\frac{\pi}{3}\right)\right) - \left(\frac{1}{24}\sin 0 + \frac{3}{8}\sin 0\right)$$

$$= \left(\frac{1}{24}(0) + \frac{3}{8}\left(\frac{\sqrt{3}}{2}\right)\right) - (0)$$

$$= \frac{3}{16}\sqrt{3}$$

So, $\int_0^{\frac{\pi}{6}} \cos^3 2\theta \, d\theta = \frac{3}{16}\sqrt{3}$

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Exercise I, Question 4

Question:

a Use de Moivre's theorem to show that $\cos 5\theta = \cos\theta(16\cos^4\theta - 20\cos^2\theta + 5)$.

b By solving the equation $\cos 5\theta = 0$, deduce that $\cos^2\left(\frac{\pi}{10}\right) = \frac{5+\sqrt{5}}{8}$.

c Hence, or otherwise, write down the exact values of $\cos^2\left(\frac{3\pi}{10}\right)$, $\cos^2\left(\frac{7\pi}{10}\right)$ and $\cos^2\left(\frac{9\pi}{10}\right)$.

Solution:

a $(\cos\theta + \mathrm{i}\sin\theta)^5 = \cos 5\theta + \mathrm{i}\sin 5\theta$

de Moivre's Theorem.

$= \cos^5\theta + {}^5C_1\cos^4\theta(\mathrm{i}\sin\theta) + {}^5C_2\cos^3\theta(\mathrm{i}\sin\theta)^2$
$\quad + {}^5C_3\cos^2\theta(\mathrm{i}\sin\theta)^3 + {}^5C_4\cos^3\theta(\mathrm{i}\sin\theta)^4 + (\mathrm{i}\sin\theta)^5$

$= \cos^5\theta + 5\mathrm{i}\cos^4\theta\sin\theta + 10\mathrm{i}^2\cos^3\theta\sin^2\theta$
$\quad + 10\mathrm{i}^3\cos^2\theta\sin^3\theta + 5\mathrm{i}^4\cos\theta\sin^4\theta + \mathrm{i}^5\sin^5\theta$

Binomial expansion.

Hence,

$\cos 5\theta + \mathrm{i}\sin 5\theta = \cos^5\theta + 5\mathrm{i}\cos^4\theta\sin\theta - 10\cos^3\theta\sin\theta$
$\quad - 10\mathrm{i}\cos^2\theta\sin^3\theta + 5\cos\theta\sin^4\theta + \mathrm{i}\sin^5\theta$

Equating the real parts gives,

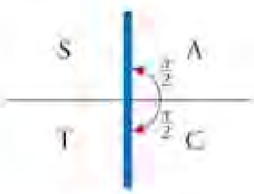
$\cos 5\theta = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$
$= \cos\theta(\cos^4\theta - 10\cos^2\theta\sin^2\theta + 5\sin^4\theta)$
$= \cos\theta(\cos^4\theta - 10\cos^2\theta(1-\cos^2\theta) + 5(1-\cos^2\theta)^2)$
$= \cos\theta(\cos^4\theta - 10\cos^2\theta + 10\cos^4\theta + 5(1 - 2\cos^2\theta + \cos^4\theta))$
$= \cos\theta(\cos^4\theta - 10\cos^2\theta + 10\cos^4\theta + 5 - 10\cos^2\theta + 5\cos^4\theta)$
$= \cos\theta(16\cos^4\theta - 20\cos^2\theta + 5)$

Applying $\sin^2\theta = 1 - \cos^2\theta$.

Hence, $\cos 5\theta = \cos\theta(16\cos^4\theta - 20\cos^2\theta + 5)$ (as required)

b $\cos 5\theta = 0$

$\alpha = \frac{\pi}{2}$



So $\quad 5\theta = \left\{\frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \frac{9\pi}{2}\right\}$

$\theta = \left\{\frac{\pi}{10}, \frac{3\pi}{10}, \frac{5\pi}{10}, \frac{7\pi}{10}, \frac{9\pi}{10}\right\}$

$\theta = \left\{\frac{\pi}{10}, \frac{3\pi}{10}, \frac{5\pi}{10}, \frac{7\pi}{10}, \frac{9\pi}{10}\right\}$ for $0 < \theta \leqslant \pi$

$\cos 5\theta = 0 \Rightarrow \cos\theta(16\cos^4\theta - 20\cos^2\theta + 5) = 0$

Five solutions must come from: $\quad \cos\theta(16\cos^4\theta - 20\cos^2\theta + 5) = 0$

Solution ① $\quad \cos\theta = 0$

$\alpha = \frac{\pi}{2}$

For $0 < \theta \leqslant \pi$, $\theta = \dfrac{\pi}{2}$ (as found earlier)

The final 4 solutions come from: $16\cos^4\theta - 20\cos^2\theta + 5 = 0$

$$\cos^2\theta = \frac{20 \pm \sqrt{400 - 4(16)(5)}}{32}$$

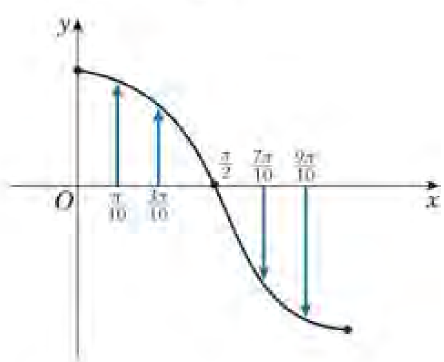
$$= \frac{20 \pm \sqrt{400 - 320}}{32}$$

$$= \frac{20 \pm \sqrt{80}}{32}$$

$$= \frac{20 \pm \sqrt{16}\sqrt{5}}{32}$$

$$= \frac{20 \pm 4\sqrt{5}}{32}$$

$$\therefore \cos^2\theta = \frac{5 \pm \sqrt{5}}{8}$$



Due to symmetry and as $\cos\left(\dfrac{\pi}{10}\right) > \cos\left(\dfrac{3\pi}{10}\right)$

$$\cos^2\left(\frac{\pi}{10}\right) = \cos^2\left(\frac{9\pi}{10}\right) > \cos^2\left(\frac{3\pi}{10}\right) = \cos^2\left(\frac{7\pi}{10}\right)$$

$$\therefore \cos^2\left(\frac{7\pi}{10}\right) = \frac{5 + \sqrt{5}}{8}$$

c $\cos^2\left(\dfrac{3\pi}{10}\right) = \dfrac{5 - \sqrt{5}}{8}$

$$\cos^2\left(\frac{7\pi}{10}\right) = \cos^2\left(\frac{3\pi}{10}\right) = \frac{5 - \sqrt{5}}{8}$$

$$\cos^2\left(\frac{9\pi}{10}\right) = \cos^2\left(\frac{\pi}{10}\right) = \frac{5 + \sqrt{5}}{8}$$

Therefore, $\cos^2\left(\dfrac{3\pi}{10}\right) = \dfrac{5 - \sqrt{5}}{8}$, $\cos^2\left(\dfrac{7\pi}{10}\right) = \dfrac{5 - \sqrt{5}}{8}$, $\cos^2\left(\dfrac{9\pi}{10}\right) = \dfrac{5 + \sqrt{5}}{8}$

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Exercise I, Question 5

Question:

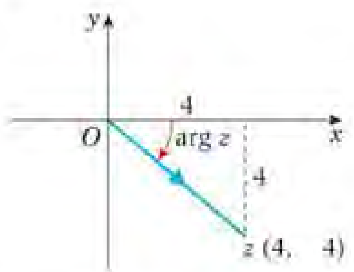
a Express $4 - 4\text{i}$ in the form $r(\cos\theta + \text{i}\sin\theta)$, where $r > 0$, $-\pi < \theta \leqslant \pi$, where r and θ are exact values.

b Hence, or otherwise, solve the equation $z^5 = 4 - 4\text{i}$ leaving your answers in the form $z = R\text{e}^{\text{i}k\pi}$, where R is the modulus of z and k is a rational number such that $-1 \leqslant k \leqslant 1$.

c Show on an Argand diagram the points representing your solutions.

Solution:

a $4 - 4\text{i}$



modulus $r = \sqrt{4^2 + (-4)^2} = \sqrt{16 + 16} = \sqrt{32} = \sqrt{16}\sqrt{2} = 4\sqrt{2}$

argument $= \theta = -\tan^{-1}\left(\frac{4}{4}\right) = -\frac{\pi}{4}$

$\therefore\ 4 - 4\text{i} = 4\sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + \text{i}\sin\left(-\frac{\pi}{4}\right)\right)$

b $z^5 = 4 - 4\text{i}$

for $4 - 4\text{i}$, $r = 4\sqrt{2}$, $\theta = -\frac{\pi}{4}$

So, $z^5 = 4\sqrt{2}\,\text{e}^{\text{i}\left(-\frac{\pi}{4}\right)}$

$z^5 = 4\sqrt{2}\,\text{e}^{\text{i}\left(-\frac{\pi}{4} + 2k\pi\right)},\ k \in \mathbb{Z}$

Hence, $z = \left[4\sqrt{2}\,\text{e}^{\text{i}\left(-\frac{\pi}{4} + 2k\pi\right)}\right]^{\frac{1}{5}}$

$= (4\sqrt{2})^{\frac{1}{5}}\,\text{e}^{\text{i}\left(\frac{-\frac{\pi}{4} + 2k\pi}{5}\right)}$

de Moivre's Theorem.

$= \sqrt{2}\,\text{e}^{\text{i}\left(-\frac{\pi}{20} + \frac{2k\pi}{5}\right)}$

$4\sqrt{2} = 2^{\frac{5}{2}}$

So, $(4\sqrt{2})^{\frac{1}{5}} = (2^{\frac{5}{2}})^{\frac{1}{5}}$

$= 2^{\frac{1}{2}} = \sqrt{2}$

$k = 0,\ z_1 = \sqrt{2}\,\text{e}^{\text{i}\left(-\frac{\pi}{20}\right)}$

$k = 1,\ z_2 = \sqrt{2}\,\text{e}^{\text{i}\left(\frac{7\pi}{20}\right)}$

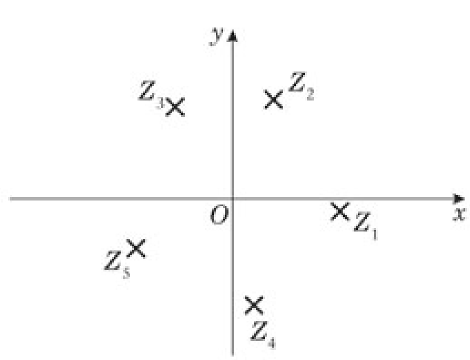
$k = 2,\ z_3 = \sqrt{2}\,\text{e}^{\text{i}\left(\frac{3\pi}{4}\right)}$

$k = -1,\ z_4 = \sqrt{2}\,\text{e}^{\text{i}\left(-\frac{9\pi}{20}\right)}$

$k = -2,\ z_5 = \sqrt{2}\,\text{e}^{\text{i}\left(-\frac{17\pi}{20}\right)}$

Therefore, $z = \sqrt{2}\,\text{e}^{-\frac{\pi\text{i}}{20}},\ \sqrt{2}\,\text{e}^{\frac{7\pi\text{i}}{20}},\ \sqrt{2}\,\text{e}^{\frac{3\pi\text{i}}{4}},\ \sqrt{2}\,\text{e}^{-\frac{9\pi\text{i}}{20}},\ \sqrt{2}\,\text{e}^{-\frac{17\pi\text{i}}{20}}$

c



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Exercise I, Question 6

Question:

a Find the Cartesian equations of

i the locus of points representing $|z-3+i|=|z-1-i|$,

ii the locus of points representing $|z-2|=2\sqrt{2}$.

b Find the two values of z that satisfy both $|z-3+i|=|z-1-i|$ and $|z-2|=2\sqrt{2}$.

c Hence on the same Argand diagram sketch:

i the locus of points representing $|z-3+i|=|z-1-i|$,

ii the locus of points representing $|z-2|=2\sqrt{2}$.

The region R is defined by the inequalities $|z-3+i| \geqslant |z-1-i|$ and $|z+2| \leqslant 2\sqrt{2}$.

d On your sketch in part **c**, identify, by shading, the region R.

Solution:

a i Let $|z - 3 + i| = |z - 1 - i|$

$\Rightarrow |x + iy - 3 + i| = |x + iy - 1 - i|$

$\Rightarrow |(x - 3) + i(y + 1)| = |(x - 1) + i(y - 1)|$

$\Rightarrow |(x - 3) + i(y + 1)|^2 = |(x - 1) + i(y - 1)|^2$

$\Rightarrow (x - 3)^2 + (y + 1)^2 = (x - 1)^2 + (y - 1)^2$

$\Rightarrow x^2 - 6x + 9 + y^2 + 2y + 1 = x^2 - 2x + 1 + y^2 - 2y + 1$

$\Rightarrow - 6x + 2y + 10 = -2x - 2y + 2$

$\Rightarrow - 4x + 4y + 8 = 0$

$\Rightarrow 4y = 4x - 8$

$\Rightarrow y = x - 2$

The Cartesian equation of the locus of points representing $|z - 3 + i| = |z - 1 - i|$ is $y = x - 2$.

METHOD ① **i** $|z - 3 + i| = |z - 1 - i|$

As $|z - 3 + i| = |z - 1 - i|$ is a perpendicular bisector of the line joining $A(3, -1)$ to $B(1, 1)$,

then $m_{AB} = \frac{1 - -1}{1 - 3} = \frac{2}{-2} = -1$

and perpendicular gradient $= \frac{-1}{-1} = 1$

mid-point of AB is $\left(\frac{3 + 1}{2}, \frac{-1 + 1}{2}\right)$

$= (2, 0)$

$\Rightarrow y - 0 = 1(x - 2)$

$y = x - 2$

The Cartesian equation of the locus of points representing $|z - 3 + i| = |z - 1 - i|$ is $y = x - 2$.

ii $|z - 2| = 2\sqrt{2}$

$\Rightarrow$ circle centre $(2, 0)$, radius $2\sqrt{2}$.

$\Rightarrow$ equation of circle is $(x - 2)^2 + y^2 = (2\sqrt{2})^2$

$\Rightarrow (x - 2)^2 + y^2 = 8$

The Cartesian equation of the locus of points representing $|z - 2| = 2\sqrt{2}$ is $(x - 2)^2 + y^2 = 8$.

b $|z - 3 + i| = |z - 1 + i| \Rightarrow y = x - 2$ ①

$|z - 2| = 2\sqrt{2} \Rightarrow (x - 2)^2 + y^2 = 8$ ②

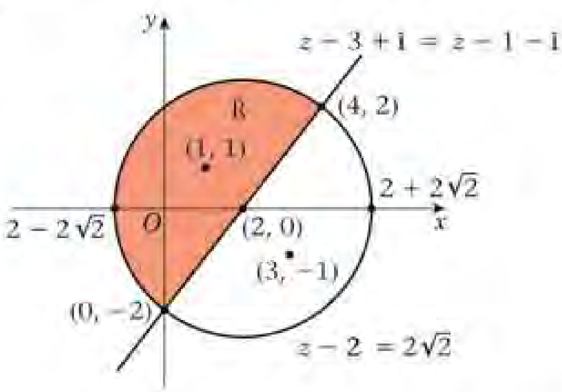
$$
\begin{aligned}
①\cap② &\Rightarrow (x-2)^2 + (x-2)^2 = 8 \\
&\Rightarrow 2(x-2)^2 = 8 \\
&\Rightarrow (x-2)^2 = 4 \\
&\Rightarrow x - 2 = \pm\sqrt{4} \\
&\Rightarrow x - 2 = \pm 2 \\
&\Rightarrow x = 2 \pm 2 \\
&\Rightarrow x = 0, 4
\end{aligned}
$$

when $x = 0$, $y = 0 - 2 = -2 \Rightarrow z = 0 - 2i$

when $x = 4$, $y = 4 - 2 = 2 \Rightarrow z = 4 + 2i$

The values of z are $-2i$ and $4 + 2i$

c



Note that $|z - 3 + i| = |z - 1 + i| \Rightarrow y = x - 2$ goes through the point $(2, 0)$ and so is a diameter of $|z - 2| = 2\sqrt{2}$.

d The region R is shaded on the Argand diagram in part **i**, which satisfies $|z - 3 + i| \geqslant |z - 1 - i|$ and $|z - 2| \leqslant 2\sqrt{2}$.

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Exercise I, Question 7

Question:

a Find the Cartesian equation of the locus of points representing $|z + 2| = |2z - 1|$.

b Find the value of z which satisfies both $|z + 2| = |2z - 1|$ and $\arg z = \frac{\pi}{4}$.

c Hence shade in the region R on an Argand diagram which satisfies both $|z + 2| \geqslant |2z - 1|$ and $\frac{\pi}{4} \leqslant \arg z \leqslant \pi$.

Solution:

a $|z+2| = |2z-1|$

$\Rightarrow |x+\mathrm{i}y+2| = |2(x+\mathrm{i}y)-1|$

$\Rightarrow |x+\mathrm{i}y+2| = |2x+2\mathrm{i}y-1|$

$\Rightarrow |(x+2)+\mathrm{i}y| = |(2x-1)+\mathrm{i}(2y)|$

$\Rightarrow |(x+2)+\mathrm{i}y|^2 = |(2x-1)+\mathrm{i}(2y)|^2$

$\Rightarrow (x+2)^2+y^2 = (2x-1)^2+\mathrm{i}(2y)^2$

$\Rightarrow x^2+4x+4+y^2 = 4x^2-4x+1+4y^2$

$\Rightarrow 0 = 3x^2-8x+3y^2+1-4$

$\Rightarrow 3x^2-8x+3y^2-3=0$

$\Rightarrow x^2-\frac{8}{3}x+y^2-1=0$

$\Rightarrow \left(x-\frac{4}{3}\right)^2-\frac{16}{9}+y^2-1=0$

$\Rightarrow \left(x-\frac{4}{3}\right)^2+y^2=\frac{16}{9}+1$

$\Rightarrow \left(x-\frac{4}{3}\right)^2+y^2=\frac{25}{9}$

$\Rightarrow \left(x-\frac{4}{3}\right)^2+y^2=\left(\frac{5}{3}\right)^2$

This is a circle, centre $\left(\frac{4}{3}, 0\right)$, radius $\frac{5}{3}$.

The Cartesian equation of the locus of points representing $|z+2| = |2z-1|$ is

$\left(x-\frac{4}{3}\right)^2+y^2=\frac{25}{9}$.

b $|z+2| = |2z-1| \Rightarrow \left(x-\frac{4}{3}\right)^2+y^2=\frac{25}{9}$ ①

$\arg z = \frac{\pi}{4} \Rightarrow \arg(x+\mathrm{i}y) = \frac{\pi}{4}$

$\Rightarrow \frac{y}{x} = \tan\frac{\pi}{4}$

$\Rightarrow \frac{y}{x} = 1$

$\Rightarrow y = x$ where $x > 0, y > 0$ ②

②^①: $\left(x - \frac{4}{3}\right)^2 + x^2 = \frac{25}{9}$

$$\Rightarrow x^2 - \frac{4}{3}x - \frac{4}{3}x + \frac{16}{9} + x^2 = \frac{25}{9}$$

$$\Rightarrow 2x^2 - \frac{8}{3}x = \frac{25}{9} - \frac{16}{9}$$

$$\Rightarrow 2x^2 - \frac{8}{3}x = \frac{9}{9}$$

$$\Rightarrow 2x^2 - \frac{8}{3}x = 1 \qquad (\times 3)$$

$$\Rightarrow 6x^2 - 8x = 3$$

$$\Rightarrow 6x^2 - 8x - 3 = 0$$

$$\Rightarrow x = \frac{8 \pm \sqrt{64 - 4(6)(-3)}}{2(6)}$$

$$\Rightarrow x = \frac{8 \pm \sqrt{136}}{12}$$

$$\Rightarrow x = \frac{8 \pm 2\sqrt{34}}{12}$$

$$\Rightarrow x = \frac{4 \pm \sqrt{34}}{6}$$

As $x > 0$ then we reject $x = \frac{4 - \sqrt{34}}{6}$

and accept $x = \frac{4 + \sqrt{34}}{6}$

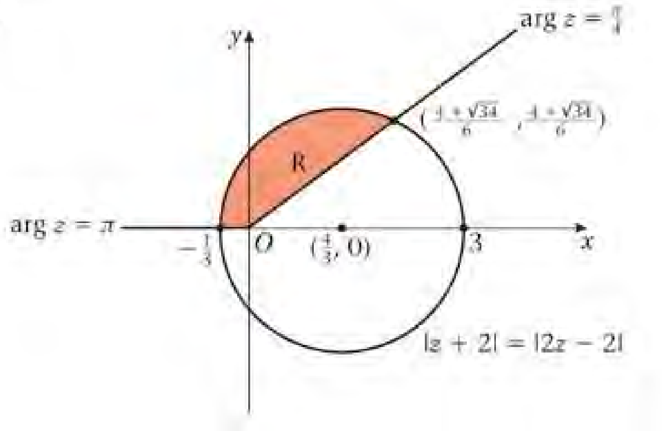
as $y = x$, then $y = \frac{4 + \sqrt{34}}{6}$

So $z = \left(\frac{4 + \sqrt{34}}{6}\right) + \left(\frac{4 + \sqrt{34}}{6}\right)\mathrm{i}$

The value of z satisfying $|z + 2| = |2z - 1|$ and $\arg z = \frac{\pi}{4}$

is $z = \left(\frac{4 + \sqrt{34}}{6}\right) + \left(\frac{4 + \sqrt{34}}{6}\right)\mathrm{i}$ OR $z = 1.64 + 1.64\mathrm{i}$ (2 d.p.)

c



The region R (shaded) satisfies both $|z+2| \geqslant |2z-1|$ and $\frac{\pi}{4} \leqslant \arg z \leqslant \pi$.

Note that $|z+2| \geqslant |2z-1|$

$\Rightarrow (x+2)^2 + y^2 \geqslant (2x-1)^2 + (2y)^2$

$\Rightarrow 0 \geqslant 3x^2 - 8x + 3y^2 - 3$

$\Rightarrow 0 \geqslant \left(x - \frac{4}{3}\right)^2 - \frac{16}{9} + y^2 - 1$

$\Rightarrow \frac{25}{9} \geqslant \left(x - \frac{4}{3}\right)^2 + y^2$

$\Rightarrow \left(x - \frac{4}{3}\right)^2 + y^2 \leqslant \frac{25}{9}$

represents region inside and bounded by the circle, centre $\left(\frac{4}{3}, 0\right)$, radius $\frac{5}{3}$.

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Exercise I, Question 8

Question:

The point P represents a complex number z in an Argand diagram.
Given that $|z + 1 - i| = 1$

a find a Cartesian equation for the locus of P,

b sketch the locus of P on an Argand diagram,

c find the greatest and least values of $|z|$,

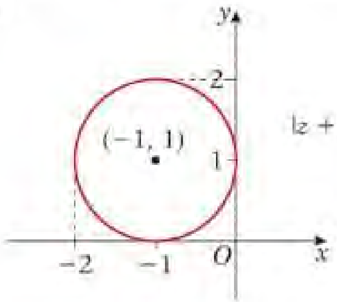
d find the greatest and least values of $|z - 1|$.

Solution:

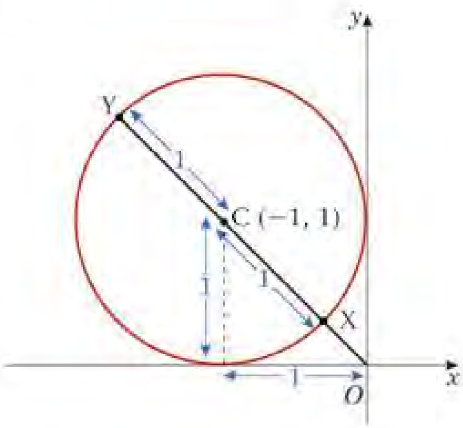
a $|z + 1 - i| = 1$ is a circle, centre $(-1, 1)$, radius 1.

The Cartesian equation for the locus of P is $(x + 1)^2 + (y - 1)^2 = 1$.

b



c



$|z|$ is the distance from $(0, 0)$ to the locus of points.

From the Argand diagram,

$|z|_{max}$ is the distance OY

$|z|_{min}$ is the distance OX

Note that radius $= CX = CY = 1$

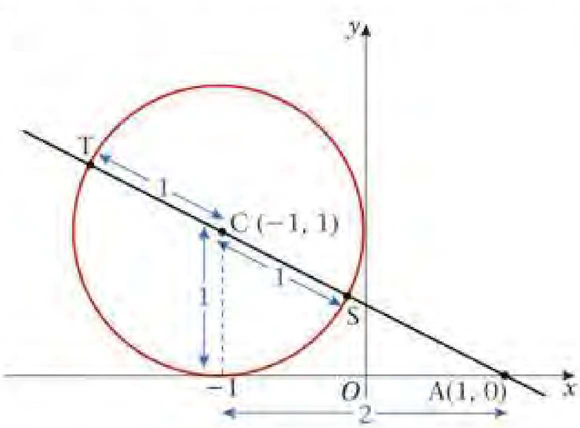
and $OC = \sqrt{1^2 + 1^2} = \sqrt{2}$

$|z|_{max} = OC + CY = \sqrt{2} + 1$

$|z|_{min} = OC - CX = \sqrt{2} - 1$

The greatest value of $|z|$ is $\sqrt{2} + 1$ and the least value of $|z|$ is $\sqrt{2} - 1$.

d



$|z - 1|$ is the distance from $A(1, 0)$ to the locus of points.

From the Argand diagram,

$|z - 1|_{max}$ is the distance AS

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Exercise I, Question 9

Question:

Given that $\arg\left(\frac{z-4-2\mathrm{i}}{z-6\mathrm{i}}\right)=\frac{\pi}{2}$,

a sketch the locus of $P(x, y)$ which represents z on an Argand diagram,

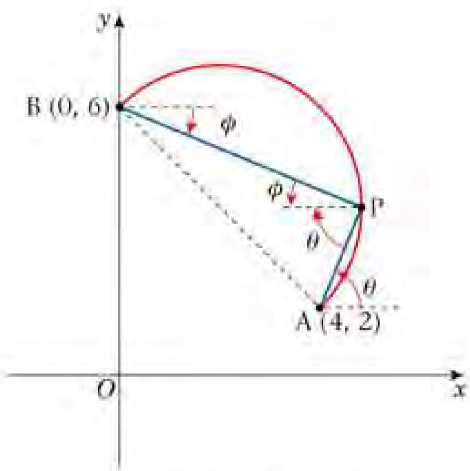
b deduce the exact value of $|z-2-4\mathrm{i}|$.

Solution:

a $\arg\left(\frac{z-4-2\mathrm{i}}{z-6\mathrm{i}}\right)=\frac{\pi}{2}$

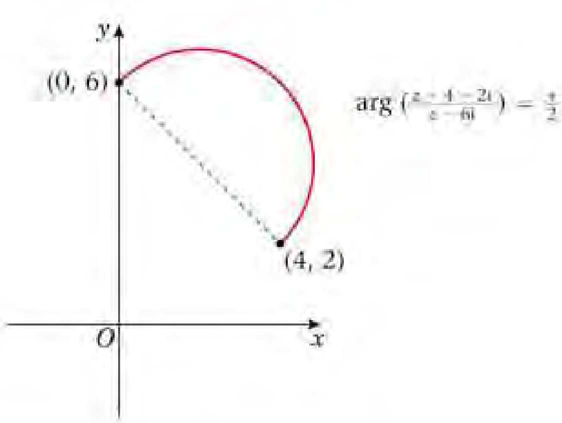
$\Rightarrow \arg(z-4-2\mathrm{i})-\arg(z-6\mathrm{i})=\frac{\pi}{2}$

$\Rightarrow \theta-\phi=\frac{\pi}{2}$, where $\arg(z-4-2\mathrm{i})=\theta$ and $\arg(z-6\mathrm{i})=\phi$.



Using geometry,
$\Rightarrow A\hat{P}B=-\phi+\theta$
$\Rightarrow A\hat{P}B=\theta-\phi$
$\Rightarrow A\hat{P}B=\frac{\pi}{2}$

The locus of z is the arc of a circle (in this case, a semi-circle) cut off at $(4, 2)$ and $(0, 6)$ as shown below.



b $|z - 2 - 4\text{i}|$ is the distance from the point (2, 4) to the locus of points P.

Note, as the locus is a semi-circle, its centre is $\left(\frac{4 + 0}{2}, \frac{2 + 6}{2}\right) = (2, 4)$.

Therefore $|z - 2 - 4\text{i}|$ is the distance from the centre of the semi-circle to points on the locus of points P.

Hence $|z - 2 - 4\text{i}|$ = radius of semi-circle

$$\begin{aligned} &= \sqrt{(0-2)^2 + (6-4)^2} \\ &= \sqrt{4+4} \\ &= \sqrt{8} \\ &= 2\sqrt{2} \end{aligned}$$

The exact value of $|z - 2 - 4\text{i}|$ is $2\sqrt{2}$

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Exercise I, Question 10

Question:

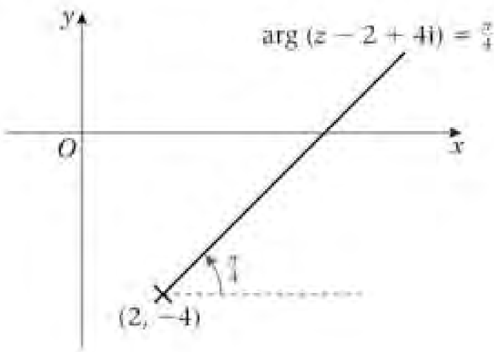
Given that $\arg(z - 2 + 4\mathrm{i}) = \frac{\pi}{4}$,

a sketch the locus of $P(x, y)$ which represents z on an Argand diagram,

b find the minimum value of $|z|$ for points on this locus.

Solution:

a $\arg(z-2+4\text{i}) = \frac{\pi}{4}$ is a half-line from $(2, -4)$ as shown



b $\arg(z-2+4\text{i}) = \frac{\pi}{4} \Rightarrow \arg(x+\text{i}y-2+4\text{i}) = \frac{\pi}{4}$

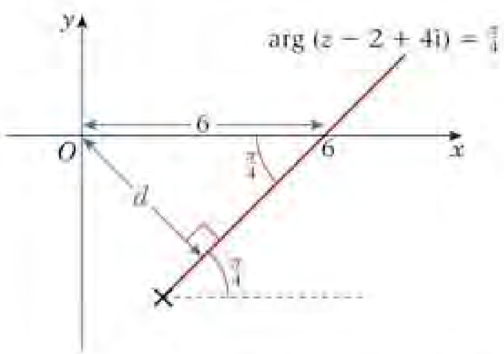
$$\Rightarrow \arg((x-2)+\text{i}(y+4)) = \frac{\pi}{4}$$

$$\Rightarrow \frac{y+4}{x-2} = \tan\frac{\pi}{4} = 1$$

$$\Rightarrow y+4 = x-2$$

$$\Rightarrow y = x-6, x > 0, y > 0$$

Half-line cuts x-axis at $0 = x-6 \Rightarrow x = 6$.



$|z|$ is the distance from $(0, 0)$ to the locus of points.

$|z|_{\min} = d \Rightarrow \frac{d}{6} = \sin\left(\frac{\pi}{4}\right) \Rightarrow d = 6\sin\left(\frac{\pi}{4}\right) = 6\left(\frac{1}{\sqrt{2}}\right) = \frac{6\sqrt{2}}{2} = 3\sqrt{2}$.

Therefore the minimum value of $|z|$ is $3\sqrt{2}$.

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Exercise I, Question 11

Question:

The transformation T from the z-plane, where $z = x + iy$, to the w-plane where $w = u + iv$, is given by $w = \frac{1}{z}$, $z \neq 0$.

a Show that the image, under T, of the line with equation $x = \frac{1}{2}$ in the z-plane is a circle C in the w-plane. Find the equation of C.

b Hence, or otherwise, shade and label on an Argand diagram the region R of the w-plane which is the image of $x \geqslant \frac{1}{2}$ under T.

Solution:

T: $w = \frac{1}{z}$

a line $x = \frac{1}{2}$ in the z-plane

$w = \frac{1}{z}$

$\Rightarrow wz = 1$

$\Rightarrow z = \frac{1}{w}$

$\Rightarrow z = \frac{1}{u + iv}$

$\Rightarrow z = \frac{1}{(u + iv)} \times \frac{(u - iv)}{(u - iv)}$

$\Rightarrow z = \frac{u - iv}{u^2 + v^2}$

$\Rightarrow z = \frac{u}{u^2 + v^2} + i\left(\frac{-v}{u^2 + v^2}\right)$

So, $x + iy = \frac{u}{u^2 + v^2} + i\left(\frac{-v}{u^2 + v^2}\right)$

$\Rightarrow x = \frac{u}{u^2 + v^2}$ and $y = \frac{-v}{u^2 + v^2}$

As $x = \frac{1}{2}$, then $\frac{1}{2} = \frac{u}{u^2 + v^2}$

$\Rightarrow u^2 + v^2 = 2u$

$\Rightarrow u^2 - 2u + v^2 = 0$

$\Rightarrow (u - 1)^2 - 1 + v^2 = 0$

$\Rightarrow (u - 1)^2 + v^2 = 1$

Therefore the transformation T maps the line $x = \frac{1}{2}$ in the z-plane to a circle C, with centre $(1, 0)$, radius 1. The equation of C is $(u - 1)^2 + v^2 = 1$.

b $x \geqslant \frac{1}{2}$ $\frac{u}{u^2 + v^2} \geqslant \frac{1}{2}$

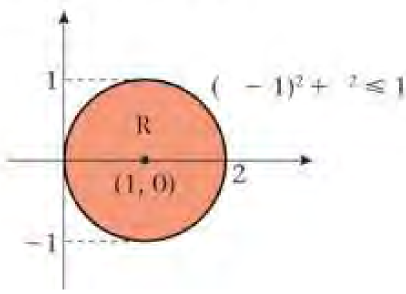
$\Rightarrow 2u \geqslant u^2 + v^2$

$\Rightarrow 0 \geqslant u^2 + v^2 - 2u$

$\Rightarrow 0 \geqslant (u - 1)^2 + v^2 - 1$

$\Rightarrow 1 \geqslant (u - 1)^2 + v^2$

$\Rightarrow (u - 1)^2 + v^2 \leqslant 1$



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Exercise I, Question 12

Question:

The point P represents the complex number z on an Argand diagram.
Given that $|z + 4\text{i}| = 2$,

a sketch the locus of P on an Argand diagram.

b Hence find the maximum value of $|z|$.

T_1, T_2, T_3 and T_4 represent transformations from the z-plane to the w-plane. Describe the locus of the image of P under the transformations

c T_1: $w = 2z$,

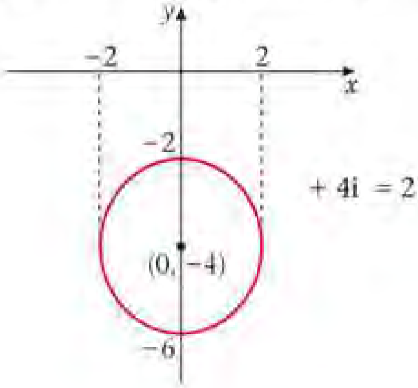
d T_2: $w = \text{i}z$,

e T_3: $w = -\text{i}z$,

f T_4: $w = z^*$

Solution:

a $|z + 4\mathrm{i}| = 2$ is represented by a circle centre $(0, -4)$, radius 2.



b $|z|$ represents the distance from $(0, 0)$ to points on the locus of P. Hence $|z|_{max}$ is the distance OY.
$|z|_{max} = OY = 6$.

c T_1: $w = 2z$

METHOD ① z lies on circle with equation $|z + 4\mathrm{i}| = 2$

$$\Rightarrow w = 2z$$
$$\Rightarrow \frac{w}{2} = z$$
$$\Rightarrow \frac{w}{2} + 4\mathrm{i} = z + 4\mathrm{i}$$
$$\Rightarrow \frac{w + 8\mathrm{i}}{2} = z + 4\mathrm{i}$$
$$\Rightarrow \left|\frac{w + 8\mathrm{i}}{2}\right| = |z + 4\mathrm{i}|$$
$$\Rightarrow \frac{|w + 8\mathrm{i}|}{|2|} = |z + 4\mathrm{i}|$$
$$\Rightarrow \frac{|w + 8\mathrm{i}|}{2} = 2$$
$$\Rightarrow |w + 8\mathrm{i}| = 4$$

So the locus of the image of P under T_1 is a circle centre $(0, -8)$, radius 4, with equation $u^2 + (v + 8)^2 = 16$.

METHOD ② z lies on circle centre $(0, -4)$, radius 2

$\downarrow$ $\boxed{\text{enlargement scale factor 2, centre 0.}}$

$w = 2z$ lies on a circle centre $(0, -8)$, radius 4.

So the locus of the image of P under T_1 is a circle centre $(0, -8)$, radius 4, with equation $u^2 + (v + 8)^2 = 16$.

d T_2: $w = \mathrm{i}z$

z lies on a circle with equation $|z + 4\mathrm{i}| = 2$

$w = \mathrm{i}z$

$\Rightarrow \frac{w}{\mathrm{i}} = z$

$\Rightarrow \frac{w}{\mathrm{i}}\left(\frac{\mathrm{i}}{\mathrm{i}}\right) = z$

$\Rightarrow \frac{w\mathrm{i}}{(-1)} = z$

$\Rightarrow -w\mathrm{i} = z$

$\Rightarrow z = -w\mathrm{i}$

Hence $|z + 4\mathrm{i}| = 2 \Rightarrow |-w\mathrm{i} + 4\mathrm{i}| = 2$

$\Rightarrow |(-\mathrm{i})(w - 4)| = 2$

$\Rightarrow |(-\mathrm{i})|\,|w - 4| = 2$

$\Rightarrow |w - 4| = 2$

So the locus of the image of P under T_2 is a circle centre $(4, 0)$, radius 2, with equation $(u - 4)^2 + v^2 = 4$.

e T_3: $w = -\mathrm{i}z$

z lies on a circle with equation $|z + 4\mathrm{i}| = 2$

$w = -\mathrm{i}z$

$\Rightarrow \mathrm{i}w = \mathrm{i}(-\mathrm{i}z)$

$\Rightarrow \mathrm{i}w = z$

$\Rightarrow z = \mathrm{i}w$

Hence $|z + 4\mathrm{i}| = 2 \Rightarrow |\mathrm{i}w + 4\mathrm{i}| = 2$

$\Rightarrow |\mathrm{i}(w + 4)| = 2$

$\Rightarrow |\mathrm{i}||w + 4| = 2$

$\Rightarrow |w + 4| = 2$ $\quad$ $\boxed{|\mathrm{i}| = 1}$

So the locus of the image of P under T_3 is a circle centre $(-4, 0)$, radius 2, with equation $(u + 4)^2 + v^2 = 4$.

f T_4: $w = z^*$

z lies on a circle with equation $|z + 4\mathrm{i}| = 2$

$w = z^* \Rightarrow u + \mathrm{i}v = x - \mathrm{i}y$ $\quad$ $\boxed{z = x + \mathrm{i}y \\ \Rightarrow z^* = x - \mathrm{i}y}$

So $\quad u = x,\ v = -y \quad$ and $\quad x = u$ and $y = -v$

$|z + 4\mathrm{i}| = 2 \Rightarrow |x + \mathrm{i}y + 4\mathrm{i}| = 2$

$\Rightarrow |x + \mathrm{i}(y + 4)| = 2$

$\Rightarrow |u + \mathrm{i}(-v + 4)| = 2$

$\Rightarrow |u + \mathrm{i}(4 - v)| = 2$

$\Rightarrow |u + \mathrm{i}(4 - v)|^2 = 2^2$

$\Rightarrow u^2 + (4 - v)^2 = 4$

$\Rightarrow u^2 + (v - 4)^2 = 4$

So the locus of the image of P under T_4 is a circle centre $(0, 4)$, radius 2, with equation $u^2 + (v - 4)^2 = 4$.

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Exercise I, Question 13

Question:

The transformation T from the z-plane, where $z = x + \mathrm{i}y$, to the w-plane where $w = u + \mathrm{i}v$, is given by $w = \frac{z + 2}{z + \mathrm{i}}$, $z \neq -\mathrm{i}$.

a Show that the image, under T, of the imaginary axis in the z-plane is a line l in the w-plane. Find the equation of l.

b Show that the image, under T, of the line $y = x$ in the z-plane is a circle C in the w-plane. Find the centre of C and show that the radius of C is $\frac{1}{2}\sqrt{10}$.

Solution:

T: $w = \dfrac{z+2}{z+i}, z \neq -i$

a the imaginary axis in z-plane $\Rightarrow x = 0$

$$w = \frac{z+2}{z+i}$$

$$\Rightarrow w(z+i) = z+2$$

$$\Rightarrow wz + iw = z+2$$

$$\Rightarrow wz - z = 2 - iw$$

$$\Rightarrow z(w-1) = 2 - iw$$

$$\Rightarrow z = \frac{2 - iw}{w - i}$$

$$\Rightarrow z = \frac{2 - i(u+iv)}{u + iv - 1}$$

$$\Rightarrow z = \frac{2 - iu + v}{(u-1) + iv}$$

$$\Rightarrow z = \left[\frac{(2+v) - iu}{(u-1) + iv}\right] \times \left[\frac{(u-1) - iv}{(u-1) - iv}\right]$$

$$\Rightarrow z = \frac{(2+v)(u-1) - uv - iv(2+v) - iu(u-1)}{(u-1)^2 + v^2}$$

$$\Rightarrow z = \frac{(2+v)(u-1) - uv}{(u-1)^2 + v^2} - i\left(\frac{v(2+v) + u(u-1)}{(u-1)^2 + v^2}\right)$$

So $x + iy = \dfrac{(2+v)(u-1) - uv}{(u-1)^2 + v^2} - i\left(\dfrac{v(2+v) + u(u-1)}{(u-1)^2 + v^2}\right)$

$$\Rightarrow x = \frac{(2+v)(u-1) - uv}{(u-1)^2 + v^2} \text{ and } y = \frac{-v(2+v) - u(u-1)}{(u-1)^2 + v^2}$$

As $x = 0$, then

$$\frac{(2+v)(u-1) - uv}{(u-1)^2 + v^2} = 0$$

$$\Rightarrow (2+v)(u-1) - uv = 0$$

$$\Rightarrow 2u - 2 + vu - v - uv = 0$$

$$\Rightarrow 2u - 2 - v = 0$$

$$\Rightarrow v = 2u - 2$$

The transformation T maps the imaginary axis in the z-plane to the line l with equation $v = 2u - 2$ in the w-plane.

b As $y = x$, then

$$\frac{-v(2+v) - u(u-1)}{(u-1)^2 + v^2} = \frac{(2+v)(u-1) - uv}{(u-1)^2 + v^2}$$

$$\Rightarrow -v(2+v) - u(u-1) = (2+v)(u-1) - uv$$

$$\Rightarrow -2v - v^2 - u^2 + u = 2u - 2 + vu - v - uv$$

$$\Rightarrow -2v - v^2 - u^2 + u = 2u - 2 - v$$

$$\Rightarrow 0 = u^2 + v^2 + u + v - 2$$

$$\Rightarrow \left(u + \tfrac{1}{2}\right)^2 - \tfrac{1}{4} + \left(v + \tfrac{1}{2}\right)^2 - \tfrac{1}{4} - 2 = 0$$

$$\sqrt{\frac{5}{2}} = \frac{\sqrt{5}}{\sqrt{2}}$$

$$\sqrt{5}\,\sqrt{2}$$

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Exercise I, Question 14

Question:

The transformation T from the z-plane, where $z = x + \mathrm{i}y$ to the w-plane where $w = u + \mathrm{i}v$, is given by $w = \frac{4 - z}{z + \mathrm{i}}$, $z \neq -\mathrm{i}$.

The circle $|z| = 1$ is mapped by T onto a line l. Show that l can be written in the form $au + bv + c = 0$, where a, b and c are integers to be determined.

Solution:

T: $\quad w = \frac{4 - z}{z + \mathrm{i}} \quad z \neq -\mathrm{i}$

circle with equation $|z| = 1$ in the z-plane.

$w = \frac{4 - z}{z + \mathrm{i}}$

$\Rightarrow w(z + \mathrm{i}) = 4 - z$

$\Rightarrow wz + \mathrm{i}w = 4 - z$

$\Rightarrow wz + z = 4 - \mathrm{i}w$

$\Rightarrow z(w + 1) = 4 - \mathrm{i}w$

$\Rightarrow z = \frac{4 - \mathrm{i}w}{w + 1}$

$\Rightarrow |z| = \left|\frac{4 - \mathrm{i}w}{w + 1}\right|$

$\Rightarrow |z| = \frac{|4 - \mathrm{i}w|}{|w + 1|}$

Applying $|z| = 1$ gives $1 = \frac{|4 - \mathrm{i}w|}{|w + 1|}$

$\Rightarrow |w + 1| = |4 - \mathrm{i}w|$

$\Rightarrow |w + 1| = |-\mathrm{i}(w + 4\mathrm{i})|$

$\Rightarrow |w + 1| = |-\mathrm{i}|\,|w + 4\mathrm{i}|$

$\Rightarrow |w + 1| = |w + 4\mathrm{i}|$

$\Rightarrow |u + \mathrm{i}v + 1| = |u + \mathrm{i}v + 4\mathrm{i}|$

$\Rightarrow |(u + 1) + \mathrm{i}v| = |u + \mathrm{i}(v + 4)|$

$\Rightarrow |(u + 1) + \mathrm{i}v|^2 = |u + \mathrm{i}(v + 4)|^2$

$\Rightarrow (u + 1)^2 + v^2 = u^2 + (v + 4)^2$

$\Rightarrow u^2 + 2u + 1 + v^2 = u^2 + v^2 + 8v + 16$

$\Rightarrow 2u + 1 = 8v + 16$

$\Rightarrow 2u - 8v - 15 = 0$

The circle $|z| = 1$ is mapped by T onto the line l: $2u - 8v - 15 = 0$
(i.e. $a = 2$, $b = -8$, $c = -15$).

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Exercise I, Question 15

Question:

The transformation T from the z-plane, where $z = x + iy$, to the w-plane where $w = u + iv$, is given by $w = \frac{3iz + 6}{1 - z}$, $z \neq 1$.

Show that the circle $|z| = 2$ is mapped by T onto a circle C. State the centre of C and show that the radius of C can be expressed in the form $k\sqrt{5}$ where k is an integer to be determined.

Solution:

T: $w = \dfrac{3\mathrm{i}z + 6}{1 - z}$; $z \neq 1$

circle with equation $|z| = 2$

$$w = \frac{3\mathrm{i}z + 6}{1 - z}$$
$$\Rightarrow w(1 - z) = 3\mathrm{i}z + 6$$
$$\Rightarrow w - wz = 3\mathrm{i}z + 6$$
$$\Rightarrow w - 6 = 3\mathrm{i}z + wz$$
$$\Rightarrow w - 6 = z(3\mathrm{i} + w)$$
$$\Rightarrow \frac{w - 6}{w + 3\mathrm{i}} = z$$
$$\Rightarrow \left|\frac{w - 6}{w + 3\mathrm{i}}\right| = |z|$$
$$\Rightarrow \frac{|w - 6|}{|w + 3\mathrm{i}|} = |z|$$

Applying $|z| = 2 \Rightarrow \dfrac{|w - 6|}{|w + 3\mathrm{i}|} = 2$

$$\Rightarrow |w - 6| = 2|w + 3\mathrm{i}|$$
$$\Rightarrow |u + \mathrm{i}v - 6| = 2|u + \mathrm{i}v + 3\mathrm{i}|$$
$$\Rightarrow |(u - 6) + \mathrm{i}v| = 2|u + \mathrm{i}(v + 3)|$$
$$\Rightarrow |(u - 6) + \mathrm{i}v|^2 = 2^2|u + \mathrm{i}(v + 3)|^2$$
$$\Rightarrow (u - 6)^2 + v^2 = 4[u^2 + (v + 3)^2]$$
$$\Rightarrow u^2 - 12u + 36 + v^2 = 4[u^2 + v^2 + 6v + 9]$$
$$\Rightarrow u^2 - 12u + 36 + v^2 = 4u^2 + 4v^2 + 24v + 36$$
$$\Rightarrow 0 = 3u^2 + 12u + 3v^2 + 24v$$
$$\Rightarrow 0 = u^2 + 4u + v^2 + 8v$$
$$\Rightarrow 0 = (u + 2)^2 - 4 + (v + 4)^2 - 16$$
$$\Rightarrow 20 = (u + 2)^2 + (v + 4)^2$$
$$\Rightarrow (u + 2)^2 + (v + 4)^2 = (2\sqrt{5})^2$$

$\boxed{\sqrt{20} = \sqrt{4}\sqrt{5} = 2\sqrt{5}}$

Therefore the circle with equation $|z| = 2$ is mapped onto a circle C, centre $(-2, -4)$, radius $2\sqrt{5}$. So $k = 2$.

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Exercise I, Question 16

Question:

A transformation from the z-plane to the w-plane is defined by $w = \frac{az + b}{z + c}$,
where $a, b, c \in \mathbb{R}$.

Given that $w = 1$ when $z = 0$ and that $w = 3 - 2\text{i}$ when $z = 2 + 3\text{i}$,

a find the values of a, b and c,

b find the exact values of the two points in the complex plane which remain invariant under the transformation.

Solution:

a $w = \dfrac{az + b}{z + c} \quad a, b, c \in \mathbb{R}.$

$w = 1$ when $z = 0$ ①

$w = 3 - 2\text{i}$ when $z = 2 + 3\text{i}$ ②

① $\Rightarrow 1 = \dfrac{a(0) + b}{0 + c} \Rightarrow 1 = \dfrac{b}{c} \Rightarrow c = 6$ ③

③ $\Rightarrow w = \dfrac{az + b}{z + b}$

② $\Rightarrow 3 - 2\text{i} = \dfrac{a(2 + 3\text{i}) + b}{2 + 3\text{i} + b}$

$3 - 2\text{i} = \dfrac{(2a + b) + 3a\text{i}}{(2 + b) + 3\text{i}}$

$(3 - 2\text{i})[(2 + b) + 3\text{i}] = 2a + b + 3a\text{i}$

$6 + 3b + 9\text{i} - 4\text{i} - 2b\text{i} + 6 = 2a + b + 3a\text{i}$

$(12 + 3b) + (5 - 2b) = (2a + b) + 3a\text{i}$

Equate real parts: $12 + 3b = 2a + b$

$\Rightarrow \quad 12 = 2a - 2b$ ④

Equate imaginary parts: $5 - 2b = 3a$

$\Rightarrow \quad 5 = 3a + 2b$ ⑤

④ + ⑤: $\quad 17 = 5a$

$\Rightarrow \quad \dfrac{17}{5} = a$

⑤ $\quad \Rightarrow \quad 5 = \dfrac{51}{5} + 2b$

$\Rightarrow \quad \dfrac{-26}{5} = 2b$

$\Rightarrow \quad \dfrac{-13}{5} = b$

As $b = c$ then $c = \dfrac{-13}{5}$

The values are $a = \dfrac{17}{5}, b = \dfrac{-13}{5}, c = \dfrac{-13}{5}$

b $w = \dfrac{\frac{17}{5}z - \frac{13}{5}}{z - \frac{13}{5}} \quad \begin{matrix}(\times 5)\\(\times 5)\end{matrix}$

$w = \dfrac{17z - 13}{5z - 13}$

invariant points $\Rightarrow z = \dfrac{17z - 13}{5z - 13}$

$z(5z - 13) = 17z - 13$

$5z^2 - 13z = 17z - 13$

$5z^2 - 30z + 13 = 0$

$z = \dfrac{30 \pm \sqrt{900 - 4(5)(13)}}{10}$

$z = \dfrac{30 \pm \sqrt{900 - 260}}{}$

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Exercise I, Question 17

Question:

The transformation T from the z-plane, where $z = x + \mathrm{i}y$, to the w-plane where $w = u + \mathrm{i}v$, is given by

$$w = \frac{z + \mathrm{i}}{z}, z \neq 0.$$

a The transformation T maps the points on the line with equation $y = x$ in the z-plane other than $(0, 0)$, to points on the l in the w-plane. Find an equation of l.

b Show that the image, under T, of the line with equation $x + y + 1 = 0$ in the z-plane is a circle C in the w-plane, where C has equation $u^2 + v^2 - u + v = 0$.

c On the same Argand diagram, sketch l and C.

Solution:

$T: \quad w = \frac{z + i}{z}, \quad z \neq 0.$

a the line $y = x$ in the z-plane other than $(0, 0)$

$$w = \frac{z + i}{z}$$

$$\Rightarrow wz = z + i$$

$$\Rightarrow wz - z = i$$

$$\Rightarrow z(w - 1) = i$$

$$\Rightarrow z = \frac{i}{w - 1}$$

$$\Rightarrow z = \frac{i}{(u + iv) - 1} = \frac{i}{(u - 1) + iv}$$

$$\Rightarrow z = \left[\frac{i}{(u - 1) + iv}\right]\left[\frac{(u - 1) - iv}{(u - 1) - iv}\right]$$

$$\Rightarrow z = \frac{i(u - 1) + v}{(u - 1)^2 + v^2}$$

$$\Rightarrow z = \frac{v}{(u - 1)^2 + v^2} + i\,\frac{(u - 1)}{(u - 1)^2 + v^2}$$

So $x + iy = \frac{v}{(u - 1)^2 + v^2} + i\,\frac{(u - 1)}{(u - 1)^2 + v^2}$

$\Rightarrow x = \frac{v}{(u - 1)^2 + v^2}$ and $y = \frac{u - 1}{(u - 1)^2 + v^2}$

Applying $y = x$, gives $\frac{u - 1}{(u - 1)^2 + v^2} = \frac{v}{(u - 1)^2 + v^2}$

$$\Rightarrow u - 1 = v$$

$$\Rightarrow v = u - 1$$

Therefore the line l has equation $v = u - 1$.

b the line with equation $x + y + 1 = 0$ in the z-plane

$x + y + 1 = 0 \Rightarrow \frac{v}{(u - 1)^2 + v^2} + \frac{u - 1}{(u - 1)^2 + v^2} + 1 = 0 \; [\times(u - 1)^2 + v^2]$

$$\Rightarrow v + (u - 1) + (u - 1)^2 + v^2 = 0$$

$$\Rightarrow v + u - 1 + u^2 - 2u + 1 + v^2 = 0$$

$$\Rightarrow u^2 + v^2 - u + v = 0$$

$$\Rightarrow \left(u - \frac{1}{2}\right)^2 - \frac{1}{4} + \left(v + \frac{1}{2}\right)^2 - \frac{1}{4} = 0$$

$$\Rightarrow \left(u - \frac{1}{2}\right)^2 + \left(v + \frac{1}{2}\right)^2 = \frac{1}{2}$$

$$\Rightarrow \left(u - \frac{1}{2}\right)^2 + \left(v + \frac{1}{2}\right)^2 = \left(\frac{\sqrt{2}}{2}\right)^2$$

The image of $x + y + 1 = 0$ under T is a circle C, centre $\left(\frac{1}{2}, \frac{-1}{2}\right)$, radius $\frac{\sqrt{2}}{2}$ with equation $u^2 + v^2 - u + v = 0$, as required.

c

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Exercise A, Question 1

Question:

$$\frac{dy}{dx} = 2x$$

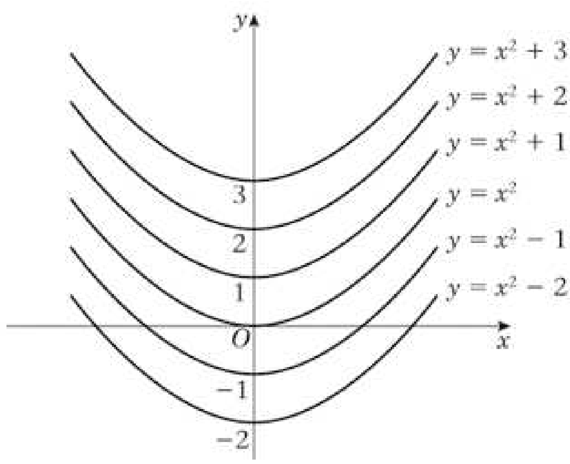
Solution:

$$\frac{dy}{dx} = 2x$$

$$\therefore \quad y = \int 2x \, dx$$

Integrate and include the constant of integration.
Let the constant take values 1, 2, 3, 0, −1, −2 and draw solution curves.

$$\therefore \quad y = x^2 + c \quad \text{where } c \text{ is constant}$$



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Exercise A, Question 2

Question:

$$\frac{dy}{dx} = y$$

Solution:

$$\frac{dy}{dx} = y$$

$$\therefore \int \frac{1}{y}\, dy = \int 1\, dx$$

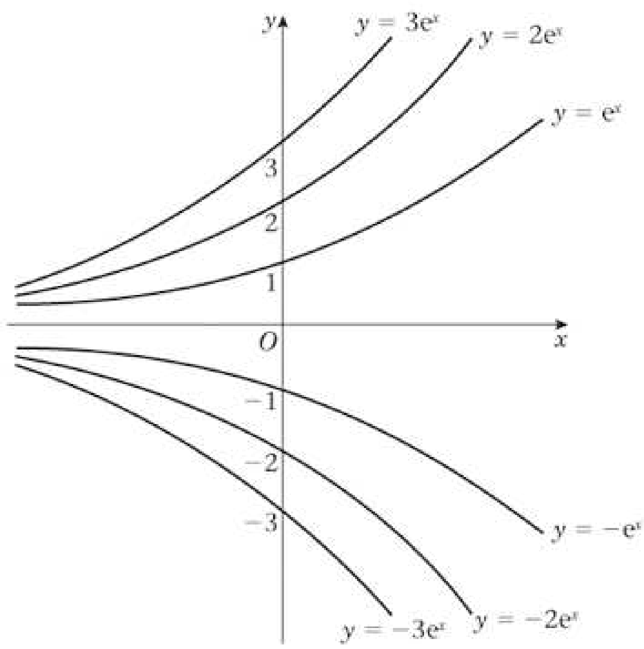
Separate the variables and integrate. Include a constant of integration on one side of the equation.

$\therefore \quad \ln y = x + c \quad$ where c is constant

$$y = e^{x+c}$$

$$= e^c \times e^x$$

$y = Ae^x \quad$ where A is constant $(A = e^c)$



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Exercise A, Question 3

Question:

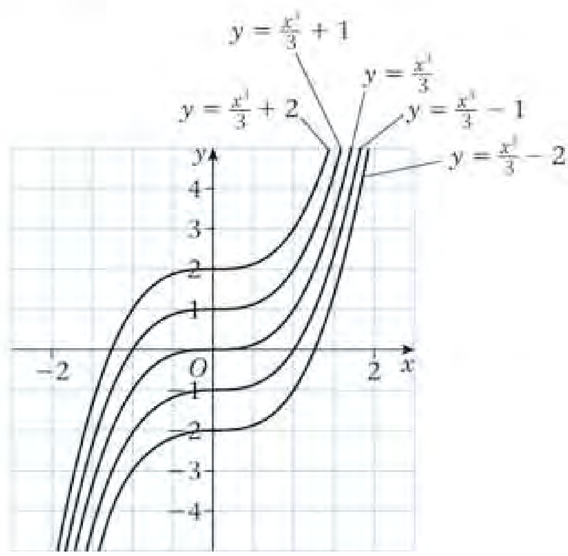
$$\frac{dy}{dx} = x^2$$

Solution:

$$\frac{dy}{dx} = x^2$$

$$y = \int x^2 \, dx$$

$$y = \frac{x^3}{3} + c \quad \text{where } c \text{ is constant}$$



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Exercise A, Question 4

Question:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{x},\ x > 0$$

Solution:

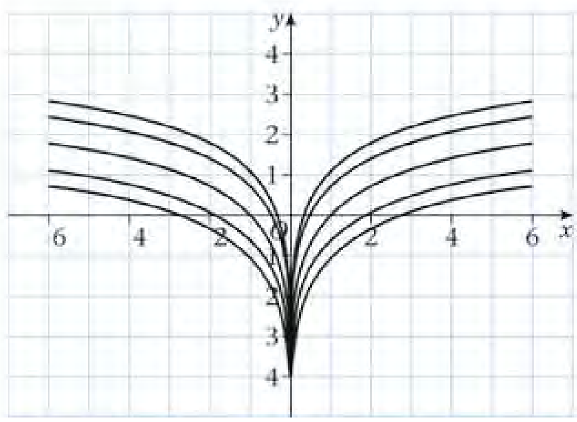
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{x}$$

$$\therefore \quad y = \int \frac{1}{x}\,\mathrm{d}x$$

$$= \ln x + c$$

$$= \ln x + \ln A$$

$$\therefore \quad y = \ln Ax$$



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Exercise A, Question 5

Question:

$$\frac{dy}{dx}=\frac{2y}{x}$$

Solution:

$$\frac{dy}{dx}=\frac{2y}{x}$$

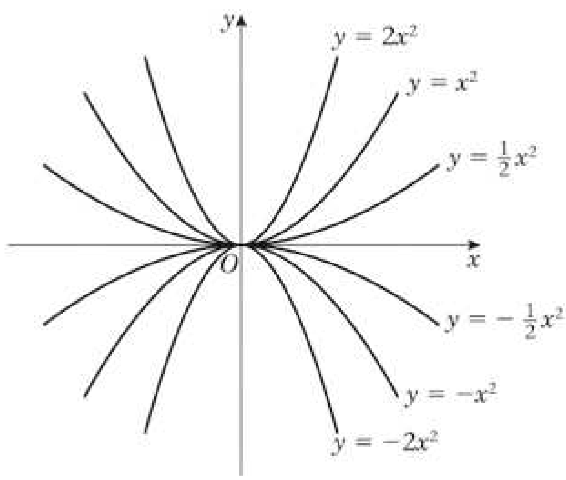
$\therefore \int \frac{1}{y}\,dy=\int \frac{2}{x}\,dx$ — Separate the variables and integrate.

$\therefore \quad \ln y = 2\ln x + c$

$\therefore \quad \ln y = \ln x^2 + \ln A$ — Express the constant of integration as $\ln A$ where A is constant and use laws of logs to simplify your answer.

$\quad = \ln Ax^2$

$\therefore \quad y = Ax^2$



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Exercise A, Question 6

Question:

$$\frac{dy}{dx} = \frac{x}{y}$$

Solution:

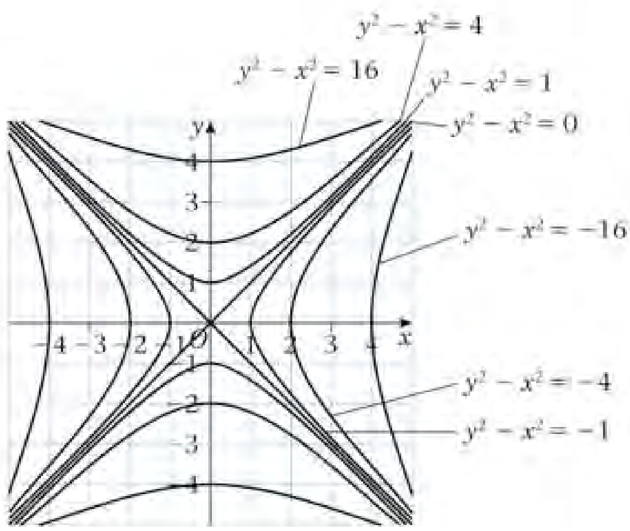
$$\frac{dy}{dx} = \frac{x}{y}$$

$$\therefore \quad \int y\,dy = \int x\,dx$$

$$\therefore \quad \frac{y^2}{2} = \frac{x^2}{2} + c$$

or $y^2 - x^2 = 2c$

$y^2 - x^2 = 0$ is a pair of straight lines. These are $y = x$ and $y = -x$
$y^2 - x^2 = 2c$, $c \neq 0$ is a hyperbola.



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Exercise A, Question 7

Question:

$$\frac{dy}{dx} = e^{y}$$

Solution:

$$\frac{dy}{dx} = e^{y}$$

$$\therefore \int \frac{1}{e^{y}} dy = \int 1 \, dx$$

To integrate $\frac{1}{e^{y}}$, express it as e^{-y}.

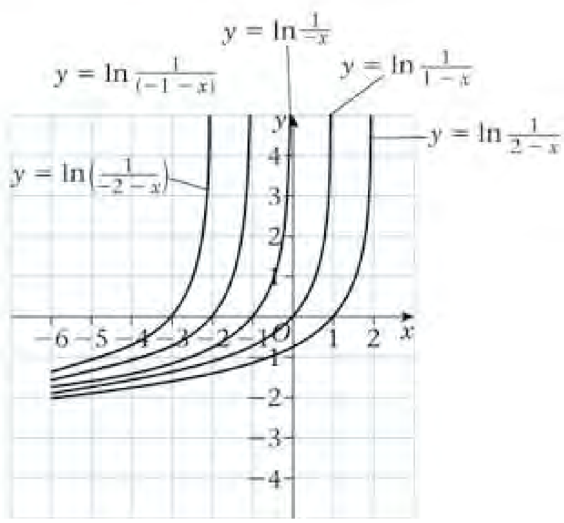
$$\therefore \int e^{-y} dy = \int 1 \, dx$$

$$\therefore \quad -e^{-y} = x + c$$

$$\therefore \quad -e^{-y} = -x - c$$

$$\therefore \quad -y = \ln[-x - c]$$

$$y = -\ln[-x - c] \text{ or } \ln\frac{1}{(-x - c)}$$



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Exercise A, Question 8

Question:

$$\frac{dy}{dx} = \frac{y}{x(x+1)}, \quad x > 0$$

Solution:

$$\frac{dy}{dx} = \frac{y}{x(x+1)}$$

$$\therefore \int \frac{1}{y}\,dy = \int \frac{1}{x(x+1)}\,dx$$

Separate the variables, then use partial fractions to integrate the function of x.

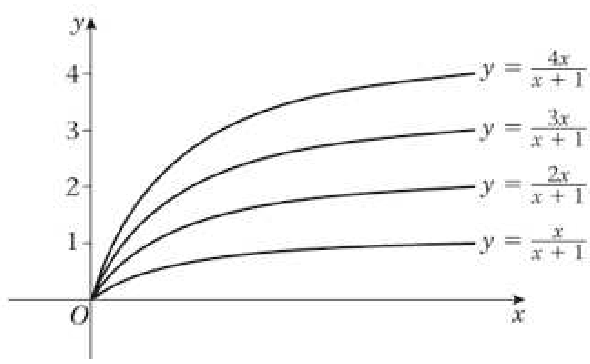
$$\therefore \quad \ln y = \int \left(\frac{1}{x} - \frac{1}{(x+1)}\right) dx$$

$$= \ln x - \ln(x+1) + c$$

$$\therefore \quad \ln y = \ln \frac{x}{x+1} + \ln A$$

$$= \ln \frac{Ax}{x+1}$$

$$\therefore \quad y = \frac{Ax}{x+1} \quad x > 0$$



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Exercise A, Question 9

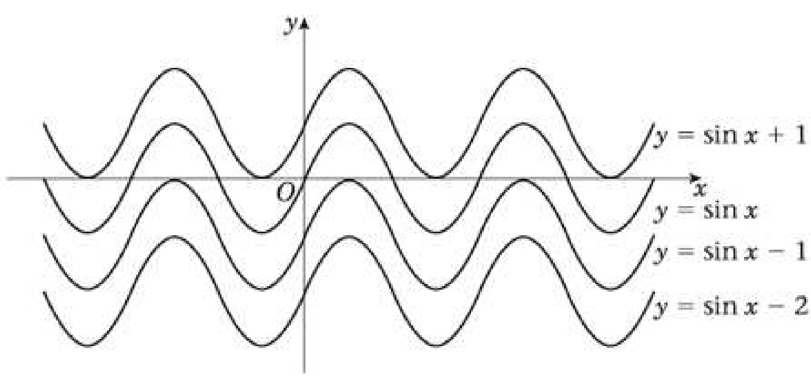
Question:

$$\frac{dy}{dx} = \cos x$$

Solution:

$$\frac{dy}{dx} = \cos x$$

$$\therefore \quad y = \sin x + c$$



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Exercise A, Question 10

Question:

$$\frac{dy}{dx} = y \cot x, \quad 0 < x < \pi$$

Solution:

$$\frac{dy}{dx} = y \cot x \qquad 0 < x < \pi$$

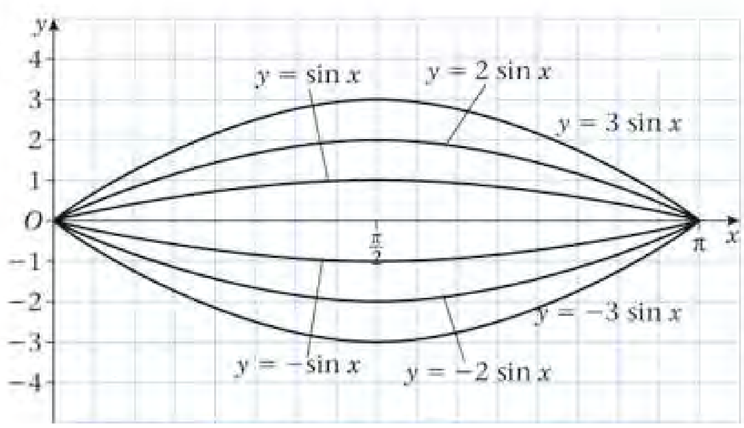
$$\therefore \int \frac{1}{y}\, dy = \int \frac{\cos x}{\sin x}\, dx$$

$$\therefore \quad \ln|y| = \ln|\sin x| + \ln|A|$$

$$= \ln|A \sin x|$$

$$\therefore \quad y = A \sin x$$

Express the constant of integration as $\ln|A|$ and combine logs to simplify your solution



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Exercise A, Question 11

Question:

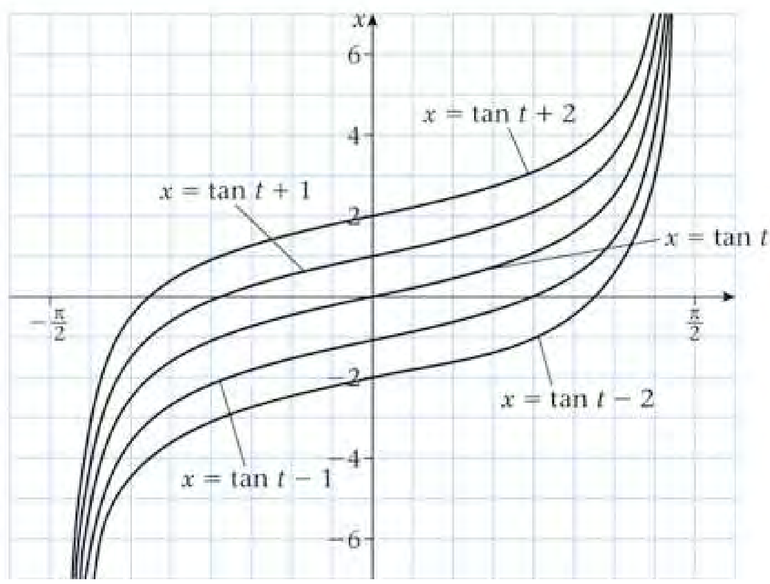
$$\frac{dy}{dx} = \sec^2 t, \ -\frac{\pi}{2} < t < \frac{\pi}{2}$$

Solution:

$$\frac{dx}{dt} = \sec^2 t \qquad -\frac{\pi}{2} < t < \frac{\pi}{2}$$

$$\therefore \quad x = \int \sec^2 t \, dt$$

i.e. $x = \tan t + c$ for $\quad -\frac{\pi}{2} < t < \frac{\pi}{2}$



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Exercise A, Question 12

Question:

$$\frac{dy}{dx} = x(1-x), \quad 0 < x < 1$$

Solution:

$$\frac{dy}{dx} = x(1-x)$$

$$\therefore \quad \int \frac{1}{x(1-x)}\,dx = \int 1\,dt$$

$$\therefore \int \left(\frac{1}{x} + \frac{1}{1-x}\right) dx = \int 1\,dt$$

$$\therefore \ln x - \ln(1-x) = t + c$$

$$\therefore \quad \ln\frac{x}{1-x} = t + c$$

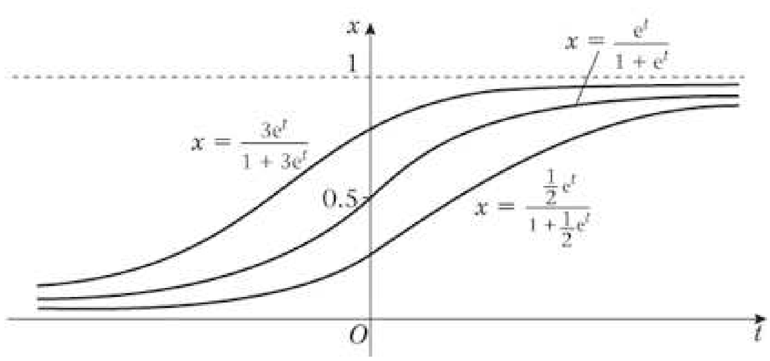
$$\therefore \quad \frac{x}{1-x} = e^{t+c} = Ae^t$$

$0 < x < 1$ implies that A is a positive constant.

$$\therefore \quad x = Ae^t - xAe^t$$

$$\therefore \quad x(1 + Ae^t) = Ae^t$$

$$x = \frac{Ae^t}{1 + Ae^t}$$



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Exercise A, Question 13

Question:

Given that a is an arbitrary constant, show that $y^2 = 4ax$ is the general solution of the differential equation $\frac{dy}{dx} = \frac{y}{2x}$.

a Sketch the members of the family of solution curves for which $a = \frac{1}{4}$, 1 and 4.

b Find also the particular solution, which passes through the point (1, 3), and add this curve to your diagram of solution curves.

Solution:

$$\frac{dy}{dx} = \frac{y}{2x}$$

$$\therefore \int \frac{1}{y}\,dy = \frac{1}{2}\int \frac{1}{x}\,dx$$

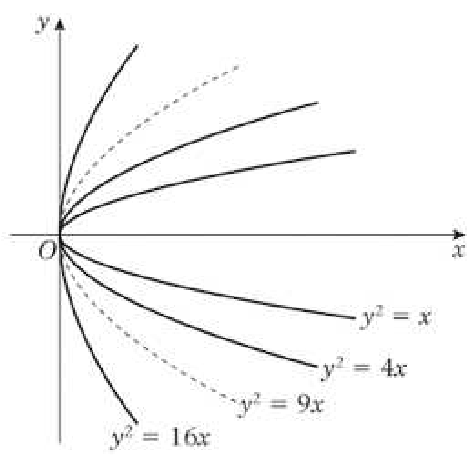
$$\therefore \quad \ln y = \frac{1}{2}\ln x + c$$

$$\text{or} \quad \ln y = \frac{1}{2}\ln x + \ln A$$

$$\therefore \quad \ln y = \ln A\sqrt{x}$$

$$\text{i.e.} \quad y = A\sqrt{x} \quad \text{or} \quad y^2 = A^2x \quad \text{or} \quad y^2 = 4ax$$

a Sketch $y^2 = x$, $y^2 = 4x$ and $y^2 = 16x$



b $y^2 = 4ax$ passes through (1, 3)

$$\therefore \quad 9 = 4a$$

$$\text{i.e.} \quad a = \frac{9}{4} \text{ and } y^2 = 9x$$

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Exercise A, Question 14

Question:

Given that k is an arbitrary positive constant, show that $y^2 + kx^2 = 9k$ is the general solution of the differential equation $\frac{dy}{dx} = \frac{-xy}{9 - x^2}$ $|x| \leq 3$.

a Find the particular solution, which passes through the point (2, 5).

b Sketch the family of solution curves for $k = \frac{1}{9}, \frac{4}{9}, 1$ and include your particular solution in the diagram.

Solution:

$\frac{dy}{dx} = \frac{-xy}{9-x^2}$

$\therefore \int \frac{1}{y}\,dy = -\int \frac{x}{9-x^2}\,dx$

$\therefore \quad \ln y = \frac{1}{2}\ln(9-x^2) + \ln A$

$\therefore \; 2\ln y = \ln A^2(9-x^2)$

$\therefore \quad \ln y^2 = \ln A^2(9-x^2)$

$\therefore \quad y^2 = 9A^2 - A^2x^2$

The solution curves are all ellipses, except when $k = 1$ when the curve is a circle.

Let $A^2 = k$

Then $y^2 + kx^2 = 9k$

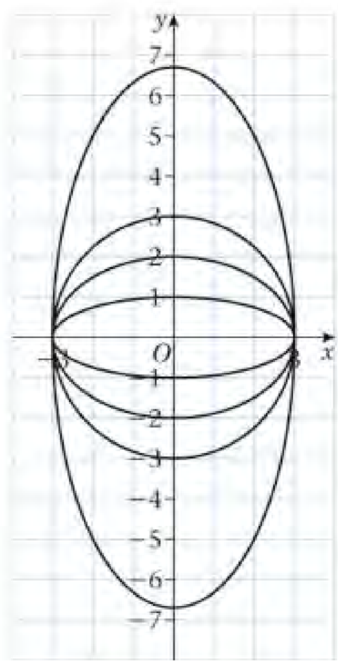
a If this curve passes through (2, 5) then

$25 + 4k = 9k$

$\therefore \quad 25 = 5k \rightarrow k = 5$

i.e. $y^2 + 5x^2 = 45$

b When $y = 0$ $x = \pm 3$, when $x = 0$ $y = \pm\sqrt{9k}$



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Exercise B, Question 1

Question:

$$x\frac{\mathrm{d}y}{\mathrm{d}x} + y = \cos x$$

Solution:

$$x\frac{\mathrm{d}y}{\mathrm{d}x} + y = \cos x$$

So $\frac{\mathrm{d}}{\mathrm{d}x}(xy) = \cos x$

$$\therefore \quad xy = \int \cos x \, \mathrm{d}x$$

$$= \sin x + c$$

Remember to add the constant of integration when you integrate – not at the end of the process.

$$\therefore \quad y = \frac{1}{x}\sin x + \frac{c}{x}$$

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Exercise B, Question 2

Question:

$$e^{-x}\frac{dy}{dx} - e^{-x}y = xe^{x}$$

Solution:

$$e^{-x}\frac{dy}{dx} - e^{-x}y = xe^{x}$$

$$\therefore \quad \frac{d}{dx}(e^{-x}y) = xe^{x}$$

$$\therefore \quad e^{-x}y = \int xe^{x}\,dx$$

Use integration by parts to integrate xe^x.

$$= xe^{x} - \int e^{x}\,dx$$

$$= xe^{x} - e^{x} + c$$

$$\therefore \quad y = xe^{2x} - e^{2x} + ce^{x}$$

Multiply integration by e^x.

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Exercise B, Question 3

Question:

$$\sin x \frac{dy}{dx} + y \cos x = 3$$

Solution:

$$\sin x \frac{dy}{dx} + y \cos x = 3$$

$$\therefore \quad \frac{d}{dx}(y \sin x) = 3$$

$$\therefore \quad y \sin x = \int 3 \, dx$$

$$\therefore \quad y \sin x = 3x + c$$

$$\therefore \quad y = \frac{3x}{\sin x} + \frac{c}{\sin x}$$

$$= 3x \operatorname{cosec} x + c \operatorname{cosec} x$$

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Exercise B, Question 4

Question:

$$\frac{1}{x}\frac{dy}{dx}-\frac{1}{x^2}y=e^x$$

Solution:

$$\frac{1}{x}\frac{dy}{dx}-\frac{1}{x^2}y=e^x$$

$$\therefore\ \frac{d}{dx}\left(\frac{1}{x}y\right)=e^x$$

$$\therefore\ \frac{1}{x}y=\int e^x\,dx$$

$$=e^x+c$$

$$\therefore\ y=xe^x+cx$$

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Exercise B, Question 5

Question:

$$x^2e^y\frac{dy}{dx} + 2xe^y = x$$

Solution:

$$x^2e^y\frac{dy}{dx} + 2xe^y = x$$

This time the left hand side is $\frac{d}{dx}(x^2\,f(y))$ not just $\frac{d}{dx}(x^2y)$.

$$\therefore \quad \frac{d}{dx}(x^2e^y) = x$$

$$\therefore \quad x^2e^y = \int x\,dx$$

$$= \frac{x^2}{2} + c$$

$$\therefore \quad e^y = \tfrac{1}{2} + \frac{c}{x^2}$$

or $\quad y = \ln\left[\tfrac{1}{2} + \frac{c}{x^2}\right]$

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Exercise B, Question 6

Question:

$$4xy\frac{dy}{dx} + 2y^2 = x^2$$

Solution:

$$4xy\frac{dy}{dx} + 2y^2 = x^2$$

Again the left hand side of the equation can be written $\frac{d}{dx}(2x\,f(y))$.

$$\therefore \quad \frac{d}{dx}(2xy^2) = x^2$$

$$\therefore \quad 2xy^2 = \int x^2\,dx$$

$$= \tfrac{1}{3}x^3 + c$$

$$\therefore \quad y^2 = \tfrac{1}{6}x^2 + \frac{c}{2x}$$

Divide both sides by $2x$.

$$\text{or} \quad y = \pm\sqrt{\left(\tfrac{1}{6}x^2 + \frac{c}{2x}\right)}$$

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Exercise B, Question 7

Question:

a Find the general solution of the differential equation

$$x^2 \frac{dy}{dx} + 2xy = 2x + 1.$$

b Find the three particular solutions which pass through the points with coordinates $(-\frac{1}{2}, 0)$, $(-\frac{1}{2}, 3)$ and $(-\frac{1}{2}, 19)$ respectively and sketch their solution curves for $x < 0$.

Solution:

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Exercise B, Question 8

Question:

a Find the general solution of the differential equation

$$\ln x \frac{dy}{dx} + \frac{y}{x} = \frac{1}{(x+1)(x+2)}, \qquad x > 1.$$

b Find the specific solution which passes through the point (2, 2).

Solution:

a $$\ln x \frac{dy}{dx} + \frac{y}{x} = \frac{1}{(x+1)(x+2)}$$

$$\therefore \frac{d}{dx}(\ln x \times y) = \frac{1}{(x+1)(x+2)}$$

$$\therefore \quad y \ln x = \int \frac{1}{(x+1)(x+2)}\,dx$$

You will need to use partial fractions to do the integration.

$$= \int \left(\frac{1}{x+1} - \frac{1}{x+2}\right) dx$$

$$= \ln(x+1) - \ln(x+2) + c$$

$$\therefore \quad y = \frac{\ln(x+1) - \ln(x+2) + \ln A}{\ln x}$$

$$\therefore \quad y = \frac{\ln \dfrac{A(x+1)}{(x+2)}}{\ln x}$$ is the general solution

b When $x = 2$, $y = 2$

$$\therefore \quad 2 = \frac{\ln \frac{3}{4}A}{\ln 2}$$

$$\therefore \quad \ln \tfrac{3}{4}A = 2\ln 2 = \ln 4$$

$$\therefore \quad A = \tfrac{16}{3}$$

So $$y = \frac{\ln \dfrac{16(x+1)}{3(x+2)}}{\ln x}$$

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Exercise C, Question 1

Question:

$$\frac{dy}{dx} + 2y = e^x$$

Solution:

$$\frac{dy}{dx} + 2y = e^x$$

The integrating factor is $e^{\int 2dx} = e^{2x}$

Find the integral factor $e^{\int p dx}$ and multiply the differential equation by it to give an exact equation.

$$\therefore e^{2x}\frac{dy}{dx} + 2e^{2x}y = e^{3x}$$

$$\therefore \quad \frac{d}{dx}(e^{2x}y) = e^{3x}$$

$$\therefore \quad e^{2x}y = \int e^{3x}\,dx$$

$$= \frac{1}{3}e^{3x} + c$$

$$\therefore \quad y = \frac{1}{3}e^{x} + ce^{-2x}$$

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Exercise C, Question 2

Question:

$$\frac{dy}{dx} + y \cot x = 1$$

Solution:

$$\frac{dy}{dx} + y \cot x = 1$$

The integrating factor is $e^{\int P dx} = e^{\int \cot x \, dx}$

$$= e^{\ln \sin x}$$

$$= \sin x$$

The integrating factor $e^{\ln f(x)}$ can be simplified to $f(x)$.

Multiply differential equation by $\sin x$.

$$\therefore \sin x \frac{dy}{dx} + y \cos x = \sin x$$

$$\therefore \quad \frac{d}{dx}(y \sin x) = \sin x$$

$$\therefore \quad y \sin x = \int \sin x \, dx$$

$$= -\cos x + c$$

$$\therefore \quad y = -\cot x + c \operatorname{cosec} x$$

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Exercise C, Question 3

Question:

$$\frac{dy}{dx} + y \sin x = e^{\cos x}$$

Solution:

$$\frac{dy}{dx} + y \sin x = e^{\cos x}$$

The integrating factor is $e^{\int \sin x \, dx} = e^{-\cos x}$

$\therefore \; e^{-\cos x}\frac{dy}{dx} + y \sin x e^{-\cos x} = 1$

$\therefore \quad \frac{d}{dx}(ye^{-\cos x}) = 1$

$\therefore \quad ye^{-\cos x} = x + c$

$\therefore \quad y = xe^{\cos x} + ce^{\cos x}$

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Exercise C, Question 4

Question:

$$\frac{dy}{dx} - y = e^{2x}$$

Solution:

$$\frac{dy}{dx} - y = e^{2x}$$

The integrating factor is $e^{\int -1\,dx} = e^{-x}$

Remember that $P(x) = -1$ and the minus sign is important.

$\therefore\ e^{-x}\frac{dy}{dx} - ye^{-x} = e^{2x} \times e^{-x}$

$\therefore\quad \frac{d}{dx}(ye^{-x}) = e^{x}$

$\therefore\quad ye^{-x} = \int e^{x}\,dx$

$= e^{x} + c$

$\therefore\quad y = e^{2x} + ce^{x}$

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Exercise C, Question 5

Question:

$$\frac{dy}{dx} + y\tan x = x\cos x$$

Solution:

$$\frac{dy}{dx} + y\tan x = x\cos x$$

The integrating factor is $e^{\int \tan x\, dx} = e^{\ln\sec x}$

$= \sec x$

Find the integrating factor and simplify $e^{\ln f(x)}$ to give $f(x)$.

$$\therefore\ \sec x\frac{dy}{dx} + y\sec x\tan x = x$$

$$\therefore\quad \frac{d}{dx}(y\sec x) = x$$

$$\therefore\quad y\sec x = \int x\, dx$$

$$= \tfrac{1}{2}x^2 + c$$

$$\therefore\quad y = \left(\tfrac{1}{2}x^2 + c\right)\cos x$$

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Exercise C, Question 6

Question:

$$\frac{dy}{dx}+\frac{y}{x}=\frac{1}{x^2}$$

Solution:

$$\frac{dy}{dx}+\frac{y}{x}=\frac{1}{x^2}$$

The integrating factor is $e^{\int \frac{1}{x}dx}=e^{\ln x}=x$

$$\therefore x\frac{dy}{dx}+y=\frac{1}{x}$$

$$\therefore \quad \frac{d}{dx}(xy)=\frac{1}{x}$$

$$\therefore \quad xy=\int\frac{1}{x}\,dx$$

$$=\ln x+c$$

$$\therefore \quad y=\frac{1}{x}\ln x+\frac{c}{x}$$

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Exercise C, Question 7

Question:

$$x^2 \frac{dy}{dx} - xy = \frac{x^3}{x+2} \quad x > -2$$

Solution:

$$x^2 \frac{dy}{dx} - xy = \frac{x^3}{x+2}$$

Divide by x^2

First divide the equation through by x^2, to give the correct form of equation.

$$\therefore \frac{dy}{dx} - \frac{1}{x} y = \frac{x}{x+2}$$

The integrating factor is $e^{\int -\frac{1}{x} dx} = e^{-\ln x} = e^{\ln \frac{1}{x}} = \frac{1}{x}$

Multiply the new equation by $\frac{1}{x}$

$$\therefore \frac{1}{x} \frac{dy}{dx} - \frac{1}{x^2} y = \frac{1}{x+2}$$

$$\therefore \quad \frac{d}{dx}\left(\frac{1}{x} y\right) = \frac{1}{x+2}$$

$$\therefore \quad \frac{1}{x} y = \int \frac{1}{x+2} dx$$

$$= \ln(x+2) + c$$

$$\therefore \quad y = x\ln(x+2) + cx$$

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Exercise C, Question 8

Question:

$$3x\frac{dy}{dx}+y=x$$

Solution:

$$3x\frac{dy}{dx}+y=x$$

$$\therefore \frac{dy}{dx}+\frac{1}{3x}y=\frac{1}{3} \quad *$$

First divide equation through by $3x$, to get an equation of the correct form.

The integrating factor is $e^{\int\frac{1}{3x}dx}=e^{\frac{1}{3}\ln x}$

$$=e^{\ln x^{\frac{1}{3}}}=x^{\frac{1}{3}}$$

Multiply equation $*$ by $x^{\frac{1}{3}}$

$$\therefore x^{\frac{1}{3}}\frac{dy}{dx}+\frac{1}{3}x^{-\frac{2}{3}}y=\frac{1}{3}x^{\frac{1}{3}}$$

$$\therefore \quad \frac{d}{dx}\left(x^{\frac{1}{3}}y\right)=\frac{1}{3}x^{\frac{1}{3}}$$

$$\therefore \quad x^{\frac{1}{3}}y=\int\frac{1}{3}x^{\frac{1}{3}}\,dx$$

$$=\frac{1}{4}x^{\frac{4}{3}}+c$$

$$\therefore \quad y=\frac{1}{4}x+cx^{-\frac{1}{3}}$$

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Exercise C, Question 9

Question:

$$(x+2)\frac{dy}{dx} - y = (x+2)$$

Solution:

$$(x+2)\frac{dy}{dx} - y = (x+2)$$

$$\therefore \frac{dy}{dx} - \frac{1}{(x+2)}y = 1 \quad *$$

Divide equation by $(x + 2)$ before finding integrating factor.

The integrating factor is $e^{\int \frac{-1}{(x+2)} dx} = e^{-\ln(x+2)} = e^{\ln \frac{1}{x+2}}$

$$= \frac{1}{x+2}$$

Multiply differential equation $*$ by integrating factor.

$$\therefore \frac{1}{(x+2)}\frac{dy}{dx} - \frac{1}{(x+2)^2}y = \frac{1}{(x+2)}$$

$$\therefore \quad \frac{d}{dx}\left[\frac{1}{(x+2)}y\right] = \frac{1}{x+2}$$

$$\therefore \quad \frac{1}{(x+2)}y = \int \frac{1}{x+2}\,dx$$

$$= \ln(x+2) + c$$

$$\therefore \quad y = (x+2)\ln(x+2) + c(x+2)$$

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Exercise C, Question 10

Question:

$$x\frac{dy}{dx}+4y=\frac{e^x}{x^2}$$

Solution:

$$x\frac{dy}{dx}+4y=\frac{e^x}{x^2}$$

Divide throughout by x

Then $\frac{dy}{dx}+\frac{4}{x}y=\frac{e^x}{x^3}$ $*$

The integrating factor is $e^{\int\frac{4}{x}dx}=e^{4\ln x}=e^{\ln x^4}=x^4$

$\therefore x^4\frac{dy}{dx}+4x^3y=xe^x$ [having multiplied $*$ by x^4]

Integrate xe^x using integration by parts.

$$\therefore \quad \frac{d}{dx}(x^4y)=xe^x$$

$$\therefore \quad x^4y=\int xe^x\,dx$$

$$=xe^x-\int e^x\,dx$$

$$=xe^x-e^x+c$$

$$\therefore \quad y=\frac{1}{x^3}e^x-\frac{1}{x^4}e^x+\frac{c}{x^4}$$

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Exercise C, Question 11

Question:

Find y in terms of x given that
$x\dfrac{dy}{dx} + 2y = e^x$ and that $y = 1$ when $x = 1$.

Solution:

$$x\frac{dy}{dx} + 2y = e^x$$

Divide throughout by x

Then $\dfrac{dy}{dx} + \dfrac{2}{x}y = \dfrac{1}{x}e^x$ $\quad *$

The integrating factor is $e^{\int \frac{2}{x}dx} = e^{2\ln x} = e^{\ln x^2} = x^2$

Multiply equation $*$ by x^2

Then $x^2\dfrac{dy}{dx} + 2xy = xe^x$

$\therefore \quad \dfrac{d}{dx}(x^2y) = xe^x$

$\therefore \quad x^2y = \displaystyle\int xe^x\,dx$

$= xe^x - \displaystyle\int e^x\,dx$

$= xe^x - e^x + c$

$\therefore \quad y = \dfrac{1}{x}e^x - \dfrac{1}{x^2}e^x + \dfrac{c}{x^2}$

Solve the differential equation then use the boundary condition $y = 1$ when $x = 1$ to find the constant of integration.

Given also that $y = 1$ when $x = 1$

Then $1 = e - e + c$

$\therefore \quad c = 1$

$\therefore \quad y = \dfrac{1}{x}e^x - \dfrac{1}{x^2}e^x + \dfrac{1}{x^2}$

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Exercise C, Question 12

Question:

Solve the differential equation, giving y in terms of x, where
$x^3 \frac{dy}{dx} - x^2 y = 1$ and $y = 1$ at $x = 1$.

Solution:

$$x^3 \frac{dy}{dx} - x^2 y = 1$$

Divide throughout by x^3

$$\frac{dy}{dx} - \frac{1}{x} y = \frac{1}{x^3} \quad *$$

The integrating factor is $e^{-\int \frac{1}{x} dx} = e^{-\ln x} = e^{\ln \frac{1}{x}} = \frac{1}{x}$

Multiply equation $*$ by $\frac{1}{x}$

Then $\frac{1}{x} \frac{dy}{dx} - \frac{1}{x^2} y = \frac{1}{x^4}$

$\therefore \quad \frac{d}{dx}\left(\frac{1}{x} y\right) = \frac{1}{x^4}$

$\therefore \quad \frac{1}{x} y = \int \frac{1}{x^4} dx$

$= \int x^{-4} dx$

$= -\frac{1}{3} x^{-3} + c$

$\therefore \quad y = -\frac{1}{3} x^{-2} + cx$

So $\quad y = -\frac{1}{3x^2} + cx$

But $y = 1$, when $x = 1$

$\therefore \quad 1 = -\frac{1}{3} + c$

$\therefore \quad c = \frac{4}{3}$

$\therefore \quad y = -\frac{1}{3x^2} + \frac{4x}{3}$

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Exercise C, Question 13

Question:

Find the general solution of the differential equation

$$\left(x+\frac{1}{x}\right)\frac{dy}{dx}+2y=2(x^2+1)^2,$$

giving y in terms of x.

Find the particular solution which satisfies the condition that $y = 1$ at $x = 1$.

Solution:

$$\left(x+\frac{1}{x}\right)\frac{dy}{dx}+2y=2\,(x^2+1)^2$$

Divide equation by $\left(x+\frac{1}{x}\right)$.

$$\therefore \frac{dy}{dx}+\frac{2}{\left(x+\frac{1}{x}\right)}y=\frac{2(x^2+1)^2}{\left(x+\frac{1}{x}\right)}$$

i.e. $\dfrac{dy}{dx}+\dfrac{2x}{x^2+1}\times y=2x(x^2+1)$ $*$

The integrating factor is $e^{\int\frac{2x}{x^2+1}dx}=e^{\ln(x^2+1)}=(x^2+1)$

Multiply $*$ by (x^2+1)

Then $(x^2+1)\dfrac{dy}{dx}+2xy=2x(x^2+1)^2$

$$\therefore \quad \frac{d}{dx}\left[(x^2+1)\,y\right]=2x\,(x^2+1)^2$$

$$\therefore \quad y\,(x^2+1)=\int 2x\,(x^2+1)^2\,dx$$

$$=\tfrac{1}{3}(x^2+1)^3+c$$

$$\therefore \quad y=\tfrac{1}{3}(x^2+1)^2+\frac{c}{(x^2+1)}$$

But $y = 1$, when $x = 1$

$\therefore \quad 1=\frac{1}{3}\times 4+\frac{1}{2}c$

$\therefore \quad c=-\frac{2}{3}$

$\therefore \quad y=\frac{1}{3}(x^2+1)^2-\dfrac{2}{3(x^2+1)}$

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Exercise C, Question 14

Question:

Find the general solution of the differential equation

$\cos x \frac{dy}{dx} + y = 1, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}.$

Find the particular solution which satisfies the condition that $y = 2$ at $x = 0$.

Solution:

$$\cos x \frac{dy}{dx} + y = 1$$

Divide throughout by $\cos x$

$$\therefore \quad \frac{dy}{dx} + \sec x \, y = \sec x$$

The integrating factor is $e^{\int \sec x \, dx} = e^{\ln(\sec x + \tan x)}$

$$= \sec x + \tan x$$

$\int \sec x \, dx = \ln(\sec x + \tan x)$

$$\therefore (\sec x + \tan x)\frac{dy}{dx} + (\sec^2 x + \sec x \tan x)\, y = \sec^2 x + \sec x \tan x$$

$$\therefore \quad \frac{d}{dx}\left[(\sec x + \tan x)y\right] = \sec^2 x + \sec x \tan x$$

$$\therefore \quad (\sec x + \tan x)y = \int \sec^2 x + \sec x \tan x \, dx$$

$$= \tan x + \sec x + c$$

$$\therefore \quad y = 1 + \frac{c}{\sec x + \tan x}$$

Given also that $y = 2$, when $x = 0$

$$\therefore \quad 2 = 1 + \frac{c}{1 + 0}$$

$$\therefore \quad c = 1$$

So $\quad y = 1 + \frac{1}{\sec x + \tan x}$ or $y = 1 + \frac{\cos x}{1 + \sin x}$

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Exercise D, Question 1

Question:

$$\frac{dy}{dx}=\frac{y}{x}+\frac{x}{y}, \quad x>0, y>0$$

Solution:

$$z=\frac{y}{x} \quad \Rightarrow \quad y=xz$$

$$\therefore \quad \frac{dy}{dx}=z+x\frac{dz}{dx}$$

Use the given substitution to express $\frac{dy}{dx}$ in terms of z, x and $\frac{dz}{dx}$.

Substitute into the equation:

$$\frac{dy}{dx}=\frac{y}{x}+\frac{x}{y}$$

$$\therefore \quad z+x\frac{dz}{dx}=z+\frac{1}{z}$$

$$\therefore \quad x\frac{dz}{dx}=\frac{1}{z}$$

Separate the variables:

Then $\int z\,dz=\int\frac{1}{x}\,dx$

$$\therefore \quad \frac{z^2}{2}=\ln x+c$$

$$\therefore \quad \frac{y^2}{2x^2}=\ln x+c, \text{ as } z=\frac{y}{x}$$

$$\therefore \quad y^2=2x^2(\ln x+c)$$

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Exercise D, Question 2

Question:

$$\frac{dy}{dx} = \frac{y}{x} + \frac{x^2}{y^2}, \quad x > 0$$

Solution:

As $z = \frac{y}{x}$, $y = zx$ and $\frac{dy}{dx} = z + x\frac{dz}{dx}$

$\therefore \frac{dy}{dx} = \frac{y}{x} + \frac{x^2}{y^2} \Rightarrow z + x\frac{dz}{dx} = z + \frac{1}{z^2}$

$\therefore \quad x\frac{dz}{dx} = \frac{1}{z^2}$

Separate the variables:

Then $\int z^2 \, dz = \int \frac{1}{x} \, dx$

$\therefore \quad \frac{z^3}{3} = \ln x + c$

But $z = \frac{y}{x}$

$\therefore \frac{y^3}{3x^3} = \ln x + c$

$\therefore \quad y^3 = 3x^3 (\ln x + c)$

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Exercise D, Question 3

Question:

$$\frac{dy}{dx} = \frac{y}{x} + \frac{y^2}{x^2}, \quad x > 0$$

Solution:

As $z = \frac{y}{x}$, $y = zx$ and $\frac{dy}{dx} = z + x\frac{dz}{dx}$

$\therefore \quad \frac{dy}{dx} = \frac{y}{x} + \frac{y^2}{x^2} \Rightarrow z + x\frac{dz}{dx} = z + z^2$

$\therefore \quad x\frac{dz}{dx} = z^2$

Separate the variables:

$\therefore \int \frac{1}{z^2}\,dz = \int \frac{1}{x}\,dx$

$\therefore \quad -\frac{1}{z} = \ln x + c$

$\therefore \quad z = \frac{-1}{\ln x + c}$

But $z = \frac{y}{x}$

$\therefore \quad y = \frac{-x}{\ln x + c}$

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Exercise D, Question 4

Question:

$$\frac{dy}{dx} = \frac{x^3 + 4y^3}{3xy^2}, x > 0$$

Solution:

$$z = \frac{y}{x} \Rightarrow y = zx \text{ and } \frac{dy}{dx} = z + x\frac{dz}{dx}$$

$$\therefore \frac{dy}{dx} = \frac{x^3 + 4y^3}{3xy^2} \Rightarrow z + x\frac{dz}{dx} = \frac{x^3 + 4z^3x^3}{3xz^2x^2}$$

$$\therefore \quad x\frac{dz}{dx} = \frac{1 + 4z^3}{3z^2} - z$$

$$= \frac{1 + z^3}{3z^2}$$

Separate the variables:

$$\therefore \int \frac{3z^2}{1 + z^3}\,dz = \int \frac{1}{x}\,dx$$

$\therefore$ $\ln(1 + z^3) = \ln x + \ln A$, where A is constant

$\therefore$ $\ln(1 + z^3) = \ln Ax$

So $1 + z^3 = Ax$

And $z^3 = Ax - 1$. But $z = \frac{y}{x}$

$$\therefore \quad \frac{y^3}{x^3} = Ax - 1$$

$\therefore$ $y^3 = x^3(Ax - 1)$, where A is a positive constant

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Exercise D, Question 5

Question:

Use the substitution $z = y^{-2}$ to transform the differential equation

$\frac{dy}{dx} + (\frac{1}{2}\tan x)\,y = -(2\sec x)y^3, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$

into a differential equation in z and x. By first solving the transformed equation, find the general solution of the original equation, giving y in terms of x.

Solution:

Given $z = y^{-2}$ $\therefore$ $y = z^{-\frac{1}{2}}$

and $\frac{dy}{dx} = -\frac{1}{2}z^{-\frac{3}{2}}\frac{dz}{dx}$

Find $\frac{dy}{dx}$ in terms of $\frac{dz}{dx}$ and z.

$\therefore$ $\frac{dy}{dx} + \left(\frac{1}{2}\tan x\right)y = -(2\sec x)\,y^3$

$\Rightarrow -\frac{1}{2}z^{-\frac{3}{2}}\frac{dz}{dx} + \left(\frac{1}{2}\tan x\right)z^{-\frac{1}{2}} = -2\sec x\,z^{-\frac{3}{2}}$

$\therefore$ $\frac{dz}{dx} - z\tan x = 4\sec x$ $*$

This is a first order equation which can be solved by using an integrating factor.

The integrating factor is $e^{-\int \tan x\,dx} = e^{\ln\cos x}$

$= \cos x$

The equation that you obtain needs an integrating factor to solve it.

Multiply the equation $*$ by $\cos x$

Then $\cos x \times \frac{dz}{dx} - z\sin x = 4$

$\therefore$ $\frac{d}{dx}(z\cos x) = 4$

$\therefore$ $z\cos x = \int 4\,dx$

$= 4x + c$

$\therefore$ $z = \frac{4x + c}{\cos x}$

As $y = z^{-\frac{1}{2}}$, $y = \sqrt{\frac{\cos x}{4x + c}}$

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Exercise D, Question 6

Question:

Use the substitution $z = x^{\frac{1}{2}}$ to transform the differential equation

$$\frac{dx}{dt} + t^2 x = t^2 x^{\frac{1}{2}}$$

into a differential equation in z and t. By first solving the transformed equation, find the general solution of the original equation, giving x in terms of t.

Solution:

Given that $z = x^{\frac{1}{2}}$, $x = z^2$ and $\frac{dx}{dt} = 2z\frac{dz}{dt}$

$\therefore$ The equation $\frac{dx}{dt} + t^2 x = t^2 x^{\frac{1}{2}}$ becomes

$$2z\frac{dz}{dt} + t^2 z^2 = t^2 z$$

Divide through by $2z$

Then $\frac{dz}{dt} + \frac{1}{2}t^2 z = \frac{1}{2}t^2$

The integrating factor is $e^{\int \frac{1}{2}t^2\,dt} = e^{\frac{1}{6}t^3}$

$$\therefore \quad e^{\frac{1}{6}t^3}\frac{dz}{dt} + \frac{1}{2}t^2 e^{\frac{1}{6}t^3} z = \frac{1}{2}t^2 e^{\frac{1}{6}t^3}$$

$$\therefore \quad \frac{d}{dt}\left(ze^{\frac{1}{6}t^3}\right) = \frac{1}{2}t^2 e^{\frac{1}{6}t^3}$$

$$\therefore \quad z e^{\frac{1}{6}t^3} = \int \frac{1}{2}t^2 e^{\frac{1}{6}t^3}\,dt$$

$$= e^{\frac{1}{6}t^3} + c$$

$$\therefore \quad z = 1 + ce^{-\frac{1}{6}t^3}$$

But $\quad x = z^2 \quad \therefore x = \left(1 + ce^{-\frac{1}{6}t^3}\right)^2$

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Exercise D, Question 7

Question:

Use the substitution $z = y^{-1}$ to transform the differential equation

$$\frac{dy}{dx} - \frac{1}{x}y = \frac{(x+1)^3}{x}y^2$$

into a differential equation in z and x. By first solving the transformed equation, find the general solution of the original equation, giving y in terms of x.

Solution:

Let $z = y^{-1}$, then $y = z^{-1}$ and $\frac{dy}{dx} = -z^{-2}\frac{dz}{dx}$

So $\frac{dy}{dx} - \frac{1}{x}y = \frac{(x+1)^3}{x}y^2$ becomes:

$$-z^{-2}\frac{dz}{dx} - \frac{1}{x}z^{-1} = \frac{(x+1)^3}{x}z^{-2}$$

Multiply through by $-z^2$

Then $\frac{dz}{dx} + \frac{1}{x}z = -\frac{(x+1)^3}{x}$

The integrating factor is $e^{\int P\,dx} = e^{\int \frac{1}{x}\,dx} = e^{\int \ln x} = x$

$\therefore\ x\frac{dz}{dx} + z = -(x+1)^3$

i.e. $\frac{d}{dx}(xz) = -(x+1)^3$

$$\therefore \quad xz = -\int (x+1)^3\,dx$$

$$= -\tfrac{1}{4}(x+1)^4 + c$$

$$\therefore \quad z = -\frac{1}{4x}(x+1)^4 + \frac{c}{x}$$

$$\therefore \quad y = -\frac{4x}{4c - (x+1)^4}$$

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Exercise D, Question 8

Question:

Use the substitution $z = y^2$ to transform the differential equation

$$2(1 + x^2)\frac{dy}{dx} + 2xy = \frac{1}{y}$$

into a differential equation in z and x. By first solving the transformed equation,

a find the general solution of the original equation, giving y in terms of x.

b Find the particular solution for which $y = 2$ when $x = 0$.

Solution:

a Given that $z = y^2$, and so $y = z^{\frac{1}{2}}$ and $\frac{dy}{dx} = \frac{1}{2}z^{-\frac{1}{2}}\frac{dz}{dx}$

The equation $2(1 + x^2)\frac{dy}{dx} + 2xy = \frac{1}{y}$ becomes

$$2(1 + x^2) \times \frac{1}{2}z^{-\frac{1}{2}}\frac{dz}{dx} + 2xz^{\frac{1}{2}} = z^{-\frac{1}{2}}$$

Multiply the equation by $\frac{z^{\frac{1}{2}}}{1 + x^2}$

Then $\frac{dz}{dx} + \frac{2x}{1 + x^2}z = \frac{1}{1 + x^2}$

The integrating factor is $e^{\int \frac{2x}{1 + x^2}dx} = e^{\ln(1 + x^2)} = 1 + x^2$

$\therefore (1 + x^2)\frac{dz}{dx} + 2xz = 1$

$\therefore \quad \frac{d}{dx}[(1 + x^2)z] = 1$

$\therefore \quad (1 + x^2)z = \int 1\, dx$

$= x + c$

$\therefore \quad z = \frac{x + c}{(1 + x^2)}$

As $y = z^{\frac{1}{2}}$, $\quad y = \sqrt{\frac{x + c}{(1 + x^2)}}$

b When $x = 0$, $y = 2$ $\quad \therefore 2 = \sqrt{c} \Rightarrow c = 4$

$\therefore y = \sqrt{\frac{x + 4}{1 + x^2}}$

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Exercise D, Question 9

Question:

Show that the substitution $z = y^{-(n-1)}$ transforms the general equation

$\frac{dy}{dx} + Py = Qy^n$,

where P and Q are functions of x, into the linear equation $\frac{dz}{dx} - P(n-1)z = -Q(n-1)$

(Bernoulli's equation)

Solution:

Given $z = y^{-(n-1)}$

$\therefore \quad y = z^{-\frac{1}{(n-1)}}$

$$\frac{dy}{dx} = \frac{-1}{n-1} z^{-\frac{1}{n-1}-1} \frac{dz}{dx}$$

$$= \frac{-1}{n-1} z^{-\frac{n}{n-1}} \frac{dz}{dx}$$

$\therefore \frac{dy}{dx} + Py = Q\,y^n$ becomes

$$\frac{-1}{n-1} z^{-\frac{n}{n-1}} \frac{dz}{dx} + P\,z^{-\frac{1}{n-1}} = Q\,z^{-\frac{n}{n-1}}$$

Multiply each term by $-(n-1)\,z^{\frac{n}{n-1}}$

Then $\quad \frac{dz}{dz} - P(n-1)\,z^{\frac{n}{n-1}} z^{-\frac{1}{n-1}} = -Q(n-1)\,z^{\frac{n}{n-1}} z^{-\frac{n}{n-1}}$

i.e. $\quad \frac{dz}{dz} - P(n-1)\,z = -Q(n-1)$

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Exercise D, Question 10

Question:

Use the substitution $u = y + 2x$ to transform the differential equation

$$\frac{dy}{dx} = \frac{-(1 + 2y + 4x)}{1 + y + 2x}$$

into a differential equation in u and x. By first solving this new equation, show that the general solution of the original equation may be written $4x^2 + 4xy + y^2 + 2y + 2x = k$, where k is a constant

Solution:

Given $u = y + 2x$ and so $y = u - 2x$ and $\frac{dy}{dx} = \frac{du}{dx} - 2$

$\therefore$ the differential equation $\frac{dy}{dx} = -\frac{(1 + 2y + 4x)}{1 + y + 2x}$ becomes

Rearrange the given substitution to give y in terms of u and x, and $\frac{dy}{dx}$ in terms of $\frac{du}{dx}$.

$$\frac{du}{dx} - 2 = -\frac{1 + 2u}{1 + u}$$

$$\therefore \quad \frac{du}{dx} = \frac{-(1 + 2u) + 2(1 + u)}{1 + u}$$

$$\therefore \quad \frac{du}{dx} = \frac{1}{1 + u}$$

Separate the variables

$$\int (1 + u)\, du = \int 1 \times dx$$

$$\therefore \quad u + \frac{u^2}{2} = x + c, \text{ where } c \text{ is constant}$$

And $\quad (y + 2x) + \frac{(y + 2x)^2}{2} = x + c$

$$2y + 4x + y^2 + 4xy + 4x^2 = 2x + 2c$$

i.e. $4x^2 + 4xy + y^2 + 2y + 2x = k$, where $k = 2c$

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Exercise E, Question 1

Question:

Solve the equation $\frac{dy}{dx} = \frac{x}{\sqrt{x^2 + 16}}$ and sketch three solution curves.

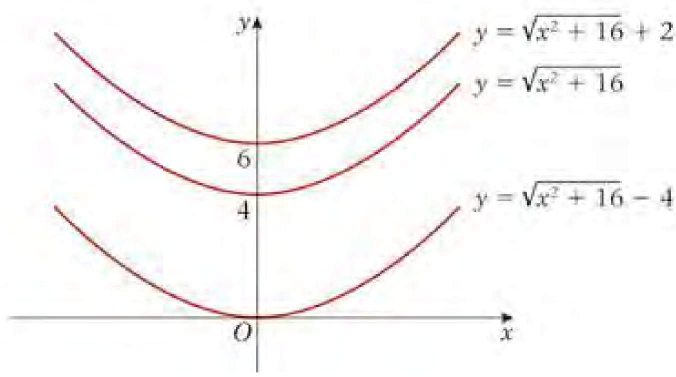
Solution:

$$\frac{dy}{dx} = \frac{x}{\sqrt{x^2 + 16}}$$

$$\therefore y = \int \frac{x}{\sqrt{x^2 + 16}} \, dx$$

$$= (x^2 + 16)^{\frac{1}{2}} + c$$

The integral is of the type $\int [f(x)]^n f'(x)\, dx$, which integrates to give $[f(x)]^{n+1} + c$.



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Exercise E, Question 2

Question:

Solve the equation $\frac{dy}{dx} = xy$ and sketch the solution curves which pass through

a (0, 1) **b** (0, 2) **c** (0, 3)

Solution:

$$\frac{dy}{dx} = xy$$

Separate the variables and integrate.

$$\therefore \int \frac{1}{y}\,dy = \int x\,dx$$

$$\therefore \quad \ln y = \tfrac{1}{2}x^2 + c, \text{ where } c \text{ is constant}$$

$$\therefore \quad y = e^{\frac{1}{2}x^2 + c}$$

$$= e^c e^{\frac{1}{2}x^2} \qquad = Ae^{\frac{1}{2}x^2}, \text{ where } A \text{ is } e^c$$

a The solution which satisfies $x = 0$ when $y = 1$

is $y = Ae^{\frac{1}{2}x^2}$ where $1 = Ae^0$ i.e. $A = 1$

$$\therefore \quad y = e^{\frac{1}{2}x^2}$$

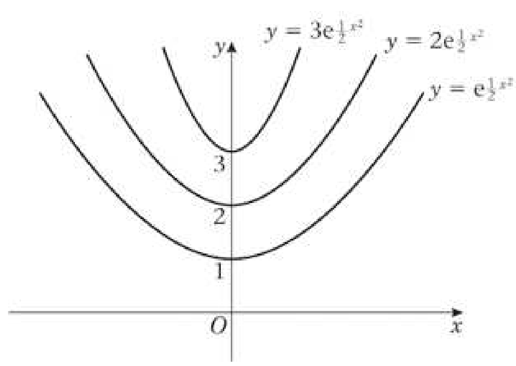
b The solution for which $y = 2$ when $x = 0$ is $y = Ae^{\frac{1}{2}x^2}$

with $2 = Ae^0$ i.e. $A = 2$

$$\therefore \quad y = 2e^{\frac{1}{2}x^2}$$

c The solution for which $y = 3$ when $x = 0$ is $y = 3e^{\frac{1}{2}x^2}$

The solution curves are shown in the sketch.



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Exercise E, Question 3

Question:

Solve the equation $\frac{dv}{dx} = -g - kv$ given that $v = u$ when $t = 0$, and that u, g and k are positive constants. Sketch the solution curve indicating the velocity which v approaches as t becomes large.

Solution:

$$\frac{dv}{dt} = -g - kv$$

You can separate the variables by dividing both sides by $(g + kv)$, or you could rearrange the equation as $\frac{dv}{dt} + kv = g$ and use the integrating factor e^{kt}.

$$\therefore \quad \int \frac{dv}{g + kv} = -\int 1 \, dt$$

$\therefore \frac{1}{k} \ln |g + kv| = -t + c$ where c is a constant $*$

When $t = 0$, $v = u$

$$\therefore \frac{1}{k} \ln |g + ku| = c$$

$\therefore$ Substituting c back into the equation $*$

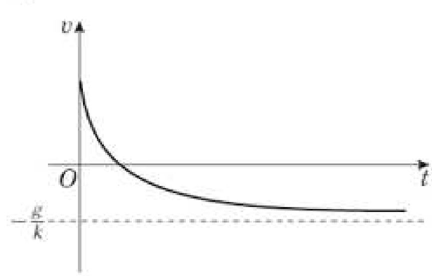
$$\frac{1}{k} \ln |g + kv| = -t + \frac{1}{k} \ln |g + ku|$$

$$\therefore \frac{1}{k} [\ln |g + kv| - \ln |g + ku|] = -t$$

$$\therefore \quad \ln \frac{g + kv}{g + ku} = -kt$$

$$\therefore \quad g + kv = (g + ku) e^{-kt}$$

$$\therefore \quad v = \frac{1}{k} [(g + ku) e^{-kt} - g]$$



The required velocity is $-\frac{g}{k}$ m s^{-1}

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Exercise E, Question 4

Question:

Solve the equation $\frac{dy}{dx} + y \tan x = 2 \sec x$

Solution:

$$\frac{dy}{dx} + y\tan x = 2\sec x$$

Use an integrating factor to solve this equation.

Use the integrating factor $e^{\int \tan x \, dx} = e^{\ln \sec x} = \sec x$

$$\therefore \quad \sec x \frac{dy}{dx} + y \sec x \tan x = 2\sec^2 x$$

$$\therefore \quad \frac{d}{dx}(y \sec x) = 2\sec^2 x$$

$$\therefore \quad y \sec x = \int 2\sec^2 x \, dx$$

$$= 2\tan x + c$$

$$\therefore \quad y = 2\sin x + c\cos x$$

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Exercise E, Question 5

Question:

Solve the equation $(1 - x^2)\frac{dy}{dx} + xy = 5x \qquad -1 < x < 1$

Solution:

$$(1 - x^2)\frac{dy}{dx} + xy = 5x$$

Divide through by $(1 - x^2)$, then find the integrating factor.

Divide through by $(1 - x^2)$

$$\therefore \quad \frac{dy}{dx} + \frac{x}{1 - x^2}y = \frac{5x}{1 - x^2}$$

Use the integrating factor $e^{\int \frac{x}{1-x^2}dx} = e^{-\frac{1}{2}\ln(1-x^2)}$

$$= e^{\ln(1-x^2)^{-\frac{1}{2}}} = \frac{1}{\sqrt{1 - x^2}}$$

$$\therefore \quad \frac{1}{\sqrt{1-x^2}}\frac{dy}{dx} + \frac{x}{(1-x^2)^{\frac{3}{2}}}y = \frac{5x}{(1-x^2)^{\frac{3}{2}}}$$

$$\therefore \quad \frac{d}{dx}\left[(1-x^2)^{-\frac{1}{2}}y\right] = \frac{5x}{(1-x^2)^{\frac{3}{2}}}$$

$$\therefore \quad (1-x^2)^{-\frac{1}{2}}y = \int \frac{5x}{(1-x^2)^{\frac{3}{2}}}dx$$

$$= 5(1-x^2)^{-\frac{1}{2}} + c$$

$$\therefore \quad y = 5 + c(1-x^2)^{\frac{1}{2}}$$

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Exercise E, Question 6

Question:

Solve the equation $x\frac{\mathrm{d}y}{\mathrm{d}x} + x + y = 0$

Solution:

$$x\frac{\mathrm{d}y}{\mathrm{d}x} + x + y = 0$$

Take the 'x' term to the other side of the equation.

$$\therefore \quad x\frac{\mathrm{d}y}{\mathrm{d}x} + y = -x$$

This is an exact equation.

So $\quad \frac{\mathrm{d}}{\mathrm{d}x}(xy) = -x$

$$\therefore \quad xy = -\int x\,\mathrm{d}x$$

$$= -\tfrac{1}{2}x^2 + c$$

$$\therefore \quad y = -\tfrac{1}{2}x + \frac{c}{x}$$

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Exercise E, Question 7

Question:

Solve the equation $\frac{dy}{dx} + \frac{y}{x} = \sqrt{x}$

Solution:

$$\frac{dy}{dx} + \frac{y}{x} = \sqrt{x}$$

The integrating factor is $e^{\int \frac{1}{x} dx} = e^{\ln x} = x$

Multiply the differential equation by the integrating factor:

$$x\frac{dy}{dx} + y = x\sqrt{x}$$

$$\therefore \quad \frac{d}{dx}(xy) = x^{\frac{3}{2}}$$

$$\therefore \quad xy = \int x^{\frac{3}{2}} dx$$

$$= \frac{2}{5}x^{\frac{5}{2}} + c$$

$$\therefore \quad y = \frac{2}{5}x^{\frac{3}{2}} + \frac{c}{x}$$

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Exercise E, Question 8

Question:

Solve the equation $\frac{dy}{dx} + 2xy = x$

Solution:

$$\frac{dy}{dx} + 2xy = x$$

The integrating factor is $e^{\int 2x\,dx} = e^{x^2}$

Multiply the differential equation by e^{x^2}

$$\therefore \quad e^{x^2}\frac{dy}{dx} + 2xe^{x^2}y = xe^{x^2}$$

$$\therefore \quad \frac{d}{dx}(e^{x^2}y) = xe^{x^2}$$

$$\therefore \quad ye^{x^2} = \int xe^{x^2}\,dx$$

$$= \tfrac{1}{2}e^{x^2} + c$$

$$\therefore \quad y = \tfrac{1}{2} + ce^{-x^2}$$

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Exercise E, Question 9

Question:

Solve the equation $x(1-x^2)\frac{dy}{dx}+(2x^2-1)y=2x^3 \qquad 0<x<1$

Solution:

$$x(1-x^2)\frac{dy}{dx}+(2x^2-1)y=2x^3$$

Divide through by $x(1-x^2)$

$$\therefore \quad \frac{dy}{dx}+\frac{2x^2-1}{x(1-x^2)}y=\frac{2x^3}{x(1-x^2)} \quad *$$

You will need to use partial fractions to integrate $\frac{2x^2-1}{x(1-x^2)}$ and to find the integrating factor.

The integrating factor is $e^{\int\frac{2x^2-1}{x(1-x^2)}dx}$

$$\int\frac{2x^2-1}{x(1-x)(1+x)}dx=\int\left(-\frac{1}{x}+\frac{1}{2(1-x)}-\frac{1}{2(1+x)}\right)dx$$

$$=-\ln x-\tfrac{1}{2}\ln(1-x)-\tfrac{1}{2}\ln(1+x)$$

$$=-\ln x\sqrt{1-x^2}$$

So the integrating factor is $e^{-\ln x\sqrt{1-x^2}}=e^{\ln\frac{1}{x\sqrt{1-x^2}}}=\frac{1}{x\sqrt{1-x^2}}$

Multiply the differential equation $*$ by $\frac{1}{x\sqrt{1-x^2}}$

$$\therefore \quad \frac{1}{x\sqrt{1-x^2}}\frac{dy}{dx}+\frac{2x^2-1}{x^2(1-x^2)^{\frac{3}{2}}}y=\frac{2x}{(1-x^2)^{\frac{3}{2}}}$$

$$\therefore \quad \frac{d}{dx}\left[\frac{1}{x\sqrt{1-x^2}}y\right]=\frac{2x}{(1-x^2)^{\frac{3}{2}}}$$

$$\therefore \quad \frac{y}{x\sqrt{1-x^2}}=\int\frac{2x}{(1-x^2)^{\frac{3}{2}}}dx$$

$$=2(1-x^2)^{-\frac{1}{2}}+c$$

$$\therefore \quad y=2x+cx\sqrt{1-x^2}$$

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Exercise E, Question 10

Question:

Solve the equation $R\frac{dq}{dt} + \frac{q}{c} = E$ when

a $E = 0$ **b** $E = \text{constant}$ **c** $E = \cos pt$

(R, c and p are constants)

Solution:

$$R\frac{dq}{dt} + \frac{q}{c} = E$$

$$\therefore \quad \frac{dq}{dt} + \frac{1}{Rc}q = \frac{E}{R}$$

This is a difficult question – particularly part **c**. You may decide to omit this question, unless you want a challenge.

The integrating factor is $e^{\int \frac{1}{Rc}dt} = e^{\frac{t}{Rc}}$

$$\therefore \quad e^{\frac{t}{Rc}}\frac{dq}{dt} + \frac{1}{Rc}e^{\frac{t}{Rc}}q = \frac{E}{R}e^{\frac{t}{Rc}}$$

$$\therefore \quad \frac{d}{dt}\left(qe^{\frac{t}{Rc}}\right) = \frac{E}{R}e^{\frac{t}{Rc}}$$

$$\therefore \quad qe^{\frac{t}{Rc}} = \int \frac{E}{R}e^{\frac{t}{Rc}}\,dt$$

a When $E = 0$

$\therefore \quad qe^{\frac{t}{Rc}} = k$, where k is constant.

$$\therefore \quad q = ke^{-\frac{t}{Rc}}$$

b When E = constant

$$qe^{\frac{t}{Rc}} = \int \frac{E}{R}e^{\frac{t}{Rc}}\,dt$$

$= Ece^{\frac{t}{Rc}} + k$, where k is constant

$$\therefore \quad q = Ec + ke^{-\frac{t}{Rc}}$$

c When $E = \cos pt$

$$qe^{\frac{t}{Rc}} = \int \frac{1}{R}\cos pt\, e^{\frac{t}{Rc}}\,dt \quad *$$

i.e. $\int \frac{1}{R}\cos pt\, e^{\frac{t}{Rc}} = ce^{\frac{t}{Rc}}\cos pt + \int cpe^{\frac{t}{Rc}}\sin pt\,dt$

Use integration by parts.

$$\int \frac{1}{R}\cos pt e^{\frac{t}{Rc}}\,dt = ce^{\frac{t}{Rc}}\cos pt + Rpc^2 e^{\frac{t}{Rc}}\sin pt - \int Rp^2c^2e^{\frac{t}{Rc}}\cos pt\,dt$$

Use 'parts' again.

$\therefore \quad \int\left(\frac{1}{R} + Rp^2c^2\right)e^{\frac{t}{Rc}}\cos pt\,dt = ce^{\frac{t}{Rc}}(\cos pt + Rpc\sin pt) + k$, where k is a constant

$$\therefore \quad \frac{1}{R}\int e^{\frac{t}{Rc}}\cos pt\,dt = \frac{c}{(1 + R^2p^2c^2)}e^{\frac{t}{Rc}}(\cos pt + Rpc\sin pt) + \frac{k}{(1 + R^2p^2c^2)}$$

From $*$

$$qe^{\frac{t}{Rc}} = \frac{c}{(1 + R^2p^2c^2)}e^{\frac{t}{Rc}}(\cos pt + Rpc\sin pt) + \frac{k}{(1 + R^2p^2c^2)}$$

$\therefore \quad q = \frac{c}{(1 + R^2p^2c^2)}(\cos pt + Rpc\sin pt) + k'e^{-\frac{t}{Rc}}$, where $k' = \frac{k}{1 + R^2p^2c^2}$ is constant

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Exercise E, Question 11

Question:

Find the general solution of the equation $\frac{dy}{dx} - ay = Q$, where a is a constant, giving your answer in terms of a, when

a $Q = ke^{\lambda x}$ **b** $Q = ke^{ax}$ **c** $Q = kx^n e^{ax}$.

(k, λ and n are constants).

Solution:

Given that $\frac{dy}{dx} - ay = Q$

The integrating factor is $e^{\int -a\,dx} = e^{-ax}$

Then $e^{-ax}\frac{dy}{dx} - ae^{-ax}y = Qe^{-ax}$

$\therefore \quad \frac{d}{dx}(ye^{-ax}) = Qe^{-ax}$

$\therefore \quad ye^{-ax} = \int Qe^{-ax}\,dx$

a When $Q = ke^{\lambda x}$

$$ye^{-ax} = \int ke^{(\lambda - a)x}\,dx$$

$$= \frac{k}{\lambda - a}e^{(\lambda - a)x} + c, \text{ where } c \text{ is constant}$$

$$\therefore \quad y = \frac{k}{\lambda - a}e^{\lambda x} + ce^{ax}$$

When $\lambda \neq a$.
For $\lambda = a$, see part **b**.

b When $Q = ke^{ax}$

$$ye^{-ax} = \int k\,dx$$

$$= kx + c, \text{ where } c \text{ is constant}$$

$$\therefore \quad y = (kx + c)e^{ax}$$

c When $Q = kx^n e^{ax}$

$$ye^{-ax} = \int kx^n\,dx$$

$$= \frac{kx^{n+1}}{n+1} + c, \text{ where } c \text{ is constant}$$

$$\therefore \quad y = \frac{kx^{n+1}}{n+1}e^{ax} + ce^{ax}$$

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Exercise E, Question 12

Question:

Use the substitution $z = y^{-1}$ to transform the differential equation $x\frac{\mathrm{d}y}{\mathrm{d}x} + y = y^2 \ln x$, into a linear equation. Hence obtain the general solution of the original equation.

Solution:

Given that $z = y^{-1}$, then $y = z^{-1}$ so $\frac{\mathrm{d}y}{\mathrm{d}x} = -z^{-2}\frac{\mathrm{d}z}{\mathrm{d}x}$.

Use the substitution to express y in terms of z and $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of z and $\frac{\mathrm{d}z}{\mathrm{d}x}$.

The equation $x\frac{\mathrm{d}y}{\mathrm{d}x} + y = y^2 \ln x$ becomes

$$-xz^{-2}\frac{\mathrm{d}z}{\mathrm{d}x} + z^{-1} = z^{-2}\ln x$$

Divide through by $-xz^{-2}$

$$\therefore \quad \frac{\mathrm{d}z}{\mathrm{d}y} - \frac{z}{x} = -\frac{\ln x}{x}$$

The integrating factor is $\mathrm{e}^{-\int \frac{1}{x}\mathrm{d}x} = \mathrm{e}^{-\ln x} = \mathrm{e}^{\ln \frac{1}{x}} = \frac{1}{x}$

$$\therefore \quad \frac{1}{x}\frac{\mathrm{d}z}{\mathrm{d}x} - \frac{z}{x^2} = -\frac{\ln x}{x^2}$$

$$\therefore \quad \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{x}z\right) = -\frac{\ln x}{x^2}$$

$$\therefore \quad \frac{1}{x}z = -\int \frac{1}{x^2}\ln x\,\mathrm{d}x$$

$$= -\left[-\frac{1}{x}\ln x + \int \frac{1}{x^2}\mathrm{d}x\right]$$

$$= \frac{1}{x}\ln x + \frac{1}{x} + c$$

$$\therefore \quad z = \ln x + 1 + cx.$$

As $y = z^{-1}$ $\therefore$ $y = \frac{1}{1 + cx + \ln x}$

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Exercise E, Question 13

Question:

Use the substitution $z = y^2$ to transform the differential equation $2\cos x \frac{dy}{dx} - y \sin x + y^{-1} = 0$, into a linear equation. Hence obtain the general solution of the original equation.

Solution:

Given that $z = y^2$, $y = z^{\frac{1}{2}}$ and $\frac{dy}{dx} = \frac{1}{2} z^{-\frac{1}{2}} \frac{dz}{dx}$

The differential equation

$$2\cos x \frac{dy}{dx} - y\sin x + y^{-1} = 0 \text{ becomes}$$

$$\cos x \, z^{-\frac{1}{2}} \frac{dz}{dx} - z^{\frac{1}{2}} \sin x + z^{-\frac{1}{2}} = 0$$

Divide through by $z^{-\frac{1}{2}}$

then $\cos x \frac{dz}{dx} - z\sin x = -1$

This becomes an exact equation which can be solved directly.

$$\therefore \quad \frac{d}{dx}(z\cos x) = -1$$

$$\therefore \quad z\cos x = -\int 1 \, dx$$

$$= -x + c$$

$$\therefore \quad z = \frac{c - x}{\cos x}$$

$$\therefore \quad y = \sqrt{\frac{c - x}{\cos x}}$$

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Exercise E, Question 14

Question:

Use the substitution $z = \frac{y}{x}$ to transform the differential equation $(x^2 - y^2)\frac{dy}{dx} - xy = 0$, into a linear equation. Hence obtain the general solution of the original equation.

Solution:

Given that $z = \frac{y}{x}$, $y = zx$ so $\frac{dy}{dx} = z + x\frac{dz}{dx}$

The equation $(x^2 - y^2)\frac{dy}{dx} - xy = 0$ becomes

$$(x^2 - z^2x^2)\left(z + x\frac{dz}{dx}\right) - xzx = 0$$

$$\therefore \quad (1 - z^2)z + (1 - z^2)x\frac{dz}{dx} - z = 0$$

$$\therefore \quad x\frac{dz}{dx} = \frac{z}{1 - z^2} - z$$

i.e. $$x\frac{dz}{dx} = \frac{z^3}{1 - z^2}$$

Separate the variables to give

$$\int \frac{1 - z^2}{z^3}\,dz = \int \frac{1}{x}\,dx$$

$$\therefore \quad \int (z^{-3} - z^{-1})\,dz = \int x^{-1}\,dx$$

$$\therefore \quad \frac{z^{-2}}{-2} - \ln z = \ln x + c$$

$$\therefore \quad -\frac{1}{2z^2} = \ln x + \ln z + c$$

$$= \ln xz + c$$

But $$y = zx$$

$$\therefore \quad (c + \ln y) = -\frac{x^2}{2y^2}$$

$$\therefore \quad 2y^2(\ln y + c) + x^2 = 0$$

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Exercise E, Question 15

Question:

Use the substitution $z = \frac{y}{x}$ to transform the differential equation $\frac{dy}{dx} = \frac{y(x+y)}{x(y-x)}$, into a linear equation. Hence obtain the general solution of the original equation.

Solution:

$z = \frac{y}{x}$, $\therefore$ $y = xz$ and $\frac{dy}{dx} = z + x\frac{dz}{dx}$

$\therefore$ $\frac{dy}{dx} = \frac{y(x+y)}{x(y-x)}$ becomes $z + x\frac{dz}{dx} = \frac{xz(x+xz)}{x(xz-x)}$

$\therefore$ $z + x\frac{dz}{dx} = \frac{z(1+z)}{(z-1)}$

So $x\frac{dz}{dx} = \frac{z(1+z)}{z-1} - z$

$= \frac{2z}{z-1}$

Separating the variables

$$\int \frac{(z-1)}{2z} dz = \int \frac{1}{x} dx$$

$\therefore$ $\int \left(\frac{1}{2} - \frac{1}{2z}\right) dz = \int \frac{1}{x} dx$

$\therefore$ $\frac{1}{2}z - \frac{1}{2}\ln z = \ln x + c$

As $z = \frac{y}{x}$ $\therefore$ $\frac{y}{2x} - \frac{1}{2}\ln\frac{y}{x} = \ln x + c$

$\therefore$ $\frac{y}{2x} - \frac{1}{2}\ln y + \frac{1}{2}\ln x = \ln x + c$

$\therefore$ $\frac{y}{2x} - \frac{1}{2}\ln y = \frac{1}{2}\ln x + c$

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Exercise E, Question 16

Question:

Use the substitution $z = \frac{y}{x}$ to transform the differential equation $\frac{dy}{dx} = \frac{-3xy}{(y^2 - 3x^2)}$, into a linear equation. Hence obtain the general solution of the original equation.

Solution:

Given that $z = \frac{y}{x}$, so $y = zx$ and $\frac{dy}{dx} = z + x\frac{dz}{dx}$

The equation $\frac{dy}{dx} = \frac{-3xy}{y^2 - 3x^2}$ becomes

$$z + x\frac{dz}{dx} = \frac{-3x^2z}{z^2x^2 - 3x^2}$$

i.e. $$x\frac{dz}{dx} = \frac{-3z}{z^2 - 3} - z$$

$$= \frac{-z^3}{z^2 - 3}$$

Separate the variables:

Then $\int\left(\frac{z^2 - 3}{z^3}\right) dz = -\int \frac{1}{x} dx.$

$\therefore \int\left(\frac{1}{z} - 3z^{-3}\right) dz = -\ln x + c$

$\therefore \quad \ln z + \frac{3}{2}z^{-2} = -\ln x + c$

$\therefore \quad \ln zx + \frac{3}{2z^2} = c$

But $zx = y$ and $z = \frac{y}{x}$

$\therefore \quad \ln y + \frac{3x^2}{2y^2} = c$

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Exercise E, Question 17

Question:

Use the substitution $u = x + y$ to transform the differential equation $\frac{dy}{dx} = (x + y + 1)(x + y - 1)$ into a differential equation in u and x. By first solving this new equation, find the general solution of the original equation, giving y in terms of x.

Solution:

Let $u = x + y$, then $\frac{du}{dx} = 1 + \frac{dy}{dx}$ and so $\frac{dy}{dx} = (x + y + 1)(x + y - 1)$ becomes

$$\frac{du}{dx} - 1 = (u + 1)(u - 1)$$
$$= u^2 - 1$$
$$\therefore \quad \frac{du}{dx} = u^2$$

Separate the variables.

Then $\quad \int \frac{1}{u^2} du = \int 1 \, dx$

$\therefore \quad -\frac{1}{u} = x + c$

But $u = x + y \quad \therefore \quad -\frac{1}{x + y} = x + c$

$\therefore \quad y + x = \frac{-1}{x + c}$

$$y = \frac{-1}{x + c} - x$$

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Exercise E, Question 18

Question:

Use the substitution $u = y - x - 2$ to transform the differential equation $\frac{dy}{dx} = (y - x - 2)^2$ into a differential equation in u and x. By first solving this new equation, find the general solution of the original equation, giving y in terms of x.

Solution:

Given that $u = y - x - 2$, and so $\frac{du}{dx} = \frac{dy}{dx} - 1$

$\therefore \quad \frac{dy}{dx} = (y - x - 2)^2$ becomes $\frac{du}{dx} + 1 = u^2$

i.e. $$\frac{du}{dx} = u^2 - 1$$

$$\therefore \quad \int \frac{1}{u^2 - 1}\,du = \int 1\,dx$$

Factorise $\frac{1}{u^2 - 1}$ into $\frac{1}{(u-1)(u+1)}$ and use partial fractions.

$$\therefore \quad \int \left(\frac{1}{2(u-1)} - \frac{1}{2(u+1)} \right) du = x + c \text{ where } c \text{ is constant}$$

$$\therefore \quad \tfrac{1}{2}\ln(u-1) - \tfrac{1}{2}\ln(u+1) = x + c$$

$$\therefore \quad \tfrac{1}{2}\ln\frac{u-1}{u+1} = x + c$$

$$\therefore \quad \frac{u-1}{u+1} = e^{2c + 2x} = Ae^{2x} \text{ where } A = e^{2c} \text{ is a constant}$$

$$\therefore \quad u - 1 = Aue^{2x} + Ae^{2x}$$

$$\therefore \quad u(1 - Ae^{2x}) = (1 + Ae^{2x})$$

$$\therefore \quad u = \frac{1 + Ae^{2x}}{1 - Ae^{2x}}$$

But $\quad u = y - x - 2$

$$\therefore \quad y = x + 2 + \frac{1 + Ae^{2x}}{1 - Ae^{2x}}$$

Solutionbank FP2
Edexcel AS and A Level Modular Mathematics

Exercise A, Question 1

Question:

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6x = 0$$

Solution:

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 0$$

The auxiliary equation is

$$m^2 + 5m + 6 = 0$$

$\therefore \quad (m + 3)(m + 2) = 0$

$\therefore \quad m = -3 \text{ or } -2$

So the general solution is $y = Ae^{-3x} + Be^{-2x}$.

The auxiliary equation of $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$ is $am^2 + bm^2 + c = 0$. If α and β are roots of this quadratic then $y = Ae^{\alpha x} + Be^{\beta x}$ is the general solution of the differential equation.

Solutionbank FP2
Edexcel AS and A Level Modular Mathematics

Exercise A, Question 2

Question:

$$\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 12y = 0$$

Solution:

$$\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 12y = 0$$

The auxiliary equation is

$$m^2 - 8m + 12 = 0$$

$$\therefore \quad (m-6)(m-2) = 0$$

$$\therefore \quad m = 2 \text{ or } 6$$

So the general solution is $y = Ae^{2x} + Be^{6x}$.

Find the auxiliary equation $am^2 + bm + c = 0$ and solve to give two real roots α and β. General solution is $Ae^{\alpha x} + Be^{\beta x}$.

Solutionbank FP2

Edexcel AS and A Level Modular Mathematics

Exercise A, Question 3

Question:

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 15y = 0$$

Solution:

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 15y = 0$$

The auxiliary equation is

$$m^2 + 2m - 15 = 0$$

$\therefore \quad (m+5)(m-3) = 0$

$\therefore \quad m = -5 \text{ or } 3$

So the general solution is $y = Ae^{-5x} + Be^{3x}$.

Find the auxiliary equation and solve to give 2 real roots α and β. General solution is $Ae^{\alpha x} + Be^{\beta x}$.

Solutionbank FP2
Edexcel AS and A Level Modular Mathematics

Exercise A, Question 4

Question:

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} - 28y = 0$$

Solution:

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} - 28y = 0$$

The auxiliary equation is

$$m^2 - 3m - 28 = 0$$

$$\therefore \quad (m-7)(m+4) = 0$$

$$\therefore \quad m = 7 \text{ or } -4$$

So the general solution is $y = Ae^{7x} + Be^{-4x}$.

Find the auxiliary equation and solve to give 2 real roots α and β. General solution is $Ae^{\alpha x} + Be^{\beta x}$.

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Exercise A, Question 5

Question:

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 12y = 0$$

Solution:

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 12y = 0$$

The auxiliary equation is

$$m^2 + m - 12 = 0$$

$\therefore \quad (m + 4)(m - 3) = 0$

$\therefore \quad m = -4 \text{ or } 3$

So the general solution is $y = Ae^{-4x} + Be^{3x}$.

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Exercise A, Question 6

Question:

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} = 0$$

Solution:

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} = 0$$

The auxiliary equation is

$$m^2 + 5m = 0$$

$\therefore \quad m(m + 5) = 0$

$\therefore \quad m = 0 \text{ or } -5$

The auxiliary equation has two real roots, but one of them is zero. As $Ae^{0x} = A$, the general solution is $A + Be^{\beta x}$.

So the general solution is

$$y = Ae^{0x} + Be^{-5x}$$
$$= A + Be^{-5x}.$$

NB. There are other methods of solving this differential equation.

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Exercise A, Question 7

Question:

$$3\frac{d^2y}{dx^2}+7\frac{dy}{dx}+2y=0$$

Solution:

$$3\frac{d^2y}{dx^2}+7\frac{dy}{dx}+2y=0$$

The auxiliary equation is

$$3m^2+7m+2=0$$

$\therefore$ $(3m+1)(m+2)=0$

$\therefore$ $m=-\frac{1}{3}$ or -2

$\therefore$ $y=Ae^{-\frac{1}{3}x}+Be^{-2x}$ is the general solution.

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Exercise A, Question 8

Question:

$$4\frac{d^2y}{dx^2} - 7\frac{dy}{dx} - 2y = 0$$

Solution:

$$4\frac{d^2y}{dx^2} - 7\frac{dy}{dx} - 2y = 0$$

The auxiliary equation is

$$4m^2 - 7m - 2 = 0$$

$$\therefore \quad (4m + 1)(m - 2) = 0$$

$$\therefore \quad m = -\tfrac{1}{4} \text{ or } 2$$

So the general solution is $y = Ae^{-\frac{1}{4}x} + Be^{2x}$.

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Exercise A, Question 9

Question:

$$6\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 0$$

Solution:

$$6\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 0$$

Find the auxiliary equation and solve to give two distinct real roots α and β. The general solution is $y = Ae^{\alpha x} + Be^{\beta x}$.

The auxiliary equation is

$$6m^2 - m - 2 = 0$$

$\therefore \quad (3m - 2)(2m + 1) = 0$

$\therefore \quad m = \frac{2}{3} \text{ or } -\frac{1}{2}$

So the general solution is $y = Ae^{\frac{2}{3}x} + Be^{-\frac{1}{2}x}$.

Solutionbank FP2
Edexcel AS and A Level Modular Mathematics

Exercise A, Question 10

Question:

$$15\frac{d^2y}{dx^2} - 7\frac{dy}{dx} - 2y = 0$$

Solution:

$$15\frac{d^2y}{dx^2} - 7\frac{dy}{dx} - 2y = 0$$

The auxiliary equation is

$$15m^2 - 7m - 2 = 0$$

$$\therefore\ (5m + 1)(3m - 2) = 0$$

$$\therefore\quad m = -\tfrac{1}{5} \text{ or } \tfrac{2}{3}$$

So the general solution is

$$y = Ae^{-\frac{1}{5}x} + Be^{\frac{2}{3}x}.$$

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Exercise B, Question 1

Question:

$$\frac{d^2y}{dx^2} + 10\frac{dy}{dx} + 25y = 0$$

Solution:

$$\frac{d^2y}{dx^2} + 10\frac{dy}{dx} + 25y = 0$$

The auxiliary equation is

$$m^2 + 10m + 25 = 0$$

$\therefore \quad (m + 5)(m + 5) = 0 \quad$ or $\quad (m + 5)^2 = 0$

$\therefore \quad m = -5$ only.

So the general solution is

$$y = (A + Bx)e^{-5x}.$$

The auxiliary equation has repeated roots and so the general solution is of the form $(A + Bx)e^{\alpha x}$, where α is the repeated root.

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Edexcel AS and A Level Modular Mathematics

Exercise B, Question 2

Question:

$$\frac{d^2y}{dx^2} - 18\frac{dy}{dx} + 81y = 0$$

Solution:

$$\frac{d^2y}{dx^2} - 18\frac{dy}{dx} + 81y = 0$$

The auxiliary equation is

$$m^2 - 18m + 81 = 0$$

$$\therefore \quad (m-9)^2 = 0$$

$$\therefore \quad m = 9 \text{ only.}$$

So the general solution is

$$y = (A + Bx)e^{9x}.$$

The auxiliary equation is $m^2 - 18m + 81 = 0$, which has repeated roots and so the general solution is of the form $(A + Bx)e^{\alpha x}$, where α is the repeated root.

Solutionbank FP2
Edexcel AS and A Level Modular Mathematics

Exercise B, Question 3

Question:

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 0$$

Solution:

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 0$$

The auxiliary equation is

$$m^2 + 2m + 1 = 0$$

$$\therefore \quad (m + 1)(m + 1) = 0 \quad \text{or} \quad (m + 1)^2 = 0$$

$$\therefore \quad m = -1 \text{ only.}$$

So the general solution is

$$y = (A + Bx)e^{-x}.$$

The auxiliary equation is $m^2 + 2m + 1 = 0$, which has repeated roots and so the general solution is of the form $(A + Bx)e^{\alpha x}$, where α is the repeated root.

Solutionbank FP2
Edexcel AS and A Level Modular Mathematics

Exercise B, Question 4

Question:

$$\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 16y = 0$$

Solution:

$$\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 16y = 0$$

The auxiliary equation is

$$m^2 - 8m + 16 = 0$$

$\therefore \quad (m-4)^2 = 0$

$\therefore \quad m = 4$ only.

$\therefore$ The general solution is $y = (A + Bx)e^{4x}$.

The auxiliary equation has repeated roots and so the general solution is of the form $(A + Bx)e^{\alpha x}$, where α is the repeated root.

Solutionbank FP2

Edexcel AS and A Level Modular Mathematics

Exercise B, Question 5

Question:

$$\frac{d^2y}{dx^2} + 14\frac{dy}{dx} + 49y = 0$$

Solution:

$$\frac{d^2y}{dx^2} + 14\frac{dy}{dx} + 49y = 0$$

The auxiliary equation is

$$m^2 + 14m + 49 = 0$$

$\therefore \quad (m + 7)^2 = 0$

$\therefore \quad m = -7$ only.

So the general solution is

$$y = (A + Bx)e^{-7x}.$$

The auxiliary equation has repeated roots and so the general solution is of the form $(A + Bx)e^{\alpha x}$, where α is the repeated root.

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Edexcel AS and A Level Modular Mathematics

Exercise B, Question 6

Question:

$$16\frac{d^2y}{dx^2}+8\frac{dy}{dx}+y=0$$

Solution:

$$16\frac{d^2y}{dx^2}+8\frac{dy}{dx}+y=0$$

The auxiliary equation is

$$16m^2+8m+1=0$$

$\therefore \quad (4m+1)^2=0$

$\therefore \quad m=-\frac{1}{4}$ only.

So the general solution is

$$y=(A+Bx)e^{-\frac{1}{4}x}.$$

The auxiliary equation has repeated roots and so the general solution is of the form $(A+Bx)e^{\alpha x}$, where α is the repeated root.

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Exercise B, Question 7

Question:

$$4\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + y = 0$$

Solution:

$$4\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + y = 0$$

The auxiliary equation is

$$4m^2 - 4m + 1 = 0$$

$\therefore \quad (2m - 1)^2 = 0$

$\therefore \quad m = \frac{1}{2}$ only.

So the general solution is

$$y = (A + Bx)e^{\frac{1}{2}x}.$$

The auxiliary equation has repeated roots and so the general solution is of the form $(A + Bx)e^{\alpha x}$, where α is the repeated root.

Solutionbank FP2
Edexcel AS and A Level Modular Mathematics

Exercise B, Question 8

Question:

$$4\frac{d^2y}{dx^2} + 20\frac{dy}{dx} + 25y = 0$$

Solution:

$$4\frac{d^2y}{dx^2} + 20\frac{dy}{dx} + 25y = 0$$

The auxiliary equation is

$$4m^2 + 20m + 25 = 0$$

$$\therefore \quad (2m + 5)^2 = 0$$

$$\therefore \quad m = -2\tfrac{1}{2} = -\tfrac{5}{2} \text{ only.}$$

So the general solution is

$$y = (A + Bx)e^{-\frac{5}{2}x}.$$

The auxiliary equation has repeated roots and so the general solution is of the form $(A + Bx)e^{\alpha x}$, where α is the repeated root.

Solutionbank FP2
Edexcel AS and A Level Modular Mathematics

Exercise B, Question 9

Question:

$$16\frac{d^2y}{dx^2} - 24\frac{dy}{dx} + 9y = 0$$

Solution:

$$16\frac{d^2y}{dx^2} - 24\frac{dy}{dx} + 9y = 0$$

The auxiliary equation is

$$16m^2 - 24m + 9 = 0$$

$$\therefore \quad (4m - 3)^2 = 0$$

$$\therefore \quad m = \tfrac{3}{4} \text{ only.}$$

So the general solution is

$$y = (A + Bx)e^{\frac{3}{4}x}.$$

The auxiliary equation has repeated roots and so the general solution is of the form $(A + Bx)e^{\alpha x}$, where α is the repeated root.

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Edexcel AS and A Level Modular Mathematics

Exercise B, Question 10

Question:

$$\frac{d^2y}{dx^2} + 2\sqrt{3}\frac{dy}{dx} + 3y = 0$$

Solution:

$$\frac{d^2y}{dx^2} + 2\sqrt{3}\frac{dy}{dx} + 3y = 0$$

The auxiliary equation is

$$m^2 + 2\sqrt{3}m + 3 = 0$$

$$\therefore \quad (m + \sqrt{3})^2 = 0$$

$$\therefore \quad m = -\sqrt{3}$$

or using quadratic formula:

$$m = \frac{-2\sqrt{3} \pm \sqrt{12 - 12}}{2}$$

$$= -\sqrt{3}$$

So the general solution is

$$y = (A + Bx)e^{-\sqrt{3}x}.$$

The auxiliary equation has repeated roots and so the general solution is of the form $(A + Bx)e^{\alpha x}$, where α is the repeated root.

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Edexcel AS and A Level Modular Mathematics

Exercise C, Question 1

Question:

$$\frac{d^2y}{dx^2} + 25y = 0$$

Solution:

$$\frac{d^2y}{dx^2} + 25y = 0$$

The auxiliary equation is

$m^2 + 25 = 0$

$\therefore \quad m = \pm 5i$

The general solution is

$y = A\cos 5x + B\sin 5x.$

The auxiliary equation has imaginary roots and so the general solution has the form $A \cos \omega x + B \sin \omega x$, where A and B are constants and where $i\omega$ is a solution of the auxiliary equation.

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Exercise C, Question 2

Question:

$$\frac{d^2y}{dx^2} + 81y = 0$$

Solution:

$$\frac{d^2y}{dx^2} + 81y = 0$$

The auxiliary equation is

$$m^2 + 81 = 0$$

$$\therefore \quad m = \pm 9i$$

The general solution is

$$y = A\cos 9x + B\sin 9x.$$

The auxiliary equation has imaginary roots and so the general solution has the form $A \cos \omega x + B \sin \omega x$, where A and B are constants and where $i\omega$ is a solution of the auxiliary equation.

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Exercise C, Question 3

Question:

$$\frac{d^2y}{dx^2} + y = 0$$

Solution:

$$\frac{d^2y}{dx^2} + y = 0$$

The auxiliary equation is

$$m^2 + 1 = 0$$

$$\therefore \quad m = \pm i$$

The general solution is

$$y = A\cos x + B\sin x.$$

The auxiliary equation has imaginary roots and so the general solution has the form $A \cos \omega x + B \sin \omega x$, where A and B are constants and where $i\omega$ is a solution of the auxiliary equation.

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Exercise C, Question 4

Question:

$$9\frac{d^2y}{dx^2} + 16y = 0$$

Solution:

$$9\frac{d^2y}{dx^2} + 16y = 0$$

The auxiliary equation is

$$9m^2 + 16 = 0$$

$\therefore \quad m^2 = -\frac{16}{9}$

and $\quad m = \pm\frac{4}{3}\text{i}$

$\therefore$ The general solution is

$$y = A\cos\tfrac{4}{3}x + B\sin\tfrac{4}{3}x.$$

The auxiliary equation has imaginary roots and so the general solution has the form $A\cos\omega x + B\sin\omega x$, where A and B are constants and where $\text{i}\omega$ is a solution of the auxiliary equation.

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Exercise C, Question 5

Question:

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 13y = 0$$

Solution:

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 13y = 0$$

The auxiliary equation is

$m^2 + 4m + 13 = 0$

$\therefore \quad m = \dfrac{-4 \pm \sqrt{16 - 52}}{2}$

And $\quad m = -2 \pm 3i$

The general solution is

$$y = e^{-2x}(A\cos 3x + B\sin 3x).$$

The auxiliary equation has complex roots and so the general solution has the form $e^{px}(A \cos qx + B \sin qx)$, where A and B are constants and where $p \pm iq$ are solutions of the auxiliary equation.

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Exercise C, Question 6

Question:

$$\frac{d^2y}{dx^2} + 8\frac{dy}{dx} + 17y = 0$$

Solution:

$$\frac{d^2y}{dx^2} + 8\frac{dy}{dx} + 17y = 0$$

The auxiliary equation is

$$m^2 + 8m + 17 = 0$$

$$\therefore \quad m = \frac{-8 \pm \sqrt{64 - 4 \times 17}}{2}$$

$$= -4 \pm \tfrac{1}{2}\sqrt{-4}$$

$$= -4 \pm \mathrm{i}$$

The general solution is

$$y = e^{-4x}(A\cos x + B\sin x).$$

The auxiliary equation has complex roots and so the general solution has the form $e^{px}(A \cos qx + B \sin qx)$, where A and B are constants and where $p \pm \mathrm{i}q$ are solutions of the auxiliary equation.

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Exercise C, Question 7

Question:

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 5y = 0$$

Solution:

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 5y = 0$$

The auxiliary equation is

$$m^2 - 4m + 5 = 0$$

$$\therefore \quad m = \frac{4 \pm \sqrt{16 - 20}}{2}$$

$$= 2 \pm \tfrac{1}{2}\sqrt{-4}$$

$$= 2 \pm \mathrm{i}$$

$$\therefore \quad y = e^{2x}(A\cos x + B\sin x).$$

The auxiliary equation has complex roots and so the general solution has the form $e^{px}(A \cos qx + B \sin qx)$, where A and B are constants and where $p \pm iq$ are solutions of the auxiliary equation.

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Exercise C, Question 8

Question:

$$\frac{d^2y}{dx^2} + 20\frac{dy}{dx} + 109y = 0$$

Solution:

$$\frac{d^2y}{dx^2} + 20\frac{dy}{dx} + 109y = 0$$

The auxiliary equation is

$$m^2 + 20m + 109 = 0$$

$$\therefore \quad m = \frac{-20 \pm \sqrt{400 - 436}}{2}$$

$$= \frac{-20 \pm \sqrt{-36}}{2}$$

$$= -10 \pm 3i$$

$\therefore$ The general solution is

$$y = e^{-10x}(A\cos 3x + B\sin 3x).$$

The auxiliary equation has complex roots and so the general solution has the form $e^{px}(A\cos qx + B\sin qx)$, where A and B are constants and where $p \pm iq$ are solutions of the auxiliary equation.

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Exercise C, Question 9

Question:

$$9\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 5y = 0$$

Solution:

$$9\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 5y = 0$$

The auxiliary equation is

$$9m^2 - 6m + 5 = 0$$

$$\therefore \quad m = \frac{6 \pm \sqrt{36 - 4 \times 9 \times 5}}{2 \times 9}$$

$$= \frac{6 \pm \sqrt{36 - 180}}{18}$$

$$= \frac{6 \pm \sqrt{-144}}{18}$$

$$= \frac{1 \pm 2i}{3}$$

$\therefore$ The general solution is

$$y = e^{\frac{1}{3}x}(A\cos\tfrac{2}{3}x + B\sin\tfrac{2}{3}x).$$

The auxiliary equation has complex roots and so the general solution has the form $e^{px}(A\cos qx + B\sin qx)$, where A and B are constants and where $p \pm iq$ are solutions of the auxiliary equation.

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Exercise C, Question 10

Question:

$$\frac{d^2y}{dx^2} + \sqrt{3}\frac{dy}{dx} + 3y = 0$$

Solution:

$$\frac{d^2y}{dx^2} + \sqrt{3}\frac{dy}{dx} + 3y = 0$$

The auxiliary equation is

$$m^2 + \sqrt{3}m + 3 = 0$$

$$\therefore \quad m = \frac{-\sqrt{3} \pm \sqrt{3 - 4 \times 3}}{2}$$

$$= \frac{-\sqrt{3} \pm \sqrt{-9}}{2}$$

$$= \frac{-\sqrt{3} \pm 3\text{i}}{2}$$

$\therefore$ The general solution is

$$y = e^{-\frac{\sqrt{3}}{2}x}(A\cos\tfrac{3}{2}x + B\sin\tfrac{3}{2}x).$$

The auxiliary equation has complex roots and so the general solution has the form $e^{px}(A\cos qx + B\sin qx)$, where A and B are constants and where $p \pm iq$ are solutions of the auxiliary equation.

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Exercise D, Question 1

Question:

$$\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 5y = 10$$

Solution:

$$\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 5y = 10 \quad *$$

First consider

$$\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 5y = 0$$

The auxiliary equation is

$$m^2 + 6m + 5 = 0$$

$\therefore \quad (m + 5)(m + 1) = 0$

$\therefore \quad m = -5$ or -1

So the complementary function is $y = Ae^{-x} + Be^{-5x}$.

The particular integral is λ and so $\frac{dy}{dx} = 0$,

$\frac{d^2y}{dx^2} = 0$ and substituting into $*$ gives

$5\lambda = 10$

$\therefore \quad \lambda = 2$

The general solution is $y = Ae^{-x} + Be^{-5x} + 2$.

Find the complementary function, which is the solution of $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 5y = 0$, then try a particular integral $y = \lambda$.

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Exercise D, Question 2

Question:

$$\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 12y = 36x$$

Solution:

$$\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 12y = 36x \quad *$$

First consider the equation

$$\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 12y = 0.$$

The auxiliary equation is

$$m^2 - 8m + 12 = 0$$

$$\therefore \quad (m-6)(m-2) = 0$$

$$\therefore \quad m = 6 \text{ or } 2$$

So the complementary function is $y = Ae^{6x} + Be^{2x}$.

The particular integral is $y = \lambda + \mu x$

Try a particular integral of the form $\lambda + \mu x$.

so $\dfrac{dy}{dx} = \mu, \dfrac{d^2y}{dx^2} = 0$

Substitute into $*$.

Then $-8\mu + 12\lambda + 12\mu x = 36x$.

Comparing coefficients of x: $12\mu = 36$, and so $\mu = 3$

Comparing constant terms: $-8\mu + 12\lambda = 0$

and as $\mu = 3$ $\quad \therefore \quad -24 + 12\lambda = 0 \Rightarrow \lambda = 2$

$\therefore$ $2 + 3x$ is the particular integral.

$\therefore$ The general solution is

$$y = Ae^{6x} + Be^{2x} + 2 + 3x.$$

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Exercise D, Question 3

Question:

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 12y = 12e^{2x}$$

Solution:

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 12y = 12e^{2x} \quad *$$

First consider the equation

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 12y = 0.$$

The auxiliary equation is

$$m^2 + m - 12 = 0$$

$\therefore \quad (m+4)(m-3) = 0$

$\therefore \quad m = -4 \text{ or } 3$

So the complementary function is $y = Ae^{-4x} + Be^{3x}$.

Try a particular integral of the form λe^{2x}.

The particular integral is $y = \lambda e^{2x}$

$\therefore \quad \frac{dy}{dx} = 2\lambda e^{2x}$ and $\frac{d^2y}{dx^2} = 4\lambda e^{2x}$

Substitute into $*$.

Then $4\lambda e^{2x} + 2\lambda e^{2x} - 12\lambda e^{2x} = 12e^{2x}$

i.e. $\quad -6\lambda e^{2x} = 12e^{2x}$

$\therefore \quad \lambda = -2$

$\therefore \quad -2e^{2x}$ is a particular integral.

The general solution is

$$y = Ae^{-4x} + Be^{3x} - 2e^{2x}.$$

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Exercise D, Question 4

Question:

$$\frac{d^2y}{dx^2}+2\frac{dy}{dx}-15y=5$$

Solution:

$$\frac{d^2y}{dx^2}+2\frac{dy}{dx}-15y=5 \quad *$$

First consider the equation

$$\frac{d^2y}{dx^2}+2\frac{dy}{dx}-15y=0$$

The auxiliary equation is

$$m^2+2m-15=0$$

$\therefore \quad (m+5)(m-3)=0$

$\therefore \quad m=-5 \text{ or } 3$

So the complementary function is $y=Ae^{-5x}+Be^{3x}$.

The particular integral is $y=\lambda$

$$\therefore \quad \frac{dy}{dx}=\frac{d^2y}{dx^2}=0$$

Substitute into $*$.

Then $\quad -15\lambda=5$

i.e. $\quad \lambda=-\frac{1}{3}$

$\therefore \quad -\frac{1}{3}$ is the particular integral.

The general solution is $\quad y=Ae^{-5x}+Be^{3x}-\frac{1}{3}$.

Try a particular integral $y=\lambda$.

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Exercise D, Question 5

Question:

$$\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 16y = 8x + 12$$

Solution:

$$\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 16y = 8x + 12 \quad *$$

First consider the equation

$$\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 16y = 0$$

The auxiliary equation is

$$m^2 - 8m + 16 = 0$$

$\therefore \quad (m - 4)^2 = 0$

$\therefore \quad m = 4$ only.

So the complementary function is $y = (A + Bx)e^{4x}$.

The auxiliary equation has a repeated root so the complementary function is of the form $(A + Bx)e^{\alpha x}$.

The particular integral is $y = \lambda + \mu x$

$\therefore \quad \frac{dy}{dx} = \mu$ and $\frac{d^2y}{dx^2} = 0$

Substitute in $*$.

Then $\quad 0 - 8\mu + 16\lambda + 16\mu x = 8x + 12$

Equate coefficients of x: $\quad 16\mu = 8$

$\therefore \quad \mu = \frac{1}{2}$

Equate constant terms: $-8\mu + 16\lambda = 12$

Substitute $\mu = \frac{1}{2} \quad \therefore \quad -4 + 16\lambda = 12$

$\therefore \quad 16\lambda = 16$

and $\quad \lambda = 1$

$\therefore \quad 1 + \frac{1}{2}x$ is a particular integral

The general solution is $y = (A+ Bx)e^{4x} + 1 + \frac{1}{2}x$.

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Exercise D, Question 6

Question:

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 25\cos 2x$$

Solution:

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 25\cos 2x \quad *$$

Solve $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 0$

The auxiliary equation is

$$m^2 + 2m + 1 = 0$$

$\therefore \quad (m+1)^2 = 0$

$\therefore \quad m = -1$ only.

So the complementary function is $y = (A + Bx)e^{-x}$.

The particular integral is $y = \lambda\cos 2x + \mu\sin 2x$

$\therefore \quad \frac{dy}{dx} = -2\lambda\sin 2x + 2\mu\cos 2x$

$$\frac{d^2y}{dx^2} = -4\lambda\cos 2x - 4\mu\sin 2x$$

The complementary function is of the form $y = (A + Bx)e^{\alpha x}$. The particular integral is $\lambda\cos 2x + \mu\sin 2x$.

Substitute in $*$.

Then $(-4\lambda\cos 2x - 4\mu\sin 2x) + 2(-2\lambda\sin 2x + 2\mu\cos 2x)$
$+ (\lambda\cos 2x + \mu\sin 2x) = 25\cos 2x$

Equate coefficients of $\cos 2x$: $\quad -3\lambda + 4\mu = 25$ ①

Equate coefficients of $\sin 2x$: $\quad -3\mu - 4\lambda = 0$ ②

Solve equations ① and ②: $\quad 3 \times ① + 4 \times ② \Rightarrow -25\lambda = 75$

$\therefore \quad \lambda = -3$

Substitute into ① $9 + 4\mu = 25 \quad \therefore \quad \mu = 4$ [check in ②.]

$\therefore$ The particular integral is $y = 4\sin 2x - 3\cos 2x$

$\therefore$ General solution is $y = (A + Bx)e^{-x} + 4\sin 2x - 3\cos 2x$.

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Exercise D, Question 7

Question:

$$\frac{d^2y}{dx^2} + 81y = 15e^{3x}$$

Solution:

$$\frac{d^2y}{dx^2} + 81y = 15e^{3x} \quad *$$

First solve $\frac{d^2y}{dx^2} + 81y = 0$

This has auxiliary equation

$$m^2 + 81 = 0$$

$$\therefore \quad m = \pm 9i$$

The complementary function is $y = A\cos 9x + B\sin 9x$.

The auxiliary equation has imaginary roots, so the complementary function is of the form $A\cos \omega x + B\sin \omega x$.

The particular integral is $y = \lambda e^{3x}$

Then $\frac{dy}{dx} = 3\lambda e^{3x}$ and $\frac{d^2y}{dx^2} = 9\lambda e^{3x}$

Substitute into $*$.

Then $9\lambda e^{3x} + 81\lambda e^{3x} = 15e^{3x}$

$\therefore \quad 90\lambda e^{3x} = 15e^{3x}$

So $\lambda = \frac{15}{90} = \frac{1}{6}$

$\therefore$ The particular integral is $\frac{1}{6}e^{3x}$

$\therefore$ The general solution is $y = A\cos 9x + B\sin 9x + \frac{1}{6}e^{3x}$.

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Exercise D, Question 8

Question:

$$\frac{d^2y}{dx^2} + 4y = \sin x$$

Solution:

$$\frac{d^2y}{dx^2} + 4y = \sin x \quad *$$

First solve $\frac{d^2y}{dx^2} + 4y = 0$.

This has auxiliary equation

$$m^2 + 4 = 0$$

$$\therefore \quad m = \pm 2\mathrm{i}$$

The complementary function is $y = A\cos 2x + B\sin 2x$

The complementary function is of the form $A\cos\omega x + B\sin\omega x$, as the auxiliary equation has imaginary roots.

The particular integral is $y = \lambda\cos x + \mu\sin x$

$$\therefore \quad \frac{dy}{dx} = -\lambda\sin x + \mu\cos x$$

and $$\frac{d^2y}{dx^2} = -\lambda\cos x - \mu\sin x$$

Substitute into $*$.

Then $-\lambda\cos x - \mu\sin x + 4(\lambda\cos x + \mu\sin x) = \sin x$

Equate coefficients of $\cos x$: $3\lambda = 0$

$\therefore \quad \lambda = 0$

Equate coefficients of $\sin x$: $3\mu = 1$

$\therefore \quad \mu = \frac{1}{3}$

So the particular integral is $\frac{1}{3}\sin x$

The general solution is $y = A\cos 2x + B\sin 2x + \frac{1}{3}\sin x$.

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Exercise D, Question 9

Question:

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 5y = 25x^2 - 7$$

Solution:

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 5y = 25x^2 - 7 \quad *$$

First solve $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 5y = 0$

This has auxiliary equation

$$m^2 - 4m + 5 = 0$$

$$\therefore \quad m = \frac{4 \pm \sqrt{16 - 20}}{2}$$

$$= 2 \pm 2i$$

The complementary function is $y = e^{2x}(A\cos 2x + B\sin 2x)$

The P.I is of the form $y = \lambda + \mu x + \nu x^2$

The particular integral is $y = \lambda + \mu x + \nu x^2$

$$\therefore \quad \frac{dy}{dx} = \mu + 2\nu x$$

and $\frac{d^2y}{dx^2} = 2\nu$

Substitute into $*$.

Then $2\nu - 4\mu - 8\nu x + 5\lambda + 5\mu x + 5\nu x^2 = 25x^2 - 7$

Equate coefficients of x^2: $5\nu = 25 \Rightarrow \nu = 5$

coefficients of x: $5\mu - 8\nu = 0 \Rightarrow \mu = 8$

constant terms: $2\nu - 4\mu + 5\lambda = -7$

$$\therefore \quad 10 - 32 + 5\lambda = -7$$

$$\therefore \quad 5\lambda = 15 \Rightarrow \lambda = 3$$

So the particular integral is $3 + 8x + 5x^2$

The general solution is $y = e^{2x}(A\cos 2x + B\sin 2x) + 3 + 8x + 5x^2$.

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Exercise D, Question 10

Question:

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 26y = e^x$$

Solution:

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 26y = e^x \quad *$$

First solve $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 26y = 0$

This has auxiliary equation

$$m^2 - 2m + 26 = 0$$

$$\therefore \quad m = \frac{2 \pm \sqrt{4 - 4 \times 26}}{2}$$

$$= \frac{2 \pm \sqrt{-100}}{2}$$

$$= 1 \pm 5i$$

The auxiliary equation has complex roots and so the complementary function is of the form $e^{px}(A\cos qx + B\sin qx)$.

$\therefore$ the complementary function is $y = e^x(A\cos 5x + B\sin 5x)$.

The particular integral is λe^x, so $\frac{dy}{dx} = \lambda e^x$ and

$$\frac{d^2y}{dx^2} = \lambda e^x$$

Substitute into equation $*$.

Then $\lambda e^x - 2\lambda e^x + 26\lambda e^x = e^x$

i.e. $25\lambda e^x = e^x$

$\therefore \quad \lambda = \frac{1}{25}$

The particular integral is $\frac{1}{25}e^x$.

$\therefore$ The general solution is

$$y = e^x(A\cos 5x + B\sin 5x) + \frac{1}{25}e^x.$$

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Exercise D, Question 11

Question:

a Find the value of λ for which $\lambda x^2 e^x$ is a particular integral for the differential equation

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = e^x$$

b Hence find the general solution.

Solution:

a $$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = e^x \quad *$$

Given $y = \lambda x^2 e^x$ is a particular integral

$$\frac{dy}{dx} = \lambda x^2 e^x + 2\lambda x e^x$$

$$\frac{d^2y}{dx^2} = \lambda x^2 e^x + 2\lambda x e^x + 2\lambda x e^x + 2\lambda e^x$$

Substitute into $*$.

Then $(\lambda x^2 + 4\lambda x + 2\lambda)e^x - (2\lambda x^2 + 4\lambda x)e^x + \lambda x^2 e^x = e^x$

$\therefore \quad 2\lambda e^x = e^x$

$\therefore \quad \lambda = \frac{1}{2}$

So $y = \frac{1}{2}x^2 e^x$ is a particular integral.

b Now solve $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 0$

This has auxiliary equation $m^2 - 2m + 1 = 0$

$\therefore \quad (m-1)^2 = 0$

$\therefore \quad m = 1$ only

So the complementary function is $(A + Bx)e^x$

The general solution is $y = (A + Bx + \frac{1}{2}x^2)e^x$.

The auxiliary equation has equal roots and so the complementary function has the form $y = (A + Bx)e^{\alpha x}$

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Exercise E, Question 1

Question:

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 12e^x \qquad y = 1 \text{ and } \frac{dy}{dx} = 0 \text{ at } x = 0$$

Solution:

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 12e^x \quad *$$

Solve the equation to find the general solution, then substitute $y = 1$ when $x = 0$ to obtain an equation relating A and B. Obtain a second equation by using $\frac{dy}{dx} = 0$ at $x = 0$, and solve to find A and B.

Find complementary function.

Auxiliary equation is $m^2 + 5m + 6 = 0$

$\therefore$ $(m + 3)(m + 2) = 0$

$\therefore$ $m = -3$ or -2

$\therefore$ complementary function is $y = Ae^{-3x} + Be^{-2x}$

Then find particular integral

Let $y = \lambda e^x$

Then $\frac{dy}{dx} = \lambda e^x$ and $\frac{d^2y}{dx^2} = \lambda e^x$

Substitute into $*$. Then $(\lambda + 5\lambda + 6\lambda)e^x = 12e^x$

$\therefore$ $12\lambda e^x = 12e^x$

$\therefore$ $\lambda = 1$

So particular integral is $y = e^x$

$\therefore$ General solution is $Ae^{-3x} + Be^{-2x} + e^x = y$ †

But $y = 1$ when $x = 0$ $\therefore$ $A + B + 1 = 1$

i.e. $A + B = 0$ ①

$$\frac{dy}{dx} = -3Ae^{-3x} - 2Be^{-2x} + e^x$$

$\frac{dy}{dx} = 0$ when $x = 0$ $\therefore$ $-3A - 2B + 1 = 0$

$\therefore$ $3A + 2B = 1$ ②

From ① $B = -A$, substitute into equation ②

$3A - 2A = 1 \Rightarrow A = 1$

$\therefore$ $B = -1$

Substitute these values into †

The particuar solution is $y = e^{-3x} - e^{-2x} + e^x$

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Exercise E, Question 2

Question:

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 12e^{2x} \qquad y = 2 \text{ and } \frac{dy}{dx} = 6 \text{ at } x = 0$$

Solution:

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 12e^{2x} \quad *$$

Find complementary function (c.f.):

Auxiliary equation is $m^2 + 2m = 0$

$\therefore \quad m(m+2) = 0$

$\therefore \quad m = 0 \text{ or } -2$

$\therefore$ c.f. is $y = Ae^{0x} + Be^{-2x}$

$= A + Be^{-2x}$

Particular integral (p.i.) is of the form $y = \lambda e^{2x}$

$\therefore \quad \frac{dy}{dx} = 2\lambda e^{2x}, \quad \frac{d^2y}{dx^2} = 4\lambda e^{2x}$

Substitute into $*$.

Then $(4\lambda + 4\lambda)e^{2x} = 12e^{2x}$

i.e. $8\lambda e^{2x} = 12e^{2x} \Rightarrow \lambda = \frac{12}{8} = \frac{3}{2}$

$\therefore$ p.i. is $\frac{3}{2}e^{2x}$

$\therefore$ General solution is $y = A + Be^{-2x} + \frac{3}{2}e^{2x} \quad †$

But $y = 2$ when $x = 0 \quad \therefore \quad 2 = A + B + \frac{3}{2}$

i.e. $A + B = \frac{1}{2}$ ①

The general solution is $y = A + Be^{-2x} + \frac{3}{2}e^{2x}$.

$\frac{dy}{dx} = -2Be^{-2x} + 3e^{2x}$

$\frac{dy}{dx} = 6$ when $x = 0 \quad \therefore \quad 6 = -2B + 3$

$\therefore \quad -2B = 3 \Rightarrow B = -\frac{3}{2}$

Substitute into equation ① $\quad A - \frac{3}{2} = \frac{1}{2}$

$\therefore \quad A = 2$

Substitute A and B into †

$\therefore$ The particular solution is $\quad y = 2 - \frac{3}{2}e^{-2x} + \frac{3}{2}e^{2x}$

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Exercise E, Question 3

Question:

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 42y = 14 \qquad y = 0 \text{ and } \frac{dy}{dx} = \frac{1}{6} \text{ at } x = 0$$

Solution:

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 42y = 14 \quad *$$

Find the general solution, then use the boundary conditions to find the constants A and B.

Find c.f.: The auxiliary equation is

$m^2 - m - 42 = 0$

$\therefore \quad (m-7)(m+6) = 0$

$\therefore \quad m = -6$ or 7

$\therefore$ c.f. is $y = Ae^{-6x} + Be^{7x}$

Find p.i.: The particular integral is $y = \lambda$. Substitute in $*$.

$\therefore \quad -42\lambda = 14$

$\therefore \quad \lambda = -\frac{1}{3}$

$\therefore$ The general solution is $y = Ae^{-6x} + Be^{7x} - \frac{1}{3}$ †

When $x = 0, y = 0 \quad \therefore \quad 0 = A + B - \frac{1}{3}$

$\therefore \quad A + B = \frac{1}{3}$ ①

$$\frac{dy}{dx} = -6Ae^{-6x} + 7Be^{7x}$$

When $x = 0$, $\frac{dy}{dx} = \frac{1}{6} \quad \therefore \quad \frac{1}{6} = -6A + 7B$

i.e: $-6A + 7B = \frac{1}{6}$ ②

Solve equations ① and ② by forming $6 \times$ ① + ②

$\therefore \quad 13B = 2\frac{1}{6}$

$B = \frac{1}{6}$

Substitute into ① $\quad \therefore \quad A + \frac{1}{6} = \frac{1}{3} \Rightarrow A = \frac{1}{6}$

Substitute values of A and B into †

$\therefore \quad y = \frac{1}{6}e^{-6x} + \frac{1}{6}e^{7x} - \frac{1}{3}$ is required solution

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Exercise E, Question 4

Question:

$\frac{d^2y}{dx^2} + 9y = 16 \sin x$ $\qquad$ $y = 1$ and $\frac{dy}{dx} = 0$ at $x = 0$

Solution:

$$\frac{d^2y}{dx^2} + 9y = 16 \sin x \quad *$$

Find c.f.: The auxiliary equation is

$m^2 + 9 = 0$

$\therefore \quad m = \pm 3\text{i}$

$\therefore$ The c.f. is $y = A\cos 3x + B\sin 3x$

The auxiliary equation has imaginary roots and so the complementary function has the form $y = A\cos \omega x + B\sin \omega x$.

Find p.i. use $y = \lambda\cos x + \mu\sin x$

$$\frac{dy}{dx} = -\lambda\sin x + \mu\cos x$$

$$\frac{d^2y}{dx^2} = -\lambda\cos x - \mu\sin x$$

$\therefore$ Substituting into $*$ gives

$$-\lambda\cos x - \mu\sin x + 9\lambda\cos x + 9\mu\sin x = 16\sin x$$

Equating coefficients of $\cos x$: $8\lambda = 0 \Rightarrow \lambda = 0$

$\sin x$: $8\mu = 16 \Rightarrow \mu = 2$

$\therefore$ The particular integral is $y = 2\sin x$

$\therefore$ The general solution is $y = A\cos 3x + B\sin 3x + 2\sin x \quad †$

Given also that $y = 1$ at $x = 0 \quad \therefore \quad 1 = A$

$$\frac{dy}{dx} = -3A\sin 3x + 3B\cos 3x + 2\cos x$$

Using $\frac{dy}{dx} = 8$ at $x = 0 \quad \therefore \quad 8 = 3B + 2 \quad \therefore \quad B = 2$

Substituting A and B into †

$y = \cos 3x + 2\sin 3x + 2\sin x$ is the required solution.

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Exercise E, Question 5

Question:

$$4\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 5y = \sin x + 4\cos x \qquad y = 0 \text{ and } \frac{dy}{dx} = 0 \text{ at } x = 0$$

Solution:

$$4\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 5y = \sin x + 4\cos x \quad *$$

The auxiliary equation has complex roots and so the complementary function has the form $y = e^{px}(A\cos qx + B\sin qx)$.

Find c.f.: the auxiliary equation is

$$4m^2 + 4m + 5 = 0$$

$\therefore \quad m = \frac{-4 \pm \sqrt{16 - 80}}{8} = \frac{-4 \pm \sqrt{-64}}{8} = \frac{-4 \pm 8i}{8}$

$\therefore \quad m = -\frac{1}{2} \pm i$

$\therefore$ The c.f. is $y = e^{-\frac{1}{2}x}(A\cos x + B\sin x)$

The p.i. is $y = \lambda\cos x + \mu\sin x$

$\therefore \quad \frac{dy}{dx} = -\lambda\sin x + \mu\cos x$

$\frac{d^2y}{dx^2} = -\lambda\cos x - \mu\sin x$

Substitute into $*$

Then $-4\lambda\cos x - 4\mu\sin x - 4\lambda\sin x + 4\mu\cos x + 5\lambda\cos x + 5\mu\sin x = \sin x + 4\cos x$

Equating coefficients of $\cos x$: $\lambda + 4\mu = 4$ ①

$\sin x$: $\mu - 4\lambda = 1$ ②

Add equation ② to 4 times equation ①

$\therefore \quad 17\mu = 17 \Rightarrow \mu = 1$

Substitute into equation ① $\therefore \lambda + 4 = 4 \Rightarrow \lambda = 0$

$\therefore$ p.i. is $y = \sin x$

$\therefore$ The general solution is

$y = e^{-\frac{1}{2}x}(A\cos x + B\sin x) + \sin x \quad \dagger$

As $y = 0$ when $x = 0$

$\therefore \quad 0 = A$

$\therefore \quad y = Be^{-\frac{1}{2}x}\sin x + \sin x$

$\therefore \quad \frac{dy}{dx} = Be^{-\frac{1}{2}x}\cos x - \frac{1}{2}Be^{-\frac{1}{2}x}\sin x + \cos x$

As $\frac{dy}{dx} = 0$ when $x = 0$

$0 = B + 1 \Rightarrow B = -1$

Substituting these values for A and B into $\dagger$

$\therefore \quad y = \sin x\,(1 - e^{-\frac{1}{2}x})$ is the required solution.

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Exercise E, Question 6

Question:

$$\frac{d^2x}{dt^2} - 3\frac{dx}{dt} + 2x = 2t - 3 \qquad x = 2 \text{ and } \frac{dx}{dt} = 4 \text{ when } t = 0$$

Solution:

$$\frac{d^2x}{dt^2} - 3\frac{dx}{dt} + 2x = 2t - 3 \quad *$$

This time t is the independent variable, and x the dependent variable. The method of solution is the same as in the questions connecting x and y.

Find c.f.: the auxiliary equation is

$$m^2 - 3m + 2 = 0$$

$\therefore \quad (m-2)(m-1) = 0$

$\therefore \quad m = 1 \text{ or } 2$

$\therefore$ c.f. is $x = Ae^t + Be^{2t}$

The p.i. is $x = \lambda + \mu t$, $\frac{dx}{dt} = \mu$, $\frac{d^2x}{dt^2} = 0$

Substitute into $*$ to give $-3\mu + 2\lambda + 2\mu t = 2t - 3$

Equate coefficients of t: $\quad 2\mu = 2 \Rightarrow \mu = 1$

Equate constant terms: $\quad 2\lambda - 3\mu = -3 \quad \therefore \quad \lambda = 0$

The particular integral is t.

$\therefore$ The general solution is $x = Ae^t + Be^{2t} + t \quad \dagger$

Given that $x = 2$ when $t = 0 \quad \therefore \quad 2 = A + B \quad ①$

Also $\frac{dx}{dt} = Ae^t + 2Be^{2t} + 1$

As $\frac{dx}{dt} = 4$ when $t = 0 \quad \therefore \quad 4 = A + 2B + 1$

$\therefore \quad A + 2B = 3 \quad ②$

Subtract $\quad ② - ① \Rightarrow B = 1$

Substitute into $\quad \therefore \quad A = 1$

Substituting the values of A and B back into $\dagger$

$$x = e^t + e^{2t} + t$$

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Exercise E, Question 7

Question:

$\frac{d^2x}{dt^2} - 9x = 10 \sin t$ $\qquad$ $x = 2$ and $\frac{dx}{dt} = -1$ when $t = 0$

Solution:

$\frac{d^2x}{dt^2} - 9x = 10 \sin t \quad *$

Find c.f.: auxiliary equation is

$m^2 - 9 = 0$

$\therefore \quad m = \pm 3$

$\therefore$ c.f. is $x = Ae^{3t} + Be^{-3t}$

The particular integral is of the form $\lambda \cos t + \mu \sin t$.

p.i. is of the form $x = \lambda \cos t + \mu \sin t$

$\therefore \quad \frac{dx}{dt} = -\lambda \sin t + \mu \cos t$

$\frac{d^2x}{dt^2} = -\lambda \cos t - \mu \sin t$

Substitute into equation $*$.

Then $-\lambda \cos t - \mu \sin t - 9\lambda \cos t - 9\mu \sin t = 10 \sin t$

Equate coefficients of $\cos t$: $\therefore \quad -10\lambda = 0 \Rightarrow \lambda = 0$

Equate coefficients of $\sin t$: $\therefore \quad -10\mu = 10 \Rightarrow \mu = -1$

$\therefore$ p.i. is $-\sin t$

$\therefore$ General solution is $x = Ae^{3t} + Be^{-3t} - \sin t \quad \dagger$

When $t = 0, x = 2 \qquad \therefore \quad 2 = A + B \quad ①$

$\frac{dx}{dt} = 3Ae^{3t} - 3Be^{-3t} - \cos t$

When $t = 0, \frac{dx}{dt} = -1 \quad \therefore \quad -1 = 3A - 3B - 1$

$\therefore \quad 0 = 3A - 3B \quad ②$

Solving equations ① and ②, $A = B = 1$

$\therefore$ Substitute values of A and B into $\dagger$

$\therefore \quad x = e^{3t} + e^{-3t} - \sin t$ is the required solution.

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Exercise E, Question 8

Question:

$$\frac{d^2x}{dt^2} - 4\frac{dx}{dt} + 4x = 3te^{2t} \qquad x = 0 \text{ and } \frac{dx}{dt} = 1 \text{ when } t = 0$$

Solution:

$$\frac{d^2x}{dt^2} - 4\frac{dx}{dt} + 4x = 3te^{2t} \quad *$$

The complementary function has the form $x = (A + Bt)e^{\alpha t}$.

Find c.f.: auxiliary equation is

$m^2 - 4m + 4 = 0$

$\therefore \quad (m - 2)^2 = 0$

$\therefore \quad m = 2$ only

$\therefore \quad$ c.f. is $x = (A + Bt)e^{2t}$

Find p.i.: Let p.i. be $x = \lambda t^3 e^{2t}$

Then $\quad \dfrac{dx}{dt} = 2\lambda t^3 e^{2t} + 3\lambda t^2 e^{2t}$

$$\frac{d^2x}{dt^2} = 4\lambda t^3 e^{2t} + 6\lambda t^2 e^{2t} + 6\lambda t^2 e^{2t} + 6\lambda t e^{2t}$$

Substitute into $*$.

Then $(4\lambda t^3 + 12\lambda t^2 + 6\lambda t - 8\lambda t^3 - 12\lambda t^2 + 4\lambda t^3)e^{2t} = 3te^{2t}$

$\therefore \quad 6\lambda = 3 \Rightarrow \lambda = \frac{1}{2}$

$\therefore \quad$ p.i. is $x = \frac{1}{2}t^3 e^{2t}$

$\therefore \quad$ General solution is $x = \left((A + Bt) + \frac{1}{2}t^3\right)e^{2t} \quad \dagger$

But $\quad x = 0$ when $t = 0 \quad \therefore \quad 0 = A$

$$\frac{dx}{dt} = 2\left[A + Bt + \frac{1}{2}t^3\right]e^{2t} + \left[B + \frac{3}{2}t^2\right]e^{2t}$$

As $\quad \dfrac{dx}{dt} = 1$ when $t = 0$ and $A = 0$

$\therefore \quad 1 = B$

Substitute $A = 0$ and $B = 1$ into $\dagger$

Then $x = \left(t + \frac{1}{2}t^3\right)e^{2t}$ is the required solution.

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Exercise E, Question 9

Question:

$25\frac{d^2x}{dt^2} + 36x = 18$ $\quad\quad x = 1$ and $\frac{dx}{dt} = 0.6$ when $t = 0$

Solution:

$25\frac{d^2x}{dt^2} + 36x = 18$ $\quad *$

Find c.f.: auxiliary equation is

$25m^2 + 36 = 0$

$\therefore \quad m^2 = -\frac{36}{25}$ and $m = \pm\frac{6}{5}\mathrm{i}$

$\therefore \quad$ c.f. is $x = A\cos\frac{6}{5}t + B\sin\frac{6}{5}t$

The auxiliary equation has imaginary roots and so $x = A\cos\omega t + B\sin\omega t$ is the form of the complementary function.

Let p.i. be $x = \lambda$. Substitute into $*$

Then $\quad 36\lambda = 18$

$\therefore \quad \lambda = \frac{18}{36} = \frac{1}{2}$

$\therefore \quad$ General solution is $x = A\cos\frac{6}{5}t + B\sin\frac{6}{5}t + \frac{1}{2}$ $\quad †$

When $t = 0, x = 1 \quad \therefore \quad 1 = A + \frac{1}{2} \Rightarrow A = \frac{1}{2} = 0.5$

$$\frac{dx}{dt} = -\frac{6}{5}A\sin\frac{6}{5}t + \frac{6}{5}B\cos\frac{6}{5}t$$

When $t = 0, \frac{dx}{dt} = 0.6 \quad \therefore \quad 0.6 = \frac{6}{5}B$

$\therefore \quad B = 0.5 = \frac{1}{2}$

Substitute values for A and B into $†$

Then $\quad x = \frac{1}{2}\left(\cos\frac{6}{5}t + \sin\frac{6}{5}t + 1\right)$

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Exercise E, Question 10

Question:

$$\frac{d^2x}{dt^2} - 2\frac{dx}{dt} + 2x = 2t^2 \qquad x = 1 \text{ and } \frac{dx}{dt} = 3 \text{ when } t = 0$$

Solution:

$$\frac{d^2x}{dt^2} - 2\frac{dx}{dt} + 2x = 2t^2 \quad *$$

Find c.f.: auxiliary equation is

$m^2 - 2m + 2 = 0$

$\therefore \qquad m = \frac{2 \pm \sqrt{4-8}}{2} = 1 \pm i$

$\therefore$ c.f. is $x = e^t (A \cos t + B \sin t)$

Let p.i. be $x = \lambda + \mu t + \nu t^2$

The particular integral has the form $x = \lambda + \mu t + \nu t^2$.

then $\frac{dx}{dt} = \mu + 2\nu t$

$\frac{d^2x}{dt^2} = 2\nu$

Substitute into $*$

Then $2\nu - 2(\mu + 2\nu t) + 2(\lambda + \mu t + \nu t^2) = 2t^2$

Equate coefficients of t^2: $2\nu = 2 \Rightarrow \nu = 1$

coefficients of t: $-4\nu + 2\mu = 0 \Rightarrow \mu = 2$

constants: $2\nu - 2\mu + 2\lambda = 0 \Rightarrow \lambda = 1$

$\therefore$ p.i. is $x = 1 + 2t + t^2$

$\therefore$ General solution is $x = e^t (A \cos t + B \sin t) + 1 + 2t + t^2 \quad †$

But $x = 1$ when $t = 0 \quad \therefore \quad 1 = A + 1 \quad \therefore \quad A = 0$

As $x = Be^t \sin t + 1 + 2t + t^2$

$\frac{dx}{dt} = Be^t \cos t + Be^t \sin t + 2 + 2t$

As $\frac{dx}{dt} = 3$ when $t = 0$

$\therefore \quad 3 = B + 2$

$\therefore \quad B = 1$

Substitute $A = 0$ and $B = 1$ into the general solution $†$

$\therefore \quad x = e^t \sin t + 1 + 2t + t^2 \quad$ or $\quad x = e^t \sin t + (1 + t)^2$

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Exercise F, Question 1

Question:

$$x^2\frac{d^2y}{dx^2} + 6x\frac{dy}{dx} + 4y = 0$$

Solution:

$$x^2\frac{d^2y}{dx^2}+6x\frac{dy}{dx}+4y=0 \quad *$$

First express $x\frac{dy}{dx}$ as $\frac{dy}{du}$ and $x\frac{d^2y}{dx^2}$ as $\frac{d^2y}{du^2}-\frac{dy}{du}$.

As $x=e^u$, $\frac{dx}{du}=e^u=x$

From the chain rule $\quad \frac{dy}{du}=\frac{dy}{dx}\times\frac{dx}{du}$

$\therefore \quad \frac{dy}{du}=x\frac{dy}{dx} \quad ①$

Also $\quad \frac{d^2y}{du^2}=\frac{d}{du}\left(x\frac{dy}{dx}\right)$

$$=\frac{dx}{du}\times\frac{dy}{dx}+x\frac{d^2y}{dx^2}\times\frac{dx}{du}$$

$$=\frac{dy}{du}+x^2\frac{d^2y}{dx^2}$$

$\therefore \quad x^2\frac{d^2y}{dx^2}=\frac{d^2y}{du^2}-\frac{dy}{du} \quad ②$

Use the results ① and ② to change the variable in $*$

$\therefore \quad \frac{d^2y}{du^2}-\frac{dy}{du}+6\frac{dy}{du}+4y=0$

i.e. $\quad \frac{d^2y}{du^2}+5\frac{dy}{du}+4y=0 \quad †$

This has auxiliary equation

$$m^2+5m+4=0$$

$\therefore \quad (m+4)(m+1)=0$

i.e. $\quad m=-4$ or -1

$\therefore$ The solution of the differential equation † is

$$y=Ae^{-4u}+Be^{-u}$$

But $\quad e^u=x$

$\therefore \quad e^{-u}=x^{-1}=\frac{1}{x}$

and $e^{-4u}=x^{-4}=\frac{1}{x^4}$

$\therefore \quad y=\frac{A}{x^4}+\frac{B}{x}$

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Exercise F, Question 2

Question:

$$x^2\frac{d^2y}{dx^2} + 5x\frac{dy}{dx} + 4y = 0$$

Solution:

$$x^2\frac{d^2y}{dx^2} + 5x\frac{dy}{dx} + 4y = 0 \quad *$$

As $x = e^u$, $x\frac{dy}{dx} = \frac{dy}{du}$ and $x^2\frac{d^2y}{dx^2} = \frac{d^2y}{du^2} - \frac{dy}{du}$

(See solution to question 1 for proof of this.)

Use these results to change the variable in $*$.

$\therefore \quad \frac{d^2y}{du^2} - \frac{dy}{du} + 5\frac{dy}{du} + 4y = 0.$

$\therefore \quad \frac{d^2y}{du^2} + 4\frac{dy}{du} + 4y = 0 \quad \dagger$

This has auxiliary equation

$m^2 + 4m + 4 = 0$

$\therefore \quad (m + 2)^2 = 0$

$\therefore \quad m = -2$ only

The solution of the differential equation $\dagger$ is thus

$$y = (A + Bu)e^{-2u}$$

As $x = e^u$ $\therefore$ $e^{-2u} = x^{-2} = \frac{1}{x^2}$

and $u = \ln x$

$\therefore \quad y = (A + B\ln x) \times \frac{1}{x^2}$

Use $x\frac{dy}{dx} = \frac{dy}{du}$ and $x^2\frac{d^2y}{dx^2} = \frac{d^2y}{du^2} - \frac{dy}{du}$.

Ensure that you can prove these two results.

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Exercise F, Question 3

Question:

$$x^2\frac{d^2y}{dx^2} + 6x\frac{dy}{dx} + 6y = 0$$

Solution:

$$x^2\frac{d^2y}{dx^2} + 6x\frac{dy}{dx} + 6y = 0 \quad *$$

As $x = e^u$, $x\frac{dy}{dx} = \frac{dy}{du}$ and $x^2\frac{d^2y}{dx^2} = \frac{d^2y}{du^2} - \frac{dy}{du}$

(See solution to question 1 for proof of this.)

Use these results to change the variable in $*$.

$$\therefore \quad \frac{d^2y}{du^2} - \frac{dy}{du} + 6\frac{dy}{du} + 6y = 0$$

$$\therefore \quad \frac{d^2y}{du^2} + 5\frac{dy}{du} + 6y = 0 \quad \dagger$$

This has auxiliary equation

$$m^2 + 5m + 6 = 0$$

$$\therefore \quad (m+2)(m+3) = 0$$

$$\therefore \quad m = -2 \text{ or } -3$$

The solution of the differential equation $\dagger$ is thus

$$y = Ae^{-2u} + Be^{-3u}$$

As $\quad x = e^u$, $e^{-2u} = x^{-2} = \frac{1}{x^2}$

and $\quad e^{-3u} = x^{-3} = \frac{1}{x^3}$

$$\therefore \quad y = \frac{A}{x^2} + \frac{B}{x^3}$$

Use $x\frac{dy}{dx} = \frac{dy}{du}$ and $x^2\frac{d^2y}{dx^2} = \frac{d^2y}{du^2} - \frac{dy}{du}$.

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Exercise F, Question 4

Question:

$$x^2\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} - 28y = 0$$

Solution:

$$x^2\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} - 28y = 0 \quad *$$

Use $x\frac{dy}{dx} = \frac{dy}{du}$ and $x^2\frac{d^2y}{dx^2} = \frac{d^2y}{du^2} - \frac{dy}{du}$.

As $x = e^u$, $x\frac{dy}{dx} = \frac{dy}{du}$ and $x^2\frac{d^2y}{dx^2} = \frac{d^2y}{du^2} - \frac{dy}{du}$

Substitute these results into equation $*$

$\therefore \quad \frac{d^2y}{du^2} - \frac{dy}{du} + 4\frac{dy}{du} - 28y = 0$

$\therefore \quad \frac{d^2y}{du^2} + 3\frac{dy}{du} - 28y = 0 \quad \dagger$

This has auxiliary equation:

$m^2 + 3m - 28 = 0$

$\therefore \quad (m + 7)(m - 4) = 0$

$\therefore \quad m = -7 \text{ or } 4$

$\therefore \quad y = Ae^{-7u} + Be^{4u}$ is the solution to $\dagger$.

As $x = e^u$, $\therefore \quad e^{-7u} = \frac{1}{x^7}$

and $\quad e^{4u} = x^4$

$\therefore \quad y = \frac{A}{x^7} + Bx^4$

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Exercise F, Question 5

Question:

$$x^2\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} - 14y = 0$$

Solution:

$$x^2\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} - 14y = 0 \quad *$$

As $x = e^u$, $x\frac{dy}{dx} = \frac{dy}{du}$ and $x^2\frac{d^2y}{dx^2} = \frac{d^2y}{du^2} - \frac{dy}{du}$

Use $x\frac{dy}{dx} = \frac{dy}{du}$ and $x^2\frac{d^2y}{dx^2} = \frac{d^2y}{du^2} - \frac{dy}{du}$.

Substituting these results into $*$ gives

$$\frac{d^2y}{du^2} - \frac{dy}{du} - 4\frac{dy}{du} - 14y = 0$$

i.e. $\frac{d^2y}{du^2} - 5\frac{dy}{du} - 14y = 0 \quad \dagger$

This has auxiliary equation:

$$m^2 - 5m - 14 = 0$$

i.e. $(m - 7)(m + 2) = 0$

$\therefore \quad m = 7$ or -2

$\therefore$ The solution of the differential equation $\dagger$ is

$$y = Ae^{7u} + Be^{-2u}$$

But $x = e^u, \quad \therefore \quad e^{7u} = x^7$

and $e^{-2u} = x^{-2} = \frac{1}{x^2}$

$\therefore \quad y = Ax^7 + \frac{B}{x^2}$

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Exercise F, Question 6

Question:

$$x^2\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} + 2y = 0$$

Solution:

$$x^2\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} + 2y = 0 \quad *$$

As $x = e^u$, $x\frac{dy}{dx} = \frac{dy}{du}$ and $x^2\frac{d^2y}{dx^2} = \frac{d^2y}{du^2} - \frac{dy}{du}$

Substitute these results into $*$ to give:

$$\frac{d^2y}{du^2} - \frac{dy}{du} + 3\frac{dy}{du} + 2y = 0$$

i.e. $\frac{d^2y}{du^2} + 2\frac{dy}{du} + 2y = 0 \quad \dagger$

This has auxiliary equation:

$$m^2 + 2m + 2 = 0$$

$$\therefore \quad m = \frac{-2 \pm \sqrt{4-8}}{2}$$

$$= -1 \pm i$$

The solution of the differential equation $\dagger$ is thus

$$y = e^{-u}[A\cos u + B\sin u]$$

As $x = e^u$, $e^{-u} = x^{-1} = \frac{1}{x}$

and $u = \ln x$

$$\therefore \quad y = \frac{1}{x}[A\cos\ln x + B\sin\ln x]$$

Use $x\frac{dy}{dx} = \frac{dy}{du}$ and $x^2\frac{d^2y}{dx^2} = \frac{d^2y}{du^2} - \frac{dy}{du}$. A proof of these results is given in the book in Section 5.6.

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Exercise F, Question 7

Question:

Use the substitution $y = \frac{z}{x}$ to transform the differential equation

$x\frac{d^2y}{dx^2} + (2 - 4x)\frac{dy}{dx} - 4y = 0$ into the equation $\frac{d^2z}{dx^2} - 4\frac{dz}{dx} = 0$.

Hence solve the equation $x\frac{d^2y}{dx^2} + (2 - 4x)\frac{dy}{dx} - 4y = 0$, giving y in terms of x.

Solution:

Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in terms of $\frac{dz}{dx}$ and $\frac{d^2z}{dx^2}$.

$y = \frac{z}{x}$ implies $xy = z$

$\therefore \quad x\frac{dy}{dx} + y = \frac{dz}{dx}$

Also $x\frac{d^2y}{dx^2} + \frac{dy}{dx} + \frac{dy}{dx} = \frac{d^2z}{dx^2}$

$\therefore$ The equation $x\frac{d^2y}{dx^2} + (2 - 4x)\frac{dy}{dx} - 4y = 0$

becomes $\frac{d^2z}{dx^2} - 4\left(\frac{dz}{dx} - y\right) - 4y = 0$

i.e. $\frac{d^2z}{dx^2} - 4\frac{dz}{dx} = 0 \quad *$

The equation $*$ has auxiliary equation

$m^2 - 4m = 0$

$\therefore \quad m(m - 4) = 0$

i.e. $m = 0$ or 4

$\therefore \quad z = A + Be^{4x}$ is the solution of $*$

But $z = xy$

$\therefore \quad xy = A + Be^{4x}$

$\therefore \quad y = \frac{A}{x} + \frac{B}{x}e^{4x}$

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Exercise F, Question 8

Question:

Use the substitution $y = \frac{z}{x^2}$ to transform the differential equation

$x^2\frac{d^2y}{dx^2} + 2x(x + 2)\frac{dy}{dx} + 2(x + 1)^2 y = e^{-x}$ into the equation $\frac{d^2z}{dx^2} + 2\frac{dz}{dx} + 2z = e^{-x}$.

Hence solve the equation $x^2\frac{d^2y}{dx^2} + 2x(x + 2)\frac{dy}{dx} + 2(x + 1)^2 y = e^{-x}$, giving y in terms of x.

Solution:

Express $\frac{dz}{dx}$ and $\frac{d^2z}{dx^2}$ in terms of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ respectively.

$y = \frac{z}{x^2}$ implies $z = yx^2$ or $x^2y = z$

$\therefore \quad x^2\frac{dy}{dx} + 2xy = \frac{dz}{dx} \quad ①$

Also $\quad x^2\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} + 2x\frac{dy}{dx} + 2y = \frac{d^2z}{dx^2} \quad ②$

The differential equation:

$x^2\frac{d^2y}{dx^2} + 2x(x+2)\frac{dy}{dx} + 2(x+1)^2y = e^{-x}$ can be written

$$\left(x^2\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + 2y\right) + \left(2x^2\frac{dy}{dx} + 4xy\right) + 2x^2y = e^{-x}$$

Using the results ① and ②

$$\frac{d^2z}{dx^2} + 2\frac{dz}{dx} + 2z = e^{-x} \quad †$$

This has auxiliary equation

$m^2 + 2m + 2 = 0$

$\therefore \quad m = \frac{-2 \pm \sqrt{4-8}}{2}$

$m = -1 \pm i$

$\therefore \quad z = e^{-x}(A\cos x + B\sin x)$ is the complementary function

A particular integral of † is $z = \lambda e^{-x}$

$\therefore \quad \frac{dz}{dx} = -\lambda e^{-x}$ and $\frac{d^2z}{dx^2} = \lambda e^{-x}$

Substituting into †

$(\lambda - 2\lambda + 2\lambda)e^{-x} = e^{-x}$

$\therefore \quad \lambda = 1$

So $z = e^{-x}$ is a particular integral.

$\therefore$ The general solution of † is

$z = e^{-x}(A\cos x + B\sin x + 1)$

But $z = x^2y$

$\therefore \quad y = \frac{e^{-x}}{x^2}(A\cos x + B\sin x + 1)$ is the general solution of the given differential equation.

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Exercise F, Question 9

Question:

Use the substitution $z = \sin x$ to transform the differential equation

$\cos x\frac{d^2y}{dx^2} + \sin x\frac{dy}{dx} - 2y\cos^3 x = 2\cos^5 x$ into the equation $\frac{d^2y}{dz^2} - 2y = 2(1 - z^2)$.

Hence solve the equation $\cos x\frac{d^2y}{dx^2} + \sin x\frac{dy}{dx} - 2y\cos^3 x = 2\cos^5 x$, giving y in terms of x.

Solution:

Find $\frac{dy}{dx}$ in terms of $\frac{dy}{dx}$ and find $\frac{d^2y}{dx^2}$ in terms of $\frac{d^2y}{dz^2}$ and $\frac{dy}{dz}$.

$z = \sin x$ implies $\frac{dz}{dx} = \cos x$

$\therefore \quad \frac{dy}{dx} = \frac{dy}{dz} \times \cos x$

And $\frac{d^2y}{dx^2} = \frac{d^2y}{dz^2}\cos^2 x - \frac{dy}{dz}\sin x$

$\therefore$ The equation $\cos x \frac{d^2y}{dx^2} + \sin x \frac{dy}{dx} - 2y\cos^3 x = 2\cos^5 x$

becomes $\cos^3 x \frac{d^2y}{dz^2} - \cos x \sin x \frac{dy}{dz} + \cos x \sin x \frac{dy}{dz} - 2y\cos^3 x = 2\cos^5 x$

$\therefore$ Divide by $\cos^3 x$ gives:

$$\frac{d^2y}{dz^2} - 2y = 2\cos^2 x$$
$$= 2(1 - z^2) \dagger \qquad [\text{as } \cos^2 x = 1 - \sin^2 x = 1 - z^2]$$

First solve $\frac{d^2y}{dz^2} - 2y = 0$

This has auxiliary equation

$m^2 - 2 = 0$

$\therefore \quad m = \pm\sqrt{2}$

$\therefore$ The complementary function is $y = Ae^{\sqrt{2}z} + Be^{-\sqrt{2}z}$.

Let $y = \lambda z^2 + \mu z + \nu$ be a particular integral of the differential equation $\dagger$.

Then $\frac{dy}{dz} = 2\lambda z + \mu$ and $\frac{d^2y}{dz^2} = 2\lambda$

Substitute into $\dagger$

Then $2\lambda - 2(\lambda z^2 + \mu z + \nu) = 2(1 - z^2)$

Compare coefficients of z^2: $-2\lambda = -2 \quad \therefore \quad \lambda = 1$

Compare coefficients of z: $-2\mu = 0 \quad \therefore \quad \mu = 0$

Compare constants: $2\lambda - 2\nu = 2 \quad \therefore \quad \nu = 0$

$\therefore$ z^2 is the particular integral.

$\therefore$ The general solution of $\dagger$ is

$y = Ae^{\sqrt{2}z} + Be^{-\sqrt{2}z} + z^2$.

But $z = \sin x$

$\therefore \quad y = Ae^{\sqrt{2}\sin x} + Be^{-\sqrt{2}\sin x} + \sin^2 x$

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Exercise G, Question 1

Question:

Find the general solution of the differential equation $\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = 0$

Solution:

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = 0$$

Auxiliary equation is

$$m^2 + m + 1 = 0$$

$$\therefore \quad m = \frac{-1 \pm \sqrt{1-4}}{2}$$

$$= -\tfrac{1}{2} \pm \frac{\sqrt{3}}{2}\mathrm{i}$$

$\therefore$ The solution of the equation is

$$y = e^{-\frac{1}{2}x}\left(A\cos\tfrac{\sqrt{3}}{2}x + B\sin\tfrac{\sqrt{3}}{2}x\right)$$

The auxiliary equation has complex roots and so the solution is of the form $y = e^{px}(A\cos qx + B\sin qx)$.

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Exercise G, Question 2

Question:

Find the general solution of the differential equation $\frac{d^2y}{dx^2} - 12\frac{dy}{dx} + 36y = 0$

Solution:

$$\frac{d^2y}{dx^2} - 12\frac{dy}{dx} + 36y = 0$$

The auxiliary equation is

$m^2 - 12m + 36 = 0$

$\therefore \quad (m - 6)^2 = 0$

$\therefore \quad m = 6$ only

$\therefore$ The solution of the equation is

$y = (A + Bx)e^{6x}$.

The auxiliary equation has a repeated solution and so the solution is of the form $y = (A + Bx)e^{\alpha x}$.

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Exercise G, Question 3

Question:

Find the general solution of the differential equation $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} = 0$

Solution:

$$\frac{d^2y}{dx^2} - \frac{4\,dy}{dx} = 0$$

The auxiliary equation is

$m^2 - 4m = 0$

$\therefore \quad m(m-4) = 0$

$\therefore \quad m = 0$ or 4

$\therefore$ The solution of the equation is

$$y = Ae^{0x} + Be^{4x}$$
$$= A + Be^{4x}$$

The auxiliary equation has two distinct roots, but one of them is zero.

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Exercise G, Question 4

Question:

Find y in terms of k and x, given that $\frac{d^2y}{dx^2} + k^2y = 0$ where k is a constant, and $y = 1$ and $\frac{dy}{dx} = 1$ at $x = 0$.

Solution:

$$\frac{d^2y}{dx^2} + k^2y = 0$$

The auxiliary equation is

$m^2 + k^2 = 0$

$\therefore \quad m = \pm \text{i}k$

The solution of the equation is

$y = A\cos kx + B\sin kx$. [This is the general solution.]

But $\quad y = 1$ when $x = 0$

$\therefore \quad 1 = A + 0 \Rightarrow A = 1$

$\therefore \quad y = \cos kx + B\sin kx$

$$\frac{dy}{dx} = -k\sin kx + Bk\cos kx$$

Also $\quad \frac{dy}{dx} = 1$ when $x = 0$

$\therefore \quad 1 = Bk \Rightarrow B = \frac{1}{k}$

$\therefore \quad y = \cos kx + \frac{1}{k}\sin kx.$

The auxiliary equation has imaginary solutions and so the general solution has the form $y = A\cos \omega x + B\sin \omega x$. A and B can be found by using the boundary conditions.

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Exercise G, Question 5

Question:

Find the solution of the differential equation $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 10y = 0$ for which $y = 0$ and $\frac{dy}{dx} = 3$ at $x = 0$.

Solution:

$$\frac{d^2y}{dx^2} - \frac{2dy}{dx} + 10y = 0$$

This has auxiliary equation

$$m^2 - 2m + 10 = 0$$

$$\therefore \quad m = \frac{2 \pm \sqrt{4 - 40}}{2}$$

$$= 1 \pm 3i$$

The auxiliary equation has complex roots and so the general solution is of the form $y = e^{px}(A\cos qx + B\sin qx)$.

The general solution of the equation is

$$y = e^x(A\cos 3x + B\sin 3x)$$

As $y = 0$ when $x = 0$,

$\therefore \quad 0 = A + 0 \Rightarrow A = 0$

$\therefore \quad y = Be^x \sin 3x$

$$\frac{dy}{dx} = 3Be^x \cos 3x + Be^x \sin 3x$$

Also $\frac{dy}{dx} = 3$ when $x = 0$

$\therefore \quad 3 = 3B + 0 \Rightarrow B = 1$

$\therefore \quad y = e^x \sin 3x$ is the required solution.

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Exercise G, Question 6

Question:

Given that the differential equation $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 13y = e^{2x}$ has a particular integral of the form ke^{2x}, determine the value of the constant k and find the general solution of the equation.

Solution:

$$\frac{d^2y}{dx^2} - \frac{4\,dy}{dx} + 13y = e^{2x} \quad *$$

First find the complementary function (c.f.):

Use the fact that the general solution = complementary function + particular integral.

the auxiliary equation is

$$m^2 - 4m + 13 = 0$$

$$\therefore \quad m = \frac{4 \pm \sqrt{16 - 52}}{2}$$

$$= 2 \pm 3i$$

$\therefore$ The c.f. is $y = e^{2x}(A\cos 3x + B\sin 3x)$

Let the particular integral (p.i.) be $y = ke^{2x}$

Then $\frac{dy}{dx} = 2ke^{2x}$ and $\frac{d^2y}{dx^2} = 4ke^{2x}$.

Substitute in $*$ to give

$$(4k - 8k + 13k)e^{2x} = e^{2x}$$

i.e. $9ke^{2x} = e^{2x}$

$\therefore$ $k = \frac{1}{9}$

$\therefore$ The general solution of $*$ is y = c.f. + p.i.

i.e. $y = e^{2x}(A\cos 3x + B\sin 3x) + \frac{1}{9}e^{2x}$.

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Exercise G, Question 7

Question:

Given that the differential equation $\frac{d^2y}{dx^2} - y = 4e^x$ has a particular integral of the form kxe^x, determine the value of the constant k and find the general solution of the equation.

Solution:

$$\frac{d^2y}{dx^2} - y = 4e^x \quad *$$

First find the c.f.

Use general solution = complementary function + particular integral.

The auxiliary equation is

$m^2 - 1 = 0$

$\therefore \quad m = \pm 1$

$\therefore$ The c.f. is $y = Ae^x + Be^{-x}$

Let the p.i. be $y = kxe^x$

Then $\frac{dy}{dx} = kxe^x + ke^x$

$\frac{d^2y}{dx^2} = kxe^x + ke^x + ke^x$

Substitute into $*$.

Then $kxe^x + 2ke^x - kxe^x = 4e^x$

$\therefore \quad k = 2$

So the p.i. is $y = 2xe^x$

The general solution is $y = \text{c.f.} + \text{p.i.}$

$\therefore \quad y = Ae^x + Be^{-x} + 2xe^x.$

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Exercise G, Question 8

Question:

The differential equation $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 4e^{2x}$ is to be solved.

a Find the complementary function.

b Explain why **neither** λe^{2x} **nor** $\lambda x e^{2x}$ can be a particular integral for this equation.

c Determine the value of the constant k and find the general solution of the equation.

Solution:

$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 4e^{2x}$ $*$

a First find the c.f.

The auxiliary equation is

$m^2 - 4m + 4 = 0$

$\therefore \quad (m-2)^2 = 0$

i.e. $\quad m = 2$ only

$\therefore$ The c.f. is $y = (A + Bx)e^{2x}$

The auxiliary equation has a repeated root and so the c.f. is of the form $y = (A + Bx)e^{\alpha x}$.

b Ae^{2x} and Bxe^{2x} are part of the c.f. so satisfy the equation $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 0$.

The p.i. must satisfy $*$.

c Let $y = kx^2e^{2x}$

$$\frac{dy}{dx} = 2kx^2e^{2x} + 2kxe^{2x}$$

$$\frac{d^2y}{dx^2} = 4kx^2e^{2x} + 4kxe^{2x} + 2kx \times 2e^{2x} + 2ke^{2x}$$

Substitute into $*$

$\therefore \quad (4kx^2 + 8kx + 2k - 8kx^2 - 8kx + 4kx^2)e^{2x} = 4e^{2x}$

$\therefore \quad 2ke^{2x} = 4e^{2x}$

$\therefore \quad k = 2$

So the p.i. is $2x^2e^{2x}$

$\therefore$ The general solution is $y = (A + Bx + 2x^2)e^{2x}$.

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Exercise G, Question 9

Question:

Given that the differential equation $\frac{d^2y}{dt^2} + 4y = 5\cos 3t$ has a particular integral of the form $k\cos 3t$, determine the value of the constant k and find the general solution of the equation.

Find the solution which satisfies the initial conditions that when $t = 0$, $y = 1$ and $\frac{dy}{dt} = 2$.

Solution:

$\frac{d^2y}{dt^2} + 4y = 5\cos 3t$ $\ast$

The p.i. is $y = k\cos 3t$.

$$\frac{dy}{dt} = -3k\sin 3t$$

$$\frac{d^2y}{dt^2} = -9k\cos 3t$$

Substitute into $\ast$

Then $-9k\cos 3t + 4k\cos 3t = 5\cos 3t$

$\therefore \quad -5k\cos 3t = 5\cos 3t$

$\therefore \quad k = -1$

$\therefore$ The p.i. is $-\cos 3t$.

The c.f. is found next.

The auxiliary equation is $m^2 + 4 = 0$.

$\therefore \quad m = \pm 2\text{i}$

$\therefore$ The c.f. is $y = A\cos 2t + B\sin 2t$

$\therefore$ The general solution is $y = A\cos 2t + B\sin 2t - \cos 3t$

When $t = 0$, $y = 1$ $\quad \therefore \quad 1 = A - 1 \Rightarrow A = 2$

$$\frac{dy}{dt} = -2A\sin 2t + 2B\cos 2t + 3\sin 3t$$

When $t = 0$, $\frac{dy}{dt} = 2$ $\quad \therefore \quad 2 = 2B \Rightarrow B = 1$

$\therefore \quad y = 2\cos 2t + \sin 2t - \cos 3t.$

The auxiliary equation has imaginary roots so the c.f. is $y = A\cos\omega t + B\sin\omega t$. '$t$' is the independent variable in this question.

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Exercise G, Question 10

Question:

Given that the differential equation $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 4x + e^{2x}$ has a particular integral of the form $\lambda + \mu x + kxe^{2x}$, determine the values of the constants λ, μ and k and find the general solution of the equation.

Solution:

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 4x + e^{2x} \quad *$$

P.I. is $y = \lambda + \mu x + kxe^{2x}$

$\therefore \quad \frac{dy}{dx} = \mu + 2kxe^{2x} + ke^{2x}$

$\frac{d^2y}{dx^2} = 2kx \times 2e^{2x} + 2ke^{2x} + 2ke^{2x}$

Find the complementary function and add to the particular integral to give the general solution.

Substitute into $*$.

Then $(4kx + 4k)e^{2x} - 3\mu - (6kx + 3k)e^{2x} + 2\lambda + 2\mu x + 2kxe^{2x} = 4x + e^{2x}$

$\therefore \quad ke^{2x} + (2\lambda - 3\mu) + 2\mu x = 4x + e^{2x}.$

Equating coefficients of e^{2x}: $\quad k = 1$

x: $2\mu = 4 \Rightarrow \mu = 2$

constants: $2\lambda - 3\mu = 0 \Rightarrow \lambda = 3$

$\therefore \quad y = 3 + 2x + xe^{2x}$ is the particular integral.

The auxiliary equation for $*$ is

$m^2 - 3m + 2 = 0$

$\therefore \quad (m - 2)(m - 1) = 0$

$\therefore \quad m = 1$ or 2

$\therefore \quad$ The c.f. is $y = Ae^{x} + Be^{2x}$

$\therefore \quad$ The general solution is $y = Ae^{x} + Be^{2x} + 3 + 2x + xe^{2x}$.

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Exercise G, Question 11

Question:

Find the solution of the differential equation $16\frac{d^2y}{dx^2} + 8\frac{dy}{dx} + 5y = 5x + 23$ for which $y = 3$ and $\frac{dy}{dx} = 3$ at $x = 0$. Show that $y \approx x + 3$ for large values of x.

Solution:

$16\frac{d^2y}{dx^2} + 8\frac{dy}{dx} + 5y = 5x + 23$ ①

The particular integral is of the form $y = \lambda x + \mu$.

The auxiliary equation is

$16m^2 + 8m + 5 = 0$

$\therefore \quad m = \frac{-8 \pm \sqrt{64 - 320}}{32}$

$= -\frac{1}{4} \pm \frac{\sqrt{-256}}{32}$

$= -\frac{1}{4} \pm \frac{1}{2}\mathrm{i}$

$\therefore$ The c.f. is $y = e^{-\frac{1}{4}x}(A\cos\frac{1}{2}x + B\sin\frac{1}{2}x)$

Let the p.i. be $y = \lambda x + \mu$.

$\therefore \quad \frac{dy}{dx} = \lambda, \quad \frac{d^2y}{dx^2} = 0$

Substitute into ①

$\therefore \quad 8\lambda + 5\lambda x + 5\mu = 5x + 23$

Equate coefficients of x: $\therefore \quad 5\lambda = 5 \Rightarrow \lambda = 1$

constant terms: $8\lambda + 5\mu = 23 \Rightarrow \mu = 3$

$\therefore$ The p.i. is $y = x + 3$

The general solution is c.f. + p.i.

i.e. $y = e^{-\frac{1}{4}x}(A\cos\frac{1}{2}x + B\sin\frac{1}{2}x) + x + 3$.

As $y = 3$, when $x = 0$

$\therefore \quad 3 = A + 3 \Rightarrow A = 0$

$\therefore \quad y = Be^{-\frac{1}{4}x}\sin\frac{1}{2}x + x + 3$

$\therefore \quad \frac{dy}{dx} = \frac{1}{2}Be^{-\frac{1}{4}x}\cos\frac{1}{2}x - \frac{1}{4}Be^{-\frac{1}{4}x}\sin\frac{1}{2}x + 1$

As $\frac{dy}{dx} = 3$ when $x = 0$

$3 = \frac{1}{2}B + 1 \Rightarrow B = 4$

$\therefore \quad y = 4e^{-\frac{1}{4}x}\sin\frac{1}{2}x + x + 3$

As $x \to \infty$, $e^{-\frac{1}{4}x} \to 0$; $\therefore \quad y \to x + 3$

$\therefore \quad y \approx x + 3$ for large values of x.

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Exercise G, Question 12

Question:

Find the solution of the differential equation $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 6y = 3\sin 3x - 2\cos 3x$ for which $y = 1$ at $x = 0$ and for which y remains finite for large values of x.

Solution:

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 6y = 3\sin 3x - 2\cos 3x \quad *$$

The auxiliary equation is

$m^2 - m - 6 = 0$

$\therefore \quad (m-3)(m+2) = 0$

$\therefore \quad m = 3$ or -2

$\therefore$ The c.f. is $y = Ae^{3x} + Be^{-2x}$.

Let the particular integral be $y = \lambda \sin 3x + \mu \cos 3x$.

The particular inegral is $y = \lambda \sin 3x + \mu \cos 3x$.

Then $\frac{dy}{dx} = 3\lambda \cos 3x - 3\mu \sin 3x$

$\frac{d^2y}{dx^2} = -9\lambda \sin 3x - 9\mu \cos 3x$

Substitute into $*$

Then $-9\lambda \sin 3x - 9\mu \cos 3x - 3\lambda \cos 3x + 3\mu \sin 3x - 6\lambda \sin 3x - 6\mu \cos 3x$
$= 3\sin 3x - 2\cos 3x$.

Equate coefficients of $\sin 3x$:

$-9\lambda + 3\mu - 6\lambda = 3$ i.e. $3\mu - 15\lambda = 3$ ①

Equate coefficients of $\cos 3x$:

$-9\mu - 3\lambda - 6\mu = -2$ i.e. $-15\mu - 3\lambda = -2$ ②

Solve equations ① and ② to give $\lambda = -\frac{1}{6} \quad \mu = \frac{1}{6}$

$\therefore$ P.I. is $y = \frac{1}{6}(\cos 3x - \sin 3x)$

$\therefore$ The general solution is

$y = Ae^{3x} + Be^{-2x} + \frac{1}{6}(\cos 3x - \sin 3x)$

As $y = 1$ when $x = 0$, $1 = A + B + \frac{1}{6}$

$\therefore \quad A + B = \frac{5}{6}$

As y remains finite for large values of x,

$A = 0$

$\therefore \quad B = \frac{5}{6}$

$\therefore \quad y = \frac{5}{6}e^{-2x} + \frac{1}{6}(\cos 3x - \sin 3x)$

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Exercise G, Question 13

Question:

Find the general solution of the differential equation $\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 10x = 27\cos t - 6\sin t$.

The equation is used to model water flow in a reservoir. At time t days, the level of the water above a fixed level is x m. When $t = 0$, $x = 3$ and the water level is rising at 6 metres per day.

a Find an expression for x in terms of t.

b Show that after about a week, the difference between the lowest and highest water level is approximately 6 m.

Solution:

a $\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 10x = 27\cos t - 6\sin t$ $*$

The auxiliary equation is

$m^2 + 2m + 10 = 0$

$\therefore \quad m = \frac{-2 \pm \sqrt{4 - 40}}{2}$

$= -1 \pm 3i$

$\therefore$ The c.f. is $x = e^{-t}(A\cos 3t + B\sin 3t)$

The p.i. is $x = \lambda\cos t + \mu\sin t$

$\frac{dx}{dt} = -\lambda\sin t + \mu\cos t$

$\frac{d^2x}{dt^2} = -\lambda\cos t - \mu\sin t$

Substitute into $*$

$\therefore \quad -\lambda\cos t - \mu\sin t - 2\lambda\sin t + 2\mu\cos t + 10\lambda\cos t + 10\mu\sin t = 27\cos t - 6\sin t$

Equate coefficients of $\cos t$: $9\lambda + 2\mu = 27$ ①

$\sin t$: $9\mu - 2\lambda = -6.$ ②

Solve equations ① and ② to give $\lambda = 3$, $\mu = 0$.

$\therefore$ The p.i. is $x = 3\cos t$.

$\therefore$ The general solution is $x = 3\cos t + e^{-t}(A\cos 3t + B\sin 3t)$

But $x = 3$ when $t = 0$: $\therefore \quad 3 = 3 + A \Rightarrow A = 0$

$\therefore \quad x = 3\cos t + Be^{-t}\sin 3t$

$\therefore \quad \frac{dx}{dt} = -3\sin t + 3Be^{-t}\cos 3t - Be^{-t}\sin 3t$

When $t = 0$, $\frac{dx}{dt} = 6$

$\therefore \quad 6 = 3B \Rightarrow B = 2$

$\therefore \quad x = 3\cos t + 2e^{-t}\sin 3t.$

b After a week $t \approx 7$ days. $\therefore \quad e^{-t} \to 0.$

$\therefore \quad x \approx 3\cos t$

In part **b** if t is large, then $e^{-t} \to 0$.

The distance between highest and lowest water level is $3 - (-3) = 6$ m.

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Exercise G, Question 14

Question:

a Find the general solution of the differential equation

$$x^2\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + 2y = \ln x, \qquad x > 0,$$

using the substitution $x = e^u$, where u is a function of x.

b Find the equation of the solution curve passing through the point (1, 1) with gradient 1.

Solution:

a Let $x = e^u$, then $\frac{dx}{du} = e^u$

and $\frac{dy}{du} = \frac{dy}{dx} \times \frac{dx}{du} = e^u \frac{dy}{dx} = x\frac{dy}{dx}$

$\frac{d^2y}{du^2} = \frac{dx}{du} \times \frac{dy}{dx} + x\frac{d^2y}{dx^2} \times \frac{dx}{du}$

$= x\frac{dy}{dx} + x^2\frac{d^2y}{dx^2}$

Find $\frac{dy}{du}$ in terms of x and $\frac{dy}{dx}$, and show that $\frac{d^2y}{du^2} = x\frac{dy}{dx} + x^2\frac{d^2y}{dx^2}$ then substitute into the differential equation.

$\therefore \quad x^2\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + 2y = \ln x \Rightarrow \frac{d^2y}{du^2} + 3\frac{dy}{du} + 2y = \ln x = u \quad *$

The auxiliary equation is

$m^2 + 3m + 2 = 0$

$\therefore \quad (m+2)(m+1) = 0$

$\Rightarrow \quad m = -1$ or -2

$\therefore$ The c.f. is $y = Ae^{-u} + Be^{-2u}$

Let the p.i. be $y = \lambda u + \mu \Rightarrow \frac{dy}{du} = \lambda, \frac{d^2y}{du^2} = 0$

Substitute into $*$

$\therefore \quad 3\lambda + 2\lambda u + 2\mu = u$

Equate coefficients of u: $2\lambda = 1 \Rightarrow \lambda = \frac{1}{2}$

constants: $3\lambda + 2\mu = 0 \quad \therefore \quad \mu = -\frac{3}{4}$

$\therefore$ The p.i. is $y = \frac{1}{2}u - \frac{3}{4}$

The general solution is $y = Ae^{-u} + Be^{-2u} + \frac{1}{2}u - \frac{3}{4}$.

But $x = e^u \rightarrow u = \ln x$.

Also $e^{-u} = x^{-1} = \frac{1}{x}$ and $e^{-2u} = x^{-2} = \frac{1}{x^2}$

$\therefore$ The general solution of the original equation is $y = \frac{A}{x} + \frac{B}{x^2} + \frac{1}{2}\ln x - \frac{3}{4}$.

b But $y = 1$ when $x = 1$

$\therefore \quad 1 = A + B - \frac{3}{4} \Rightarrow A + B = 1\frac{3}{4}$ ①

$\frac{dy}{dx} = -\frac{A}{x^2} - \frac{2B}{x^3} + \frac{1}{2x}$

When $x = 1$, $\frac{dy}{dx} = 1$

$\therefore \quad 1 = -A - 2B + \frac{1}{2} \Rightarrow A + 2B = -\frac{1}{2}$ ②

Solve the simultaneous equations ① and ② to give $B = -2\frac{1}{4}$ and $A = 4$.

$\therefore$ The equation of the solution curve described is $y = \frac{4}{x} - \frac{9}{4x^2} + \frac{1}{2}\ln x - \frac{3}{4}$.

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Exercise G, Question 15

Question:

Solve the equation $\frac{d^2y}{dx^2} + \tan x\frac{dy}{dx} + y\cos^2 x = \cos^2 x\, e^{\sin x}$, by putting $z = \sin x$, finding the solution for which $y = 1$ and $\frac{dy}{dx} = 3$ at $x = 0$.

Solution:

$z = \sin x \quad \therefore \quad \frac{dz}{dx} = \cos x$ and $\frac{dy}{dx} = \frac{dy}{dz} \times \cos x$

$$\therefore \quad \frac{d^2y}{dx^2} = -\frac{dy}{dz}\sin x + \cos x \frac{d^2y}{dz^2} \times \frac{dz}{dx}$$

$$= -\frac{dy}{dz}\sin x + \cos^2 x \frac{d^2y}{dz^2}$$

$$\therefore \quad \frac{d^2y}{dx^2} + \tan x \frac{dy}{dx} + y\cos^2 x = \cos^2 x\, e^{\sin x} \quad \dagger$$

$$\Rightarrow \cos^2 x \frac{d^2y}{dz^2} - \sin x \frac{dy}{dz} + \tan x \cos x \frac{dy}{dz} + y\cos^2 x = \cos^2 x\, e^z$$

$$\Rightarrow \frac{d^2y}{dz^2} + y = e^z \quad *$$

The auxiliary equation is $m^2 + 1 = 0 \Rightarrow m = \pm i$

$\therefore$ The c.f. is $y = A\cos z + B\sin z$

The p.i. is $y = \lambda e^z \Rightarrow \frac{dy}{dz} = \lambda e^z$ and $\frac{d^2y}{dz^2} = \lambda e^z$

Substitute in $*$ to give

$$2\lambda e^z = e^z \Rightarrow \lambda = \tfrac{1}{2}$$

$\therefore$ The general solution of $*$ is $y = A\cos z + B\sin z + \frac{1}{2}e^z$.

The original equation $\dagger$ has solution

$$y = A\cos(\sin x) + B\sin(\sin x) + \tfrac{1}{2}e^{\sin x}$$

But $\quad y = 1$ when $x = 0$

$$\therefore \quad 1 = A + \tfrac{1}{2} \Rightarrow A = \tfrac{1}{2}$$

$$\frac{dy}{dx} = \cos x\,(-A\sin(\sin x)) + \cos x(B\cos(\sin x)) + \tfrac{1}{2}\cos x e^{\sin x}$$

As $\quad \frac{dy}{dx} = 3$ when $x = 0$

$$\therefore \quad 3 = B + \tfrac{1}{2} \Rightarrow B = 2\tfrac{1}{2}$$

$$\therefore \quad y = \tfrac{1}{2}\cos(\sin x) + \tfrac{5}{2}\sin(\sin x) + \tfrac{1}{2}e^{\sin x}$$

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Exercise A, Question 1

Question:

For each of the following functions, f(x), find $f'(x)$, $f''(x)$, $f'''(x)$ and $f^{(n)}(x)$.

a e^{2x} **b** $(1+x)^n$ **c** xe^x **d** $\ln(1+x)$

Solution:

	$f'(x)$	$f''(x)$	$f'''(x)$	$f^{(n)}(x)$
a	$2e^{2x}$	$2^2e^{2x} = 4e^{2x}$	$2^3e^{2x} = 8e^{2x}$	2^ne^{2x}
b	$n(1+x)^{n-1}$	$n(n-1)(1+x)^{n-2}$	$n(n-1)(n-2)(1+x)^{n-3}$	$n!$
c	$e^x + xe^x$	$e^x + (e^x + xe^x)$ $= 2e^x + xe^x$	$2e^x + (e^x + xe^x) = 3e^x + xe^x$	$ne^x + xe^x$
d	$(1+x)^{-1}$	$-(1+x)^{-2}$	$(-1)(-2)(1+x)^{-3} = 2(1+x)^{-3}$	$(-1)^{n-1}(n-1)!(1+x)^{-n}$

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Exercise A, Question 2

Question:

a Given that $y = e^{2+3x}$, find an expression, in terms of y, for $\frac{d^n y}{dx^n}$.

b Hence show that $\left(\frac{d^6 y}{dx^6}\right)_{\ln\left(\frac{1}{9}\right)} = e^2$

Solution:

a $y = e^{2+3x}$, so $\frac{dy}{dx} = 3\,e^{2+3x}$, $\frac{d^2y}{dx^2} = 3^2\,e^{2+3x}$, $\frac{d^3y}{dx^3} = 3^3\,e^{2+3x}$, and so on.

It follows that $\frac{d^n y}{dx^n} = 3^n e^{2+3x} = 3^n y$ as $y = e^{2+3x}$.

b $\frac{d^6 y}{dx^6} = 3^6 y$

When $x = \ln\left(\frac{1}{9}\right) = \ln 3^{-2}$, $y = e^{2+3\ln 3^{-2}} = e^2 \times e^{3\ln 3^{-2}} = e^2 \times e^{\ln 3^{-6}} = \frac{e^2}{3^6}$

As $e^{\ln a} = a$

So $\left(\frac{d^6 y}{dx^6}\right)_{\ln\left(\frac{1}{9}\right)} = 3^6 \times \frac{e^2}{3^6} = e^2$.

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Exercise A, Question 3

Question:

Given that $y = \sin^2 3x$,

a show that $\frac{dy}{dx} = 3 \sin 6x$.

b Find expressions for $\frac{d^2y}{dx^2}$, $\frac{d^3y}{dx^3}$ and $\frac{d^4y}{dx^4}$.

c Hence evaluate $\left(\frac{d^4y}{dx^4}\right)_{\frac{\pi}{6}}$.

Solution:

a $y = \sin^2 3x = (\sin 3x)^2$, so $\frac{dy}{dx} = 2(\sin 3x)(3\cos 3x)$

$= 3(2\sin 3x \cos 3x)$

$= 3\sin 6x$

Use $\frac{du^n}{dx} = nu^{n-1}\frac{du}{dx}$.

Use $\sin 2A = 2\sin A \cos A$.

b $\frac{d^2y}{dx^2} = 18\cos 6x$, $\frac{d^3y}{dx^3} = -108\sin 6x$, $\frac{d^4y}{dx^4} = -648\cos 6x$

c $\left(\frac{d^4y}{dx^4}\right)_{\frac{\pi}{6}} = -648\cos\pi = 648$

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Exercise A, Question 4

Question:

$f(x) = x^2e^{-x}$.

a Show that $f'''(x) = (6x - 6 - x^2)e^{-x}$.

b Show that $f''''(2) = 0$.

Solution:

a $f'(x) = 2xe^{-x} - x^2e^{-x}$

$f''(x) = (2e^{-x} - 2xe^{-x}) - (2xe^{-x} - x^2e^{-x}) = e^{-x}(2 - 4x + x^2)$

$f'''(x) = e^{-x}(-4 + 2x) - e^{-x}(2 - 4x + x^2) = e^{-x}(-6 + 6x - x^2)$

b $f''''(x) = e^{-x}(6 - 2x) - e^{-x}(-6 + 6x - x^2) = e^{-x}(12 - 8x + x^2)$

so $f''''(2) = e^{-2}(12 - 16 + 4) = 0$

Solutionbank FP2
Edexcel AS and A Level Modular Mathematics

Exercise A, Question 5

Question:

Given that $y = \sec x$, show that

a $\frac{d^2y}{dx^2} = 2\sec^3 x - \sec x$,

b $\left(\frac{d^3y}{dx^3}\right)_{\frac{\pi}{4}} = 11\sqrt{2}$.

Solution:

a Given that $y = \sec x$, so $\frac{dy}{dx} = \sec x \tan x$

$$\frac{d^2y}{dx^2} = \sec x(\sec^2 x) + (\sec x \tan x)\tan x$$

Use the product rule.

$$= \sec x(\sec^2 x + \tan^2 x)$$

$$= \sec x(\sec^2 x + \sec^2 x - 1)$$

Use $1 + \tan^2 A = \sec^2 A$.

$$= 2\sec^3 x - \sec x$$

b $\frac{d^3y}{dx^3} = 6\sec^2 x(\sec x \tan x) - \sec x \tan x$

$$= \sec x \tan x(6\sec^2 x - 1)$$

Substituting $x = \frac{\pi}{4}$ in $\frac{d^3y}{dx^3}$

$$\left(\frac{d^3y}{dx^3}\right)_{\frac{\pi}{4}} = (\sqrt{2})(1)\{6(2) - 1\} = 11\sqrt{2}$$

Solutionbank FP2

Edexcel AS and A Level Modular Mathematics

Exercise A, Question 6

Question:

Given that y is a function of x, show that

a $\dfrac{d^2}{dx^2}(y^2) = 2y\dfrac{d^2y}{dx^2} + 2\left(\dfrac{dy}{dx}\right)^2$

b Find an expression, in terms of y, $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$, for $\dfrac{d^3}{dx^3}(y^2)$.

Solution:

a $\dfrac{d}{dx}(y^2) = \dfrac{d}{dy}(y^2)\dfrac{dy}{dx} = 2y\dfrac{dy}{dx}$

Use the chain rule.

$$\frac{d^2}{dx^2}(y^2) = \frac{d}{dx}\left(2y\frac{dy}{dx}\right) = 2y\frac{d^2y}{dx^2} + 2\frac{dy}{dx}\frac{dy}{dx} = 2y\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2$$

Use the product rule.

b $\dfrac{d^3}{dx^3}(y^2) = \dfrac{d}{dx}\left(2y\dfrac{d^2y}{dx^2} + 2\left(\dfrac{dy}{dx}\right)^2\right)$

$$= 2\left\{y\frac{d^3y}{dx^3} + \frac{dy}{dx}\times\frac{d^2y}{dx^2} + 2\frac{dy}{dx}\times\frac{d^2y}{dx^2}\right\}$$

$$= 2\left\{y\frac{d^3y}{dx^3} + 3\frac{dy}{dx}\times\frac{d^2y}{dx^2}\right\}$$

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Exercise A, Question 7

Question:

Given that $f(x) = \ln\{x + \sqrt{1 + x^2}\}$, show that

a $\sqrt{1 + x^2}\, f'(x) = 1$,

b $(1 + x^2)\, f''(x) + xf'(x) = 0$,

c $(1 + x^2)\, f'''(x) + 3xf''(x) + f'(x) = 0$.

d Deduce the values of $f'(0)$, $f''(0)$ and $f'''(0)$.

Solution:

$f(x) = \ln\{x + \sqrt{1 + x^2}\}$

a $f'(x) = \dfrac{1}{x + \sqrt{(1 + x^2)}} \times \left\{1 + \dfrac{x}{\sqrt{(1 + x^2)}}\right\},$

Use $\dfrac{d}{dx}(\ln u) = \dfrac{1}{u}\dfrac{du}{dx}$.

$= \dfrac{1}{\cancel{x + \sqrt{(1 + x^2)}}} \times \left\{\dfrac{\cancel{\sqrt{(1 + x^2)} + x}}{\sqrt{(1 + x^2)}}\right\} = \dfrac{1}{\sqrt{(1 + x^2)}}$

So $\sqrt{(1 + x^2)}\, f'(x) = 1$

b Differentiating this equation w.r.t. x, using the product rule

$\sqrt{(1 + x^2)}\, f''(x) + \dfrac{x}{\sqrt{(1 + x^2)}} f'(x) = 0$

So $(1 + x^2)f''(x) + xf'(x) = 0$

Multiply through by $\sqrt{(1 + x^2)}$.

c Differentiating this result w.r.t. x

$\{(1 + x^2)f'''(x) + 2xf''(x)\} + \{f'(x) + xf''(x)\} = 0$

giving

$(1 + x^2)f'''(x) + 3xf''(x) + f'(x) = 0$

d $f'(0) = \dfrac{1}{\sqrt{1 + 0}} = 1$

Using $(1 + x^2)f''(x) + xf'(x) = 0$ with $x = 0$ and $f'(0) = 1$

$f''(0) + (0)(1) = 0 \Rightarrow f''(0) = 0$

Using $(1 + x^2)f'''(x) + 3xf''(x) + f'(x) = 0$ with $x = 0$, $f'(0) = 1$ and $f''(0) = 0$

$f'''(0) + (0)(0) + 1 = 0 \Rightarrow f'''(0) = -1$

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Exercise B, Question 1

Question:

Use the formula for the Maclaurin expansion and differentiation to show that

a $(1-x)^{-1} = 1 + x + x^2 + \ldots + x^r + \ldots$

b $\sqrt{(1+x)} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \ldots$

Solution:

a $f(x) = (1-x)^{-1} \quad \Rightarrow f(0) = 1$

$f'(x) = -(1-x)^{-2}(-1) = (1-x)^{-2} \quad \Rightarrow f'(0) = 1$

$f''(x) = -2(1-x)^{-3}(-1) = 2(1-x)^{-3} \quad \Rightarrow f''(0) = 2$

$f'''(x) = -3.2(1-x)^{-4}(-1) = 3.2(1-x)^{-4} \quad \Rightarrow f'''(0) = 3!$

General term: The pattern here is such that $f^{(r)}(x)$ can be written down

$f^{(r)}(x) = r(r-1)\ldots 2(1-x)^{-(r+1)} = r!(1-x)^{-(r+1)} \quad \Rightarrow f^{(r)}(0) = r!$

Using $f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \ldots + \frac{f^{(r)}(0)}{r!}x^r + \ldots$

$$(1-x)^{-1} = 1 + x + \frac{2}{2!}x^2 + \ldots + \frac{r!}{r!}x^r + \ldots = 1 + x + x^2 + \ldots + x^r + \ldots$$

b $f(x) = \sqrt{(1+x)} = (1+x)^{\frac{1}{2}} \quad \Rightarrow f(0) = 1$

$f'(x) = \frac{1}{2}(1+x)^{-\frac{1}{2}} \quad \Rightarrow f'(0) = \frac{1}{2}$

$f''(x) = \frac{1}{2}(-\frac{1}{2})(1+x)^{-\frac{3}{2}} \quad \Rightarrow f''(0) = -\frac{1}{4}$

$f'''(x) = \frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})(1+x)^{-\frac{5}{2}} \quad \Rightarrow f'''(0) = \frac{3}{8}$

Using Maclaurin's expansion

$$\sqrt{(1+x)} = 1 + \tfrac{1}{2}x + \frac{\left(-\frac{1}{4}\right)}{2!}x^2 + \frac{\left(\frac{3}{8}\right)}{3!}x^3 - \ldots$$

$$= 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \ldots$$

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Exercise B, Question 2

Question:

Use Maclaurin's expansion and differentiation to show that the first three terms in the series expansion of $e^{\sin x}$ are $1 + x + \frac{x^2}{2}$.

Solution:

a $f(x) = e^{\sin x} \quad \Rightarrow f(0) = 1$

$f'(x) = \cos x \, e^{\sin x} \quad \Rightarrow f'(0) = 1$

$f''(x) = \cos^2 x \, e^{\sin x} - \sin x \, e^{\sin x} \quad \Rightarrow f''(0) = 1$

Substituting into Maclaurin's expansion gives

$$e^{\sin x} = 1 + 1x + \frac{1}{2!}x^2 + \ldots$$
$$= 1 + x + \frac{1}{2}x^2 + \ldots$$

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Exercise B, Question 3

Question:

a Show that the Maclaurin expansion for $\cos x$ is $1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots + (-1)^r \frac{x^{2r}}{(2r)!} + \ldots$

b Using the first 3 terms of the series, show that it gives a value for $\cos 30°$ correct to 3 decimal places.

Solution:

a $f(x) = \cos x \quad \Rightarrow f(0) = 1$
$f'(x) = -\sin x \quad \Rightarrow f'(0) = 0$
$f''(x) = -\cos x \quad \Rightarrow f''(0) = -1$
$f'''(x) = \sin x \quad \Rightarrow f'''(0) = 0$
$f''''(x) = \cos x \quad \Rightarrow f''''(0) = 1$

The process repeats itself after every 4th derivative, like $\sin x$ does (see Example 5).
Using Maclaurin's expansion, only even powers of x are produced.

$$\cos x = 1 + \frac{(-1)}{2!}x^2 + \frac{1}{4!}x^4 + \ldots + \frac{(-1)^{r+1}}{(2r)!}x^{2r} + \ldots$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots + (-1)^r \frac{x^{2r}}{(2r)!} + \ldots$$

b Using $\cos x \approx 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$ with $x = \frac{\pi}{6}$ (must be in radians)

$$\cos x \approx 1 - \frac{\pi^2}{72} + \frac{\pi^4}{31\,104} = 0.86605 \ldots \text{ which is correct to 3 d.p.}$$

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Exercise B, Question 4

Question:

Using the series expansions for e^x and $\ln(1 + x)$ respectively, find, correct to 3 decimal places, the value of

a e **b** $\ln\left(\frac{6}{5}\right)$

Solution:

a Substituting $x = 1$ into the Maclaurin expansion of e^x, gives

$$e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} + \frac{1}{7!} + \ldots$$

The approximations, to 4 d.p. where necessary, using n terms of the series are

n	1	2	3	4	5	6	7	8	9	10
Approx.	1	2	2.5	2.6667	2.7083	2.7167	2.7181	2.7183	2.7183	2.7183

So e = 2.718 (3 d.p.)

b Substituting $x = 0.2$ into the Maclaurin expansion of $\ln(1 + x)$, gives

$$\ln\left(\frac{6}{5}\right) = 0.2 - \frac{(0.2)^2}{2} + \frac{(0.2)^3}{3} - \frac{(0.2)^4}{4} + \frac{(0.2)^5}{5} - \frac{(0.2)^6}{6} + \frac{(0.2)^7}{7} - \ldots$$

The approximations, to 4 d.p. where necessary, using n terms of the series are

n	1	2	3	4	5
Approximation	0.2	0.18	0.1827	0.1823	0.1823

So $\ln\left(\frac{6}{5}\right) = 0.182$ (3 d.p.)

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Exercise B, Question 5

Question:

Use Maclaurin's expansion and differentiation to expand, in ascending powers of x up to and including the term in x^4,

a e^{3x} **b** $\ln(1 + 2x)$ **c** $\sin^2 x$

Solution:

a $f(x) = e^{3x}$, $f^{(n)}(x) = 3^n e^{3x}$

So $f(0) = 1$, $f'(0) = 3$, $f''(0) = 3^2$, $f'''(0) = 3^3$, $f''''(0) = 3^4$

$$f(x) = e^{3x} = 1 + 3x + \frac{3^2}{2!}x^2 + \frac{3^3}{3!}x^3 + \frac{3^4}{4!}x^4 + \ldots$$

$$= 1 + 3x + \frac{9x^2}{2} + \frac{9x^3}{2} + \frac{27}{8}x^4 + \ldots \left[\text{Note: this is } 1 + 3x + \frac{(3x)^2}{2!} + \frac{(3x)^3}{3!} + \frac{(3x)^4}{4!} + \ldots\right]$$

b As $f(x) = \ln(1 + 2x)$, $f(0) = \ln 1 = 0$

$f'(x) = \frac{2}{1 + 2x} = 2(1 + 2x)^{-1}$, $f'(0) = 2$

$f''(x) = -4(1 + 2x)^{-2}$, $f''(0) = -4$

$f'''(x) = 16(1 + 2x)^{-3}$, $f'''(0) = 16$

$f''''(x) = -96(1 + 2x)^{-4}$, $f''''(0) = -96$

$$\text{So } \ln(1 + 2x) = 0 + 2x + \frac{(-4)}{2!}x^2 + \frac{(16)}{3!}x^3 + \frac{(-96)}{4!}x^4 + \ldots$$

$$= 2x - 2x^2 + \frac{8x^3}{3} - 4x^4 + \ldots \left[\text{Note: this is } 2x - \frac{(2x)^2}{2} + \frac{(2x)^3}{3} - \frac{(2x)^4}{4} + \ldots\right]$$

c $f(x) = \sin^2 x$ $f(0) = 0$

$f'(x) = 2\sin x \cos x = \sin 2x$ $f'(0) = 0$

$f''(x) = 2\cos 2x$ $f''(0) = 2$

$f'''(x) = -4\sin 2x$ $f'''(0) = 0$

$f''''(x) = -8\cos 2x$ $f''''(0) = -8$

$$\text{So } f(x) = \sin^2 x = 0 + 0x + \frac{2}{2!}x^2 + 0x^3 + \frac{(-8)}{4!}x^4 + \ldots = x^2 - \frac{x^4}{3} + \ldots$$

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Exercise B, Question 6

Question:

Using the addition formula for cos $(A - B)$ and the series expansions of $\sin x$ and $\cos x$, show that

$$\cos\left(x - \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}\left(1 + x - \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \ldots\right)$$

Solution:

a $\cos\left(x - \frac{\pi}{4}\right) = \cos x \cos\left(\frac{\pi}{4}\right) + \sin x \sin\left(\frac{\pi}{4}\right)$

Use $\cos(A - B) = \cos A \cos B + \sin A \sin B$.

$$= \frac{1}{\sqrt{2}}(\cos x + \sin x)$$

$$= \frac{1}{\sqrt{2}}\left\{\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots\right) + \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\right)\right\}$$

$$= \frac{1}{\sqrt{2}}\left(1 + x - \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \ldots\right)$$

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Exercise B, Question 7

Question:

Given that $f(x) = (1-x)^2 \ln(1-x)$

a Show that $f''(x) = 3 + 2\ln(1-x)$.

b Find the values of $f(0)$, $f'(0)$, $f''(0)$, and $f'''(0)$.

c Express $(1-x)^2 \ln(1-x)$ in ascending powers of x up to and including the term in x^3.

Solution:

a $f(x) = (1-x)^2 \ln(1-x)$

$$f'(x) = (1-x)^2 \times \frac{(-1)}{1-x} + 2(1-x)(-1)\ln(1-x)$$

Use the product rule.

$$= x - 1 - 2(1-x)\ln(1-x)$$

$$f''(x) = 1 - 2\left[(1-x) \times \frac{(-1)}{1-x} - \ln(1-x)\right] = 1 + 2 + 2\ln(1-x) = 3 + 2\ln(1-x)$$

b $f'''(x) = \dfrac{-2}{1-x}$

Substituting $x = 0$ in all the results gives

$f(0) = 0,\ f'(0) = -1,\ f''(0) = 3,\ f'''(0) = -2$

c $f(x) = (1-x)^2 \ln(1-x) = 0 + (-1)x + \dfrac{3}{2!}x^2 + \dfrac{(-2)}{3!}x^3 + \ldots$

$$= -x + \frac{3x^2}{2} - \frac{1}{3}x^3$$

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Exercise B, Question 8

Question:

a Using the series expansions of $\sin x$ and $\cos x$, show that

$3\sin x - 4x\cos x + x = \frac{3}{2}x^3 - \frac{17}{120}x^5 + \ldots$

b Hence, find the limit, as $x \to 0$, of $\frac{3\sin x - 4x\cos x + x}{x^3}$.

Solution:

a Using the series expansions for $\sin x$ and $\cos x$ as far as the term in x^5,

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots = x - \tfrac{1}{6}x^3 + \tfrac{1}{120}x^5 - \ldots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots = 1 - \tfrac{1}{2}x^2 + \tfrac{1}{24}x^4 - \ldots$$

$$\text{so } 3\sin x - 4x\cos x + x = 3\left(x - \tfrac{1}{6}x^3 + \tfrac{1}{120}x^5 - \ldots\right) - 4x\left(1 - \tfrac{1}{2}x^2 + \tfrac{1}{24}x^4 - \ldots\right) + x$$

$$= 3x - \tfrac{1}{2}x^3 + \tfrac{1}{40}x^5 - 4x + 2x^3 - \tfrac{1}{6}x^5 + x + \ldots$$

$$3\sin x - 4x\cos x + x = \tfrac{3}{2}x^3 - \tfrac{17}{120}x^5 + \ldots$$

b $\frac{3\sin x - 4x\cos x + x}{x^3} = \frac{3}{2} - \frac{17}{120}x^2$ + higher powers in x using **a**

Hence, the limit, as $x \to 0$, is $\frac{3}{2}$.

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Exercise B, Question 9

Question:

Given that $f(x) = \ln \cos x$,

a Show that $f'(x) = -\tan x$

b Find the values of $f'(0)$, $f''(0)$, $f'''(0)$ and $f''''(0)$.

c Express $\ln \cos x$ as a series in ascending powers of x up to and including the term in x^4.

d Show that, using the first two terms of the series for $\ln \cos x$, with $x = \frac{\pi}{4}$, gives a value for $\ln 2$ of $\frac{\pi^2}{16}\left(1 + \frac{\pi^2}{96}\right)$.

Solution:

a $f(x) = \ln \cos x \qquad \Rightarrow f(0) = 0$

$$f'(x) = \frac{1}{\cos x} \times (-\sin x) \left[\frac{d}{dx}(\ln u) = \frac{1}{u}\frac{du}{dx}\right] \qquad \Rightarrow f'(0) = 0$$

$$= -\tan x$$

b $f''(x) = -\sec^2 x \qquad \Rightarrow f''(0) = -1$

$f'''(x) = -2\sec x(\sec x \tan x) = -2\sec^2 x \tan x \qquad \Rightarrow f'''(0) = 0$

$f''''(x) = -2\{\sec^2 x(\sec^2 x) + \tan x(2\sec^2 x \tan x)\} \qquad \Rightarrow f''''(0) = -2$

c Substituting into Maclaurin's expansion

$$\ln \cos x = 0 + 0x + \frac{(-1)}{2!}x^2 + 0x^3 + \frac{(-2)}{4!}x^4 + \ldots$$

$$= -\frac{x^2}{2} - \frac{x^4}{12} + \ldots$$

d Substituting $x = \frac{\pi}{4}$ gives $\ln\left(\frac{1}{\sqrt{2}}\right) = -\frac{1}{2}\left(\frac{\pi^2}{16}\right) - \frac{1}{12}\left(\frac{\pi^4}{256}\right)$

but $\ln\left(\frac{1}{\sqrt{2}}\right) = \ln 2^{-\frac{1}{2}} = -\frac{1}{2}\ln 2$,

so $-\frac{1}{2}\ln 2 = -\frac{\pi^2}{2.16} - \frac{\pi^4}{12.256} + \ldots$

$\Rightarrow \ln 2 = \frac{\pi^2}{16} + \frac{\pi^4}{6.256}$, using only first two terms.

$$= \frac{\pi^2}{16}\left(1 + \frac{\pi^2}{96}\right)$$

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Exercise B, Question 10

Question:

Show that the Maclaurin series for $\tan x$, as far as the term in x^5, is $x + \frac{1}{3}x^3 + \frac{2}{15}x^5$.

Solution:

a $f(x) = \tan x \quad \Rightarrow f(0) = 0$

$f'(x) = \sec^2 x \quad \Rightarrow f'(0) = 1$

$f''(x) = 2\sec x(\sec x \tan x) = 2\sec^2 x \tan x \quad \Rightarrow f''(0) = 0$

$f'''(x) = 2[\sec^2 x(\sec^2 x) + \tan x(2\sec^2 x \tan x)] \quad \Rightarrow f'''(0) = 2$

$= 2(\sec^4 x + 2\sec^2 x \tan^2 x)$

$f''''(x) = 2(\{4\sec^3 x(\sec x \tan x)\} + 2\{\sec^2 x(2\tan x \sec^2 x) + \tan^2 x(2\sec^2 x \tan x)\} \quad \Rightarrow f''''(0) = 0$ as $\tan(0) = 0$

$= 16\sec^4 x \tan x + 8\sec^2 x \tan^3 x$

$= 8\sec^2 x \tan x(2\sec^2 x + \tan^2 x)$

$f''''' (x) = 8\sec^2 x \tan x(4\sec^2 x \tan x + 2\tan x \sec^2 x) + 8(\sec^4 x + 2\sec^2 x \tan^2 x)(2\sec^2 x + \tan^2 x)$

$\Rightarrow f'''''(0) = 16$ as $\tan(0) = 0$, $\sec(0) = 1$

Substitute into Maclaurin's expansion gives

$$\tan x = 0 + 1x + \frac{0}{2!}x^2 + \frac{2}{3!}x^3 + \frac{0}{4!}x^3 + \frac{16}{5!}x^5 + \ldots$$

$$= x + \frac{x^3}{3} + \frac{2}{15}x^5 + \ldots$$

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Exercise C, Question 1

Question:

Use the series expansions of e^x, $\ln(1 + x)$ and $\sin x$ to expand the following functions as far as the fourth non-zero term. In each case state the interval in x for which the expansion is valid.

a $\frac{1}{e^x}$ **b** $\frac{e^{2x} \times e^{3x}}{e^x}$

c e^{1+x} **d** $\ln(1 - x)$

e $\sin\left(\frac{x}{2}\right)$ **f** $\ln(2 + 3x)$

Solution:

a $\frac{1}{e^x} = e^{-x} = 1 + (-x) + \frac{(-x)^2}{2!} + \frac{(-x)^3}{3!} +$

$= 1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \ldots$ valid for all values of x

b $\frac{e^{2x} \times e^{3x}}{e^x} = e^{4x} = 1 + (4x) + \frac{(4x)^2}{2!} + \frac{(4x)^3}{3!} +$

$= 1 + 4x + 8x^2 + \frac{32x^3}{3} + \ldots$ valid for all values of x

c $e^{1+x} = e \times e^x = e\left\{1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \ldots\right\}$ valid for all values of x

d $\ln(1 - x) = (-x) - \frac{(-x)^2}{2} + \frac{(-x)^3}{3} + \frac{(-x)^4}{4} + \ldots$ $[-1 < -x \leqslant 1]$

$\Rightarrow 1 > x \geqslant -1$

$= -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} -$ $-1 \leqslant x < 1$

e $\sin\left(\frac{x}{2}\right) = \left(\frac{x}{2}\right) - \frac{\left(\frac{x}{2}\right)^3}{3!} + \frac{\left(\frac{x}{2}\right)^5}{5!} - \frac{\left(\frac{x}{2}\right)^7}{7!} + \ldots$

$= \frac{x}{2} - \frac{x^3}{48} + \frac{x^5}{3840} - \frac{x^7}{645\,120} +$ valid for all values of x

f $\ln(2 + 3x) = \ln\left\{2\left(1 + \frac{3x}{2}\right)\right\} = \ln 2 + \ln\left(1 + \frac{3x}{2}\right)$

$= \ln 2 + \frac{3x}{2} - \frac{\left(\frac{3x}{2}\right)^2}{2} + \frac{\left(\frac{3x}{2}\right)^3}{3} +$ $\left[-1 < \frac{3x}{2} \leqslant 1\right]$

$= \ln 2 + \frac{3x}{2} - \frac{9x^2}{8} + \frac{9x^3}{8} + \ldots$ $-\frac{2}{3} < x \leqslant \frac{2}{3}$

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Exercise C, Question 2

Question:

a Using the Maclaurin expansion of $\ln(1+x)$, show that

$$\ln\left(\frac{1+x}{1-x}\right) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \dots\right),\ -1 < x < 1.$$

b Deduce the series expansion for $\ln\sqrt{\left(\frac{1+x}{1-x}\right)}$, $-1 < x < 1$.

c By choosing a suitable value of x, and using only the first three terms of the series in **a**, find an approximation for $\ln\left(\frac{2}{3}\right)$, giving your answer to 4 decimal places.

d Show that the first three terms of your series in **b**, with $x = \frac{3}{5}$, gives an approximation for $\ln 2$, which is correct to 2 decimal places.

Solution:

a $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \ldots, \qquad -1 < x \leqslant 1$

$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} - \ldots, \qquad -1 \leqslant x < 1$

$$\ln\left(\frac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x)$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \ldots - \left(-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} - \ldots\right)$$

$$= 2x + \frac{2x^3}{3} + \frac{2x^5}{5} + \ldots$$

$$= 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots\right)$$

As x must be in both the intervals $-1 < x \leqslant 1$ and $-1 \leqslant x < 1$ this expansion requires x to be in the interval $-1 < x < 1$.

b $\ln\sqrt{\left(\frac{1+x}{1-x}\right)} = \ln\left(\frac{1+x}{1-x}\right)^{\frac{1}{2}} = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$

so $\ln\sqrt{\left(\frac{1+x}{1-x}\right)} = \left(x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots\right), -1 < x < 1.$

c Solving $\left(\frac{1+x}{1-x}\right) = \frac{2}{3}$ gives $3 + 3x = 2 - 2x$

$5x = -1$

$x = -0.2$

This is a valid value of x.

So an approximation to $\ln\left(\frac{2}{3}\right)$ is $2\left(-0.2 - \frac{0.008}{3} - \frac{0.00032}{5}\right)$

$= 2(-0.2 - 0.0026666 - 0.000064)$

$= -0.4055$ (4 d.p.)

This is accurate to 4 d.p.

d $\ln\sqrt{\left(\frac{1+x}{1-x}\right)}$ with $x = \frac{3}{5}$ gives $\ln\sqrt{4} = \ln 2$

so $\ln 2 \approx 0.6 + \frac{(0.6)^3}{3} + \frac{(0.6)^5}{5}$

Use the result in **b**.

$\approx 0.687552\ldots = 0.69$ (2 d.p.)

[Using the series in **a** gives $\ln 2 = 0.7424\ldots$]

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Exercise C, Question 3

Question:

Show that for small values of x, $\quad e^{2x} - e^{-x} \approx 3x + \frac{3}{2}x^2$.

Solution:

$$e^{2x} = 1 + 2x + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} + \ldots = 1 + 2x + 2x^2 + \frac{4x^3}{3} + \ldots$$

$$e^{-x} = 1 - x + \frac{(-x)^2}{2!} + \frac{(-x)^3}{3!} + \ldots = 1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \ldots$$

So $e^{2x} - e^{-x} \approx 3x + \frac{3}{2}x^2$, if terms x^3 and above may be neglected.

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Exercise C, Question 4

Question:

a Show that $3x \sin 2x - \cos 3x = -1 + \frac{21}{2}x^2 - \frac{59}{8}x^4 - \ldots$

b Hence find the limit, as $x \to 0$, of $\left(\frac{3x \sin 2x - \cos 3x + 1}{x^2}\right)$.

Solution:

a $3x \sin 2x = 3x\left\{(2x) - \frac{(2x)^3}{3!} + \ldots\right\} = 6x^2 - 4x^4 + \ldots$

$\cos 3x = \left\{1 - \frac{(3x)^2}{2!} + \frac{(3x)^4}{4!} - \right\} = 1 - \frac{9}{2}x^2 + \frac{27}{8}x^4 - \ldots$

So $3x \sin 2x - \cos 3x = 6x^2 - 4x^4 + \ldots - \left(1 - \frac{9}{2}x^2 + \frac{27}{8}x^4 - \ldots\right)$

$$= -1 + \frac{21}{2}x^2 - \frac{59}{8}x^4 + \ldots$$

b $\frac{3x \sin 2x - \cos 3x + 1}{x^2} = \frac{21}{2} - \frac{59}{8}x^2 +$ terms in higher powers of x

As $x \to 0$, so $\frac{3x \sin 2x - \cos 3x + 1}{x^2}$ tends to $\frac{21}{2}$.

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Exercise C, Question 5

Question:

Find the series expansions, up to and including the term in x^4, of

a $\ln(1+x-2x^2)$

b $\ln(9+6x+x^2)$.

and in each case give the range of values of x for which the expansion is valid.

Solution:

a $\ln(1+x-2x^2)=\ln(1-x)(1+2x)=\ln(1-x)+\ln(1+2x)$

$$\ln(1-x)=-x-\frac{x^2}{2}-\frac{x^3}{3}-\frac{x^4}{4}-\ldots,\qquad -1\leqslant x<1$$

$$\ln(1+2x)=(2x)-\frac{(2x)^2}{2}+\frac{(2x)^3}{3}-\frac{(2x)^4}{4}+\ldots,\qquad -\tfrac{1}{2}<x\leqslant\tfrac{1}{2}$$

$$=2x-2x^2+\frac{8x^3}{3}-4x^4$$

So $\ln(1+x-2x^2)=\ln(1-x)+\ln(1+2x)$

$$=x-\frac{5x^2}{2}+\frac{7x^3}{3}-\frac{17x^4}{4}+\ldots,\qquad -\tfrac{1}{2}<x\leqslant\tfrac{1}{2}\text{ (smaller interval)}$$

b $\ln(9+6x+x^2)=\ln(3+x)^2=2\ln(3+x)=2\ln3\left(1+\frac{x}{3}\right)=2\left\{\ln3+\ln\left(1+\frac{x}{3}\right)\right\}$

The expansion of $\ln\left(1+\frac{x}{3}\right)$ is $=\left(\frac{x}{3}\right)-\frac{\left(\frac{x}{3}\right)^2}{2}+\frac{\left(\frac{x}{3}\right)^3}{3}-\frac{\left(\frac{x}{3}\right)^4}{4}+\ldots,\qquad \left[-1<\frac{x}{3}\leqslant1\right]$

$$=\frac{x}{3}-\frac{x^2}{18}+\frac{x^3}{81}-\frac{x^4}{324}+\ldots,\qquad -3<x\leqslant3$$

So $\ln(9+6x+x^2)=2\left\{\ln3+\ln\left(1+\frac{x}{3}\right)\right\}$

$$=2\ln3+\frac{2x}{3}-\frac{x^2}{9}+\frac{2x^3}{81}-\frac{x^4}{162}+\ldots,\qquad -3<x\leqslant3$$

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Exercise C, Question 6

Question:

a Write down the series expansion of $\cos 2x$ in ascending powers of x, up to and including the term in x^8.

b Hence, or otherwise, find the first 4 non-zero terms in the power series for $\sin^2 x$.

Solution:

a $\cos 2x = \left\{1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} + \frac{(2x)^8}{8!} - \ldots\right\}$

$$= 1 - 2x^2 + \frac{2x^4}{3} - \frac{4x^6}{45} + \frac{2x^8}{315} - \ldots$$

b Using $\cos 2x = 1 - 2\sin^2 x$,

$$2\sin^2 x = 1 - \cos 2x = 2x^2 - \frac{2x^4}{3} + \frac{4x^6}{45} - \frac{2x^8}{315} + \ldots$$

So $\sin^2 x = x^2 - \frac{x^4}{3} + \frac{2x^6}{45} - \frac{x^8}{315} + \ldots$

[Alternative: write out expansion of $\sin x$ as far as term in x^7, square it, and collect together appropriate terms!]

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Exercise C, Question 7

Question:

Show that the first two non-zero terms of the series expansion, in ascending powers of x, of $\ln(1 + x) + (x - 1)(e^x - 1)$ are px^3 and qx^4, where p and q are constants to be found.

Solution:

a $\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$

$$(x - 1)(e^x - 1) = (x - 1)\left(x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots\right)$$

$$= x^2 + \frac{x^3}{2} + \frac{x^4}{6} \ldots - x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \ldots$$

$$= -x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{8} + \ldots$$

So $\ln(1 + x) + (x - 1)(e^x - 1) = \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\right) + \left(-x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{8} + \ldots\right)$

$$= \frac{2x^3}{3} - \frac{x^4}{8} + \ldots \qquad \Rightarrow p = \frac{2}{3}, q = -\frac{1}{8}$$

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Exercise C, Question 8

Question:

a Expand $\frac{\sin x}{(1-x)^2}$ in ascending powers of x as far as the term in x^4, by considering the product of the expansions of $\sin x$ and $(1-x)^{-2}$.

b Deduce the gradient of the tangent, at the origin, to the curve with equation $y = \frac{\sin x}{(1-x)^2}$.

Solution:

a Only terms up to and including x^4 in the product are required, so using

$$\sin x = x - \frac{x^3}{3!} + \ldots \quad \text{(next term is } kx^5\text{)}$$

and the binomial expansion of $(1-x)^{-2}$, with terms up to and including x^3.
(It is not necessary to use the term in x^4, because it will be multiplied by expansion of $\sin x$.)

$$(1-x)^{-2} = 1 + (-2)(-x) + (-2)(-3)\frac{(-x)^2}{2!} + (-2)(-3)(-4)\frac{(-x)^3}{3!} + \ldots$$

$$= 1 + 2x + 3x^2 + 4x^3 + \ldots$$

So $$\frac{\sin x}{(1-x)^2} = \left(x - \frac{x^3}{6} + \ldots\right)(1 + 2x + 3x^2 + 4x^3 + \ldots)$$

$$= x + 2x^2 + 3x^3 + 4x^4 + \ldots - \left(\frac{x^3}{6} + \frac{x^4}{3} + \ldots\right)$$

$$= x + 2x^2 + \frac{17x^3}{6} + \frac{11x^4}{3} + \ldots$$

b $y = \frac{\sin x}{(1-x)^2} = x + 2x^2 + \frac{17x^3}{6} + \frac{11x^4}{3} + \ldots$

So $\frac{dy}{dx} = 1 + 4x +$ higher powers of $x \Rightarrow$ at the origin the gradient of tangent $= 1$.

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Exercise C, Question 9

Question:

Using the series given on page 112, show that

a $(1-3x)\ln(1+2x) = 2x - 8x^2 + \frac{26}{3}x^3 - 12x^4 + \ldots$

b $e^{2x}\sin x = x + 2x^2 + \frac{11}{6}x^3 + x^4 + \ldots$

c $\sqrt{(1+x^2)}e^{-x} = 1 - x + x^2 - \frac{2}{3}x^3 + \frac{1}{6}x^4 + \ldots$

Solution:

a $(1-3x)\ln(1+2x) = (1-3x)\left(2x - 2x^2 + \frac{8x^3}{3} - 4x^4 + \ldots\right)$ (see Q5a)

$$= \left(2x - 2x^2 + \frac{8x^3}{3} - 4x^4 + \ldots\right) - (6x^2 - 6x^3 + 8x^4 - \ldots)$$

$$= 2x - 8x^2 + \frac{26}{3}x^3 - 12x^4 + \ldots$$

b $e^{2x}\sin x = \left\{1 + (2x) + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} + \frac{(2x)^4}{4!} + \ldots\right\}\left\{x - \frac{x^3}{3!} + \ldots\right\}$ [only terms up to x^4]

$$= \left(1 + 2x + 2x^2 + \frac{4x^3}{3} + \frac{2x^4}{3} + \ldots\right)\left(x - \frac{x^3}{6} + \ldots\right)$$

$$= \left(x + 2x^2 + 2x^3 + \frac{4x^4}{3}\right) + \left(-\frac{x^3}{6} - \frac{x^4}{3}\right) + \ldots$$

$$= x + 2x^2 + \frac{11}{6}x^3 + x^4 + \ldots$$

c $\sqrt{(1+x^2)}\, e^{-x} = (1+x^2)^{\frac{1}{2}}e^{-x}$

$$= \left\{1 + \frac{1}{2}x^2 + \left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\frac{(x^2)^2}{2!} + \ldots\right\}\left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots\right)$$

$$= \left(1 + \frac{x^2}{2} - \frac{x^4}{8} + \ldots\right)\left(1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} + \ldots\right)$$

$$= \left\{1 - x + \left(\frac{1}{2} + \frac{1}{2}\right)x^2 + \left(-\frac{1}{2} - \frac{1}{6}\right)x^3 + \left(\frac{1}{24} + \frac{1}{4} - \frac{1}{8}\right)x^4 + \ldots\right\}$$

$$= 1 - x + x^2 - \frac{2}{3}x^3 + \frac{1}{6}x^4 + \ldots$$

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Exercise C, Question 10

Question:

a Write down the first five non-zero terms in the series expansions of $e^{-\frac{x^2}{2}}$.

b Using your result in **a**, find an approximate value for $\int_{-1}^{1} e^{-\frac{x^2}{2}}\,dx$, giving your answer to 3 decimal places.

Solution:

a $$e^{-\frac{x^2}{2}} = 1 + \left(-\frac{x^2}{2}\right) + \frac{\left(-\frac{x^2}{2}\right)^2}{2!} + \frac{\left(-\frac{x^2}{2}\right)^3}{3!} + \frac{\left(-\frac{x^2}{2}\right)^4}{4!} + \ldots$$

$$= 1 - \frac{x^2}{2} + \frac{x^4}{8} - \frac{x^6}{48} + \frac{x^8}{384} - \ldots$$

b Area under the curve $= \int_{-1}^{1} e^{-\frac{x^2}{2}}\,dx = 2\int_{0}^{1} e^{-\frac{x^2}{2}}\,dx$

$$= 2\left[x - \frac{x^3}{6} + \frac{x^5}{40} - \frac{x^7}{336} + \frac{x^9}{3456} - \ldots\right]_0^1$$

Integrate the result from **a**.

$$\approx 2\left[1 - \frac{1}{6} + \frac{1}{40} - \frac{1}{336} + \frac{1}{3456}\right]$$

$$\approx 1.711 \text{ (3 d.p.)}$$

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Exercise C, Question 11

Question:

a Show that $e^{px} \sin 3x = 3x + 3px^2 + \frac{3(p^2 - 3)}{2}x^3 + \ldots$ where p is a constant.

b Given that the first non-zero term in the expansion, in ascending powers of x, of $e^{px} \sin 3x + \ln(1 + qx) - x$ is kx^3, where k is a constant, find the values of p, q and k.

Solution:

a
$$e^{px}\sin 3x = \left\{1 + (px) + \frac{(px)^2}{2!} + \frac{(px)^3}{3!} + \ldots\right\}\left\{(3x) - \frac{(3x)^3}{3!} + \ldots\right\}$$
$$= \left(1 + px + \frac{p^2x^2}{2} + \frac{p^3x^3}{6} + \ldots\right)\left(3x - \frac{9x^3}{2} + \ldots\right)$$
$$= \left(3x + 3px^2 + \frac{3p^2x^3}{2} + \ldots\right) + \left(-\frac{9x^3}{2} + \ldots\right)$$
$$= 3x + 3px^2 + \frac{3(p^2 - 3)x^3}{2} + \ldots$$

b
$$\ln(1 + qx) = \left\{(qx) - \frac{(qx)^2}{2} + \frac{(qx)^3}{3} - \ldots\right\}$$

So
$$e^{px}\sin 3x + \ln(1 + qx) - x = 3x + 3px^2 + \frac{3(p^2 - 3)x^3}{2} + qx - \frac{q^2x^2}{2} + \frac{q^3x^3}{3} - x + \ldots$$
$$= (2 + q)x + \left(3p - \frac{q^2}{2}\right)x^2 + \left(\frac{3p^2}{2} + \frac{q^3}{3} - \frac{9}{2}\right)x^3 + \ldots$$

Coefficient of x is zero, so $q = -2$.

Coefficient of x^2 is zero, so $3p - 2 = 0 \Rightarrow p = \frac{2}{3}$

Coefficient of $x^3 = \frac{2}{3} - \frac{8}{3} - \frac{9}{2} = -\frac{13}{2}$, so $k = -\frac{13}{2}$

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Exercise C, Question 12

Question:

$f(x) = e^{x - \ln x} \sin x, \qquad x > 0.$

a Show that if x is sufficiently small so that x^4 and higher powers of x may be neglected, $f(x) \approx 1 + x + \frac{x^2}{3}$.

b Show that using $x = 0.1$ in the result in **a** gives an approximation for f(0.1) which is correct to 6 significant figures.

Solution:

a $e^{x - \ln x} = e^x \times e^{-\ln x} = e^x \times e^{\ln x^{-1}}$ Using $e^{a+b} = e^a \times e^b$

$= e^x \times x^{-1}$ using $e^{\ln k} = k$

$= \frac{e^x}{x}$

$e^{x - \ln x} \sin x = \frac{e^x \sin x}{x}$, and so, using the expansions of e^x and $\sin x$,

$$f(x) = e^{x - \ln x} \sin x = \frac{\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \ldots\right)\left(x - \frac{x^3}{6} + \ldots\right)}{x}, x > 0$$

$$= \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \ldots\right)\left(1 - \frac{x^2}{6} + \ldots\right)$$

$$= \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6}\right) - \left(\frac{x^2}{6} + \frac{x^3}{6}\right)$$ ignoring terms in x^4 and above.

$$= 1 + x + \frac{x^2}{3}$$ There is no term in x^3.

b $f(0.1) = \frac{e^{0.1} \sin 0.1}{0.1} = 1.103329\ldots$

The result in **a** gives an approximation for f(0.1) of $1 + 0.1 + 0.00333333 = 1.103333\ldots$ which is corect to 6 s.f.

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Exercise D, Question 1

Question:

a Find that Taylor series expansion of $\sqrt{x}$ in ascending powers of $(x-1)$ as far as the term in $(x-1)^4$.

b Use your answer in **a** to obtain an estimate for $\sqrt{1.2}$, giving your answer to 3 decimal places.

Solution:

a $\mathrm{f}(x)=\sqrt{x}=\mathrm{f}(a)+\mathrm{f}'(a)(x-a)+\frac{\mathrm{f}''(a)}{2!}(x-a)^2+\frac{\mathrm{f}'''(a)}{3!}(x-a)^3+\frac{\mathrm{f}''''(a)}{4!}(x-a)^4+\ldots$, where $a=1$

$\mathrm{f}(x)=\sqrt{x}$ $\qquad$ $\mathrm{f}(1)=1$

$\mathrm{f}'(x)=\frac{1}{2}x^{-\frac{1}{2}}$ $\qquad$ $\mathrm{f}'(1)=\frac{1}{2}$

$\mathrm{f}''(x)=-\frac{1}{4}x^{-\frac{3}{2}}$ $\qquad$ $\mathrm{f}''(1)=-\frac{1}{4}$

$\mathrm{f}'''(x)=\frac{3}{8}x^{-\frac{5}{2}}$ $\qquad$ $\mathrm{f}'''(1)=\frac{3}{8}$

$\mathrm{f}''''(x)=-\frac{15}{16}x^{-\frac{7}{2}}$ $\qquad$ $\mathrm{f}''''(1)=-\frac{15}{16}$

So $\sqrt{x}=1+\frac{1}{2}(x-1)-\frac{1}{4\times 2!}(x-1)^2+\frac{3}{8\times 3!}(x-1)^3-\frac{15}{16\times 4!}(x-1)^4+\ldots$

$=1+\frac{1}{2}(x-1)-\frac{1}{8}(x-1)^2+\frac{1}{16}(x-1)^3-\frac{5}{128}(x-1)^4+\ldots$

b $\sqrt{1.2}\approx 1+\frac{1}{2}(0.2)-\frac{1}{8}(0.2)^2+\frac{1}{16}(0.2)^3-\frac{5}{128}(0.2)^4$

$\approx 1+0.1-0.005+0.0005-0.0000625$

$=1.095$ (3 d.p.)

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Exercise D, Question 2

Question:

Use Taylor's expansion to express each of the following as a series in ascending powers of $(x - a)$ as far as the term in $(x - a)^k$, for the given values of a and k.

a $\ln x \quad (a = \mathrm{e}, k = 2)$ **b** $\tan x \quad \left(a = \frac{\pi}{3}, k = 3\right)$ **c** $\cos x \quad (a = 1, k = 4)$

Solution:

All solutions use the Taylor expansion in the form:

$$\mathbf{f}(x) = \mathbf{f}(a) + \mathbf{f}'(a)(x - a) + \frac{\mathbf{f}''(a)}{2!}(x - a)^2 + \frac{\mathbf{f}'''(a)}{3!}(x - a)^3 + \ldots + \frac{\mathbf{f}^{(r)}(a)}{r!}(x - a)^r + \ldots,$$

a Let $f(x) = \ln x$ then $f(a) = f(e) = \ln e = 1$

$f'(x) = \frac{1}{x}$ $\quad f'(a) = f'(e) = \frac{1}{e}$

$f''(x) = -\frac{1}{x^2}$ $\quad f''(a) = f''(e) = -\frac{1}{e^2}$

$$\text{So } f(x) = \ln x = 1 + \frac{1}{e}(x - e) + \frac{\left(-\frac{1}{e^2}\right)}{2!}(x - e)^2 + \ldots$$

$$= 1 + \frac{(x - e)}{e} - \frac{(x - e)^2}{2e^2} + \ldots$$

b Let $f(x) = \tan x$ then $f(a) = f\left(\frac{\pi}{3}\right) = \sqrt{3}$

$f'(x) = \sec^2 x$ $\quad f'(a) = f'\left(\frac{\pi}{3}\right) = 4$

$f''(x) = 2\sec^2 x \tan x$ $\quad f''(a) = f''\left(\frac{\pi}{3}\right) = 2(4)(\sqrt{3}) = 8\sqrt{3}$

$f'''(x) = 2\sec^4 x + 2\tan x(2\sec^2 x \tan x)$ $\quad f'''(a) = f'''\left(\frac{\pi}{3}\right) = 2(16) + 4(4)(3) = 80$

$$\text{So } \quad f(x) = \tan x = \sqrt{3} + 4\left(x - \frac{\pi}{3}\right) + \frac{8\sqrt{3}}{2!}\left(x - \frac{\pi}{3}\right)^2 + \frac{80}{3!}\left(x - \frac{\pi}{3}\right)^3 + \ldots$$

$$= \sqrt{3} + 4\left(x - \frac{\pi}{3}\right) + 4\sqrt{3}\left(x - \frac{\pi}{3}\right)^2 + \frac{40}{3}\left(x - \frac{\pi}{3}\right)^3 + \ldots$$

c Let $f(x) = \cos x$ then $f(a) = f(1) = \cos 1$

$f'(x) = -\sin x$ $\quad f'(a) = f'(1) = -\sin 1$

$f''(x) = -\cos x$ $\quad f''(a) = f''(1) = -\cos 1$

$f'''(x) = \sin x$ $\quad f'''(a) = f'''(1) = \sin 1$

$f''''(x) = \cos x$ $\quad f''''(a) = f''''(1) = \cos 1$

$$\text{So } \quad f(x) = \cos x = \cos 1 - \sin 1\,(x - 1) - \frac{(\cos 1)}{2}(x - 1)^2 + \frac{(\sin 1)}{6}(x - 1)^3 + \frac{(\cos 1)}{24}(x - 1)^4 + \ldots$$

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Exercise D, Question 3

Question:

a Use Taylor's expansion to express each of the following as a series in ascending powers of x as far as the term in x^4.

i $\cos\left(x + \frac{\pi}{4}\right)$ **ii** $\ln(x + 5)$ **iii** $\sin\left(x - \frac{\pi}{3}\right)$

b Use your result in **ii** to find an approximation for $\ln 5.2$, giving your answer to 6 significant figures.

Solution:

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Exercise D, Question 4

Question:

Given that $y = xe^x$,

a Show that $\frac{d^ny}{dx^n} = (n + x)e^x$.

b Find the Taylor expansion of xe^x in ascending powers of $(x + 1)$ up to and including the term in $(x + 1)^4$.

Solution:

a $y = xe^x$, $\frac{dy}{dx} = xe^x + e^x = e^x(x + 1)$

Product rule.

$$\frac{d^2y}{dx^2} = xe^x + e^x + e^x = e^x(x + 2)$$

$$\frac{d^3y}{dx^3} = xe^x + 2e^x + e^x = e^x(x + 3)$$

Each differentiation adds another e^x, so $\frac{d^ny}{dx^n} = (n + x)e^x$.

So for $f(x) = xe^x$, $f^{(n)}(x) = (n + x)e^x$.

b Using the Taylor series with $a = -1$, $f(-1) = -e^{-1}$, $f'(-1) = 0$, $f''(-1) = e^{-1}$
$f'''(-1) = 2e^{-1}$, $f''''(-1) = 3e^{-1}$

So $xe^x = e^{-1}\left\{-1 + 0(x + 1) + \frac{1}{2!}(x + 1)^2 + \frac{2}{3!}(x + 1)^3 + \frac{3}{4!}(x + 1)^4 + \ldots\right\}$

$$= e^{-1}\left\{-1 + \frac{1}{2}(x + 1)^2 + \frac{1}{3}(x + 1)^3 + \frac{1}{8}(x + 1)^4 + \ldots\right\}$$

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Exercise D, Question 5

Question:

a Find the Taylor series for $x^3 \ln x$ in ascending powers of $(x - 1)$ up to and including the term in $(x - 1)^4$.

b Using your series in **a**, find an approximation for ln 1.5, giving your answer to 4 decimal places.

Solution:

a Let $f(x) = x^3 \ln x$ then as $a = 1$ $f(a) = f(1) = 0$

$f'(x) = 3x^2 \ln x + x^3 \times \frac{1}{x} = x^2(1 + 3\ln x)$ $f'(a) = f'(1) = 1$

$f''(x) = x^2 \times \frac{3}{x} + 2x(1 + 3\ln x) = x(5 + 6\ln x)$ $f''(a) = f''(1) = 5$

$f'''(x) = x \times \frac{6}{x} + (5 + 6\ln x) = 11 + 6\ln x$ $f'''(a) = f'''(1) = 11$

$f''''(x) = \frac{6}{x}$ $f''''(a) = f''''(1) = 6$

Using Taylor, form **ii**

$$f(x) = x^3 \ln x = 0 + 1(x - 1) + \frac{5}{2!}(x - 1)^2 + \frac{11}{3!}(x - 1)^3 + \frac{6}{4!}(x - 1)^4 + \dots$$

$$= (x - 1) + \frac{5}{2}(x - 1)^2 + \frac{11}{6}(x - 1)^3 + \frac{1}{4}(x - 1)^4 + \dots$$

b Substituting $x = 1.5$ in series in **a**, gives

$$\frac{27}{8} \ln 1.5 \approx 0.5 + \frac{5}{2}(0.5)^2 + \frac{11}{6}(0.5)^3 + \frac{1}{4}(0.5)^4 + \dots$$

$$\approx 0.5 + 0.625 + 0.22916\dots + 0.015625 \ (= 1.369791\dots)$$

So this gives an approximation for ln 1.5 of $\frac{8}{27}(1.369791\dots) = 0.4059$ (4 d.p.)

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Exercise D, Question 6

Question:

Find the Taylor expansion of $\tan(x - \alpha)$, where $\alpha = \arctan\left(\frac{3}{4}\right)$, in ascending powers of x up to and including the term in x^2.

Solution:

Let $f(x + a) = \tan(x - \alpha)$, so that $f(x) = \tan x$ and $a = -\alpha$

As $\alpha = \arctan\left(\frac{3}{4}\right)$, $\tan\alpha = \frac{3}{4}$ and $\cos\alpha = \frac{4}{5}$

$f(x) = \tan x$ $\qquad f(a) = f(-\alpha) = \tan(-\alpha) = -\frac{3}{4}$

$f'(x) = \sec^2 x$ $\qquad f'(a) = f'(-\alpha) = \frac{25}{16}$

$f''(x) = 2\sec^2 x \tan x$ $\qquad f''(a) = f''(-\alpha) = 2\left(\frac{25}{16}\right)\left(-\frac{3}{4}\right) = -\left(\frac{75}{32}\right)$

Using the form **ii** of the Taylor expansion gives

$$f(x + a) = \tan\left(x - \arctan\left(\frac{3}{4}\right)\right) = -\frac{3}{4} + \frac{25}{16}x + \frac{\left(-\frac{75}{32}\right)}{2!}x^2 + \dots$$

$$= -\frac{3}{4} + \frac{25}{16}x - \frac{75}{64}x^2 + \dots$$

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Exercise D, Question 7

Question:

Find the Taylor expansion of $\sin 2x$ in ascending powers of $\left(x - \frac{\pi}{6}\right)$ up to and including the term in $\left(x - \frac{\pi}{6}\right)^4$.

Solution:

a $f(x) = \sin 2x$ and $a = \frac{\pi}{6}$

$f(x) = \sin 2x$ $\quad f(a) = f\left(\frac{\pi}{6}\right) = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$

$f'(x) = 2\cos 2x$ $\quad f'(a) = f'\left(\frac{\pi}{6}\right) = 2\cos\left(\frac{\pi}{3}\right) = 1$

$f''(x) = -4\sin 2x$ $\quad f''(a) = f''\left(\frac{\pi}{6}\right) = -4\sin\left(\frac{\pi}{3}\right) = -2\sqrt{3}$

$f'''(x) = -8\cos 2x$ $\quad f'''(a) = f'''\left(\frac{\pi}{6}\right) = -8\cos\left(\frac{\pi}{3}\right) = -4$

$f''''(x) = +16\sin 2x$ $\quad f''''(a) = f''''\left(\frac{\pi}{6}\right) = 16\sin\left(\frac{\pi}{3}\right) = 8\sqrt{3}$

$$\text{So } f(x) = \sin 2x = \frac{\sqrt{3}}{2} + 1\left(x - \frac{\pi}{6}\right) + \frac{(-2\sqrt{3})}{2!}\left(x - \frac{\pi}{6}\right)^2 + \frac{(-4)}{3!}\left(x - \frac{\pi}{6}\right)^3 + \frac{(8\sqrt{3})}{4!}\left(x - \frac{\pi}{6}\right)^4 + \ldots$$

$$= \frac{\sqrt{3}}{2} + 1\left(x - \frac{\pi}{6}\right) - \sqrt{3}\left(x - \frac{\pi}{6}\right)^2 - \frac{2}{3}\left(x - \frac{\pi}{6}\right)^3 + \frac{\sqrt{3}}{3}\left(x - \frac{\pi}{6}\right)^4 + \ldots$$

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Exercise D, Question 8

Question:

Given that $y = \frac{1}{\sqrt{(1+x)}}$,

a find the values of $\left(\frac{dy}{dx}\right)_3$ and $\left(\frac{d^2y}{dx^2}\right)_3$.

b Find the Taylor expansion of $\frac{1}{\sqrt{(1+x)}}$, in ascending powers of $(x-3)$ up to and including the the term in $(x-3)^2$.

Solution:

a Given $y = \frac{1}{\sqrt{(1+x)}} = (1+x)^{-\frac{1}{2}}$ $\qquad y_3(=\text{ value of } y \text{ when } x = 3) = \frac{1}{2}$

$$\frac{dy}{dx} = -\frac{1}{2}(1+x)^{-\frac{3}{2}} \qquad \left(\frac{dy}{dx}\right)_3 = -\frac{1}{2} \times \frac{1}{8} = -\frac{1}{16}$$

$$\frac{d^2y}{dx^2} = \frac{3}{4}(1+x)^{-\frac{5}{2}} \qquad \left(\frac{d^2y}{dx^2}\right)_3 = \frac{3}{4} \times \frac{1}{32} = \frac{3}{128}$$

b So using

$$f(x) = f(3) + f'(3)(x-3) + \frac{f''(3)}{2!}(x-3)^2 + \dots \qquad \text{with } f^{(n)}(3) \equiv \left(\frac{d^ny}{dx^n}\right)_3$$

$$y = \frac{1}{\sqrt{(1+x)}} = \frac{1}{2} - \frac{1}{16}(x-3) + \frac{3}{256}(x-3)^2 + \dots$$

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Exercise E, Question 1

Question:

Find a series solution, in ascending powers of x up to and including the term in x^4, for the differential equation $\frac{d^2y}{dx^2} = x + 2y$, given that at $x = 0$, $y = 1$ and $\frac{dy}{dx} = \frac{1}{2}$.

Solution:

Differentiating $\frac{d^2y}{dx^2} = x + 2y$, with respect to x, gives $\frac{d^3y}{dx^3} = 1 + 2\frac{dy}{dx}$ ①

Differentiating ① gives $\frac{d^4y}{dx^4} = 2\frac{d^2y}{dx^2}$ ②

Substituting $x_0 = 0$, $y_0 = 1$ into $\frac{d^2y}{dx^2} = x + 2y$, gives

$$\left(\frac{d^2y}{dx^2}\right)_0 = 0 + 2(1), \text{ so } \left(\frac{d^2y}{dx^2}\right)_0 = 2$$

Substituting $\left(\frac{dy}{dx}\right)_0 = \frac{1}{2}$ into ① gives $\left(\frac{d^3y}{dx^3}\right)_0 = 1 + 2\left(\frac{1}{2}\right) = 2$

Substituting $\left(\frac{d^2y}{dx^2}\right)_0 = 2$ into ② gives $\left(\frac{d^4y}{dx^4}\right)_0 = 2(2) = 4$

So using the Taylor expansion in the form where $x_0 = 0$, i.e. **ii**

$$y = 1 + \left(\frac{1}{2}\right)x + \frac{(2)}{2!}x^2 + \frac{(2)}{3!}x^3 + \frac{(4)}{4!}x^4 + \ldots = 1 + \frac{x}{2} + x^2 + \frac{x^3}{3} + \frac{x^4}{6} + \ldots$$

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Exercise E, Question 2

Question:

The variable y satisfies $(1+x^2)\frac{d^2y}{dx^2}+x\frac{dy}{dx}=0$ and at $x=0$, $y=0$ and $\frac{dy}{dx}=1$.
Use Taylor's method to find a series expansion for y in powers of x up to and including the term in x^3.

Solution:

Differentiating $(1+x^2)\frac{d^2y}{dx^2}+x\frac{dy}{dx}=0$, gives

$$(1+x^2)\frac{dy^3}{dx^3}+2x\frac{d^2y}{dx^2}+x\frac{d^2y}{dx^2}+\frac{dy}{dx}=0 \quad ① \qquad \text{i.e. } (1+x^2)\frac{dy^3}{dx^3}+3x\frac{d^2y}{dx^2}+\frac{dy}{dx}=0$$

Substituting $x=0$ and $\left(\frac{dy}{dx}\right)_0=1$ into $(1+x^2)\frac{d^2y}{dx^2}+x\frac{dy}{dx}=0$, gives $\left(\frac{d^2y}{dx^2}\right)_0=0$

Substituting $x=0$, $\left(\frac{dy}{dx}\right)_0=1$ and $\left(\frac{d^2y}{dx^2}\right)_0=0$ into ① gives $\left(\frac{d^3y}{dx^3}\right)_0=-1$

So using the Taylor expansion in the form **ii**,

$$y=0+1x+\frac{(0)}{2!}x^2+\frac{(-1)}{3!}x^3+\ldots=x-\frac{x^3}{6}+\ldots$$

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Exercise E, Question 3

Question:

Given that y satisfies the differential equation $\frac{dy}{dx} + y - e^x = 0$, and that $y = 2$ at $x = 0$, find a series solution for y in ascending powers of x up to and including the term in x^3.

Solution:

Differentiating $\frac{dy}{dx} + y - e^x = 0$, gives $\frac{d^2y}{dx^2} + \frac{dy}{dx} - e^x = 0$ ①

Differentiating ① gives $\frac{d^3y}{dx^3} + \frac{d^2y}{dx^2} - e^x = 0$ ②

Substituting $x_0 = 0$ and $y_0 = 2$ into $\frac{dy}{dx} + y - e^x = 0$, gives $\left(\frac{dy}{dx}\right)_0 + 2 - 1 = 0$, so $\left(\frac{dy}{dx}\right)_0 = -1$

Substituting $x = 0$, $\left(\frac{dy}{dx}\right)_0 = -1$ into ① gives $\left(\frac{d^2y}{dx^2}\right)_0 + (-1) - (1) = 0$ so $\left(\frac{d^2y}{dx^2}\right)_0 = 2$

Substituting $x = 0$, $\left(\frac{d^2y}{dx^2}\right)_0 = 2$ into ② gives $\left(\frac{d^3y}{dx^3}\right)_0 + (2) - (1) = 0$ so $\left(\frac{d^3y}{dx^3}\right)_0 = -1$

Substituting into the Taylor series with $x_0 = 0$, gives

$$y = 2 + (-1)x + \frac{(2)}{2!}x^2 + \frac{(-1)}{3!}x^3 + \ldots$$

$$= 2 - x + x^2 - \frac{x^3}{6} \ldots$$

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Exercise E, Question 4

Question:

Use the Taylor method to find a series solution for
$\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$, given that $x = 0$, $y = 1$ and $\frac{dy}{dx} = 2$,
giving your answer in ascending powers of x up to and including the term in x^4.

Solution:

Differentiating $\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$ with respect to x gives

$$\frac{d^3y}{dx^3} + x\frac{d^2y}{dx^2} + \frac{dy}{dx} + \frac{dy}{dx} = 0 \quad ①, \qquad \text{i.e. } \frac{d^3y}{dx^3} + x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 0$$

Differentiating ① gives

$$\frac{d^4y}{dx^4} + x\frac{d^3y}{dx^3} + \frac{d^2y}{dx^2} + 2\frac{d^2y}{dx^2} = 0 \quad ②, \qquad \text{i.e. } \frac{d^4y}{dx^4} + x\frac{d^3y}{dx^3} + 3\frac{d^2y}{dx^2} = 0$$

Substituting $x = 0$, $y = 1$ and $\frac{dy}{dx} = 2$ into $\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$ gives

$$\left(\frac{d^2y}{dx^2}\right)_0 + 0(2) + 1 = 0 \Rightarrow \left(\frac{d^2y}{dx^2}\right)_0 = -1$$

Substituting $x = 0$, $\left(\frac{dy}{dx}\right)_0 = 2$ and $\left(\frac{d^2y}{dx^2}\right)_0 = -1$ into ① gives

$$\left(\frac{d^3y}{dx^3}\right)_0 + 0(-1) + 2(2) = 0, \text{ so } \left(\frac{d^3y}{dx^3}\right)_0 = -4$$

Substituting $x = 0$, $\left(\frac{dy}{dx}\right)_0 = 2$, $\left(\frac{d^2y}{dx^2}\right)_0 = -1$ and $\left(\frac{d^3y}{dx^3}\right)_0 = -4$ into ② gives

$$\left(\frac{d^4y}{dx^4}\right)_0 + 0(-4) + 3(-1) = 0, \text{ so } \left(\frac{d^4y}{dx^4}\right)_0 = 3$$

Substituting into the Taylor series with form **ii**, gives

$$y = 1 + 2x + \frac{(-1)}{2!}x^2 + \frac{(-4)}{3!}x^3 + \frac{(3)}{4!}x^4 + \ldots$$

$$= 1 + 2x - \frac{1}{2}x^2 - \frac{2}{3}x^3 + \frac{1}{8}x^4 + \ldots$$

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Exercise E, Question 5

Question:

The variable y satisfies the differential equation $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 3xy$, and $y = 1$ and $\frac{dy}{dx} = -1$ at $x = 1$.

Express y as a series in powers of $(x - 1)$ up to and including the term in $(x - 1)^3$.

Solution:

Differentiating $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 3xy$ gives $\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} = 3x\frac{dy}{dx} + 3y$ ①

Substituting $x_0 = 1$, $y_0 = 1$ and $\left(\frac{dy}{dx}\right)_1 = -1$ into $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 3xy$ gives $\left(\frac{d^2y}{dx^2}\right)_1 = 5$

Substituting $x_0 = 1$, $y_0 = 1$, $\left(\frac{dy}{dx}\right)_1 = -1$ and $\left(\frac{d^2y}{dx^2}\right)_1 = 5$ into ① gives $\left(\frac{d^3y}{dx^3}\right)_1 = -10$

Substituting into the form of the Taylor series form **i**, with $x_0 = 1$, gives

$$y = 1 + (-1)(x - 1) + \frac{(5)}{2!}(x - 1)^2 + \frac{(-10)}{3!}(x - 1)^3 + \ldots$$

$$= 1 - (x - 1) + \frac{5}{2}(x - 1)^2 - \frac{5}{3}(x - 1)^3 + \ldots$$

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Exercise E, Question 6

Question:

Find a series solution, in ascending powers of x up to and including the term x^4, to the differential equation $\frac{d^2y}{dx^2} + 2y\frac{dy}{dx} + y^3 = 1 + x$, given that at $x = 0$, $y = 1$ and $\frac{dy}{dx} = 1$.

Solution:

Differentiating $\frac{d^2y}{dx^2} + 2y\frac{dy}{dx} + y^3 = 1 + x$, twice with respect to x, gives

$$\frac{d^3y}{dx^3} + 2y\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2 + 3y^2\frac{dy}{dx} = 1 \quad ①$$

$$\frac{d^4y}{dx^4} + 2y\frac{d^3y}{dx^3} + 2\frac{dy}{dx}\left(\frac{d^2y}{dx^2}\right) + 4\left(\frac{dy}{dx}\right)\left(\frac{d^2y}{dx^2}\right) + 3y^2\frac{d^2y}{dx^2} + 6y\left(\frac{dy}{dx}\right)^2 = 0 \quad ②$$

Substituting $x = 0$, $y = 1$ and $\frac{dy}{dx} = 1$ into $\frac{d^2y}{dx^2} + 2y\frac{dy}{dx} + y^3 = 1 + x$ gives $\left(\frac{d^2y}{dx^2}\right)_0 = -2$

Substituting $y = 1$, $\left(\frac{dy}{dx}\right)_0 = 1$ and $\left(\frac{d^2y}{dx^2}\right)_0 = -2$ into ① gives $\left(\frac{d^3y}{dx^3}\right)_0 = 0$

Substituting $y = 1$, $\left(\frac{dy}{dx}\right)_0 = 1$, $\left(\frac{d^2y}{dx^2}\right)_0 = -2$, $\left(\frac{d^3y}{dx^3}\right)_0 = 0$ into ② gives $\left(\frac{d^4y}{dx^4}\right)_0 = 12$

So, using the Taylor series form **ii**, $y = 1 + 1x + \frac{(-2)}{2!}x^2 + \frac{(0)}{3!}x^3 + \frac{(12)}{4!}x^4 + \ldots$

so $y = 1 + x - x^2 + \frac{1}{2}x^4 + \ldots$

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Exercise E, Question 7

Question:

$(1 + 2x)\dfrac{dy}{dx} = x + 2y^2$

a Show that $(1 + 2x)\dfrac{d^3y}{dx^3} + 4(1 - y)\dfrac{d^2y}{dx^2} = 4\left(\dfrac{dy}{dx}\right)^2$

b Given that $y = 1$ at $x = 0$, find a series solution of $(1 + 2x)\dfrac{dy}{dx} = x + 2y^2$, in ascending powers of x up to and including the term in x^3.

Solution:

a Differentiating $(1 + 2x)\dfrac{dy}{dx} = x + 2y^2$ with respect to x

$$\left\{(1 + 2x)\frac{d^2y}{dx^2} + 2\frac{dy}{dx}\right\} = 1 + 4y\frac{dy}{dx} \qquad ①$$

Differentiating ① gives

$$\left\{(1 + 2x)\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2}\right\} + \left\{2\frac{d^2y}{dx^2}\right\} = \left\{4y\frac{d^2y}{dx^2} + 4\left(\frac{dy}{dx}\right)^2\right\}$$

$$\Rightarrow (1 + 2x)\frac{d^3y}{dx^3} + 4(1 - y)\frac{d^2y}{dx^2} = 4\left(\frac{dy}{dx}\right)^2 \qquad ②$$

b Substituting $x_0 = 0$ and $y_0 = 1$ into $(1 + 2x)\dfrac{dy}{dx} = x + 2y^2$ gives $\left(\dfrac{dy}{dx}\right)_0 = 2(1) = 2$

Substituting known values into ① gives

$$\left(\frac{d^2y}{dx^2}\right)_0 + 2(2) = 1 + 4(1)(2) \Rightarrow \left(\frac{d^2y}{dx^2}\right)_0 = 5$$

Substituting known values into ② gives $\left(\dfrac{d^3y}{dx^3}\right)_0 = 4(2)^2 = 16$

So using $y = y_0 + x\left(\dfrac{dy}{dx}\right)_0 + \dfrac{x^2}{2!}\left(\dfrac{d^2y}{dx^2}\right)_0 + \dfrac{x^3}{3!}\left(\dfrac{d^3y}{dx^3}\right)_0 + \ldots$

$$y = 1 + 2x + \frac{5}{2!}x^2 + \frac{16}{3!}x^3 + \ldots = 1 + 2x + \frac{5}{2}x^2 + \frac{8}{3}x^3 + \ldots$$

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Exercise E, Question 8

Question:

Find the series solution in ascending powers of $\left(x - \frac{\pi}{4}\right)$ up to and including the term in $\left(x - \frac{\pi}{4}\right)^2$ for the differential equation $\sin x \frac{dy}{dx} + y \cos x = y^2$ given that $y = \sqrt{2}$ at $x = \frac{\pi}{4}$.

Solution:

Differentiating $\sin x \frac{dy}{dx} + y \cos x = y^2$ with respect to x, gives

$$\left(\sin x \frac{d^2y}{dx^2} + \cos x \frac{dy}{dx}\right) + \left(-y \sin x + \cos x \frac{dy}{dx}\right) = 2y \frac{dy}{dx} \qquad ①$$

$$\text{or } \sin x \frac{d^2y}{dx^2} + 2 \cos x \frac{dy}{dx} - y \sin x = 2y \frac{dy}{dx}$$

Substituting $x_0 = \frac{\pi}{4}$, $y_0 = \sqrt{2}$ into $\sin x \frac{dy}{dx} + y \cos x = y^2$ gives $\frac{1}{\sqrt{2}}\left(\frac{dy}{dx}\right)_{\frac{\pi}{4}} + \sqrt{2} \times \frac{1}{\sqrt{2}} = 2$

so $\left(\frac{dy}{dx}\right)_{\frac{\pi}{4}} = \sqrt{2}$

Substituting $x_0 = \frac{\pi}{4}$, $y_0 = \sqrt{2}$, $\left(\frac{dy}{dx}\right)_{\frac{\pi}{4}} = \sqrt{2}$ into ① gives

$$\left\{\frac{1}{\sqrt{2}}\left(\frac{d^2y}{dx^2}\right)_{\frac{\pi}{4}} + 2\left(\frac{1}{\sqrt{2}}\right)(\sqrt{2}) - (\sqrt{2})\left(\frac{1}{\sqrt{2}}\right) = 2(\sqrt{2})(\sqrt{2})\right\}$$

So $\left\{\frac{1}{\sqrt{2}}\left(\frac{d^2y}{dx^2}\right)_{\frac{\pi}{4}} + 2 - 1 = 4\right\} \Rightarrow \left(\frac{d^2y}{dx^2}\right)_{\frac{\pi}{4}} = 3\sqrt{2}$

Substituting all values into $y = y_0 + (x - x_0)\left(\frac{dy}{dx}\right)_{x_0} + \frac{(x - x_0)^2}{2!}\left(\frac{d^2y}{dx^2}\right)_{x_0} + \dots$

gives the series solution $y = \sqrt{2} + \sqrt{2}\left(x - \frac{\pi}{4}\right) + \frac{3\sqrt{2}}{2}\left(x - \frac{\pi}{4}\right)^2 + \dots$

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Exercise E, Question 9

Question:

The variable y satisfies the differential equation $\frac{dy}{dx} - x^2 - y^2 = 0$.

a Show that

i $\frac{d^2y}{dx^2} - 2y\frac{dy}{dx} - 2x = 0$, **ii** $\frac{d^3y}{dx^3} - 2y\frac{d^2y}{dx^2} - 2\left(\frac{dy}{dx}\right)^2 = 2$.

b Derive a similar equation involving $\frac{d^4y}{dx^4}, \frac{d^3y}{dx^3}, \frac{d^2y}{dx^2}, \frac{dy}{dx}$, and y.

c Given also that at $x = 0$, $y = 1$, express y as a series in ascending powers of x in powers of x up to and including the term in x^4.

Solution:

a i Differentiating $\frac{dy}{dx} - x^2 - y^2 = 0$ with respect to x, gives $\frac{d^2y}{dx^2} - 2y\frac{dy}{dx} - 2x = 0$ ①

ii Differentiating ① gives $\frac{d^3y}{dx^3} - 2y\frac{d^2y}{dx^2} - 2\left(\frac{dy}{dx}\right)^2 - 2 = 0$

So $\frac{d^3y}{dx^3} - 2y\frac{d^2y}{dx^2} - 2\left(\frac{dy}{dx}\right)^2 = 2$ ②

b Differentiating ② gives $\frac{d^4y}{dx^4} - 2y\frac{d^3y}{dx^3} - 2\left(\frac{dy}{dx}\right)\left(\frac{d^2y}{dx^2}\right) - 4\left(\frac{dy}{dx}\right)\left(\frac{d^2y}{dx^2}\right) = 0$

so $\frac{d^4y}{dx^4} - 2y\frac{d^3y}{dx^3} - 6\frac{dy}{dx} \times \frac{d^2y}{dx^2} = 0$ ③

c Substituting $x_0 = 0$, $y_0 = 1$, into $\frac{dy}{dx} - x^2 - y^2 = 0$ gives

$\left(\frac{dy}{dx}\right)_0 - 0 - 1 = 0$, so $\left(\frac{dy}{dx}\right)_0 = 1$

Substituting $x_0 = 0$, $y_0 = 1$, $\left(\frac{dy}{dx}\right)_0 = 1$ into ① gives

$\left(\frac{d^2y}{dx^2}\right)_0 - 2(1)(1) - 2(0) = 0$, so $\left(\frac{d^2y}{dx^2}\right)_0 = 2$

Substituting $y_0 = 1$, $\left(\frac{dy}{dx}\right)_0 = 1$, $\left(\frac{d^2y}{dx^2}\right)_0 = 2$ into ② gives

$\left(\frac{d^3y}{dx^3}\right)_0 - 2(1)(2) - 2(1)^2 = 2$, so $\left(\frac{d^3y}{dx^3}\right)_0 = 8$

Substituting $y_0 = 1$, $\left(\frac{dy}{dx}\right)_0 = 1$, $\left(\frac{d^2y}{dx^2}\right)_0 = 2$ and $\left(\frac{d^3y}{dx^3}\right)_0 = 8$ into ③ gives

$\left(\frac{d^4y}{dx^4}\right)_0 - 2(1)(8) - 6(1)(2) = 0$, so $\left(\frac{d^4y}{dx^4}\right)_0 = 28$

Substituting these values into the form of Taylor's series form **ii**, gives

$$y = 1 + (1)x + \frac{(2)}{2!}x^2 + \frac{(8)}{3!}x^3 + \frac{(28)}{4!}x^4 + \ldots = 1 + x + x^2 + \frac{4}{3}x^3 + \frac{7}{6}x^4 + \ldots$$

Solutionbank FP2
Edexcel AS and A Level Modular Mathematics

Exercise E, Question 10

Question:

Given that $\cos x \frac{\mathrm{d}y}{\mathrm{d}x} + y \sin x + 2y^3 = 0$, and that $y = 1$ at $x = 0$, use Taylor's method to show that, close to $x = 0$, so that terms in x^4 and higher power can be ignored,
$y \approx 1 - 2x + \frac{11}{2}x^2 - \frac{56}{3}x^3$.

Solution:

Differentiating $\cos x \frac{\mathrm{d}y}{\mathrm{d}x} + y \sin x + 2y^3 = 0$, ① with respect to x, gives

$$\cos x \frac{\mathrm{d}^2y}{\mathrm{d}x^2} - \cancel{\sin x \frac{\mathrm{d}y}{\mathrm{d}x}} + y\cos x + \cancel{\sin x \frac{\mathrm{d}y}{\mathrm{d}x}} + 6y^2 \frac{\mathrm{d}y}{\mathrm{d}x} = 0, \qquad ②$$

Differentiating again

$$\cos x \frac{\mathrm{d}^3y}{\mathrm{d}x^3} - \sin x \frac{\mathrm{d}^2y}{\mathrm{d}x^2} - y\sin x + \cos x \frac{\mathrm{d}y}{\mathrm{d}x} + 6y^2 \frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 12y\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 = 0, \qquad ③$$

Substituting $x_0 = 0$, $y_0 = 1$ into ① gives $\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_0 + 2(1) = 0$, so $\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_0 = -2$

Substituting $x_0 = 0$, $y_0 = 1$, $\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_0 = -2$ into ② gives

$$\left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)_0 + 1 + 6(1)(-2) = 0, \text{ so } \left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)_0 = 11$$

Substituting $x = 0$, $y = 1$, $\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_0 = -2$, $\left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)_0 = 11$ into ③ gives

$$\left(\frac{\mathrm{d}^3y}{\mathrm{d}x^3}\right)_0 + (1)(-2) + 6(1)(11) + 12(1)(-2)^2, \text{ so } \left(\frac{\mathrm{d}^3y}{\mathrm{d}x^3}\right)_0 = -112$$

Substituting these values into the form of Taylor's series form **ii**,

$$\text{gives } y = 1 + (-2)x + \frac{11}{2!}x^2 + \frac{(-112)}{3!}x^3 + \ldots$$

$$y = 1 - 2x + \frac{11}{2}x^2 - \frac{56}{3}x^3 + \ldots$$

Ignoring terms in x^4 and higher powers, $y \approx 1 - 2x + \frac{11}{2}x^2 - \frac{56}{3}x^3$.

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Exercise F, Question 1

Question:

Using Taylor's series show that the first three terms in the expansion of $\left(x-\frac{\pi}{4}\right)\cot x$, in powers of $\left(x-\frac{\pi}{4}\right)$, are $\left(x-\frac{\pi}{4}\right)-2\left(x-\frac{\pi}{4}\right)^2+2\left(x-\frac{\pi}{4}\right)^3$.

Solution:

$f(x)=\cot x$ and $a=\frac{\pi}{4}$.

$f(x)=\cot x$ so $f\left(\frac{\pi}{4}\right)=1$

$f'(x)=-\text{cosec}^2 x$ $f'\left(\frac{\pi}{4}\right)=-2$

$f''(x)=-2\,\text{cosec}\,x(-\text{cosec}\,x\cot x)$

$=2\,\text{cosec}\,2x\cot x$ $f''\left(\frac{\pi}{4}\right)=4$

Substituting in the form of Taylor

$$f(x)=f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\ldots$$

$$\cot x=1+(-2)\left(x-\frac{\pi}{4}\right)+\frac{4}{2!}\left(x-\frac{\pi}{4}\right)^2+\ldots$$

So $\left(x-\frac{\pi}{4}\right)\cot x=\left(x-\frac{\pi}{4}\right)-2\left(x-\frac{\pi}{4}\right)^2+2\left(x-\frac{\pi}{4}\right)^3+\ldots$

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Exercise F, Question 2

Question:

a For the functions $f(x) = \ln(1 + e^x)$, find the values of $f'(0)$ and $f''(0)$.

b Show that $f'''(0) = 0$.

c Find the series expansion of $\ln(1 + e^x)$, in ascending powers of x up to and including the term in x^2, and state the range of values of x for which the expansion is valid.

Solution:

a $f(x) = \ln(1 + e^x)$ so $f(0) = \ln 2$

$f'(x) = \dfrac{e^x}{1 + e^x}$ $\boxed{= 1 - \dfrac{1}{1 + e^x} = 1 - (1 + e^x)^{-1}}$ $f'(0) = \dfrac{1}{2}$

So $f''(x) = \dfrac{e^x}{(1 + e^x)^2}$ or use the quotient rule $f''(0) = \dfrac{1}{4}$

b $f'''(x) = \dfrac{(1 + e^x)^2 e^x - e^x 2(1 + e^x)e^x}{(1 + e^x)^4}$ Use the quotient rule and chain rule.

$= \dfrac{(1 + e^x)e^x\{(1 + e^x) - 2e^x\}}{(1 + e^x)^4} = \dfrac{e^x(1 - e^x)}{(1 + e^x)^3}$ $f'''(0) = 0$

c Using Maclaurin's expansion:

$$\ln(1 + e^x) = \ln 2 + \frac{x}{2} + \frac{x^2}{8} + \ldots$$

The expansion is valid for $-1 < e^x \leqslant 1 \Rightarrow 0, e^x \leqslant 1$ so for $x \leqslant 0$.

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Exercise F, Question 3

Question:

a Write down the series for $\cos 4x$ in ascending powers of x, up to and including the term in x^6.

b Hence, or otherwise, show that the first three non-zero terms in the series expansion of $\sin^2 2x$ are $4x^2 - \frac{16}{3}x^4 + \frac{128}{45}x^6$.

Solution:

a $\cos 4x = 1 - \dfrac{(4x)^2}{2!} + \dfrac{(4x)^4}{4!} - \dfrac{(4x)^6}{6!} + \ldots$

$= 1 - 8x^2 + \dfrac{32}{3}x^4 - \dfrac{256}{45}x^6 + \ldots$

b $\cos 4x = 1 - 2\sin^2 2x$,

so $2\sin^2 2x = 1 - \cos 4x = 8x^2 - \dfrac{32}{3}x^4 + \dfrac{256}{45}x^6 + \ldots$

$\sin^2 2x = 4x^2 - \dfrac{16}{3}x^4 + \dfrac{128}{45}x^6 + \ldots$

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Exercise F, Question 4

Question:

Given that terms in x^5 and higher power may be neglected, use the series for e^x and $\cos x$, to show that $e^{\cos x} \approx e\left(1 - \frac{x^2}{2} + \frac{x^4}{6}\right)$.

Solution:

Using $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \ldots$ and $\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \ldots$

$$e^{\cos x} = e^{\left(1 - \frac{x^2}{2} + \frac{x^4}{24}\right)} = e \times e^{-\frac{x^2}{2}} \times e^{\frac{x^4}{24}}$$

$$= e\left\{1 + \left(-\frac{x^2}{2}\right) + \frac{1}{2}\left(-\frac{x^2}{2}\right)^2 + \ldots\right\}\left\{1 + \frac{x^4}{24} + \ldots\right\} \quad \text{no other terms required}$$

$$= e\left\{1 - \frac{x^2}{2} + \frac{x^4}{8} + \ldots\right\}\left\{1 + \frac{x^4}{24} + \ldots\right\}$$

$$= e\left\{1 - \frac{x^2}{2} + \frac{x^4}{8} + \frac{x^4}{24} + \ldots\right\} = e\left\{1 - \frac{x^2}{2} + \frac{x^4}{6} + \ldots\right\}$$

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Exercise F, Question 5

Question:

$\frac{dy}{dx} = 2 + x + \sin y$ with $y = 0$ at $x = 0$.

Use the Taylor series method to obtain y as a series in ascending powers of x up to and including the term in x^3, and hence obtain an approximate value for y at $x = 0.1$.

Solution:

$\frac{dy}{dx} = 2 + x + \sin y$ and $x_0 = 0, y_0 = 0$ ① so $\left(\frac{dy}{dx}\right)_0 = 2$

Differentiating ① gives $\frac{d^2y}{dx^2} = 1 + \cos y \frac{dy}{dx}$ ②

Substituting $x_0 = 0, y_0 = 0, \left(\frac{dy}{dx}\right)_0 = 2$ into ② gives $\left(\frac{d^2y}{dx^2}\right)_0 = 3$

Differentiating ② gives $\frac{d^3y}{dx^3} = \cos y \frac{d^2y}{dx^2} - \sin y \left(\frac{dy}{dx}\right)^2$ ③

Substituting $y_0 = 0, \left(\frac{dy}{dx}\right)_0 = 2, \left(\frac{d^2y}{dx^2}\right)_0 = 3$ into ③ gives $\left(\frac{d^3y}{dx^3}\right)_0 = 3$

Substituting found values into $y = y_0 + x\left(\frac{dy}{dx}\right)_0 + \frac{x^2}{2!}\left(\frac{d^2y}{dx^2}\right)_0 + \frac{x^3}{3!}\left(\frac{d^3y}{dx^3}\right)_0 + \ldots$

$$y = 2x + \tfrac{3}{2}x^2 + \tfrac{1}{2}x^3 + \ldots$$

At $x = 0.1$, $y \approx 2(0.1) + \frac{3}{2}(0.1)^2 + \frac{1}{2}(0.1)^3 = 0.2155$

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Exercise F, Question 6

Question:

Given that $|2x| < 1$, find the first two non-zero terms in the expansion of $\ln[(1 + x)^2(1 - 2x)]$ in a series of ascending powers of x.

Solution:

$$\ln[(1 + x)^2(1 - 2x)] = 2\ln(1 + x) + \ln(1 - 2x)$$
$$= 2\left\{x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots\right\} + \left\{(-2x) - \frac{(-2x)^2}{2} + \frac{(-2x)^3}{3} - \frac{(-2x)^4}{4} + \dots\right\}$$
$$= 2x - x^2 + \tfrac{2}{3}x^3 - \tfrac{1}{2}x^4 - 2x - 2x^2 - \tfrac{8}{3}x^3 - 4x^4 + \dots$$
$$= -3x^2 - 2x^3 - \dots$$

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Exercise F, Question 7

Question:

Find the solution, in ascending powers of x up to and including the term in x^3, of the differential equation $\frac{d^2y}{dx^2} - (x + 2)\frac{dy}{dx} + 3y = 0$, given that at $x = 0$, $y = 2$ and $\frac{dy}{dx} = 4$.

Solution:

$$\frac{d^2y}{dx^2} - (x + 2)\frac{dy}{dx} + 3y = 0 \qquad ①$$

Differentiating ① gives $\frac{d^3y}{dx^2} - (x + 2)\frac{d^2y}{dx^2} - \frac{dy}{dx} + 3\frac{dy}{dx} = 0 \qquad ②$

Substituting initial data in ① gives $\left(\frac{d^2y}{dx^2}\right)_0 = 2$

Substituting known data in ② gives $\left(\frac{d^3y}{dx^3}\right)_0 = -4$

$$\text{So } \quad y = 2 + 4x + \frac{2x^2}{2!} - \frac{4x^3}{3!} + \ldots$$

$$= 2 + 4x + x^2 - \tfrac{2}{3}x^3$$

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Exercise F, Question 8

Question:

Use differentiation and the Maclaurin expansion, to express $\ln(\sec x + \tan x)$ as a series in ascending powers of x up to and including the term in x^3.

Solution:

$f(x) = \ln(\sec x + \tan x)$ $\quad f(0) = \ln 1 = 0$

$f'(x) = \dfrac{\sec x \tan x + \sec^2 x}{\sec x + \tan x} = \dfrac{\sec x\,(\tan x + \sec x)}{\sec x + \tan x} = \sec x$ $\quad f'(0) = 1$

$f''(x) = \sec x \tan x$ $\quad f''(0) = 0$

$f'''(x) = \sec x \sec^2 x + \sec x \tan x \tan x$ $\quad f'''(0) = 1$

Substituting into Maclaurin's expansion gives $y = x + \dfrac{x^3}{6} + \ldots$

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Exercise F, Question 9

Question:

Show that the results of differentiating the following series expansions

$$e^x = 1 + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^r}{r!} + \ldots,$$

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \ldots + \frac{(-1)^r}{(2r+1)!}x^{2r+1} + \ldots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots + (-1)^r\frac{x^{2r}}{(2r)!} + \ldots$$

agree with the results

a $\frac{d}{dx}(e^x) = e^x$ **b** $\frac{d}{dx}(\sin x) = \cos x$ **c** $\frac{d}{dx}(\cos x) = -\sin x$

Solution:

a $$\frac{d}{dx}(e^x) = \frac{d}{dx}\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots + \frac{x^r}{r!} + \frac{x^{r+1}}{(r+1)!} + \ldots\right)$$

$$= 1 + x + \frac{2x}{2!} + \frac{3x^2}{3!} + \frac{4x^3}{4!} + \ldots + \frac{(r+1)x^r}{(r+1)!} + \ldots$$

$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^r}{r!} + \ldots$$

$$= e^x$$

b $$\frac{d}{dx}(\sin x) = \frac{d}{dx}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^r\frac{x^{2r+1}}{(2r+1)!} + \ldots\right)$$

$$= 1 - \frac{3x^2}{3!} + \frac{5x^4}{5!} - \ldots + (-1)^r\frac{(2r+1)x^{2r}}{(2r+1)!} + \ldots$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots + (-1)^r\frac{x^{2r}}{(2r)!} + \ldots = \cos x$$

c $$\frac{d}{dx}(\cos x) = \frac{d}{dx}\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots + (-1)^r\frac{x^{2r}}{(2r)!} + (-1)^{r+1}\frac{x^{2r+2}}{(2r+2)!} + \ldots\right)$$

$$= \left(-\frac{2x}{2!} + \frac{4x^3}{4!} - \frac{6x^5}{6!} + \ldots + (-1)^r\frac{2rx^{2r-1}}{(2r)!} + (-1)^{r+1}\frac{(2r+2)x^{2r+1}}{(2r+2)!} + \ldots\right)$$

$$= -x + \frac{x^3}{3!} - \frac{x^5}{5!} + \ldots + (-1)^{r+1}\frac{x^{2r+1}}{(2r+1)!} + \ldots$$

$$= -\left(x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \ldots + \frac{(-1)^r}{(2r+1)!}x^{2r+1} + \ldots\right) = -\sin x$$

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Exercise F, Question 10

Question:

$\frac{d^2y}{dx^2} + y\frac{dy}{dx} = x$, at $x = 1$, $y = 0$, $\frac{dy}{dx} = 2$.

Find a series solution of the differential equation, in ascending powers of $(x - 1)$ up to and including the term in $(x - 1)^3$.

Solution:

$$\frac{d^2y}{dx^2} + y\frac{dy}{dx} = x \qquad ①$$

Differentiating $\frac{d^2y}{dx^2} + y\frac{dy}{dx} = x$, gives $\frac{d^3y}{dx^3} + y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 1 \qquad ②$

Substituting initial values into ① gives $\left(\frac{d^2y}{dx^2}\right)_1 = 1$

Substituting $\left(\frac{dy}{dx}\right)_1 = 2$ and $\left(\frac{d^2y}{dx^2}\right)_1 = 1$ into ② gives $\left(\frac{d^3y}{dx^3}\right) = -3$.

Using Taylor's expansion in the form with $x_0 = 1$

$$y = 0 + 2(x-1) + \frac{(1)}{2!}(x-1)^2 + \frac{(-3)}{3!}(x-1)^3 + \ldots$$

$$= 2(x-1) + \frac{1}{2}(x-1)^2 - \frac{1}{2}(x-1)^3 + \ldots$$

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Exercise F, Question 11

Question:

a Given that $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots$, show that $\sec x = 1 + \frac{x^2}{2} + \frac{5}{24}x^4 + \ldots$.

b Using the result found in **a**, and given that $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$, find the first three non-zero terms in the series expansion, in ascending powers of x, for $\tan x$.

Solution:

a You can write $\cos x = 1 - \left(\frac{x^2}{2} - \frac{x^4}{24} + \ldots\right)$; it is not necessary to have higher powers

$$\sec x = \frac{1}{\cos x} = \frac{1}{1 - \left(\frac{x^2}{2} - \frac{x^4}{24} + \ldots\right)} = \left\{1 - \left(\frac{x^2}{2} - \frac{x^4}{24} + \ldots\right)\right\}^{-1}$$

Using the binomial expansion but only requiring powers up to x^4

$$\sec x = 1 + (-1)\left\{-\left(\frac{x^2}{2} - \frac{x^4}{24}\right)\right\} + \frac{(-1)(-2)}{2!}\left\{-\left(\frac{x^2}{2} - \frac{x^4}{24}\right)\right\}^2 + \ldots$$

$$= 1 + \left(\frac{x^2}{2} - \frac{x^4}{24}\right) + \frac{x^4}{4} + \text{higher powers of } x$$

$$= 1 + \frac{x^2}{2} + \frac{5}{24}x^4 + \ldots$$

b $\tan x = \frac{\sin x}{\cos x} = \sin x \times \sec x$

$$= \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\right)\left(1 + \frac{x^2}{2} + \frac{5}{24}x^4 + \ldots\right)$$

$$= x + \frac{x^3}{2} + \frac{5}{24}x^5 - \frac{x^3}{3!} - \frac{1}{2(3!)}x^5 + \frac{x^5}{5!} + \ldots$$

$$= x + \left(\frac{1}{2} - \frac{1}{6}\right)x^3 + \left(\frac{5}{24} - \frac{1}{12} + \frac{1}{120}\right)x^5 + \ldots$$

$$= x + \frac{x^3}{3} + \frac{16}{120}x^5 + \ldots$$

$$= x + \frac{x^3}{3} + \frac{2}{15}x^5 + \ldots$$

Solutionbank FP2

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Exercise F, Question 12

Question:

By using the series expansions of e^x and $\cos x$, or otherwise, find the expansion of $e^x \cos 3x$ in ascending powers of x up to and including the term in x^3.

Solution:

Using $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$ and $\cos 3x = 1 - \frac{(3x)^2}{2!} + \ldots$

$$e^x \cos 3x = \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \ldots\right)\left(1 - \frac{9x^2}{2} + \ldots\right)$$

$$= \left\{1 + x + \left(\frac{x^2}{2} - \frac{9x^2}{2}\right) + \left(\frac{x^3}{6} - \frac{9x^3}{2}\right) + \ldots\right\}$$

$$= 1 + x - 4x^2 - \frac{13}{3}x^3 + \ldots$$

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Exercise F, Question 13

Question:

$\frac{d^2y}{dx^2} + x^2\frac{dy}{dx} + y = 0$ with $y = 2$ at $x = 0$ and $\frac{dy}{dx} = 1$ at $x = 0$.

a Use the Taylor series method to express y as a polynomial in x up to and including the term in x^3.

b Show that at $x = 0$, $\frac{d^4y}{dx^4} = 0$.

Solution:

a Differentiating $\frac{d^2y}{dx^2} + x^2\frac{dy}{dx} + y = 0$ ① with respect to x, gives

$$\frac{d^3y}{dx^3} + 2x\frac{dy}{dx} + x^2\frac{d^2y}{dx^2} + x^2\frac{dy}{dx} = 0 \qquad ②$$

Substituting given data $x_0 = 0$, $y_0 = 2$ and $\left(\frac{dy}{dx}\right)_0 = 1$ into ① gives $\left(\frac{d^2y}{dx^2}\right)_0 = -2$

Substituting $x_0 = 0$, $\left(\frac{dy}{dx}\right)_0 = 1$ and $\left(\frac{d^2y}{dx^2}\right)_0 = -2$ into ② gives $\left(\frac{d^3y}{dx^3}\right)_0 = -1$

So using Taylor series $y = y_0 + x\left(\frac{dy}{dx}\right)_0 + \frac{x^2}{2!}\left(\frac{d^2y}{dx^2}\right)_0 + \frac{x^3}{3!}\left(\frac{d^3y}{dx^3}\right)_0 + \ldots$

$$y = 2 + x - x^2 - \frac{x^3}{6} + \ldots$$

b Differentiating ② with respect to x gives

$$\frac{d^4y}{dx^4} + 2x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + x^2\frac{d^3y}{dx^3} + 2x\frac{d^2y}{dx^2} + \frac{d^2y}{dx^2} = 0 \qquad ③$$

Substituting $x = 0$, $\left(\frac{dy}{dx}\right)_0 = 1$, $\left(\frac{d^2y}{dx^2}\right)_0 = -2$ and $\left(\frac{d^3y}{dx^3}\right)_0 = -1$ into ③ gives,

at $x = 0$, $\frac{d^4y}{dx^4} + 2(1) + (-2) = 0$, so $\frac{d^4y}{dx^4} = 0$

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Exercise F, Question 14

Question:

Find the first three derivatives of $(1 + x)^2 \ln(1 + x)$. Hence, or otherwise, find the expansion of $(1 + x)^2 \ln(1 + x)$ in ascending powers of x up to and including the term in x^3.

Solution:

$f(x) = (1 + x)^2 \ln(1 + x)$.

$f'(x) = (1 + x)^2 \frac{1}{1 + x} + 2(1 + x) \ln(1 + x) = (1 + x)\{1 + 2 \ln(1 + x)\}$

$f''(x) = (1 + x)\left(\frac{2}{1 + x}\right) + \{1 + 2 \ln(1 + x)\} = 3 + 2 \ln(1 + x)$

$f'''(x) = \left(\frac{2}{1 + x}\right)$

$f(0) = 0,\ f'(0) = 1,\ f''(0) = 3,\ f'''(0) = 2$

Using Maclaurin's expansion

$$(1 + x)^2 \ln(1 + x) = 0 + (1)x + \frac{3}{2!}x^2 + \frac{2}{3!}x^3 + \ldots$$

$$= x + \frac{3}{2}x^2 + \frac{1}{3}x^3 + \ldots$$

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Exercise F, Question 15

Question:

a Expand $\ln(1 + \sin x)$ in ascending powers of x up to and including the term in x^4.

b Hence find an approximation for $\int_0^{\frac{\pi}{6}} \ln(1 + \sin x)\,dx$ giving your answer to 3 decimal places.

Solution:

a $\ln(1 + \sin x) = \ln\left\{1 + \left(x - \frac{x^3}{3!} + \dots\right)\right\}$

$$= \left(x - \frac{x^3}{3!} + \dots\right) - \frac{1}{2}\left(x - \frac{x^3}{3!} + \dots\right)^2 + \frac{1}{3}\left(x - \frac{x^3}{3!} + \dots\right)^3 - \frac{1}{4}\left(x - \frac{x^3}{3!} + \dots\right)^4 + \dots$$

$$= \left(x - \frac{x^3}{6} + \dots\right) - \frac{1}{2}\left(x^2 - \frac{x^4}{3} + \dots\right) + \frac{1}{3}(x^3 + \dots) - \frac{1}{4}(x^4 + \dots) \quad \text{no other terms necessary}$$

$$= x - \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{12} + \dots$$

b $\int_0^{\frac{\pi}{6}} \ln(1 + \sin x)dx \approx \int_0^{\frac{\pi}{6}} \left(x - \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{12}\right)dx$

$$\approx \left[\frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{60}\right]_0^{\frac{\pi}{6}} = \frac{\pi^2}{72} - \frac{\pi^3}{1296} + \frac{\pi^4}{31104} - \frac{\pi^5}{466560} = 0.116 \text{ (3 d.p.)}$$

Solutionbank FP2
Edexcel AS and A Level Modular Mathematics

Exercise F, Question 16

Question:

a Using the first two terms, $x + \frac{x^3}{3}$, in the expansion of $\tan x$, show that

$e^{\tan x} = 1 + x + \frac{x^2}{2} + \frac{x^3}{2} + \dots$

b Deduce the first four terms in the expansion of $e^{-\tan x}$, in ascending powers of x.

Solution:

a $f(x) = e^{\tan x} = e^{x + \frac{x^3}{3} + \dots} = e^x \times e^{\frac{x^3}{3}}$ (As only terms up to x^3 are required, only first two terms of $\tan x$ are needed.)

$= \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots\right)\left(1 + \frac{x^3}{3} + \dots\right)$ no other terms required.

$= \left(1 + \frac{x^3}{3} + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots\right)$

$= 1 + x + \frac{x^2}{2} + \frac{x^3}{2} + \dots$

b $e^{-\tan x} = e^{\tan(-x)}$, so replacing x by $-x$ in **a** gives

$e^{-\tan x} = 1 - x + \frac{x^2}{2} - \frac{x^3}{2} + \dots$

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Exercise F, Question 17

Question:

$$y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 + y = 0.$$

a Find an expression for $\frac{d^3y}{dx^3}$.

Given that $y = 1$ and $\frac{dy}{dx} = 1$ at $x = 0$,

b find the series solution for y, in ascending powers of x, up to an including the term in x^3.

c Comment on whether it would be sensible to use your series solution to give estimates for y at $x = 0.2$ and at $x = 50$.

Solution:

a Differentiating the given differential equation with respect to x gives

$$y\frac{d^3y}{dx^3} + \frac{dy}{dx}\frac{d^2y}{dx^2} + 2\frac{dy}{dx}\frac{d^2y}{dx^2} + \frac{dy}{dx} = 0$$

So $\frac{d^3y}{dx^3} = -\frac{1}{y}\left\{\frac{dy}{dx}\left(3\frac{d^2y}{dx^2} + 1\right)\right\}$

b Given that $y_0 = 1$, $\left(\frac{dy}{dx}\right)_0 = 1$ at $x = 0$,

$\left(\frac{d^2y}{dx^2}\right)_0 + (1)^2 + (1) = 0$, so $\left(\frac{d^2y}{dx^2}\right)_0 = -2$,

And $\left(\frac{d^3y}{dx^3}\right)_0 = -\frac{1}{(1)}\{(1)[3(-2) + 1]\}$, so $\left(\frac{d^3y}{dx^3}\right)_0 = 5$

So $y = 1 + (1)x + \frac{(-2)}{2!}x^2 + \frac{5}{3!}x^3 + \ldots = 1 + x - x^2 + \frac{5x^3}{6} + \ldots$

c The approximation is best for small values of x (close to 0): $x = 0.2$, therefore, would be acceptable, but not $x = 50$.

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Exercise F, Question 18

Question:

a Using the Maclaurin expansion, and differentiation, show that $\ln \cos x = -\frac{x^2}{2} - \frac{x^4}{12} + \ldots$

b Using $\cos x = 2\cos^2\left(\frac{x}{2}\right) - 1$, and the result in **a**, show that

$\ln(1 + \cos x) = \ln 2 - \frac{x^2}{4} - \frac{x^4}{96} + \ldots$

Solution:

a $f(x) = \ln \cos x$ $\quad f(0) = 0$

$f'(x) = \frac{-\sin x}{\cos x} = -\tan x$ $\quad f'(0) = 0$

$f''(x) = -\sec^2 x$ $\quad f''(0) = -1$

$f'''(x) = -2\sec^2 x \tan x$ $\quad f'''(0) = 0$

$f''''(x) = -2\sec^4 x - 4\sec^2 x \tan^2 x$ $\quad f''''(0) = -2$

Substituting into Maclaurin:

$$\ln \cos x = (-1)\frac{x^2}{2!} + (-2)\frac{x^4}{4!} + \ldots = -\frac{x^2}{2} - \frac{x^4}{12} - \ldots$$

b Using $1 + \cos x = 2\cos^2\left(\frac{x}{2}\right)$, $\ln(1 + \cos x) = \ln 2\cos^2\left(\frac{x}{2}\right) = \ln 2 + 2\ln\cos\left(\frac{x}{2}\right)$

so $\ln(1 + \cos x) = \ln 2 + 2\left\{-\frac{1}{2}\left(\frac{x}{2}\right)^2 - \frac{1}{12}\left(\frac{x}{2}\right)^4 - \ldots\right\} = \ln 2 - \frac{x^2}{4} - \frac{x^4}{96} - \ldots$

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Exercise F, Question 19

Question:

a Show that $3^x = e^{x \ln 3}$.

b Hence find the first four terms in the series expansion of 3^x.

c Using your result in **b**, with a suitable value of x, find an approximation for $\sqrt{3}$, giving your answer to 3 significant figures.

Solution:

a Let $y = 3^x$, then $\ln y = \ln 3^x = x \ln 3 \Rightarrow y = e^{x \ln 3}$ so $3^x = e^{x \ln 3}$

b $3^x = e^{x \ln 3} = 1 + (x \ln 3) + \dfrac{(x \ln 3)^2}{2!} + \dfrac{(x \ln 3)^3}{3!} + \ldots$

$= 1 + x \ln 3 + \dfrac{x^2 (\ln 3)^2}{2} + \dfrac{x^3 (\ln 3)^3}{6} + \ldots$

c Put $x = \dfrac{1}{2}$: $\sqrt{3} \approx 1 + \dfrac{\ln 3}{2} + \dfrac{(\ln 3)^2}{8} + \dfrac{(\ln 3)^3}{48} = 1.73$ (3 s.f.)

Solutionbank FP2
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Exercise F, Question 20

Question:

Given that $f(x) = \operatorname{cosec} x$,

a show that

i $f''(x) = \operatorname{cosec} x(2\operatorname{cosec}^2 x - 1)$

ii $f'''(x) = -\operatorname{cosec} x \cot x(6\operatorname{cosec}^2 x - 1)$

b Find the Taylor expansion of cosec x in ascending powers of $\left(x - \frac{\pi}{4}\right)$ up to and including the term $\left(x - \frac{\pi}{4}\right)^3$.

Solution:

a $f(x) = \operatorname{cosec} x$

$f'(x) = -\operatorname{cosec} x \cot x$

i $f''(x) = -\operatorname{cosec} x(-\operatorname{cosec}^2 x) + \cot x(\operatorname{cosec} x \cot x)$

$= \operatorname{cosec} x(\operatorname{cosec}^2 x + \cot^2 x)$

$= \operatorname{cosec} x\{\operatorname{cosec}^2 x + (\operatorname{cosec}^2 x - 1)\}$

$= \operatorname{cosec} x\{2\operatorname{cosec}^2 x - 1\}$

ii $f'''(x) = \operatorname{cosec} x(-4\operatorname{cosec}^2 x \cot x) - \operatorname{cosec} x \cot x(2\operatorname{cosec}^2 x - 1)$

$= -\operatorname{cosec} x \cot x(6\operatorname{cosec}^2 x - 1)$

b $f\left(\frac{\pi}{4}\right) = \sqrt{2}$, $f'\left(\frac{\pi}{4}\right) = -\sqrt{2}$, $f''\left(\frac{\pi}{4}\right) = 3\sqrt{2}$, $f'''\left(\frac{\pi}{4}\right) = -11\sqrt{2}$.

Substituting all values into $y = y_0 + (x - x_0)\left(\frac{dy}{dx}\right)_{x_0} + \frac{(x - x_0)^2}{2!}\left(\frac{d^2y}{dx^2}\right)_{x_0} + \dots$ with $x_0 = \frac{\pi}{4}$

$$\operatorname{cosec} x = \sqrt{2} + (-\sqrt{2})\left(x - \frac{\pi}{4}\right) + \frac{(3\sqrt{2})}{2!}\left(x - \frac{\pi}{4}\right)^2 + \frac{(-11\sqrt{2})}{3!}\left(x - \frac{\pi}{4}\right)^3 + \dots$$

$$= \sqrt{2} - \sqrt{2}\left(x - \frac{\pi}{4}\right) + \frac{3\sqrt{2}}{2}\left(x - \frac{\pi}{4}\right)^2 - \frac{11\sqrt{2}}{6}\left(x - \frac{\pi}{4}\right)^3 + \dots$$

Solutionbank FP2
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Exercise A, Question 1

Question:

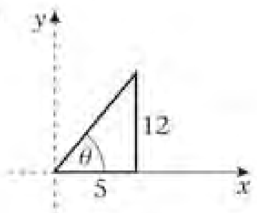
Find the polar coordinates of the following points

a $(5,12)$ **b** $(-5, 12)$ **c** $(-5, -12)$

d $(2, -3)$ **e** $(\sqrt{3}, -1)$

Solution:

a

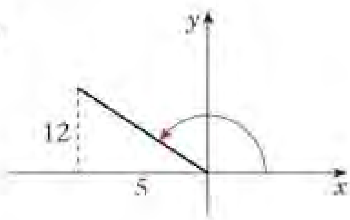


$\arctan\left(\frac{12}{5}\right) = 67.4°$

$r = \sqrt{5^2 + 12^2} = 13$

$\therefore$ point is $(13, 67.4°)$

b

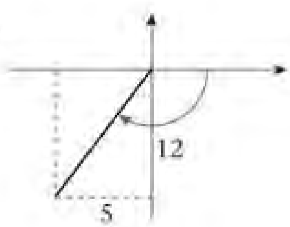


$r = \sqrt{(-5)^2 + 12^2} = 13$

$\theta = 180 - \arctan\left(\frac{12}{5}\right) = 112.6°$

$\therefore$ point is $(13, 112.6°)$

c



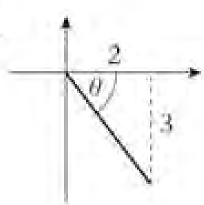
$\theta = -\left(180 - \arctan\frac{12}{5}\right)$

$= -112.6°$

$r = \sqrt{(-5)^2 + (-12)^2} = 13$

$\therefore$ point is $(13, -112.6°)$

d

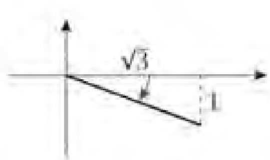


$\theta = -\arctan\frac{3}{2} = -56.3°$

$r = \sqrt{2^2 + (-3)^2} = \sqrt{13}$

$\therefore$ point is $(\sqrt{13}, -56.3°)$

e



$\theta = -\arctan\frac{1}{\sqrt{3}} = -30°$

$r = \sqrt{\sqrt{3}^2 + (-1)^2} = \sqrt{4} = 2$

$\therefore$ point is $(2, -30°)$

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Exercise A, Question 2

Question:

Find Cartesian coordinates of the following points. Angles are measured in radians.

a $\left(6, \frac{\pi}{6}\right)$ **b** $\left(6, -\frac{\pi}{6}\right)$ **c** $\left(6, \frac{3\pi}{4}\right)$

d $\left(10, \frac{5\pi}{4}\right)$ **e** $(2, \pi)$

Solution:

a $x = 6\cos\left(\frac{\pi}{6}\right) = \frac{6\sqrt{3}}{2} = 3\sqrt{3}$

$y = 6\sin\frac{\pi}{6} = 3$ $\therefore$ point is $(3\sqrt{3}, 3)$

b $x = 6\cos\left(-\frac{\pi}{6}\right) = \frac{6\sqrt{3}}{2} = 3\sqrt{3}$

$y = 6\sin\left(-\frac{\pi}{6}\right) = -3$ $\therefore$ point is $(3\sqrt{3}, -3)$

c $x = 6\cos\left(\frac{3\pi}{4}\right) = -\frac{6}{\sqrt{2}}$ or $-3\sqrt{2}$

$y = 6\sin\left(\frac{3\pi}{4}\right) = \frac{6}{\sqrt{2}} = 3\sqrt{2}$ $\therefore$ point is $(-3\sqrt{2}, 3\sqrt{2})$

d $x = 10\cos\left(\frac{5\pi}{4}\right) = -\frac{10}{\sqrt{2}} = -5\sqrt{2}$

$y = 10\sin\left(\frac{5\pi}{4}\right) = \frac{-10}{\sqrt{2}} = -5\sqrt{2}$ $\therefore$ point is $(-5\sqrt{2}, -5\sqrt{2})$

e $x = 2\cos(\pi) = -2$

$y = 2\sin(\pi) = 0$ $\therefore$ point is $(-2, 0)$

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Exercise B, Question 1

Question:

Find Cartesian equations for the following curves where a is a positive constant.

a $r = 2$ **b** $r = 3 \sec \theta$ **c** $r = 5 \operatorname{cosec} \theta$

Solution:

a $r = 2$ is $x^2 + y^2 = 4$

b $r = 3 \sec \theta$

$\Rightarrow \quad r \cos \theta = 3$ i.e. $x = 3$

c $r = 5 \operatorname{cosec} \theta$

$\Rightarrow \quad r \sin \theta = 5$ i.e. $y = 5$

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Exercise B, Question 2

Question:

Find Cartesian equations for the following curves where a is a positive constant.

a $r = 4a \tan \theta \sec \theta$ **b** $r = 2a \cos \theta$ **c** $r = 3a \sin \theta$

Solution:

a
$$r = 4a \tan \theta \sec \theta$$
$$r = \frac{4a \sin \theta}{\cos^2 \theta}$$
$$r \cos^2 \theta = 4a \sin \theta$$
Multiply by r.
$$r^2 \cos^2 \theta = 4ar \sin \theta$$
$$\therefore \quad x^2 = 4ay \quad \text{or} \quad y = \frac{x^2}{4a}$$

b
$$r = 2a \cos \theta$$
$$r^2 = 2ar \cos \theta$$
$$\therefore \quad x^2 + y^2 = 2ax \quad \text{or} \quad (x - a)^2 + y^2 = a^2$$

c
$$r = 3a \sin \theta$$
Multiply by r.
$$r^2 = 3ar \sin \theta$$
$$x^2 + y^2 = 3ay \quad \text{or} \quad x^2 + \left(y - \frac{3a}{2}\right)^2 = \frac{9a^2}{4}$$

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Exercise B, Question 3

Question:

Find Cartesian equations for the following curves where a is a positive constant.

a $r = 4(1 - \cos 2\theta)$ **b** $r = 2\cos^2\theta$ **c** $r^2 = 1 + \tan^2\theta$

Solution:

a
$$r = 4(1 - \cos 2\theta)$$
$$r = 4 \times 2\sin^2\theta$$
$$r^3 = 8r^2\sin^2\theta$$
$$\therefore \ (x^2 + y^2)^{\frac{3}{2}} = 8y^2$$

Use $\cos 2\theta = 1 - 2\sin^2\theta$
$\therefore \ 2\sin^2\theta = 1 - \cos 2\theta$

$\times r^2$

b
$$r = 2\cos^2\theta$$
$$r^3 = 2r^2\cos^2\theta$$
$$(x^2 + y^2)^{\frac{3}{2}} = 2x^2$$

$\times r^2$

c
$$r^2 = 1 + \tan^2\theta$$
$$\therefore \ r^2 = \sec^2\theta$$
$$\therefore \ r^2\cos^2\theta = 1$$
i.e. $x^2 = 1$ or $x = \pm 1$

Use $\sec^2\theta = 1 + \tan^2\theta$.

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Exercise B, Question 4

Question:

Find polar equations for the following curves:

a $x^2 + y^2 = 16$ **b** $xy = 4$ **c** $(x^2 + y^2)^2 = 2xy$

Solution:

a $x^2 + y^2 = 16$

$\Rightarrow \quad r^2 = 16 \quad$ or $\quad r = 4$

b
$$\begin{aligned} xy &= 4 \\ \Rightarrow \quad r\cos\theta\, r\sin\theta &= 4 \\ r^2 &= \frac{4}{\cos\theta\sin\theta} = \frac{8}{2\cos\theta\sin\theta} \\ \text{i.e.} \quad r^2 &= 8\operatorname{cosec}2\theta \end{aligned}$$

c
$$\begin{aligned} (x^2 + y^2)^2 &= 2xy \\ \Rightarrow \quad (r^2)^2 &= 2r\cos\theta\, r\sin\theta \\ r^4 &= 2r^2\cos\theta\sin\theta \\ r^2 &= \sin 2\theta \end{aligned}$$

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Exercise B, Question 5

Question:

Find polar equations for the following curves:

a $x^2 + y^2 - 2x = 0$ **b** $(x + y)^2 = 4$ **c** $x - y = 3$

Solution:

a
$$x^2 + y^2 - 2x = 0$$
$$\Rightarrow \quad r^2 - 2r\cos\theta = 0$$
$$r^2 = 2r\cos\theta$$
$$r = 2\cos\theta$$

b
$$(x + y)^2 = 4$$
$$\Rightarrow \quad x^2 + y^2 + 2xy = 4$$
$$\Rightarrow \quad r^2 + 2r\cos\theta\, r\sin\theta = 4$$
$$\Rightarrow \quad r^2(1 + \sin 2\theta) = 4$$
$$r^2 = \frac{4}{1 + \sin 2\theta}$$

c
$$x - y = 3$$
$$r\cos\theta - r\sin\theta = 3$$
$$r(\cos\theta - \sin\theta) = 3$$
$$r\left(\frac{1}{\sqrt{2}}\cos\theta - \frac{1}{\sqrt{2}}\sin\theta\right) = \frac{3}{\sqrt{2}}$$
$$r\cos\left(\theta + \frac{\pi}{4}\right) = \frac{3}{\sqrt{2}}$$
$$\therefore \quad r = \frac{3}{\sqrt{2}}\sec\left(\theta + \frac{\pi}{4}\right)$$

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Exercise B, Question 6

Question:

Find polar equations for the following curves:

a $y = 2x$ **b** $y = -\sqrt{3}x + a$ **c** $y = x(x - a)$

Solution:

a
$$y = 2x$$
$$\Rightarrow \quad r\sin\theta = 2r\cos\theta$$
$$\tan\theta = 2 \quad \text{or} \quad \theta = \arctan 2$$

b
$$y = -\sqrt{3}x + a$$
$$r\sin\theta = -\sqrt{3}r\cos\theta + a$$
$$r(\sin\theta + \sqrt{3}\cos\theta) = a$$
$$r\left(\tfrac{1}{2}\sin\theta + \frac{\sqrt{3}}{2}\cos\theta\right) = \frac{a}{2}$$
$$r\sin\left(\theta + \frac{\pi}{3}\right) = \frac{a}{2}$$
$$\therefore \quad r = \frac{a}{2}\operatorname{cosec}\left(\theta + \frac{\pi}{3}\right)$$

c
$$y = x(x - a)$$
$$r\sin\theta = r\cos\theta\,(r\cos\theta - a)$$
$$\tan\theta = r\cos\theta - a$$
$$r\cos\theta = \tan\theta + a$$
$$r = \tan\theta\sec\theta + a\sec\theta$$

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Exercise C, Question 1

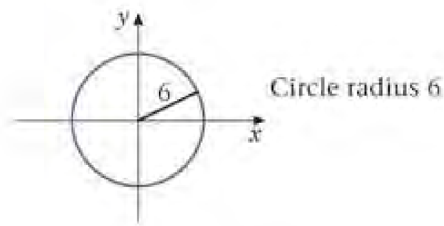
Question:

Sketch the following curves.

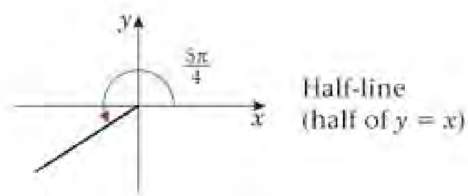
a $r = 6$ **b** $\theta = \frac{5\pi}{4}$ **c** $\theta = -\frac{\pi}{4}$

Solution:

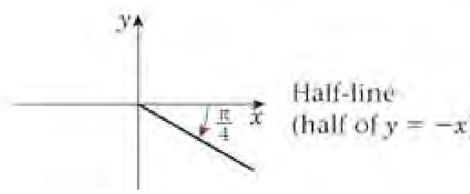
a $r = 6$



b $\theta = \frac{5\pi}{4}$



c $\theta = -\frac{\pi}{4}$



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Exercise C, Question 2

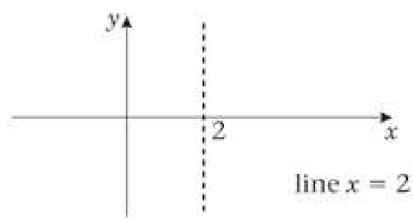
Question:

Sketch the following curves.

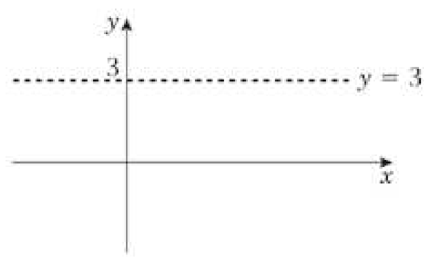
a $r = 2\sec\theta$ **b** $r = 3\operatorname{cosec}\theta$ **c** $r = 2\sec\left(\theta - \frac{\pi}{3}\right)$

Solution:

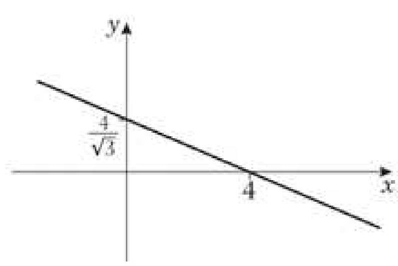
a
$$r = 2\sec\theta$$
$$\Rightarrow \quad r\cos\theta = 2$$
i.e. $\quad x = 2$



b
$$r = 3\operatorname{cosec}\theta$$
$$\Rightarrow \quad r\sin\theta = 3$$
i.e. $\quad y = 3$



c
$$r = 2\sec\left(\theta - \frac{\pi}{3}\right)$$
$$r\cos\left(\theta - \frac{\pi}{3}\right) = 2$$
$$\Rightarrow \quad r\cos\theta\cos\frac{\pi}{3} + r\sin\theta\sin\frac{\pi}{3} = 2$$
$$\Rightarrow \quad \frac{x}{2} + y\frac{\sqrt{3}}{2} = 2$$
$$x + y\sqrt{3} = 4$$
or $$y = \frac{4}{\sqrt{3}} - \frac{1}{\sqrt{3}}x$$



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Exercise C, Question 3

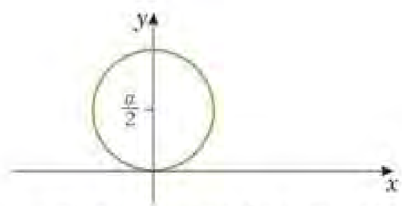
Question:

Sketch the following curves.

a $r = a\sin\theta$ **b** $r = a(1 - \cos\theta)$ **c** $r = a\cos 3\theta$

Solution:

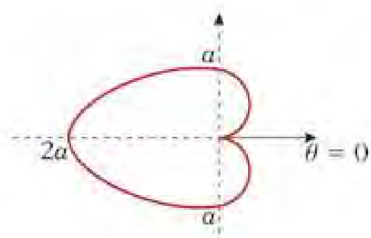
a
$$r = a\sin\theta$$
$$\Rightarrow \quad r^2 = ar\sin\theta$$
$$x^2 + y^2 = ay$$
$$x^2 + \left(y - \frac{a}{2}\right)^2 = \frac{a^2}{4}$$



Circle centre $(0, \frac{a}{2})$ radius $\frac{a}{2}$

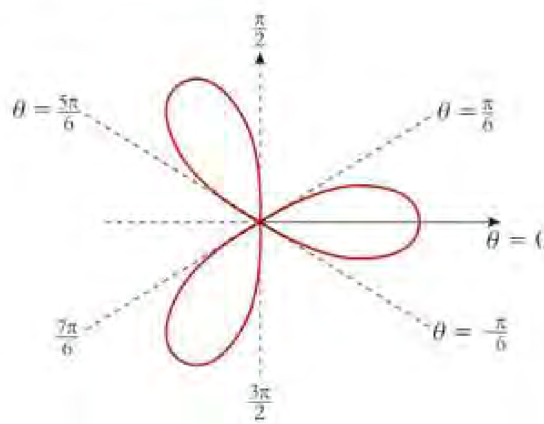
b $r = a(1 - \cos\theta)$

θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
r	0	a	$2a$	a	0



c $r = a\cos 3\theta$

θ	0	$\frac{\pi}{6}$	$-\frac{\pi}{6}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$
r	a	0	0	0	a	0	0	a	0



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Exercise C, Question 4

Question:

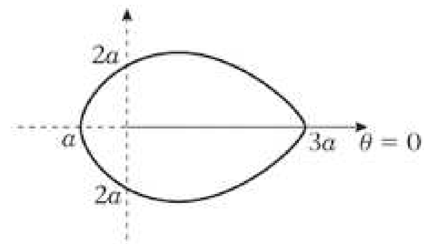
Sketch the following curves.

a $r = a(2 + \cos\theta)$ **b** $r = a(6 + \cos\theta)$ **c** $r = a(4 + 3\cos\theta)$

Solution:

a $r = a(2 + \cos\theta)$

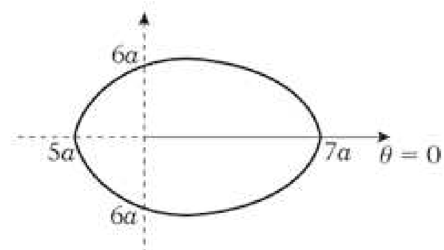
θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
r	$3a$	$2a$	a	$2a$	$3a$



$2 = 2 \times 1 \quad \therefore$ no dimple.

b $r = a(6 + \cos\theta)$

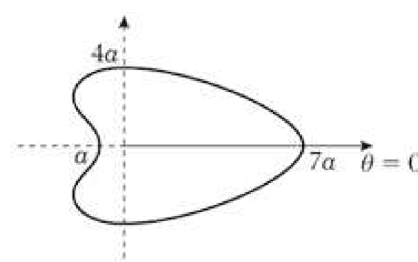
θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
r	$7a$	$6a$	$5a$	$6a$	$7a$



$6 > 2 \times 1 \quad \therefore$ no dimple.

c $r = a(4 + 3\cos\theta)$

θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
r	$7a$	$4a$	a	$4a$	$7a$



$4 < 2 \times 3 \quad \therefore$ a dimple at $\theta = \pi$.

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Exercise C, Question 5

Question:

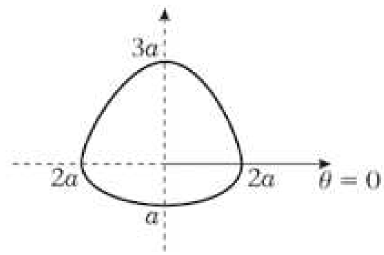
Sketch the following curves.

a $r = a(2 + \sin\theta)$ **b** $r = a(6 + \sin\theta)$ **c** $r = a(4 + 3\sin\theta)$

Solution:

a $r = a(2 + \sin\theta)$

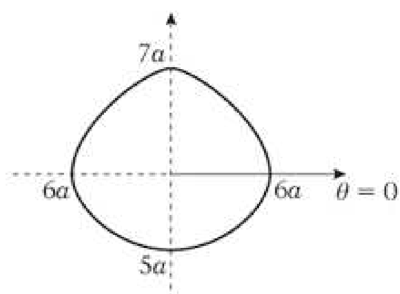
θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
r	$2a$	$3a$	$2a$	a	$2a$



$2 = 2 \times 1$ so no dimple

b $r = a(6 + \sin\theta)$

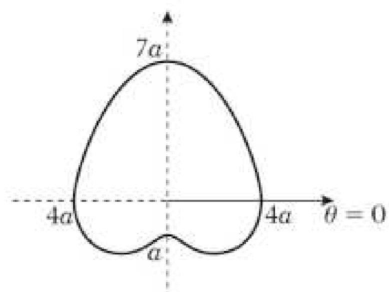
θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
r	$6a$	$7a$	$6a$	$5a$	$6a$



$6 > 2 \times 1$ so no dimple

c $r = a(4 + 3\sin\theta)$

θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
r	$4a$	$7a$	$4a$	a	$4a$



$4 < 2 \times 3 \therefore$ there is a dimple at $\theta = \frac{3\pi}{2}$

The graphs in question 5 are simply rotations of the graphs in question 4.

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Exercise C, Question 6

Question:

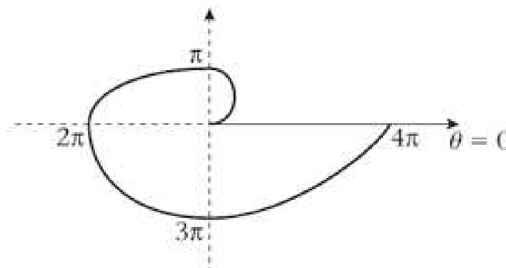
Sketch the following curves.

a $r = 2\theta$ **b** $r^2 = a^2 \sin \theta$ **c** $r^2 = a^2 \sin 2\theta$

Solution:

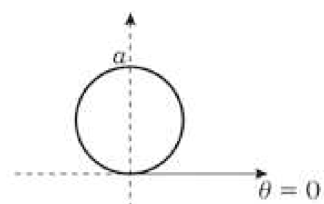
a $r = 2\theta$

$\boldsymbol{\theta}$	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\boldsymbol{r}$	0	π	2π	3π	4π



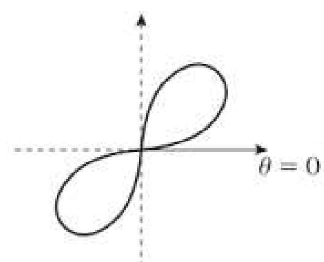
b $r^2 = a^2 \sin \theta$

$\boldsymbol{\theta}$	0	$\frac{\pi}{2}$	π
$\boldsymbol{r}$	0	a	0



c $r^2 = a^2 \sin 2\theta$

$\boldsymbol{\theta}$	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$
$\boldsymbol{r}$	0	a	0	0	a	0



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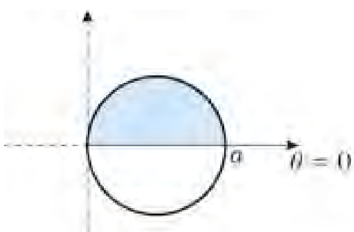
Exercise D, Question 1

Question:

Find the area of the finite region bounded by the curve with the given polar equation and the half lines $\theta = \alpha$ and $\theta = \beta$.

$r = a \cos \theta$, $\alpha = 0, \beta = \frac{\pi}{2}$

Solution:



$r = a \cos \theta$

$$\text{Area} = \tfrac{1}{2} a^2 \int_0^{\frac{\pi}{2}} \cos^2 \theta \, d\theta$$

$\boxed{\cos 2\theta = 2\cos^2 \theta - 1}$

$$= \frac{a^2}{4} \int_0^{\frac{\pi}{2}} (\cos 2\theta + 1)$$

$$= \frac{a^2}{4} \left[\tfrac{1}{2} \sin 2\theta + \theta \right]_0^{\frac{\pi}{2}}$$

$$= \frac{a^2}{4} \left[\left(0 + \frac{\pi}{2}\right) - (0) \right]$$

$$= \frac{\pi a^2}{8}$$

$r = a \cos \theta$ is a circle centre $\left(\frac{a}{2}, 0\right)$ and radius $\frac{a}{2}$.
The area of the semicircle is $\therefore$ $\frac{1}{2} \pi \frac{a^2}{4} = \frac{a^2 \pi}{8}$.

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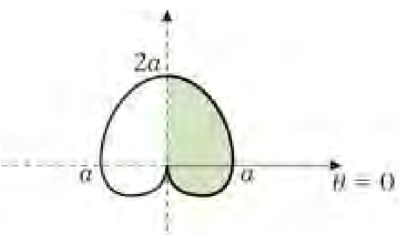
Exercise D, Question 2

Question:

Find the area of the finite region bounded by the curve with the given polar equation and the half lines $\theta = \alpha$ and $\theta = \beta$.

$r = a(1 + \sin\theta)$, $\alpha = -\frac{\pi}{2}, \beta = \frac{\pi}{2}$

Solution:



$r = a(1 + \sin\theta)$

$$\text{Area} = \frac{1}{2}a^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1 + 2\sin\theta + \sin^2\theta)\,d\theta$$

$$= \frac{1}{2}a^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(1 + 2\sin\theta + \frac{1}{2} - \frac{1}{2}\cos 2\theta\right)d\theta$$

Use $\cos 2\theta = 1 - 2\sin^2\theta$.

$$= \frac{1}{2}a^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{3}{2} + 2\sin\theta - \frac{1}{2}\cos 2\theta\right)d\theta$$

$$= \frac{1}{2}a^2 \left[\frac{3}{2}\theta - 2\cos\theta - \frac{1}{4}\sin 2\theta\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= \frac{1}{2}a^2 \left[\left(\frac{3\pi}{4} - 0 - 0\right) - \left(-\frac{3\pi}{4} - 0 - 0\right)\right]$$

$$= \frac{3\pi a^2}{4}$$

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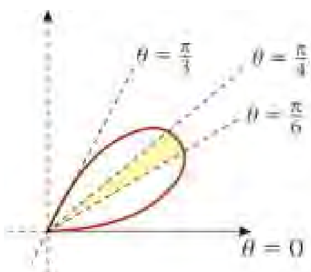
Exercise D, Question 3

Question:

Find the area of the finite region bounded by the curve with the given polar equation and the half lines $\theta = \alpha$ and $\theta = \beta$.

$r = a \sin 3\theta,\qquad \alpha = \frac{\pi}{6}, \beta = \frac{\pi}{4}$

Solution:



$r = a \sin 3\theta$

$$\text{Area} = \frac{1}{2} a^2 \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \sin^2 3\theta \, d\theta$$

$$= \frac{a^2}{4} \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} (1 - \cos 6\theta) \, d\theta$$

Use $\cos 6\theta = 1 - 2\sin^2 3\theta$.

$$= \frac{a^2}{4} \left[\theta - \frac{1}{6} \sin 6\theta \right]_{\frac{\pi}{6}}^{\frac{\pi}{4}}$$

$$= \frac{a^2}{4} \left[\left(\frac{\pi}{4} - \frac{1}{6} \sin \frac{3\pi}{2} \right) - \left(\frac{\pi}{6} - \frac{1}{6} \sin \pi \right) \right]$$

$$= \frac{a^2}{4} \left(\frac{\pi}{4} + \frac{1}{6} - \frac{\pi}{6} \right)$$

$$= \frac{a^2}{4} \left(\frac{\pi}{12} + \frac{2}{12} \right)$$

$$= \frac{(\pi + 2) a^2}{48}$$

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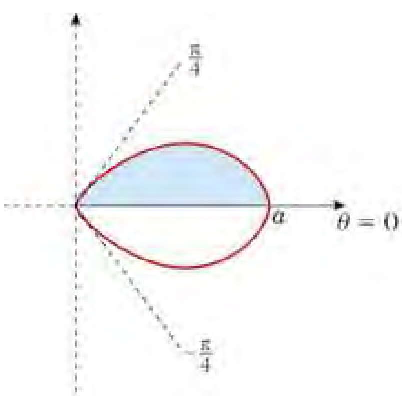
Exercise D, Question 4

Question:

Find the area of the finite region bounded by the curve with the given polar equation and the half lines $\theta = \alpha$ and $\theta = \beta$.

$r^2 = a^2 \cos 2\theta, \qquad \alpha = 0, \beta = \frac{\pi}{4}$

Solution:



$r = a^2 \cos 2\theta$

$$\text{Area} = \frac{1}{2} a^2 \int_0^{\frac{\pi}{4}} \cos 2\theta \, d\theta$$

$$= \left[\frac{a^2}{4} \sin 2\theta\right]_0^{\frac{\pi}{4}}$$

$$= \left(\frac{a^2}{4} \sin \frac{\pi}{2}\right) - (0)$$

$$= \frac{a^2}{4}$$

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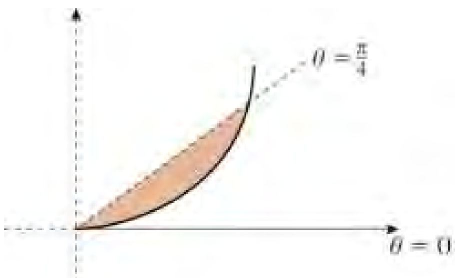
Exercise D, Question 5

Question:

Find the area of the finite region bounded by the curve with the given polar equation and the half lines $\theta = \alpha$ and $\theta = \beta$.

$r^2 = a^2 \tan \theta$, $\qquad \alpha = 0, \beta = \frac{\pi}{4}$

Solution:



$r^2 = a^2 \tan \theta$

$$\text{Area} = \frac{1}{2} a^2 \int_0^{\frac{\pi}{4}} \tan \theta \, d\theta$$

$$= \left[\frac{1}{2} a^2 \ln \sec \theta\right]_0^{\frac{\pi}{4}}$$

$$= \left(\frac{1}{2} a^2 \ln \sqrt{2}\right) - (0)$$

$$= \frac{a^2 \ln \sqrt{2}}{2} \quad \text{or} \quad \frac{a^2 \ln 2}{4}$$

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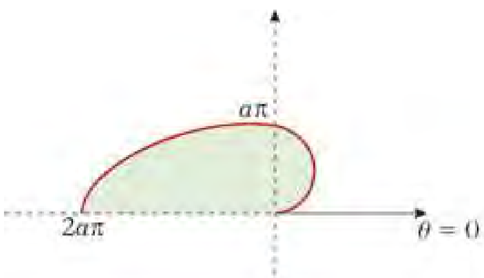
Exercise D, Question 6

Question:

Find the area of the finite region bounded by the curve with the given polar equation and the half lines $\theta = \alpha$ and $\theta = \beta$.

$r = 2a\theta$, $\alpha = 0$, $\beta = \pi$

Solution:



$r = 2a\,\theta$

$$\text{Area} = \frac{1}{2}\int_0^{\pi} 4a^2\,\theta^2\,d\theta$$

$$= 2a^2\left[\frac{\theta^3}{3}\right]_0^{\pi}$$

$$= 2a^2\left[\left(\frac{\pi^3}{3}\right) - (0)\right]$$

$$= \frac{2a^2\,\pi^3}{3}$$

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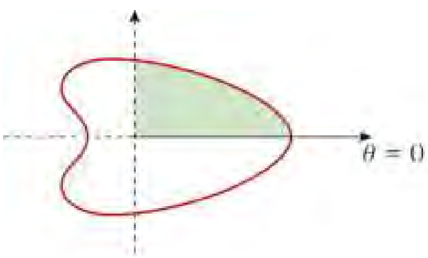
Exercise D, Question 7

Question:

Find the area of the finite region bounded by the curve with the given polar equation and the half lines $\theta = \alpha$ and $\theta = \beta$.

$r = a(3 + 2\cos\theta)$, $\alpha = 0, \beta = \frac{\pi}{2}$

Solution:



$r = a(3 + 2\cos\theta)$

$$\text{Area} = \frac{a^2}{2}\int_0^{\frac{\pi}{2}} (9 + 12\cos\theta + 4\cos^2\theta)\,\mathrm{d}\theta$$

$$= \frac{a^2}{2}\int_0^{\frac{\pi}{2}} (11 + 12\cos\theta + 2\cos 2\theta)\,\mathrm{d}\theta$$

Use $\cos 2\theta = 2\cos^2\theta - 1$.

$$= \frac{a^2}{2}\left[11\theta + 12\sin\theta + \sin 2\theta\right]_0^{\frac{\pi}{2}}$$

$$= \frac{a^2}{2}\left[\left(\frac{11\pi}{2} + 12 + 0\right) - (0)\right]$$

$$= \frac{a^2}{4}(11\pi + 24)$$

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Exercise D, Question 8

Question:

Show that the area enclosed by the curve with polar equation $r = a(p + q\cos\theta)$ is $\frac{2p^2 + q^2}{2}\pi a^2$.

Solution:

$$\text{Area} = \frac{1}{2}a^2\int_0^{2\pi}(p^2 + 2pq\cos\theta + q^2\cos^2\theta)\,d\theta$$

$$= \frac{1}{2}a^2\int_0^{2\pi}\left(p^2 + 2pq\cos\theta + \frac{q^2}{2}\cos 2\theta + \frac{q^2}{2}\right)d\theta$$

Use $\cos 2\theta = 2\cos^2\theta - 1$.

$$= \frac{1}{2}a^2\int_0^{2\pi}\left(\left[\frac{2p^2 + q^2}{2}\right] + 2pq\cos\theta + \frac{q^2}{2}\cos 2\theta\right)d\theta$$

$$= \frac{1}{2}a^2\left[\left[\frac{2p^2 + q^2}{2}\right]\theta + 2pq\sin\theta + \frac{q^2}{4}\sin 2\theta\right]_0^{2\pi}$$

$$= \frac{1}{2}a^2\left[\left(\left[\frac{2p^2 + q^2}{\cancel{2}}\right]\pi \times \cancel{2} + 0 + 0\right) - (0)\right]$$

$$= \frac{a^2(2p^2 + q^2)\pi}{2}$$

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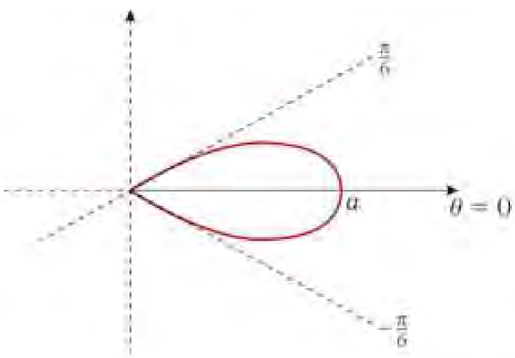
Exercise D, Question 9

Question:

Find the area of a single loop of the curve with equation $r = a\cos 3\theta$.

Solution:



$$\text{Area} = \tfrac{1}{2}a^2\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}}\cos^2 3\theta\,d\theta = 2\times\tfrac{1}{2}a^2\int_0^{\frac{\pi}{6}}\cos^2 3\theta\,d\theta$$

$$= \frac{a^2}{2}\int_0^{\frac{\pi}{6}}(1+\cos 6\theta)\,d\theta$$

Use $\cos 6\theta = 2\cos^2 3\theta - 1$.

$$= \frac{a^2}{2}\left[\theta + \tfrac{1}{6}\sin 6\theta\right]_0^{\frac{\pi}{6}}$$

$$= \frac{a^2}{2}\left[\left(\frac{\pi}{6}+0\right)-(0)\right]$$

$$= \frac{\pi a^2}{12}$$

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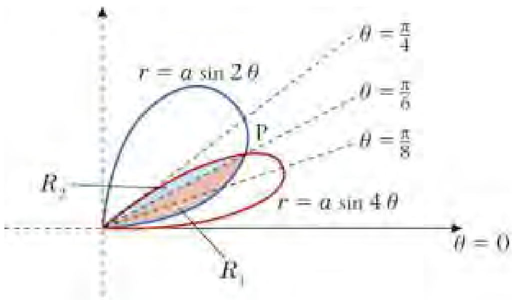
Exercise D, Question 10

Question:

Find the finite area enclosed between $r = a\sin 4\theta$ and $r = a\sin 2\theta$ for $0 \leqslant \theta \leqslant \frac{\pi}{2}$.

Solution:



Find P

$$a\sin 2\theta = a\sin 4\theta$$
$$\Rightarrow \quad \sin 2\theta = 2\sin 2\theta\cos 2\theta$$
$$\Rightarrow \quad 0 = \sin 2\theta(2\cos 2\theta - 1)$$
$$\Rightarrow \quad \sin 2\theta = 0,\ \theta = 0, \frac{\pi}{2}$$
$$\cos 2\theta = \tfrac{1}{2} \Rightarrow \theta = \frac{\pi}{6}$$

$$R_1 = \tfrac{1}{2}a^2\int_0^{\frac{\pi}{6}}\sin^2 2\theta\,\mathrm{d}\theta$$

$\boxed{\cos 2A = 1 - 2\sin^2 A.}$

$$= \frac{a^2}{4}\int_0^{\frac{\pi}{6}}(1 - \cos 4\theta)\,\mathrm{d}\theta$$
$$= \frac{a^2}{4}\left[\theta - \tfrac{1}{4}\sin 4\theta\right]_0^{\frac{\pi}{6}} = \frac{a^2}{4}\left[\left(\frac{\pi}{6} - \tfrac{1}{4}\sin\frac{2\pi}{3}\right) - (0)\right]$$
$$= \frac{a^2}{4}\left[\frac{\pi}{6} - \frac{\sqrt{3}}{8}\right]$$

$$R_2 = \tfrac{1}{2}a^2\int_{\frac{\pi}{6}}^{\frac{\pi}{4}}\sin^2 4\theta\,\mathrm{d}\theta = \frac{a^2}{4}\int_{\frac{\pi}{6}}^{\frac{\pi}{4}}(1 - \cos 8\theta)\,\mathrm{d}\theta$$
$$= \frac{a^2}{4}\left[\theta - \tfrac{1}{8}\sin 8\theta\right]_{\frac{\pi}{6}}^{\frac{\pi}{4}} = \frac{a^2}{4}\left[\left(\frac{\pi}{4} - \tfrac{1}{8}\sin 2\pi\right) - \left(\frac{\pi}{6} - \tfrac{1}{8}\sin\frac{4\pi}{3}\right)\right]$$
$$= \frac{a^2}{4}\left[\frac{\pi}{12} - \frac{\sqrt{3}}{16}\right]$$

$$\therefore \quad \text{enclosed area} = R_1 + R_2 = \frac{a^2}{4}\left[\frac{\pi}{4} - \frac{3\sqrt{3}}{16}\right]$$

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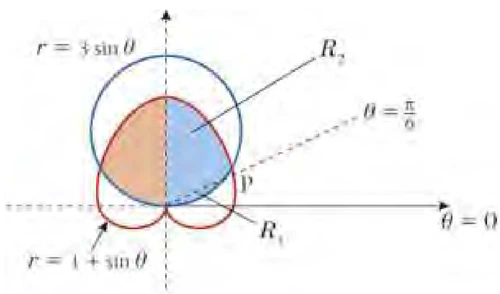
Exercise D, Question 11

Question:

Find the area of the finite region R enclosed by the curve with equation $r = (1 + \sin\theta)$ that lies entirely within the curve with equation $r = 3\sin\theta$.

Solution:



First find P:

$1 + \sin\theta = 3\sin\theta$

$\Rightarrow \quad 1 = 2\sin\theta$

$\Rightarrow \quad \sin\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6}$

Just finding RHS of the required area, so total $= 2(R_1 + R_2)$

$$R_1 = \frac{1}{2}\int_0^{\frac{\pi}{6}} (3\sin\theta)^2\,d\theta = \frac{9}{4}\int_0^{\frac{\pi}{6}} (1 - \cos 2\theta)\,d\theta$$

$$R_2 = \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (1 + \sin\theta)^2\,d\theta$$

So $$R_2 = \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (1 + 2\sin\theta + \sin^2\theta)\,d\theta = \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left(\frac{3}{2} + 2\sin\theta - \frac{1}{2}\cos 2\theta\right)d\theta$$

$$R_1 = \frac{9}{4}\left[\theta - \frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{6}} = \frac{9}{4}\left[\left(\frac{\pi}{6} - \frac{1}{2}\sin\frac{\pi}{3}\right) - (0)\right]$$

$$R_1 = \frac{3\pi}{8} - \frac{9\sqrt{3}}{16}$$

$$R_2 = \frac{1}{2}\left[\frac{3}{2}\theta - 2\cos\theta - \frac{1}{4}\sin 2\theta\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}} = \frac{1}{2}\left[\left(\frac{3\pi}{4} - 0\right) - \left(\frac{\pi}{4} - \sqrt{3} - \frac{\sqrt{3}}{8}\right)\right]$$

$$R_2 = \frac{\pi}{4} + \frac{9\sqrt{3}}{16}$$

$\therefore \quad R_1 + R_2 = \frac{5\pi}{8}$

$\therefore$ Area required is $\frac{5\pi}{4}$

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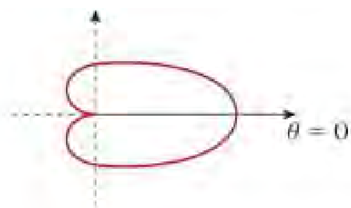
Exercise E, Question 1

Question:

Find the points on the cardioid $r = a(1 + \cos\theta)$ where the tangents are perpendicular to the initial line.

Solution:



$r = a(1 + \cos\theta)$

Require $\frac{d}{d\theta}(r\cos\theta) = 0$

i.e. $\frac{d}{d\theta}(a\cos\theta + a\cos^2\theta) = a[-\sin\theta - 2\cos\theta\sin\theta]$

So $\quad 0 = -a\sin\theta[1 + 2\cos\theta]$

$\sin\theta = 0 \Rightarrow \quad \theta = 0, \pi \quad$ (from sketch π is not allowed)

$\cos\theta = -\frac{1}{2} \Rightarrow \theta = \pm\frac{2\pi}{3} \Rightarrow r = a\left(1 - \frac{1}{2}\right) = \frac{a}{2}$

$\therefore$ points are $(2a, 0)$ and $\left(\frac{a}{2}, \frac{2\pi}{3}\right)\left(\frac{a}{2}, \frac{-2\pi}{3}\right)$

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Exercise E, Question 2

Question:

Find the points on the spiral $r = e^{2\theta}$, $0 \leqslant \theta \leqslant \pi$, where the tangents are

a perpendicular, **b** parallel

to the initial line. Give your answers to 3 s.f.

Solution:

$$r = e^{2\theta}$$

a $x = r\cos\theta = e^{2\theta}\cos\theta$

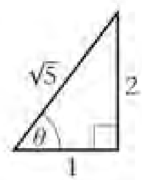
$$\frac{dx}{d\theta} = 0 \Rightarrow 0 = 2e^{2\theta}\cos\theta - \sin\theta\, e^{2\theta}$$

$$0 = e^{2\theta}(2\cos\theta - \sin\theta)$$

$$\Rightarrow \quad \tan\theta = 2$$

$$\therefore \quad \theta = 1.107 \text{ (rads)}$$

$$r = e^{2 \times 1.107} = 9.1549\ldots$$



So at (9.15, 1.11) the tangent is perpendicular to initial line.

b $y = r\sin\theta = e^{2\theta}\sin\theta$

$$\frac{dy}{d\theta} = 0 \Rightarrow 0 = 2e^{2\theta}\sin\theta + \cos\theta\, e^{2\theta}$$

$$0 = e^{2\theta}(2\sin\theta + \cos\theta)$$

$$\Rightarrow \quad \tan\theta = -\tfrac{1}{2}$$

$$\therefore \quad \theta = (-0.463\ldots,)\ 2.6779\ldots$$

$$r = e^{2 \times 2.6779\ldots} = 211.852\ldots$$

So at (212, 2.68) the tangent is parallel to initial line.

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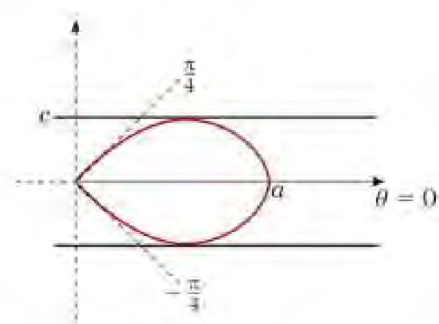
Exercise E, Question 3

Question:

a Find the points on the curve $r = a\cos 2\theta$, $-\frac{\pi}{4} \leqslant \theta \leqslant \frac{\pi}{4}$, where the tangents are parallel to the initial line, giving your answers to 3 s.f. where appropriate.

b Find the equation of these tangents.

Solution:



$$r = a\cos 2\theta$$

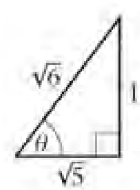
a $y = r\sin\theta = a\sin\theta\cos 2\theta$

$$\frac{\mathrm{d}y}{\mathrm{d}\theta} = 0 \Rightarrow 0 = a[\cos\theta\cos 2\theta - 2\sin 2\theta\sin\theta]$$

$$0 = a\cos\theta\,[\cos 2\theta - 4\sin^2\theta]$$

$$0 = a\cos\theta\,[\cos^2\theta - 5\sin^2\theta]$$

$\cos\theta = \Rightarrow \theta = \frac{\pi}{2}$ (outside range)



$\therefore \quad \tan^2\theta = \frac{1}{5} \Rightarrow \tan\theta = \pm\frac{1}{\sqrt{5}}$

$$\theta = \pm 0.42053\ldots$$

$$r = a[\cos^2\theta - \sin^2\theta] = a\left[\tfrac{5}{6} - \tfrac{1}{6}\right] = \frac{2a}{3}$$

$\therefore$ points are $\left(\frac{2a}{3}, \pm 0.421\right)$

b The lines are $y = \pm c$ where $c = r\sin(0.42053\ldots)$

$$= \frac{2a}{3} \times \frac{1}{\sqrt{6}} = \frac{a\sqrt{6}}{9}$$

The line $y = c$ is $r\sin\theta = \frac{a\sqrt{6}}{9}$

$\therefore$ Tangents have equations $r = \pm\frac{a\sqrt{6}}{9}\operatorname{cosec}\theta$

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Exercise E, Question 4

Question:

Find the points on the curve with equation $r = a(7 + 2\cos\theta)$ where the tangents are parallel to the initial line.

Solution:

$$r = a(7 + 2\cos\theta)$$

$$y = r\sin\theta = a(7\sin\theta + \underbrace{2\cos\theta\sin\theta}_{\sin 2\theta})$$

$$\frac{dy}{d\theta} = 0 \Rightarrow 0 = a(7\cos\theta + 2\cos 2\theta)$$

$$\Rightarrow \quad 0 = 4\cos^2\theta + 7\cos\theta - 2$$

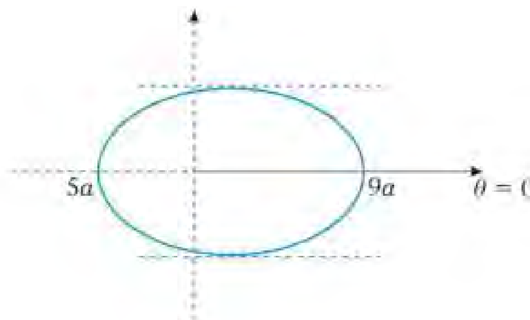
$$0 = (4\cos\theta - 1)(\cos\theta + 2)$$

$$\cos\theta = \tfrac{1}{4} \text{ (or } -2)$$

$$\Rightarrow \quad \theta = \pm 1.318\ldots$$

$$r = a\left(7 + \tfrac{2}{4}\right) = 7\tfrac{1}{2}a$$

$\therefore$ tangents are parallel at $\left(7\tfrac{1}{2}a, \pm 1.32\right)$



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Exercise E, Question 5

Question:

Find the equation of the tangents to $r = 2 + \cos\theta$ that are perpendicular to the initial line.

Solution:

$$r = 2 + \cos\theta$$

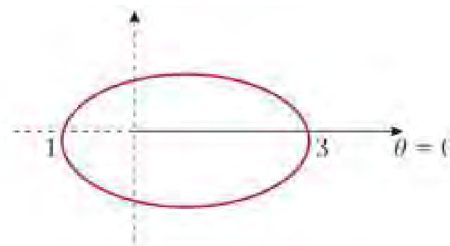
$$x = r\cos\theta = 2\cos\theta + \cos^2\theta$$

$$\frac{dx}{d\theta} = 0 \Rightarrow 0 = -2\sin\theta - 2\cos\theta\sin\theta$$

$$0 = -2\sin\theta(1 + \cos\theta)$$

$$\sin\theta = 0 \quad \Rightarrow \theta = 0, \pi$$

$$\cos\theta = -1 \Rightarrow \theta = \pi$$



$\therefore$ tangents are perpendicular to the initial line at:

$(3, 0)$ and $(1, \pi)$

The equations are

$$r\cos\theta = 3 \qquad r\cos\theta = -1$$

$$r = 3\sec\theta \qquad r = -\sec\theta$$

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Exercise E, Question 6

Question:

Find the point on the curve with equation $r = a(1 + \tan\theta)$, $0 \leqslant \theta < \frac{\pi}{2}$, where the tangent is perpendicular to the initial line.

Solution:

$$r = a(1 + \tan\theta)$$

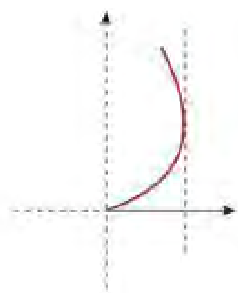
$$x = r\cos\theta = a(\cos\theta + \sin\theta)$$

$$\frac{dx}{d\theta} = 0 \Rightarrow \quad 0 = a[-\sin\theta + \cos\theta]$$

$$\Rightarrow \quad \tan\theta = 1$$

$$\Rightarrow \quad \theta = \frac{\pi}{4}$$

$\therefore$ point is $\left(2a, \frac{\pi}{4}\right)$



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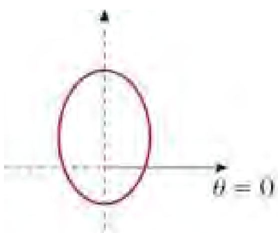
Exercise F, Question 1

Question:

Determine the area enclosed by the curve with equation
$r = a(1 + \frac{1}{2}\sin\theta)$, $a > 0$, $0 \leqslant \theta < 2\pi$,
giving your answer in terms of a and π.

Solution:



$r = a\left(1 + \frac{1}{2}\sin\theta\right)$

$$\text{Area} = \frac{1}{2}a^2\int_0^{2\pi}\left(1 + \frac{1}{2}\sin\theta\right)^2 d\theta$$

$$= \frac{a^2}{2}\int_0^{2\pi}\left(1 + \sin\theta + \frac{1}{4}\sin^2\theta\right)d\theta$$

$$= \frac{a^2}{2}\int_0^{2\pi}\left(\frac{9}{8} + \sin\theta - \frac{\cos 2\theta}{8}\right)d\theta$$

Use $\cos 2\theta = 1 - 2\sin^2\theta$.

$$= \frac{a^2}{2}\left[\frac{9}{8}\theta - \cos\theta - \frac{\sin 2\theta}{16}\right]_0^{2\pi}$$

$$= \frac{a^2}{2}\left[\left(\frac{9\pi}{4} - 1 - 0\right) - (0 - 1 - 0)\right]$$

$$= \frac{9\pi a^2}{8}$$

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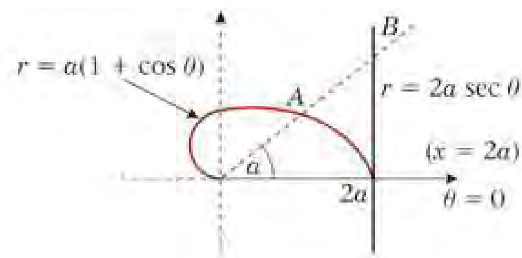
Exercise F, Question 2

Question:

Sketch the curve with equation $r = a(1 + \cos\theta)$ for $0 \leqslant \theta \leqslant \pi$, where $a > 0$.
Sketch also the line with equation $r = 2a\sec\theta$ for $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, on the same diagram.
The half-line with equation $\theta = \alpha$, $0 < \alpha < \frac{\pi}{2}$, meets the curve at A and the line with equation $r = 2a\sec\theta$ at B. If O is the pole, find the value of $\cos\alpha$ for which $OB = 2OA$.

Solution:



$OB = 2a\sec\alpha$

$OA = a(1 + \cos\alpha)$

$2OA = OB \Rightarrow 1 + \cos\alpha = \sec\alpha$

$\cos^2\alpha + \cos\alpha - 1 = 0$

$$\cos\alpha = \frac{-1 \pm \sqrt{1+4}}{2}$$

$\therefore$ α is acute.

$$\cos\alpha = \frac{\sqrt{5} - 1}{2}$$

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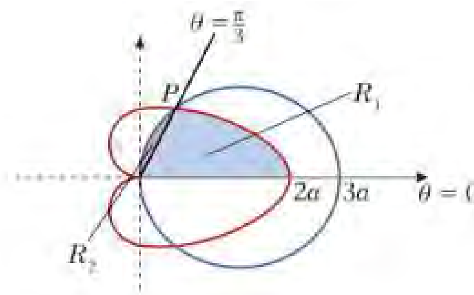
Exercise F, Question 3

Question:

Sketch, in the same diagram, the curves with equations $r = 3\cos\theta$ and $r = 1 + \cos\theta$ and find the area of the region lying inside both curves.

Solution:



First find P:

$$1 + \cos\theta = 3\cos\theta$$

$$1 = 2\cos\theta$$

$$\Rightarrow \quad \theta = \arccos\tfrac{1}{2} = \frac{\pi}{3}$$

By symmetry the required area $= 2(R_1 + R_2)$

$$R_1 = \tfrac{1}{2}\int_0^{\frac{\pi}{3}} (1 + \cos\theta)^2\,\mathrm{d}\theta = \tfrac{1}{2}\int_0^{\frac{\pi}{3}} (1 + 2\cos\theta + \cos^2\theta)\,\mathrm{d}\theta$$

$$R_1 = \tfrac{1}{2}\int_0^{\frac{\pi}{3}} \left(\tfrac{3}{2} + 2\cos\theta + \frac{\cos 2\theta}{2}\right)\mathrm{d}\theta$$

$\boxed{\text{Use } \cos 2\theta = 2\cos^2\theta - 1.}$

$$= \tfrac{1}{2}\left[\tfrac{3}{2}\theta + 2\sin\theta + \tfrac{1}{4}\sin 2\theta\right]_0^{\frac{\pi}{3}}$$

$$= \tfrac{1}{2}\left[\left(\frac{\pi}{2} + 2\sin\frac{\pi}{3} + \tfrac{1}{4}\sin\frac{2\pi}{3}\right) - (0)\right]$$

$$= \tfrac{1}{2}\left[\frac{\pi}{2} + \sqrt{3} + \frac{\sqrt{3}}{8}\right] = \frac{\pi}{4} + \frac{9\sqrt{3}}{16}$$

$$R_2 = \tfrac{9}{2}\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \cos^2\theta\,\mathrm{d}\theta = \tfrac{9}{4}\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} (1 + \cos 2\theta)\,\mathrm{d}\theta$$

$$= \tfrac{9}{4}\left[\theta + \tfrac{1}{2}\sin 2\theta\right]_{\frac{\pi}{3}}^{\frac{\pi}{2}} = \tfrac{9}{4}\left[\left(\frac{\pi}{2} + 0\right) - \left(\frac{\pi}{3} + \frac{\sqrt{3}}{4}\right)\right]$$

$$= \frac{3\pi}{8} - \frac{9\sqrt{3}}{16}$$

$$\therefore \quad \text{Area required} = 2\left(\frac{3\pi}{8} + \frac{\pi}{4}\right) = \frac{5\pi}{4}$$

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Exercise F, Question 4

Question:

Find the polar coordinates of the points on $r^2 = a^2 \sin 2\theta$ where the tangent is perpendicular to the initial line.

Solution:

$$r^2 = a^2 \sin 2\theta \qquad \left(\text{must have } 0 \leqslant \theta \leqslant \frac{\pi}{2}\right)$$

$$r = a\sqrt{\sin 2\theta}$$

$$x = r\cos\theta = a\cos\theta\sqrt{\sin 2\theta}$$

$$\frac{dx}{d\theta} = 0 \Rightarrow 0 = -\sin\theta\sqrt{\sin 2\theta} + \frac{1}{\cancel{2}}\cos\theta\frac{1}{\sqrt{\sin 2\theta}}\cancel{2}\cos 2\theta$$

i.e. $\quad 0 = -\sin\theta \times \sin 2\theta + \cos\theta\cos 2\theta$

i.e. $\quad 0 = \cos 3\theta$

$\therefore \quad 3\theta = \frac{\pi}{2}, \frac{3\pi}{2}$

$\therefore \quad \theta = \frac{\pi}{6}, \frac{\pi}{2}$

So $\left(a\sqrt{\frac{\sqrt{3}}{2}}, \frac{\pi}{6}\right)$ and $\left(0, \frac{\pi}{2}\right)$

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Exercise F, Question 5

Question:

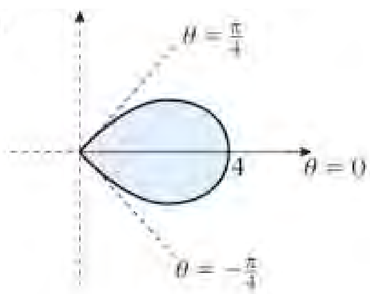
a Shade the region C for which the polar coordinates r, θ satisfy

$r \leqslant 4\cos 2\theta$ for $-\frac{\pi}{4} \leqslant \theta \leqslant \frac{\pi}{4}$

b Find the area of C.

Solution:

a



b

$$\begin{aligned}\text{Area} &= 2 \times \tfrac{1}{2}\int_0^{\frac{\pi}{4}} 16\cos^2 2\theta \, d\theta \\ &= \int_0^{\frac{\pi}{4}} (8 + 8\cos 4\theta)\, d\theta \\ &= \left[8\theta + 2\sin 4\theta\right]_0^{\frac{\pi}{4}} \\ &= 2\pi + 0 - 0 \\ &= 2\pi\end{aligned}$$

$\boxed{2\cos^2\theta = 1 + \cos 2\theta.}$

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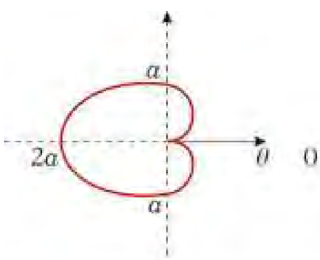
Exercise F, Question 6

Question:

Sketch the curve with polar equation $r = a(1 - \cos\theta)$, where $a > 0$, stating the polar coordinates of the point on the curve at which r has its maximum value.

Solution:



Max r is $2a$ at point $(2a, \pi)$

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Exercise F, Question 7

Question:

a On the same diagram, sketch the curve C_1 with polar equation

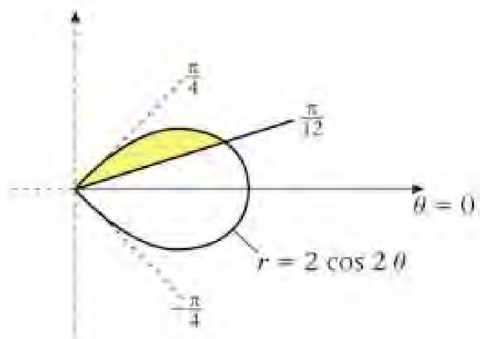
$r = 2\cos 2\theta, \quad -\frac{\pi}{4} < \theta \leqslant \frac{\pi}{4}$

and the curve C_2 with polar equation $\theta = \frac{\pi}{12}$.

b Find the area of the smaller region bounded by C_1 and C_2.

Solution:

a



b Area $= \frac{1}{2}\int_{\frac{\pi}{12}}^{\frac{\pi}{4}} 4\cos^2 2\theta$

$= \int_{\frac{\pi}{12}}^{\frac{\pi}{4}} (1 + \cos 4\theta)\, d\theta$ $\boxed{\cos 4\theta = 2\cos^2 2\theta - 1}$

$= \left[\theta + \frac{1}{4}\sin 4\theta\right]_{\frac{\pi}{12}}^{\frac{\pi}{4}}$

$= \left(\frac{\pi}{4} + 0\right) - \left(\frac{\pi}{12} + \frac{1}{4}\sin\frac{\pi}{3}\right)$

$= \frac{\pi}{6} - \frac{1}{4} \times \frac{\sqrt{3}}{2}$

$= \frac{\pi}{6} - \frac{\sqrt{3}}{8}$

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Exercise F, Question 8

Question:

a Sketch on the same diagram the circle with polar equation $r = 4\cos\theta$ and the line with polar equation $r = 2\sec\theta$.

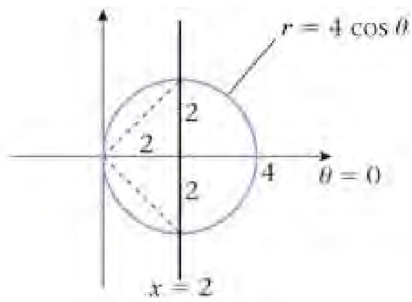
b State polar coordinates for their points of intersection.

Solution:

a $r = 2\sec\theta$

$r\cos\theta = 2$

$x = 2$



b $x = 2$ is a diameter

$r = \sqrt{2^2 + 2^2} = 2\sqrt{2}$

So polar coordinates are

$\left(2\sqrt{2}, \frac{\pi}{4}\right)$ $\left(2\sqrt{2}, -\frac{\pi}{4}\right)$

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Exercise F, Question 9

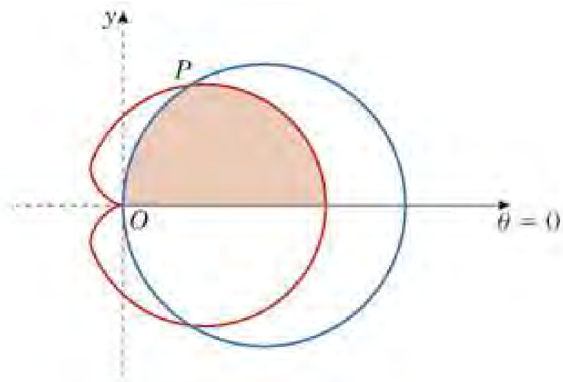
Question:

The diagram shows a sketch of the curves with polar equations

$r = a(1 + \cos\theta)$ and $r = 3a\cos\theta$, $a > 0$

a Find the polar coordinates of the point of intersection P of the two curves.

b Find the area, shaded in the figure, bounded by the two curves and by the initial line $\theta = 0$, giving your answer in terms of a and π.



Solution:

a $a(1 + \cos\theta) = 3a\cos\theta$

$$1 = 2\cos\theta$$

$$\cos\theta = \tfrac{1}{2} \Rightarrow \theta = \frac{\pi}{3}$$

So P is $\left(\frac{3}{2}a, \frac{\pi}{3}\right)$

b Area $= \dfrac{a^2}{2}\displaystyle\int_0^{\frac{\pi}{3}}\left(\tfrac{3}{2} + 2\cos\theta + \frac{\cos 2\theta}{2}\right)d\theta + \tfrac{9}{2}a^2\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \cos^2\theta\, d\theta$

$$= \frac{a^2}{2}\left[\tfrac{3}{2}\theta + 2\sin\theta + \tfrac{1}{4}\sin 2\theta\right]_0^{\frac{\pi}{3}} + \tfrac{9}{4}a^2\left[\theta + \tfrac{1}{2}\sin 2\theta\right]_{\frac{\pi}{3}}^{\frac{\pi}{2}}$$

$$= \frac{a^2}{2}\left[\frac{\pi}{2} + \sqrt{3} + \frac{\sqrt{3}}{8}\right] + \tfrac{9}{4}a^2\left[\frac{\pi}{6} - \frac{\sqrt{3}}{4}\right]$$

$$= \frac{5\pi}{8}a^2$$

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Exercise F, Question 10

Question:

Obtain a Cartesian equation for the curve with polar equation

a $r^2 = \sec 2\theta$, **b** $r^2 = \operatorname{cosec} 2\theta$.

Solution:

a
$$r^2 = \sec 2\theta$$
$$r^2 \cos 2\theta = 1$$
$$r^2(2\cos^2\theta - 1) = 1$$
$$2r^2\cos^2\theta = 1 + r^2$$
$$2x^2 = 1 + x^2 + y^2$$
$$\therefore \quad y^2 = x^2 - 1$$

b
$$r^2 = \operatorname{cosec} 2\theta$$
$$\Rightarrow \quad r^2 \sin 2\theta = 1$$
$$\Rightarrow \quad 2r\sin\theta \, r\cos\theta = 1$$
$$\Rightarrow \quad 2xy = 1$$
$$y = \frac{1}{2x}$$

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Exercise A, Question 1

Question:

Find the set of values of x for which

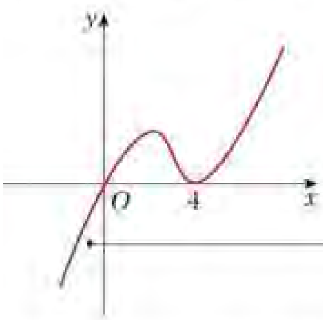
$$16x \leqslant 8x^2 - x^3.$$

Solution:

$$16x \leqslant 8x^2 - x^3$$
$$x^3 - 8x^2 + 16x \leqslant 0$$
$$x(x^2 - 8x + 16) \leqslant 0$$
$$x(x-4)^2 \leqslant 0$$

You can usually start inequality questions, if there are no modulus signs, by collecting terms together on one side of the equation, and factorising the resulting expression.

Sketching $y = x(x-4)^2$



The cubic passes through the origin and touches the x-axis at $x = 4$.

You can see from the sketch that $y = x(x-4)^2$ is negative for $x < 0$.

The solution of $16x \leqslant 8x^2 - x^3$ is

$$x \leqslant 0,\ x = 4$$

This inequality includes the equality, so you must include the solutions of $x(x-4)^2 = 0$, which are $x = 0$ and $x = 4$.

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Exercise A, Question 2

Question:

Find the set of values of x for which

$$\frac{2}{x-2} < \frac{1}{x+1}.$$

Solution:

$$\frac{2}{x-2} < \frac{1}{x+1}$$

$$\frac{2}{x-2} - \frac{1}{x+1} < 0$$

$$\frac{2(x+1) - 1(x-2)}{(x-2)(x+1)} = \frac{2x+2-x+2}{(x-2)(x+1)} < 0$$

$$\frac{x+4}{(x-2)(x+1)} < 0$$

You can start by collecting together the terms on one side reducing the expression to a single fraction. In this case no further factorisation is possible.

Considering $f(x) = \frac{x+4}{(x-2)(x+1)}$

the critical values are $x = -4, -1, 2$

You find the critical values by solving the numerator equal to zero and the denominator equal to zero. In this case the numerator = 0, gives $x = -4$ and the denominator = 0 gives $x = -1, 2$.

	$x < -4$	$-4 < x < -1$	$-1 < x < 2$	$2 < x$
Sign of $f(x)$	−	+	−	+

For example if $x < -4$, then

$$\frac{x+4}{(x-2)(x+1)} = \frac{\text{negative}}{\text{negative} \times \text{negative}},$$

which is $\frac{\text{negative}}{\text{positive}}$, which is negative.

The solution of $\frac{2}{x-2} < \frac{1}{x+1}$ is

$x < -4,\ -1 < x < 2.$

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Exercise A, Question 3

Question:

Find the set of values of x for which

$$\frac{x^2}{x-2} > 2x.$$

Solution:

$$\frac{x^2}{x-2} > 2x$$

$$\frac{x^2}{x-2} - 2x > 0$$

$$\frac{x^2 - 2x(x-2)}{x-2} > 0$$

$$\frac{4x - x^2}{x-2} > 0$$

$$\frac{x(4-x)}{x-2} > 0$$

You collect the terms together on one side of the inequality, write the expression as a single fraction and factorise the result as far as possible.

Considering $f(x) = \frac{x(4-x)}{x-2}$,

the critical values are $x = 0$, 2 and 4.

You find the critical values by solving the numerator equal to zero and the denominator equal to zero. In this case the numerator = 0, gives $x = 0$, 4 and the denominator gives $x = 2$.

	$x < 0$	$0 < x < 2$	$2 < x < 4$	$4 < x$
Sign of $f(x)$	+	−	+	−

For example if $4 < x$, then

$$\frac{x(4-x)}{x-2} = \frac{\text{positive} \times \text{negative}}{\text{positive}},$$

which is negative.

The solution of $\frac{x^2}{x-2} > 2x$ is

$x < 0, 2 < x < 4$.

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Exercise A, Question 4

Question:

Find the set of values of x for which

$$\frac{x^2-12}{x} > 1.$$

Solution:

$$\frac{x^2-12}{x} > 1$$

Multiply both sides by x^2

x cannot be zero as $\frac{x^2-12}{x}$ would be undefined, so x^2 is positive and you can multiply both sides of an inequality by a positive number or expression without changing the inequality. You could **not** multiply both sides of the inequality by x as x could be positive or negative.

$$\frac{x^2-12}{\cancel{x}} \times x^{\cancel{2}} > x^2$$

$$x(x^2-12) - x^2 > 0$$

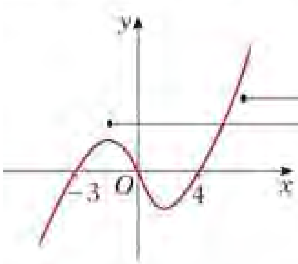
$$x^3 - 12x - x^2 > 0$$

$$x(x^2 - x - 12) > 0$$

$$x(x-4)(x+3) > 0$$

Sketching $y = x(x-4)(x+3)$

The graph of $y = x(x-4)(x+3)$ crosses the y axis at $x = -3$, 0 and 4.



You can see from the sketch that the graph is above the x-axis for $-3 < x < 0$ and $x > 4$. You can then just write down this answer.

The solution of $\frac{x^2-12}{x} > 1$ is $-3 < x < 0$, $x > 4$.

If you preferred, you could solve this question using the method illustrated in the solutions to questions 2 and 3 above.

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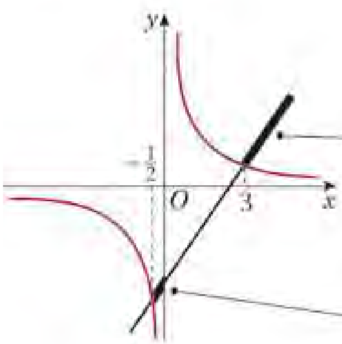
Exercise A, Question 5

Question:

Find the set of values of x for which

$$2x - 5 > \frac{3}{x}.$$

Solution:



Both $y = 2x - 5$ and $y = \frac{3}{x}$ are straightforward graphs to sketch and so this is a suitable question for a graphical method. The question, however, specifies no method and so you can use any method which gives an exact answer.

After sketching the two graphs, $2x - 5 > \frac{3}{x}$ is the set of values of x for which the line is above the curve. These parts of the line have been drawn thickly on the sketch.

$$2x - 5 = \frac{3}{x}$$

$$x(2x - 5) = 3$$

$$2x^2 - 5x - 3 = 0$$

$$(2x + 1)(x - 3) = 0$$

$$x = -\tfrac{1}{2}, 3$$

You need to find the x-coordinates of the points where the line and curve meet to find two end points of the intervals. The other end point ($x = 0$) can be seen by inspecting the sketch.

The solution to $2x - 5 > \frac{3}{x}$ is $-\frac{1}{2} < x < 0$, $x > 3$.

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Edexcel AS and A Level Modular Mathematics

Exercise A, Question 6

Question:

Given that k is a constant and that $k > 0$, find, in terms of k, the set of values of x for which $\frac{x+k}{x+4k} > \frac{k}{x}$.

Solution:

$$\frac{x+k}{x+4k} > \frac{k}{x}$$

$$\frac{x+k}{x+4k} - \frac{k}{x} > 0$$

$$\frac{(x+k)x - k(x+4k)}{(x+4k)x} > 0$$

$$\frac{x^2 - 4k^2}{(x+4k)x} > 0$$

$$\frac{(x+2k)(x-2k)}{(x+4k)x} > 0$$

Considering $f(x) = \frac{(x+2k)(x-2k)}{(x+4k)x}$,

the critical values are $x = -4k, -2k, 0$ and $2k$.

For example, when k is positive, in the interval $0 < x < 2k$, $\frac{(x+2k)(x-2k)}{(x+4k)x} = \frac{\text{positive} \times \text{negative}}{\text{positive} \times \text{positive}}$, which is negative.

	$x < -4k$	$-4k < x < -2k$	$-2k < x < 0$	$0 < x < 2k$	$2k < x$
Sign of $f(x)$	+	−	+	−	+

The solution of $\frac{x+k}{x+4k} > \frac{k}{x}$ is $x < -4k$, $-2k < x < 0$, $2k < x$.

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Edexcel AS and A Level Modular Mathematics

Exercise A, Question 7

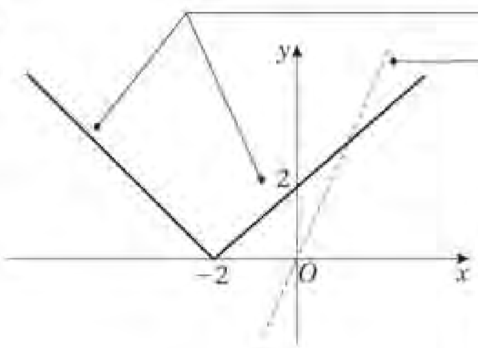
Question:

a Sketch the graph of $y = |x + 2|$.

b Use algebra to solve the inequality $2x > |x + 2|$.

Solution:

a



Inequalities which contain both an expression in x with a modulus sign and an expression in x without a modulus sign, are usually best answered by drawing a sketch. In this case, you have been instructed to draw the sketch first. The continuous line is the graph of $y = |x + 2|$. You should mark the coordinates of the points where the graph cuts the axis.

You should now add the graph of $y = 2x$ to your sketch. This has been done with a dotted line. You find the solution to the inequality by identifying the values of x where the dotted line is above the continuous line.

b The intersection occurs when $x > -2$.

When $x > -2$, $|x + 2| = x + 2$

$$2x = x + 2$$

$$x = 2$$

When $\mathrm{f}(x)$ is positive, $|\mathrm{f}(x)| = \mathrm{f}(x)$.

The solution of $2x > |x + 2|$ is $x > 2$.

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Exercise A, Question 8

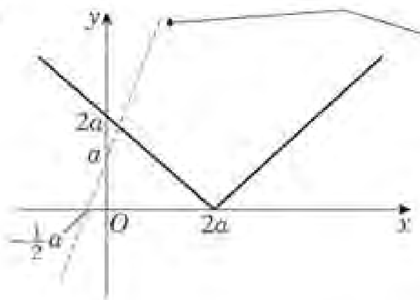
Question:

a Sketch the graph of $y = |x - 2a|$, given that $a > 0$.

b Solve $|x - 2a| > 2x + a$, where $a > 0$.

Solution:

a



The dotted line is added to the sketch in part **a** to help you to solve part **b**. The dotted line is the graph of $y = 2x + a$ and the solution to the inequality in part **b** is found by identifying where the continuous line, which corresponds to $|x - 2a|$, is above the dotted line, which corresponds to $2x + a$.

b The intersection occurs when $x < 2a$.

When $x < 2a$, $|x - 2a| = 2a - x$

If $f(x)$ is negative, then $|f(x)| = -f(x)$.

$$2a - x = 2x + a$$

$$-3x = -a \Rightarrow x = \tfrac{1}{3}a$$

The solution of $|x - 2a| > 2x + a$ is $x < \frac{1}{3}a$.

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Exercise A, Question 9

Question:

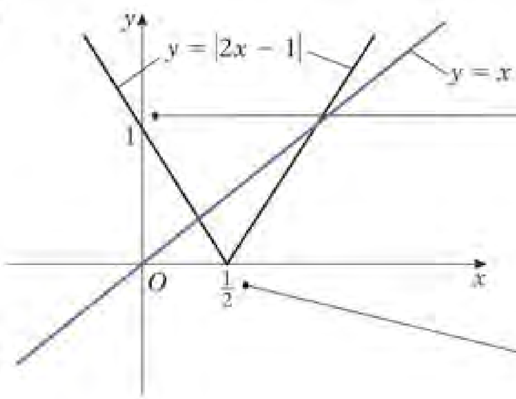
a On the same axes, sketch the graphs of $y = x$ and $y = |2x - 1|$.

b Use algebra to find the coordinates of the points of intersection of the two graphs.

c Hence, or otherwise, find the set of values of x for which $|2x - 1| > x$.

Solution:

a



At $x = 0$, $y = |-1| = 1$.

At $x = \frac{1}{2}$, $y = 0$.

b There are two points of intersection.
At the right hand point of intersection,

$x > \frac{1}{2} \Rightarrow |2x - 1| = 2x - 1$

$2x - 1 = x \Rightarrow x = 1$

At the left hand point of intersection,

$x < \frac{1}{2} \Rightarrow |2x - 1| = 1 - 2x$

$1 - 2x = x \Rightarrow x = \frac{1}{3}$

If $f(x) > 0$, then $|f(x)| = f(x)$.
If $f(x) < 0$, then $|f(x)| = -f(x)$.

The points of intersection of the two graphs are

$\left(\frac{1}{3}, \frac{1}{3}\right)$ and $(1, 1)$

You need to give both the x-coordinates and the y-coordinates.

c The solution of $|2x - 1| > x$ is $x < \frac{1}{3}$, $x > 1$.

You identify the regions on the graph where the V shape representing $y = |2x - 1|$ is above the line representing $y = x$.

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Exercise A, Question 10

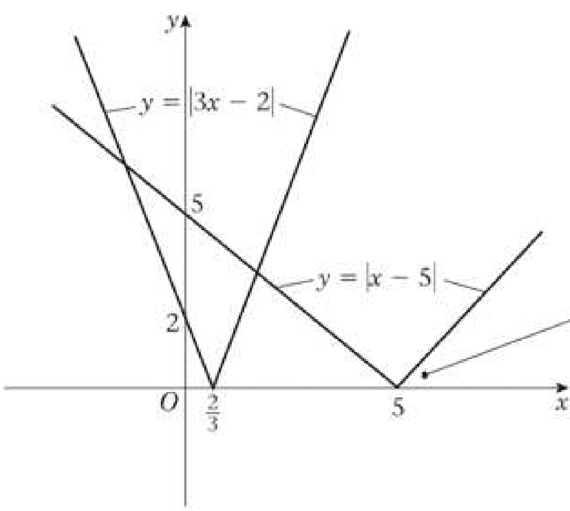
Question:

a On the same axes, sketch the graphs of $y = |x - 5|$ and $y = |3x - 2|$ distinguishing between them clearly.

b Find the set of values of x for which $|x - 5| < |3x - 2|$.

Solution:

a



You should mark the coordinates of the points where the graphs meet the axes.

b From the graph both intersections are in the region where $x < 5$ and $x - 5$ is negative.
Hence, $|x - 5| = 5 - x$

For $x > \frac{2}{3}$, $|3x - 2| = 3x - 2$

$3x - 2 = 5 - x$

$4x = 7 \Rightarrow x = \frac{7}{4}$

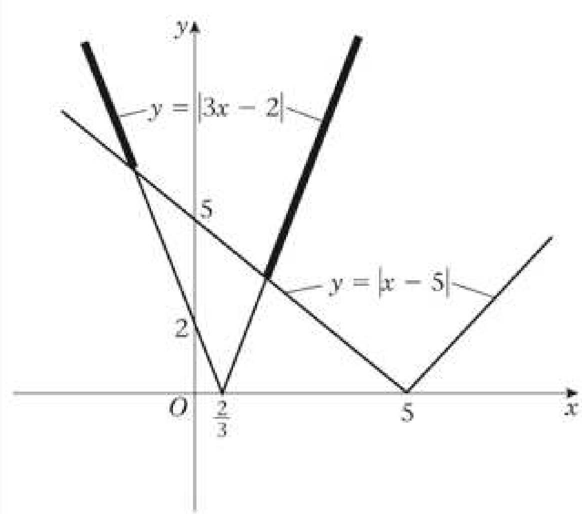
For $x < \frac{2}{3}$, $|3x - 2| = 2 - 3x$

$2 - 3x = 5 - x$

$-2x = 3 \Rightarrow x = -\frac{3}{2}$

The solution of $|x - 5| < |3x - 2|$ is

$x < -\frac{3}{2}, x > \frac{7}{4}$.



You identify the regions in which the lines representing $y = |x - 5|$ are below the lines representing $y = |3x - 2|$.
These are shown with heavy lines above.

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Exercise A, Question 11

Question:

Use algebra to find the set of real values of x for which $|x - 3| > 2|x + 1|$.

Solution:

$$|x - 3| > 2|x + 1|$$
$$(x - 3)^2 > 4(x + 1)^2$$
$$x^2 - 6x + 9 > 4x^2 + 8x + 4$$
$$0 > 3x^2 + 14x - 5$$
$$(x + 5)(3x - 1) < 0$$

As both $|x - 3|$ and $2|x + 1|$ are positive you can square both sides of the inequality without changing the direction of the inequality sign. If a and b are both positive, it is true that $a > b \Rightarrow a^2 > b^2$. You cannot make this step if either or both of a and b are negative.

Considering $f(x) = (x + 5)(3x - 1)$,
the critical values are $x = -5$ and $\frac{1}{3}$.

Alternatively you can draw a sketch of $y = (x + 5)(3x - 1)$ and identify the region where the curve is below the y-axis.

	$x < -5$	$-5 < x < \frac{1}{3}$	$\frac{1}{3} < x$
Sign of $f(x)$	+	−	+

The solution of $|x - 3| > 2|x + 1|$ is
$-5 < x < \frac{1}{3}$.

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Exercise A, Question 12

Question:

Find the set of real values of x for which

a $\frac{3x+1}{x-3} < 1$,

b $\left|\frac{3x+1}{x-3}\right| < 1$.

Solution:

a $$\frac{3x+1}{x-3} < 1$$

$$\frac{3x+1}{x-3} - 1 < 0$$

$$\frac{3x+1-1(x-3)}{x-3} < 0$$

$$\frac{2x+4}{x-3} = \frac{2(x+2)}{x-3} < 0$$

Considering $f(x) = \dfrac{2(x+2)}{x-3}$,

the critical values are $x = -2, 3$.

You should compare the solutions to parts **a** and **b**. The questions look similar but the algebraic methods of solution used here are quite different.

	$x < -2$	$-2 < x < 3$	$3 < x$
Sign of $f(x)$	+	−	+

The solution of $\dfrac{3x+1}{x-3} < 1$ is $-2 < x < 3$.

b $$\left|\frac{3x+1}{x-3}\right| < 1$$

$$\left(\frac{3x+1}{x-3}\right)^2 < 1$$

As both $\left|\dfrac{3x+1}{x-3}\right|$ and 1 are positive you can square both sides of the inequality without changing the direction of the inequality sign.

$$(3x+1)^2 < (x-3)^2$$

$$9x^2 + 6x + 1 < x^2 - 6x + 9$$

$$8x^2 + 12x - 8 < 0$$

As 4 is a positive number, you can divide throughout the inequality by 4.

$$2x^2 + 3x - 2 = (x+2)(2x-1) < 0$$

Considering $f(x) = (x+2)(2x-1)$,

the critical values are $x = -2$ and $x = \frac{1}{2}$.

	$x < -2$	$-2 < x < \frac{1}{2}$	$\frac{1}{2} < x$
Sign of $f(x)$	+	−	+

The solution of $\left|\dfrac{3x+1}{x-3}\right| < 1$ is $-2 < x < \frac{1}{2}$.

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Exercise A, Question 13

Question:

Solve, for x, the inequality $|5x + a| \leqslant |2x|$, where $a > 0$.

Solution:

$|5x + a| \leqslant |2x|$

$(5x + a)^2 \leqslant (2x)^2$

As a is positive, both $|5x + a|$ and $|2x|$ are positive and you can square both sides of the inequality.

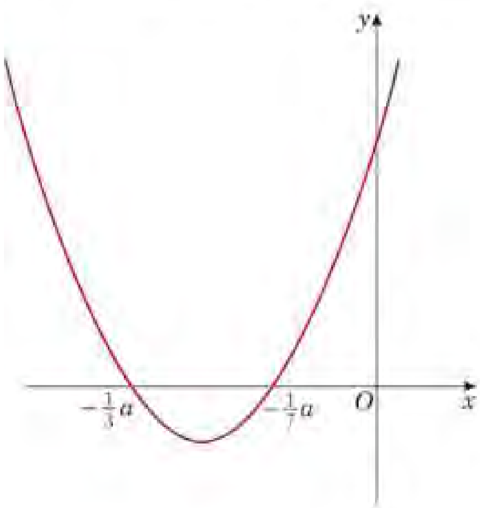
$25x^2 + 10ax + a^2 \leqslant 4x^2$

$21x^2 + 10ax + a^2 \leqslant 0$

$(3x + a)(7x + a) \leqslant 0$

Sketching $y = (3x + a)(7x + a)$

The graph is a parabola intersecting the x-axis at $x = -\frac{1}{3}a$ and $x = -\frac{1}{7}a$.



A common error here is not to realise that, for a positive a, $-\frac{1}{3}a$ is a smaller number than $-\frac{1}{7}a$. It is very easy to get the inequality the wrong way round.

The solution of $|5x + a| \leqslant |2x|$ is $-\frac{1}{3}a \leqslant x \leqslant -\frac{1}{7}a$.

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Exercise A, Question 14

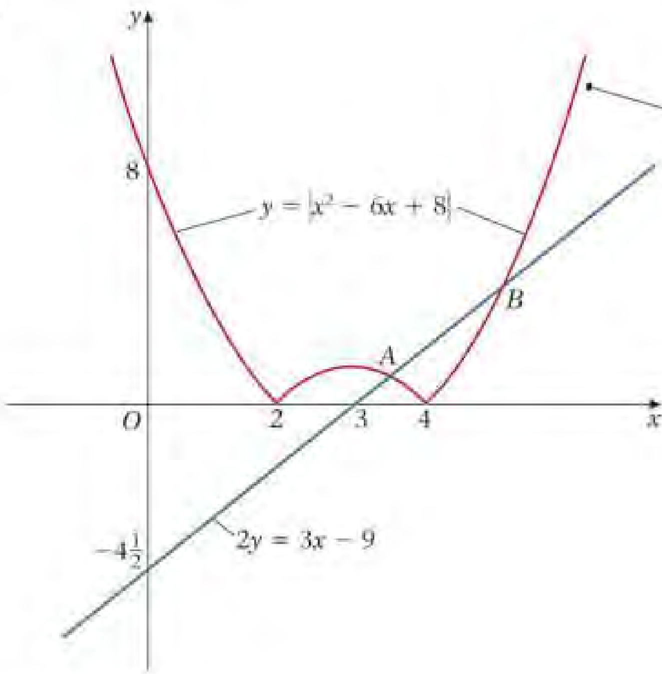
Question:

a Using the same axes, sketch the curve with equation $y = |x^2 - 6x + 8|$ and the line with equation $2y = 3x - 9$. State the coordinates of the points where the curve and the line meet the x-axis.

b Use algebra to find the coordinates of the points where the curve and the line intersect and, hence, solve the inequality $2|x^2 - 6x + 8| > 3x - 9$.

Solution:

a



As $x^2 - 6x + 8 = (x - 2)(x - 4)$ the curve meets the x-axis at $x = 2$ and $x = 4$. The sketching of the graphs of modulus functions is in Chapter 5 of book C3.

The curve meets the x-axis at $(2, 0)$ and $(4, 0)$.

The line meets the x-axis at $(3, 0)$.

b To find the coordinates of A
The x-coordinate of A is in the interval $2 < x < 4$
In this interval $x^2 - 6x + 8$ is negative and, hence,

$$|x^2 - 6x + 8| = -x^2 + 6x - 8$$

If $f(x) < 0$, then $|f(x)| = -f(x)$.

$$-x^2 + 6x - 8 = \frac{3x - 9}{2}$$

$$-2x^2 + 12x - 16 = 3x - 9$$

$$2x^2 - 9x + 7 = 0$$

$$(2x - 7)(x - 1) = 0$$

$$x = \tfrac{7}{2}, \cancel{1}$$

As the x-coordinate of A is in the interval $2 < x < 4$, the solution $x = 1$ must be rejected.

$$y = \frac{3 \times \frac{7}{2} - 9}{2} = \frac{3}{4}$$

The coordinates of A are $\left(\frac{7}{2}, \frac{3}{4}\right)$.

To find the coordinates of B
The x-coordinate of B is in the interval $x > 4$
In this interval $x^2 - 6x + 8$ is positive and, hence,

$$|x^2 - 6x + 8| = x^2 - 6x - 8$$

If $f(x) > 0$, then $|f(x)| = f(x)$.

$$x^2 + 6x + 8 = \frac{3x - 9}{2}$$

$$2x^2 - 12x + 16 = 3x - 9$$

$$2x^2 - 15x + 25 = 0$$

$$(x - 5)(2x - 5) = 0$$

$$x = 5, \cancel{2\tfrac{1}{2}}$$

As the x-coordinate of B is in the interval $x > 4$, the solution $x = 2\frac{1}{2}$ must be rejected.

$$y = \frac{3 \times 5 - 9}{2} = 3$$

The coordinates of B are $(5, 3)$.

c The solution of $2|x^2 - 6x + 8| > 3x - 9$ is $x < 3\frac{1}{2}, x > 5$.

You solve the inequality by inspecting the graphs. You look for the values of x where the curve is above the line.

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Exercise A, Question 15

Question:

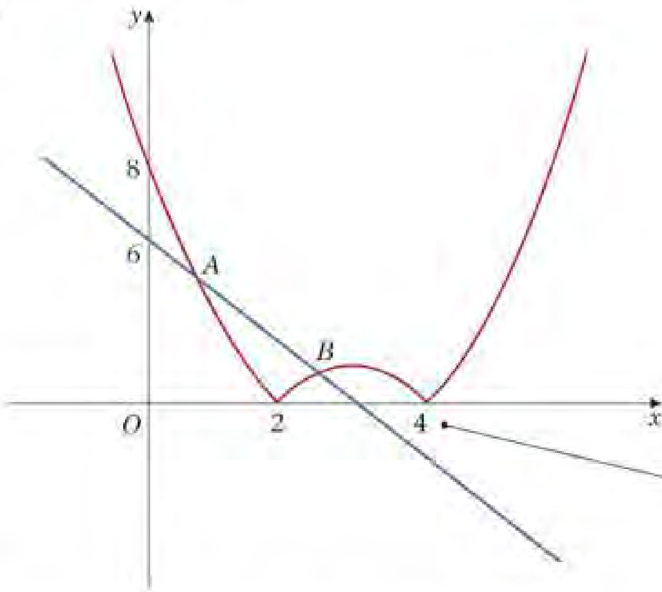
a Sketch, on the same axes, the graph of $y = |(x - 2)(x - 4)|$, and the line with equation $y = 6 - 2x$.

b Find the exact values of x for which $|(x - 2)(x - 4)| = 6 - 2x$.

c Hence solve the inequality $|(x - 2)(x - 4)| < 6 - 2x$.

Solution:

a



You should mark the coordinates of the points where the graphs meet the axes.

b Let the points where the graphs intersect be A and B.

For A, $(x-2)(x-4)$ is positive

$$(x-2)(x-4) = 6-2x$$
$$x^2 - 6x + 8 = 6 - 2x$$
$$x^2 - 4x = -2$$
$$x^2 - 4x + 4 = 2$$
$$(x-2)^2 = 2$$
$$x = 2 - \sqrt{2}$$

The quadratic equations have been solved by completing the square. You could use the formula for solving a quadratic but the conditions of the question require exact solutions and you should not use decimals.

The quadratic equation has another solution $2 + \sqrt{2}$ but the diagram shows that the x-coordinate of A is less than 2, so this solution is rejected.

For B, $(x-2)(x-4)$ is negative

$$-(x-2)(x-4) = 6-2x$$
$$-x^2 + 6x - 8 = 6 - 2x$$
$$x^2 - 8x = -14$$
$$x^2 - 8x + 16 = 2$$
$$(x-4)^2 = 2$$
$$x = 4 - \sqrt{2}$$

The quadratic equation has another solution $4 + \sqrt{2}$ but the diagram shows that the x-coordinate of B is less than 4, so this solution is rejected.

The values of x for which $|(x-2)(x-4)| = 6-2x$ are $2-\sqrt{2}$ and $4-\sqrt{2}$.

c The solution of $|(x-2)(x-4)| < 6-2x$ is $2-\sqrt{2} < x < 4-\sqrt{2}$.

You look for the values of x where the curve is below the line.

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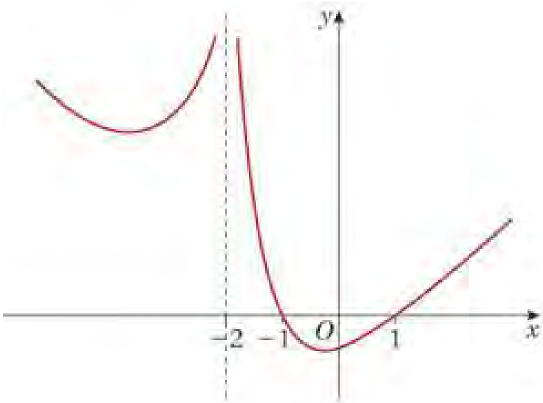
Exercise A, Question 16

Question:



The figure above shows a sketch of the curve with equation

$$y = \frac{x^2 - 1}{|x + 2|}, \quad x \neq -2.$$

The curve crosses the x-axis at $x = 1$ and $x = -1$ and the line $x = -2$ is an asymptote of the curve.

a Use algebra to solve the equation

$$\frac{x^2 - 1}{|x + 2|} = 3(1 - x).$$

b Hence, or otherwise, find the set of values of x for which

$$\frac{x^2 - 1}{|x + 2|} < 3(1 - x).$$

Solution:

a For $x > -2$, $x + 2$ is positive and the equation is

$$\frac{x^2 - 1}{x + 2} = 3(1 - x)$$

$$x^2 - 1 = 3(1 - x)(x + 2) = -3x^2 - 3x + 6$$

$$4x^2 + 3x - 7 = (4x + 7)(x - 1) = 0$$

$$x = -\tfrac{7}{4}, 1$$

As both of these answers are greater than -2 both are valid.

For $x < -2$, $x + 2$ is negative and the equation is

$$\frac{x^2 - 1}{-(x + 2)} = 3(1 - x)$$

$$x^2 - 1 = -3(1 - x)(x + 2) = 3x^2 + 3x - 6$$

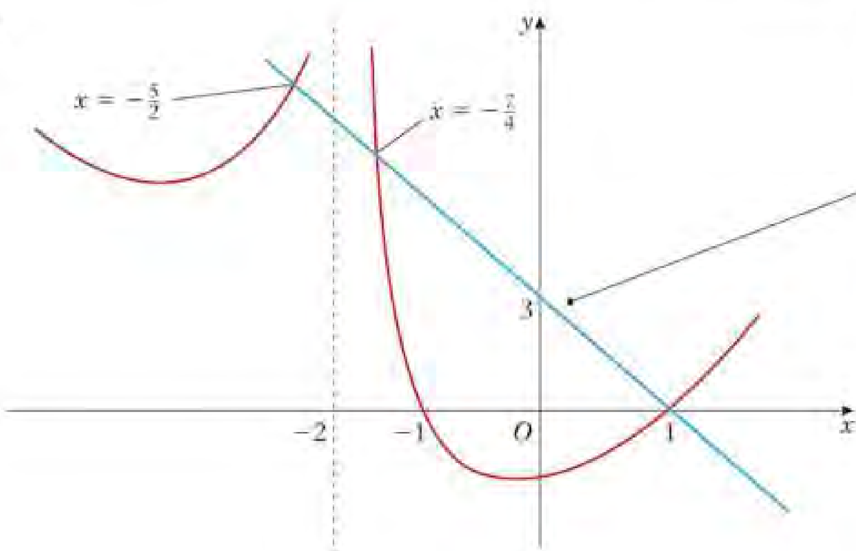
$$2x^2 + 3x - 5 = (2x + 5)(x - 1) = 0$$

$$x = -\tfrac{5}{2}, \cancel{1}$$

As 1 is not less than -2 the answer 1 should be 'rejected' here. However, the earlier working has already shown 1 to be a correct solution.

The solutions are $-\frac{5}{2}$, $-\frac{7}{4}$ and 1.

b



To complete the question, you add the graph of $y = 3(1 - x)$ to the graph which has already been drawn for you. You know the x-coordinates of the points of intersection from part **a**.

The solution of $\dfrac{x^2 - 1}{|x + 2|} < 3(1 - x)$ is

$x < -\frac{5}{2}$, $-\frac{7}{4} < x < 1$.

You look for the values of x on the graph where the curve is below the line.

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Exercise A, Question 17

Question:

a Express $\frac{1}{(x+1)(x+2)}$ in partial fractions.

b Hence, or otherwise, show that

$$\sum_{r=1}^{n} \frac{2}{(r+1)(r+2)} = \frac{n}{n+2}.$$

Solution:

a $$\frac{1}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2}$$

The methods used for partial fractions are in Chapter 1 of book C4. You may use any of the methods which can be used to split complex fractions into partial fractions. Here the substitution method is used.

Multiply throughout by $(x+1)(x+2)$

$$1 = A(x+2) + B(x+1)$$

Substitute $x = -1$

$$1 = A(-1+2) + B(-1+1) = A \Rightarrow A = 1$$

Substitute $x = -2$

$$1 = A(-2+2) + B(-2+1) = -B \Rightarrow B = -1$$

Hence

$$\frac{1}{(x+1)(x+2)} = \frac{1}{x+1} - \frac{1}{x+2}$$

b Substituting r for x and multiplying by 2

$$\frac{2}{(r+1)(r+2)} = \frac{2}{r+1} - \frac{2}{r+2}$$

The summation involves twice the fraction you worked on in part **a** with r substituted for x. So you begin by making the substitution and multiplying every term by 2.

Hence

$$\sum_{r=1}^{n} \frac{2}{(r+1)(r+2)} = \sum_{r=1}^{n} \left(\frac{2}{r+1} - \frac{2}{r+2}\right)$$

$$= \frac{2}{2} - \cancel{\frac{2}{3}}$$

This is the first term of the summation, with $r = 1$, broken up using the partial fractions. Here

$$\frac{2}{(1+1)(1+2)} = \frac{2}{1+1} - \frac{2}{1+2} = \frac{2}{2} - \frac{2}{3}.$$

$$+ \cancel{\frac{2}{3}} - \cancel{\frac{2}{4}}$$

$$+ \cancel{\frac{2}{4}} - \cancel{\frac{2}{5}}$$

This is the third term of the summation, with $r = 3$, broken up using the partial fractions. Here

$$\frac{2}{(3+1)(3+2)} = \frac{2}{3+1} - \frac{2}{3+2} = \frac{2}{4} - \frac{2}{5}.$$

$$\vdots$$

$$+ \cancel{\frac{2}{n}} - \cancel{\frac{2}{n+1}}$$

$$+ \cancel{\frac{2}{n+1}} - \frac{2}{n+2}$$

$$= \frac{2}{2} - \frac{2}{n+2} = 1 - \frac{2}{n+2}$$

$$= \frac{n+2-2}{n+2} = \frac{n}{n+2}, \text{ as required}$$

You write out two or three terms at the beginning and end of the summation and show, by crossing through the fractions, how the fractions cancel each other out. In this case all of the terms are cancelled except the first and last and these are the only two left.

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Exercise A, Question 18

Question:

a Express $\frac{2}{(r+1)(r+3)}$ in partial fractions.

b Hence prove that

$$\sum_{r=1}^{n} \frac{2}{(r+1)(r+3)} = \frac{n(5n+13)}{6(n+2)(n+3)}.$$

Solution:

a $$\frac{2}{(r+1)(r+3)} = \frac{A}{r+1} + \frac{B}{r+3}$$

Multiply throughout by $(r+1)(r+3)$

$$2 = A(r+3) + B(r+1)$$

Equating the coefficients of r

$$0 = A + B \quad ①$$

Equating the constant coefficients

$$2 = 3A + B \quad ②$$

The methods used for partial fractions are in Chapter 1 of book C4. You may use any of the methods which can be used to split complex fractions into partial fractions. Here the method used is equating coefficients and solving the resulting simultaneous equations.

Subtracting ① from ②

$$2 = 2A \Rightarrow A = 1$$

Substituting $A = 1$ into ①

$$0 = 1 + B \Rightarrow B = -1$$

Hence

$$\frac{2}{(r+1)(r+3)} = \frac{1}{r+1} - \frac{1}{r+3}$$

You use the partial fractions in part **a** to break up each term in the summation into two parts.

b $$\sum_{r=1}^{n} \frac{2}{(r+1)(r+3)} = \sum_{r=1}^{n} \left(\frac{1}{r+1} - \frac{1}{r+3}\right)$$

$$= \frac{1}{2} - \cancel{\frac{1}{4}}$$

This is the first term of the summation, with $r = 1$, broken up using the partial fractions. Here $\frac{2}{(1+1)(1+3)} = \frac{1}{1+1} - \frac{1}{1+3} = \frac{1}{2} - \frac{1}{4}$.

$$+ \frac{1}{3} - \cancel{\frac{1}{5}}$$

$$+ \cancel{\frac{1}{4}} - \cancel{\frac{1}{6}}$$

$$+ \cancel{\frac{1}{5}} - \cancel{\frac{1}{7}}$$

$$\vdots$$

$$+ \cancel{\frac{1}{n}} - \frac{1}{n+2}$$

You write out some terms at the beginning and end of the summation and show, by crossing through the fractions, how the fractions cancel each other out. In this case two terms are left at the start of the summation and two at the end.

$$+ \cancel{\frac{1}{n+1}} - \frac{1}{n+3}$$

This is the nth term of the summation, with $r = n$, broken up using the partial fractions. Here $\frac{2}{(n+1)(n+3)} = \frac{1}{n+1} - \frac{1}{n+3}$.

$$= \frac{1}{2} + \frac{1}{3} - \frac{1}{n+2} - \frac{1}{n+3}$$

$$= \frac{5}{6} - \frac{1}{n+2} - \frac{1}{n+3}$$

$$= \frac{5(n+2)(n+3) - 6(n+3) - 6(n+2)}{6(n+2)(n+3)}$$

$$= \frac{5n^2 + 25n + 30 - 6n - 18 - 6n - 12}{6(n+2)(n+3)}$$

$$= \frac{5n^2 + 13n}{6(n+2)(n+3)} = \frac{n(5n+13)}{6(n+2)(n+3)}, \text{ as required.}$$

You complete the question by expressing your answer as a single fraction and simplifying it to the answer exactly as it is printed on the question paper.

Solutionbank FP2
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Exercise A, Question 19

Question:

a Show that

$$\frac{r+1}{r+2} - \frac{r}{r+1} \equiv \frac{1}{(r+1)(r+2)}, \quad r \in \mathbb{Z}^+.$$

b Hence, or otherwise, find

$\sum_{r=1}^{n} \frac{1}{(r+1)(r+2)}$, giving your answer as a single fraction in terms of n.

Solution:

a LHS $= \dfrac{r+1}{r+2} - \dfrac{r}{r+1}$

$= \dfrac{(r+1)^2 - r(r+2)}{(r+1)(r+2)}$

$= \dfrac{r^2 + 2r + 1 - r^2 - 2r}{(r+1)(r+2)}$

$= \dfrac{1}{(r+1)(r+2)}$

= RHS, as required

To show that an algebraic identity is true, you should start from one side of the identity, here the left hand side (LHS), and use algebra to show that it is equal to the other side of the identity, here the right hand side (RHS).

b $\displaystyle\sum_{r=1}^{n} \frac{1}{(r+1)(r+2)} = \sum_{r=1}^{n} \left(\frac{r+1}{r+2} - \frac{r}{r+1}\right)$

You use the identity that you proved in part **a** to break up each term in the summation into two parts.

$= \cancel{\tfrac{2}{3}} - \tfrac{1}{2}$

This is the LHS of the identity with $r = 1$.

$+ \cancel{\tfrac{3}{4}} - \cancel{\tfrac{2}{3}}$

This is the LHS of the identity with $r = 2$.

$+ \cancel{\tfrac{4}{5}} - \cancel{\tfrac{3}{4}}$

This is the LHS of the identity with $r = 3$.

$\vdots$

$+ \cancel{\dfrac{n}{n+1}} - \cancel{\dfrac{n-1}{n}}$

This is the LHS of the identity with $r = n - 1$.

$\dfrac{r+1}{r+2} - \dfrac{r}{r+1} = \dfrac{n-1+1}{n-1+2} - \dfrac{n-1}{n-1+1}$

$= \dfrac{n}{n+1} - \dfrac{n-1}{n}$

$+ \dfrac{n+1}{n+2} - \cancel{\dfrac{n}{n+1}}$

This is the LHS of the identity with $r = n$.

$= \dfrac{n+1}{n+2} - \dfrac{1}{2}$

$= \dfrac{2(n+1) - (n+2)}{2(n+2)} = \dfrac{2n + 2 - n - 2}{2(n+2)}$

$= \dfrac{n}{2(n+2)}$

The only terms which have not cancelled one another out are the $-\frac{1}{2}$ in the first line of the summation and the $\dfrac{n+1}{n+2}$ in the last line.

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Exercise A, Question 20

Question:

$$f(x) = \frac{2}{(x+1)(x+2)(x+3)}$$

a Express f(x) in partial fractions.

b Hence find $\sum_{r=1}^{n} f(r)$.

Solution:

a Let $\dfrac{2}{(x+1)(x+2)(x+3)} = \dfrac{A}{x+1} + \dfrac{B}{x+2} + \dfrac{C}{x+3}$

Multiplying throughout by $(x+1)(x+2)(x+3)$

$2 = A(x+2)(x+3) + B(x+1)(x+3) + C(x+1)(x+2)$

Substitute $x = -1$

$2 = A \times 1 \times 2 \Rightarrow A = 1$

Substitute $x = -2$

$2 = B \times -1 \times 1 \Rightarrow B = -2$

Substitute $x = -3$

$2 = C \times -2 \times -1 \Rightarrow C = 1$

Hence

$$f(x) = \frac{1}{x+1} - \frac{2}{x+2} + \frac{1}{x+3}$$

When -1 is substituted for x then both $B(x+1)(x+3)$ and $C(x+1)(x+2)$ become zero.

b Using the result in part **a** with $x = r$

$$\sum_{r=1}^{n} f(r) = \frac{1}{r+1} - \frac{2}{r+2} + \frac{1}{r+3}$$

You use the partial fractions in part **a** to break up each term in the summation into three parts.

$$\begin{aligned}
&= \tfrac{1}{2} - \tfrac{2}{3} + \cancel{\tfrac{1}{4}} \\
&\quad + \tfrac{1}{3} - \cancel{\tfrac{2}{4}} + \cancel{\tfrac{1}{5}} \\
&\quad + \cancel{\tfrac{1}{4}} - \cancel{\tfrac{2}{5}} + \cancel{\tfrac{1}{6}} \\
&\quad \vdots \\
&\quad + \cancel{\frac{1}{n-1}} - \cancel{\frac{2}{n}} + \cancel{\frac{1}{n+1}} \\
&\quad + \cancel{\frac{1}{n}} - \cancel{\frac{2}{n+1}} + \frac{1}{n+2} \\
&\quad + \cancel{\frac{1}{n+1}} - \frac{2}{n+2} + \frac{1}{n+3}
\end{aligned}$$

Three terms at the beginning of the summation and three terms at the end have not been cancelled out.

$$= \frac{1}{2} - \frac{2}{3} + \frac{1}{3} + \frac{1}{n+2} - \frac{2}{n+2} + \frac{1}{n+3}$$

$$= \frac{1}{6} - \frac{1}{n+2} + \frac{1}{n+3}$$

This question asks for no particular form of the answer. You should collect together like terms but, otherwise, the expression can be left as it is. You do not have to express your answer as a single fraction unless the question asks you to do this.

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Exercise A, Question 21

Question:

a Express as a simplified single fraction $\frac{1}{(r-1)^2} - \frac{1}{r^2}$.

b Hence prove, by the method of differences, that

$$\sum_{r=2}^{n} \frac{2r-1}{r^2(r-1)^2} = 1 - \frac{1}{n^2}.$$

Solution:

a
$$\frac{1}{(r-1)^2} - \frac{1}{r^2} = \frac{r^2-(r-1)^2}{r^2(r-1)^2}$$

$$= \frac{r^2-(r^2-2r+1)}{r^2(r-1)^2}$$

$$= \frac{2r-1}{r^2(r-1)^2}$$

Methods for simplifying algebraic fractions can be found in Chapter 1 of book C3.

b
$$\sum_{r=2}^{n} \frac{2r-1}{r^2(r-1)^2} = \sum_{r=2}^{n}\left(\frac{1}{(r-1)^2} - \frac{1}{r^2}\right)$$

This summation starts from $r = 2$ and not from the more common $r = 1$. It could not start from $r = 1$ as $\frac{1}{(r-1)^2}$ is not defined for that value.

$$= \frac{1}{1^2} - \cancel{\frac{1}{2^2}}$$
$$+ \cancel{\frac{1}{2^2}} - \cancel{\frac{1}{3^2}}$$
$$+ \cancel{\frac{1}{3^2}} - \cancel{\frac{1}{4^2}}$$
$$\vdots$$
$$+ \cancel{\frac{1}{(n-2)^2}} - \cancel{\frac{1}{(n-1)^2}}$$
$$+ \cancel{\frac{1}{(n-1)^2}} - \frac{1}{n^2}$$

In this summation all of the terms cancel out with one another except for one term at the beginning and one term at the end.

$$= \frac{1}{1^2} - \frac{1}{n^2} = 1 - \frac{1}{n^2}, \text{ as required}$$

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Exercise A, Question 22

Question:

Find the sum of the series

$$\ln\frac{1}{2} + \ln\frac{2}{3} + \ln\frac{3}{4} + \ldots + \ln\frac{n}{n+1}.$$

Solution:

Let $S = \ln\frac{1}{2} + \ln\frac{2}{3} + \ln\frac{3}{4} + \ldots + \ln\frac{n}{n+1}$

The general term of this series is $\ln\frac{r}{r+1}$.

Using a law of logarithms

$$\ln\frac{r}{r+1} = \ln r - \ln(r+1)$$

$$S = \sum_{r=1}^{n} \ln\frac{r}{r+1} = \sum_{r=1}^{n} (\ln r - \ln(r+1))$$

For logarithms to any base $\ln\frac{a}{b} = \ln a - \ln b$. This law gives a difference and so you can use the method of differences to sum the series.

$$\begin{aligned} &= \ln 1 - \cancel{\ln 2} \\ &\quad + \cancel{\ln 2} - \cancel{\ln 3} \\ &\quad + \cancel{\ln 3} - \cancel{\ln 4} \\ &\qquad \vdots \\ &\quad + \cancel{\ln(n-1)} - \cancel{\ln n} \\ &\quad + \cancel{\ln n} - \ln(n+1) \\ &= \ln 1 - \ln(n+1) = -\ln(n+1) \end{aligned}$$

$\boxed{\ln 1 = 0}$

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Exercise A, Question 23

Question:

a Express $\frac{1}{r(r+2)}$ in partial fractions.

b Hence prove, by the method of differences, that

$$\sum_{r=1}^{n} \frac{4}{r(r+2)} = \frac{n(3n+5)}{(n+1)(n+2)}.$$

c Find the value of $\sum_{r=50}^{100} \frac{4}{r(r+2)}$, to 4 decimal places.

Solution:

a Let $\frac{1}{r(r+2)} = \frac{A}{r} + \frac{B}{r+2}$

You may use any appropriate method to find the partial fractions. If you know an abbreviated method, often called the 'cover up rule', this is accepted at this level.

Multiply throughout by $r(r+2)$

$$1 = A(r+2) + Br$$

Equating constant coefficients

$$1 = 2A \Rightarrow A = \tfrac{1}{2}$$

Equating coefficients of r

$$0 = A + B \Rightarrow B = -A = -\tfrac{1}{2}$$

Hence

$$\frac{1}{r(r+2)} = \frac{1}{2r} - \frac{1}{2(r+2)}$$

b $$\frac{4}{r(r+2)} = \frac{2}{r} - \frac{2}{r+2}$$

You need to multiply the result of part **a** throughout by 4 to apply the result to part **b**. Remember to multiply every term by 4.

$$\sum_{r=1}^{n} \frac{4}{r(r+2)} = \sum_{r=1}^{n} \left(\frac{2}{r} - \frac{2}{r+2}\right)$$

$$= \frac{2}{1} - \cancel{\frac{2}{3}}$$
$$+ \frac{2}{2} - \cancel{\frac{2}{4}}$$
$$+ \cancel{\frac{2}{3}} - \cancel{\frac{2}{5}}$$
$$\vdots$$
$$+ \cancel{\frac{2}{n-2}} - \cancel{\frac{2}{n}}$$
$$+ \cancel{\frac{2}{n-1}} - \frac{2}{n+1}$$
$$+ \cancel{\frac{2}{n}} - \frac{2}{n+2}$$

Each right hand term is cancelled out by the left hand term two rows below it.

$$= \frac{2}{1} + \frac{2}{2} - \frac{2}{n+1} - \frac{2}{n+2}$$

Four terms are left. Two from the beginning of the summation and two from the end.

$$= 3 - \frac{2}{n+1} - \frac{2}{n+2}$$

$$= \frac{3(n+1)(n+2) - 2(n+2) - 2(n+1)}{(n+1)(n+2)}$$

$$= \frac{3n^2 + 9n + 6 - 2n - 4 - 2n - 2}{(n+1)(n+2)}$$

$$= \frac{3n^2 + 5n}{(n+1)(n+2)} = \frac{n(3n+5)}{(n+1)(n+2)}, \text{ as required.}$$

You have to get your answer exactly into the form printed in the question. Put all three terms over the common denominator $(n+1)(n+2)$ and simplify the numerator.

c $$\sum_{r=50}^{100} \frac{4}{r(r+2)} = \sum_{r=1}^{100} \frac{4}{r(r+2)} - \sum_{r=1}^{49} \frac{4}{r(r+2)}$$

$$= \frac{100 \times 305}{101 \times 102} - \frac{49 \times 152}{50 \times 51}$$

$$= 2.960\,590\ldots - 2.920\,784$$

$$= 0.0398 \text{ (4 d.p.)}$$

$$\sum_{r=50}^{100} \mathrm{f}(r) = \sum_{r=1}^{100} \mathrm{f}(r) - \sum_{r=1}^{49} \mathrm{f}(r)$$

You find the sum from the 50th to the 100th term by subtracting the sum from the first to the 49th term from the sum from the first to the 100th term.
It is a common error to subtract one term too many, in this case the 50th term. The sum you are finding starts with the 50th term. You must not subtract it from the series – you have to leave it in the series.

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Exercise A, Question 24

Question:

a By expressing $\frac{2}{4r^2 - 1}$ in partial fractions, or otherwise, prove that

$$\sum_{r=1}^{n} \frac{2}{4r^2 - 1} = 1 - \frac{1}{2n + 1}.$$

b Hence find the exact value of

$$\sum_{r=11}^{20} \frac{2}{4r^2 - 1}.$$

Solution:

a $4r^2 - 1 = (2r - 1)(2r + 1)$

Let

$$\frac{2}{4r^2 - 1} = \frac{2}{(2r-1)(2r+1)} = \frac{A}{2r-1} + \frac{B}{2r+1}$$

Multiply throughout by $(2r - 1)(2r + 1)$

$$2 = A(2r + 1) + B(2r - 1)$$

Substitute $r = \frac{1}{2}$

$$2 = 2A \Rightarrow A = 1$$

Substitute $r = -\frac{1}{2}$

$$2 = -2B \Rightarrow B = -1$$

Hence

$$\frac{2}{4r^2 - 1} = \frac{1}{2r-1} - \frac{1}{2r+1}$$

This question gives you the option to choose your own method (the questions has 'or otherwise') and, as you are given the answer, you could, if you preferred, use the method of mathematical induction which you learnt in module FP1.
If the method of differences is used, you begin by factorising $4r^2 - 1$, using the difference of two squares, and then express

$\frac{2}{(2r-1)(2r+1)}$ in partial fractions.

$$\sum_{r=1}^{n} \frac{2}{4r^2 - 1} = \sum_{r=1}^{n} \left(\frac{1}{2r-1} - \frac{1}{2r+1}\right)$$

$$= \frac{1}{1} - \cancel{\frac{1}{3}}$$

$$+ \cancel{\frac{1}{3}} - \cancel{\frac{1}{5}}$$

$$+ \cancel{\frac{1}{5}} - \cancel{\frac{1}{7}}$$

$$\vdots$$

$$+ \cancel{\frac{1}{2n-3}} - \cancel{\frac{1}{2n-1}}$$

$$+ \cancel{\frac{1}{2n-1}} - \frac{1}{2n+1}$$

$$= 1 - \frac{1}{2n+1}, \text{ as required}$$

With $r = 1$,

$$\frac{1}{2r-1} - \frac{1}{2r+1} = \frac{1}{2 \times 1 - 1} - \frac{1}{2 \times 1 + 1} = \frac{1}{1} - \frac{1}{3}$$

With $r = n - 1$,

$$\frac{1}{2r-1} - \frac{1}{2r+1} = \frac{1}{2 \times (n-1) - 1} - \frac{1}{2 \times (n-1) + 1}$$

$$= \frac{1}{2n - 2 - 1} - \frac{1}{2n - 2 + 1} = \frac{1}{2n-3} - \frac{1}{2n-1}$$

The only terms which are not cancelled out in the summation are the $\frac{1}{1}$ at the beginning and the $-\frac{1}{2n+1}$ at the end.

b $$\sum_{r=11}^{20} \frac{2}{4r^2 - 1} = \sum_{r=1}^{20} \frac{2}{4r^2 - 1} - \sum_{r=1}^{10} \frac{2}{4r^2 - 1}$$

$$= \left(1 - \tfrac{1}{41} - 1 + \tfrac{1}{21}\right)$$

$$= -\frac{1}{41} + \frac{1}{21} = \frac{-21 + 41}{41 \times 21}$$

$$= \tfrac{20}{861}$$

You find the sum from the 11th to the 20th term by subtracting the sum from the first to the 10th term from the sum from the first to the 20th term.

The conditions of the question require an exact answer, so you must not use decimals.

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Exercise A, Question 25

Question:

Given that for all real values of r,

$$(2r+1)^3 - (2r-1)^3 = Ar^2 + B,$$

where A and B are constants,

a find the value of A and the value of B.

b Hence show that

$$\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1).$$

c Calculate $\sum_{r=1}^{40} (3r-1)^2$.

Solution:

a Using the binomial expansion

$(2r+1)^3 = 8r^3 + 12r^2 + 6r + 1$ ①

$(2r-1)^3 = 8r^3 - 12r^2 + 6r - 1$ ②

Subtracting ② from ①

$(2r+1)^3 - (2r-1)^3 - 24r^2 + 2$ ③

$A = 24,\ B = 2$

Subtracting the two expansions gives an expression in r^2. This enables you to sum r^2 using the method of differences.

b Using identity ③ in part **a**

$$\sum_{r=1}^{n}(24r^2+2) = \sum_{r=1}^{n}\left((2r+1)^3 - (2r-1)^3\right)$$

$$24\sum_{r=1}^{n} r^2 + \sum_{r=1}^{n} 2 = \sum_{r=1}^{n}\left((2r+1)^3 - (2r-1)^3\right)$$

$$\sum_{r=1}^{n} 2 = \underbrace{2+2+2+\ldots+2}_{n \text{ times}} = 2n$$

It is a common error to write $\sum_{r=1}^{n} 2 = 2$.

$$\begin{aligned}
24\sum_{r=1}^{n} r^2 + 2n = {} & \cancel{3^3} - 1^3 \\
& + \cancel{5^3} - \cancel{3^3} \\
& + \cancel{7^3} - \cancel{5^3} \\
& \vdots \\
& + \cancel{(2n-1)^3} - \cancel{(2n-3)^3} \\
& + (2n+1)^3 - \cancel{(2n-1)^3}
\end{aligned}$$

This expression is $(2r+1)^3 - (2r-1)^3$ with $n-1$ substituted for r.
$(2(n-1)+1)^3 - (2(n-1)-1)^3$
$= (2n-1)^3 - (2n-3)^3$

$$24\sum_{r=1}^{n} r^2 + 2n = (2n+1)^3 - 1$$

Summing gives you an equation in $\sum r^2$, which you solve. You then factorise the result to give the answer in the form required by the question.

$$24\sum_{r=1}^{n} r^2 = 8n^3 + 12n^2 + 6n + 1 - 1 - 2n$$

$$= 8n^3 + 12n^2 + 4n = 4n(2n^2 + 3n + 1)$$

$$= 4n(n+1)(2n+1)$$

$$\sum_{r=1}^{n} r^2 = \frac{4n(n+1)(2n+1)}{24} = \tfrac{1}{6}n(n+1)(2n+1), \text{ as required.}$$

c $(3r-1)^2 = 9r^2 - 6r + 1$

Hence

$$\sum_{r=1}^{40}(3r-1)^2 = 9\sum_{r=1}^{40} r^2 - 6\sum_{r=1}^{40} r + \sum_{r=1}^{40} 1$$

In the formula proved in part **b**, you replace the n by 40.

Using the result in part **b**.

$$9\sum_{r=1}^{40} r^2 = 9 \times \tfrac{1}{6} \times 40 \times 41 \times 81 = 199\,260$$

Using the standard result $\sum_{r=1}^{n} r = \frac{n(n+1)}{2}$,

This is a standard formula from the FP1 specification. The FP2 specification requires you to know the material in the FP1 specification.

$$6\sum_{r=1}^{40} r = 6 \times \frac{40 \times 41}{2} = 4920$$

$$\sum_{r=1}^{40} 1 = 40$$

$\sum_{r=1}^{40} 1 = 40$ is 40 ones added together which is, of course, 40.

Combining these results

$$\sum_{r=1}^{40}(3r-1)^2 = 199\,260 - 4920 + 40 = 194\,380$$

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Exercise A, Question 26

Question:

$f(r) = \dfrac{1}{r(r+1)}, r \in \mathbb{Z}^+$

a Show that

$$f(r) - f(r+1) = \frac{k}{r(r+1)(r+2)},$$

stating the value of k.

b Hence show, by the method of differences, that

$$\sum_{r=1}^{2n} \frac{1}{r(r+1)(r+2)} = \frac{n(2n+3)}{4(n+1)(2n+1)}.$$

Solution:

a $\mathrm{f}(r) - \mathrm{f}(r+1) = \dfrac{1}{r(r+1)} - \dfrac{1}{(r+1)(r+2)}$

$= \dfrac{r+2-r}{r(r+1)(r+2)} = \dfrac{2}{r(r+1)(r+2)}$

which is the required result with $k = 2$.

$\mathrm{f}(r+1)$ is $\mathrm{f}(r) = \dfrac{1}{r(r+1)}$ with r replaced by $r+1$. This gives $\dfrac{1}{(r+1)(r+1+1)} = \dfrac{1}{(r+1)(r+2)}$

b Using the result in part **a**

$$\sum_{r=1}^{2n} \frac{2}{r(r+1)(r+2)} = \sum_{r=1}^{2n} \left(\frac{1}{r(r+1)} - \frac{1}{(r+1)(r+2)}\right)$$

As $k = 2$, this is twice the summation you were asked to work out. You must remember to divide by 2 later.

$$= \frac{1}{1\times 2} - \cancel{\frac{1}{2\times 3}}$$
$$+ \cancel{\frac{1}{2\times 3}} - \cancel{\frac{1}{3\times 4}}$$
$$+ \cancel{\frac{1}{3\times 4}} - \cancel{\frac{1}{4\times 5}}$$
$$\vdots$$
$$+ \cancel{\frac{1}{(2n-1)2n}} - \cancel{\frac{1}{2n(2n+1)}}$$
$$+ \cancel{\frac{1}{2n(2n+1)}} - \frac{1}{(2n+1)(2n+2)}$$
$$= \frac{1}{2} - \frac{1}{(2n+1)(2n+2)}$$

Most summations in this topic have an upper limit of n but this question has an upper limit of $2n$. So the last two pairs of terms are the differences with $r = 2n - 1$ and $r = 2n$.

Hence

$$\sum_{r=1}^{2n} \frac{1}{r(r+1)(r+2)} = \frac{1}{4} - \frac{1}{2(2n+1)(2n+2)}$$

Dividing throughout by 2.

$$= \frac{1}{4} - \frac{1}{4(2n+1)(n+1)}$$
$$= \frac{(n+1)(2n+1)-1}{4(n+1)(2n+1)}$$
$$= \frac{2n^2+3n+1-1}{4(n+1)(2n+1)}$$
$$= \frac{2n^2+3n}{4(n+1)(2n+1)}$$
$$= \frac{n(2n+3)}{4(n+1)(2n+1)}, \text{ as required}$$

You complete the question by putting the fractions over a common denominator and simplifying to the printed answer.

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Exercise A, Question 27

Question:

a Show that

$$\frac{r^3 - r + 1}{r(r+1)} \equiv r - 1 + \frac{1}{r} - \frac{1}{r+1},$$

for $r \neq 0, -1$.

b Find $\sum_{r=1}^{n} \frac{r^3 - r + 1}{r(r+1)}$, expressing your answer as a single fraction in its simplest form.

Solution:

a $\text{RHS} = r - 1 + \frac{1}{r} - \frac{1}{r+1}$

$= \frac{(r-1)r(r+1) + (r+1) - r}{r(r+1)}$

$= \frac{r(r^2-1)+1}{r(r+1)}$

$= \frac{r^3 - r + 1}{r(r+1)} = \text{LHS, as required}$

To show that an algebraic identity is true, you should start from one side of the identity, here the right hand side (RHS), and use algebra to show that it is equal to the other side of the identity, here the left hand side (LHS).

b Using the result in part **a**

This summation is broken up into 3 separate summations. Only the third of these uses the method of differences.

$$\sum_{r=1}^{n} \frac{r^3 - r + 1}{r(r+1)} = \sum_{r=1}^{n} r - \sum_{r=1}^{n} 1 + \sum_{r=1}^{n}\left(\frac{1}{r} - \frac{1}{r+1}\right)$$

$$\sum_{r=1}^{n} r = \frac{n(n+1)}{2}$$

This is a standard formula from the FP1 specification. The FP2 specification requires you to know the material in the FP1 specification.

$$\sum_{r=1}^{n} 1 = n$$

$$\begin{aligned}\sum_{r=1}^{n}\left(\frac{1}{r} - \frac{1}{r+1}\right) &= \frac{1}{1} - \cancel{\frac{1}{2}} \\ &+ \cancel{\frac{1}{2}} - \cancel{\frac{1}{3}} \\ &+ \cancel{\frac{1}{3}} - \cancel{\frac{1}{4}} \\ &\vdots \\ &+ \cancel{\frac{1}{n-1}} + \cancel{\frac{1}{n}} \\ &+ \cancel{\frac{1}{n}} - \frac{1}{n+1} \\ &= 1 + \frac{1}{n+1}\end{aligned}$$

In the summation, using the method of differences, all of the terms cancel out with one another except for one term at the beginning and one term at the end.

Combining the three summations

$$\begin{aligned}\sum_{r=1}^{n} \frac{r^3 - r + 1}{r(r+1)} &= \frac{n(n+1)}{2} - n + 1 - \frac{1}{n+1} \\ &= \frac{n(n+1)^2 - 2n(n+1) + 2(n+1) - 2}{2(n+1)} \\ &= \frac{n^3 + 2n^2 + n - 2n^2 - 2n + 2n + 2 - 2}{2(n+1)} \\ &= \frac{n^3 + n}{2(n+1)} = \frac{n(n^2+1)}{2(n+1)}\end{aligned}$$

To complete the question, you put the results of the three summations over a common denominator and simplify the resulting expression as far as possible.

Solutionbank FP2
Edexcel AS and A Level Modular Mathematics

Exercise A, Question 28

Question:

a Express $\frac{2r+3}{r(r+1)}$ in partial fractions.

b Hence find $\sum_{r=1}^{n} \frac{2r+3}{r(r+1)} \times \frac{1}{3^r}$.

Solution:

a Let $\frac{2r+3}{r(r+1)} = \frac{A}{r} + \frac{B}{r+1}$

Multiply throughout by $r(r+1)$

$$2r + 3 = A(r+1) + Br$$

Substitute $r = 0$

$$3 = A$$

Substitute $r = -1$

$$1 = -B \Rightarrow B = -1$$

Hence

$$\frac{2r+3}{r(r+1)} = \frac{3}{r} - \frac{1}{r+1}$$

The partial fractions in part **a** form only part of the expression which you have to sum in part **b**; the $\frac{1}{3^r}$ is omitted. Before part **b** can be done, further work has to be carried out on the general term of the summation.

b Using the result in part **a**, the general term of the summation can be written

$$\frac{2r+3}{r(r+1)} \times \frac{1}{3^r} = \frac{3}{r} \times \frac{1}{3^r} - \frac{1}{r+1} \times \frac{1}{3^r} = \frac{1}{3^{r-1}r} - \frac{1}{3^r(r+1)}$$

$\frac{3}{3^r} = \frac{1}{3^{r-1}}$ is an important step here.

$$\sum_{r=1}^{n} \frac{2r+3}{r(r+1)} \times \frac{1}{3^r} = \frac{1}{3^0 \times 1} - \cancel{\frac{1}{3^1 \times 2}}$$
$$+ \cancel{\frac{1}{3^1 \times 2}} - \cancel{\frac{1}{3^2 \times 3}}$$
$$+ \cancel{\frac{1}{3^2 \times 3}} - \cancel{\frac{1}{3^3 \times 4}}$$
$$\vdots$$
$$+ \cancel{\frac{1}{3^{n-2}(n-1)}} - \cancel{\frac{1}{3^{n-1} \times n}}$$
$$+ \cancel{\frac{1}{3^{n-1} \times n}} - \frac{1}{3^n(n+1)}$$

This is $\frac{1}{3^{r-1}r} - \frac{1}{3^r(r+1)}$ with $r = 1$.

This is $\frac{1}{3^{r-1}r} - \frac{1}{3^r(r+1)}$ with $r = n - 1$.

$$\sum_{r=1}^{n} \frac{2r+3}{r(r+1)} \times \frac{1}{3^r} = 1 - \frac{1}{3^n(n+1)}$$

After summing, only the first and last terms are left. The first term $\frac{1}{3^0 \times 1} = 1$.

Solutionbank FP2

Edexcel AS and A Level Modular Mathematics

Exercise A, Question 29

Question:

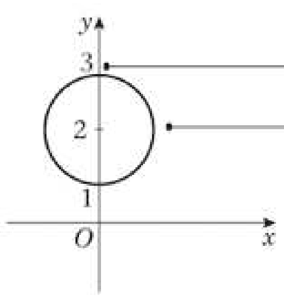
a Sketch, in an Argand diagram, the curve with equation $|z - 2i| = 1$.

Given that the point representing the complex number z lies on this curve,

b find the maximum value of $|z|$.

Solution:

a



You need to know that the locus of z in the Argand plane, when $|z - a| = b$, is a circle centre a, radius b. So this is a circle with centre (0, 2) on the complex axis and with radius 1.

b The maximum value of $|z|$ is 3.

$|z|$ is the distance of the point representing z from the origin. You can see from the diagram that the point furthest from the origin on the circle is the point representing 3i.

Solutionbank FP2
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Exercise A, Question 30

Question:

Solve the equation $z^5 = \mathrm{i}$, giving your answers in the form $\cos\theta + \mathrm{i}\sin\theta$.

Solution:

a $\mathrm{i} = \cos\frac{\pi}{2} + \mathrm{i}\sin\frac{\pi}{2} = \mathrm{e}^{\mathrm{i}\frac{\pi}{2}}$

The modulus of the complex number i is 1 and its argument is $\frac{\pi}{2}$. So $\mathrm{i} = 1\mathrm{e}^{\mathrm{i}\frac{\pi}{2}}$.

$z^5 = \mathrm{e}^{\mathrm{i}\frac{\pi}{2}}, \mathrm{e}^{\mathrm{i}\frac{5\pi}{2}}, \mathrm{e}^{\mathrm{i}\frac{9\pi}{2}}, \mathrm{e}^{\mathrm{i}\frac{13\pi}{2}}, \mathrm{e}^{\mathrm{i}\frac{17\pi}{2}}$

Additional solutions are found by increasing the arguments in steps of 2π. As the equation is of degree 5, there are exactly 5 distinct answers.

$z = \mathrm{e}^{\mathrm{i}\frac{\pi}{10}}, \mathrm{e}^{\mathrm{i}\frac{5\pi}{10}}, \mathrm{e}^{\mathrm{i}\frac{9\pi}{10}}, \mathrm{e}^{\mathrm{i}\frac{13\pi}{10}}, \mathrm{e}^{\mathrm{i}\frac{17\pi}{10}}$

For example, if $z^5 = \mathrm{e}^{\mathrm{i}\frac{9\pi}{2}}$ then $z = \left(\mathrm{e}^{\mathrm{i}\frac{9\pi}{2}}\right)^{\frac{1}{5}} = \mathrm{e}^{\mathrm{i}\frac{9\pi}{10}}$.

Hence

$z = \cos\theta + \mathrm{i}\sin\theta$, where

$\theta = \frac{\pi}{10}, \frac{5\pi}{10}\left(= \frac{\pi}{2}\right), \frac{9\pi}{10}, \frac{13\pi}{10}, \frac{17\pi}{10}$

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Exercise A, Question 31

Question:

Show that

$$\frac{\cos 2x + \mathrm{i}\sin 2x}{\cos 9x - \mathrm{i}\sin 9x}$$

can be expressed in the form $\cos nx + \mathrm{i}\sin nx$, where n is an integer to be found.

Solution:

Using Euler's solution $e^{\mathrm{i}\theta} = \cos\theta + \mathrm{i}\sin\theta$,

$$\cos 2x + \mathrm{i}\sin 2x = e^{\mathrm{i}2x}$$

$$\cos 9x - \mathrm{i}\sin 9x = \cos(-9x) + \mathrm{i}\sin(-9x) = e^{\mathrm{i}(-9x)}$$

For any angle θ, $\cos\theta = \cos(-\theta)$ and $-\sin\theta = \sin(-\theta)$

You will find these relations useful when finding the arguments of complex numbers.

Hence

$$\frac{\cos 2x + \mathrm{i}\sin 2x}{\cos 9x - \mathrm{i}\sin 9x} = \frac{e^{\mathrm{i}2x}}{e^{\mathrm{i}(-9x)}} = e^{\mathrm{i}(2x+9x)} = e^{\mathrm{i}11x}$$

$$= \cos 11x + \mathrm{i}\sin 11x$$

Manipulating the arguments in $e^{\mathrm{i}\theta}$ you use the ordinary laws of indices.

This is the required form with $n = 11$.

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Exercise A, Question 32

Question:

The transformation T from the z-plane to the w-plane is given by

$$w = \frac{z+1}{z-1}, \quad z \neq 1.$$

Find the image in the w-plane of the circle $|z| = 1$, $z \neq 1$ under the transformation.

Solution:

$$w = \frac{z+1}{z-1}$$

$$w(z-1) = wz - w = z + 1$$

$$wz - z = w + 1 \Rightarrow z(w-1) = w + 1$$

$$z = \frac{w+1}{w-1}$$

As $|z| = 1$, $\left|\frac{w+1}{w-1}\right| = 1$

and

$$\frac{|w+1|}{|w-1|} = 1 \Rightarrow |w+1| = |w-1|$$

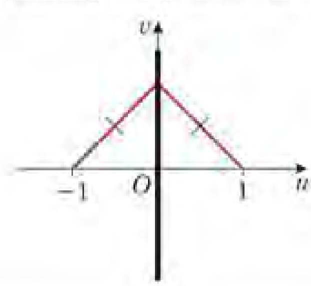
The locus of w is the line equidistant from the points representing the real numbers -1 and 1. This line is the imaginary axis.

Hence, the image of $|z| = 1$ under T is the imaginary axis.

The question gives information about $|z|$ and you are trying to show something about w. It is a good idea to change the subject of the formula to z. You can then put the modulus of the right hand side of the new formula, which contains w, equal to 1.

For any complex numbers a and b, $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$.

You need to know that the locus of z in the Argand plane, when $|z - a| = |z - b|$, is the line equidistant from the points representing the complex numbers a and b. That is the perpendicular bisector of the line joining the points. In this case;



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Exercise A, Question 33

Question:

a Express $z = 1 + i\sqrt{3}$ in the form $r(\cos\theta + i\sin\theta)$, $r > 0$, $-\pi < \theta \leqslant \pi$.

b Hence, or otherwise, show that the two solutions of

$$w^2 = (1 + i\sqrt{3})^3$$

are $(2\sqrt{2})i$ and $(-2\sqrt{2})i$.

Solution:

a $z = 1 + i\sqrt{3} = r(\cos\theta + i\sin\theta) = r\cos\theta + i\,r\sin\theta$

Equating real parts

$$1 = r\cos\theta \quad ①$$

Equating complex parts

$$\sqrt{3} = r\sin\theta \quad ②$$

Squaring both ① and ② and adding the results

$$r^2\cos^2\theta + r^2\sin^2\theta = r^2 = 1^2 + (\sqrt{3})^2 = 4$$

$$r = 2$$

Substituting into ①

$$1 = 2\cos\theta \Rightarrow \cos\theta = \tfrac{1}{2} \Rightarrow \theta = \frac{\pi}{3}$$

Unless the question clearly specifies otherwise, in this topic you should give all arguments in radians and exact answers should be given wherever possible.

Hence $1 + i\sqrt{3} = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

b From part **a**

$$1 + i\sqrt{3} = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) = 2e^{i\frac{\pi}{3}}$$

$$(1 + i\sqrt{3})^3 = \left(2e^{i\frac{\pi}{3}}\right)^3 = 8e^{i\pi}$$

You use part **a** to put the right hand side of the equation into a form from which the square roots can be found.

Hence the equation can be written

$$w^2 = 8e^{i\pi},\ 8e^{i3\pi}$$

Additional solutions are found by increasing the arguments in steps of 2π. As the equation is a quadratic, there are just 2 distinct answers.

$$w = \sqrt{8}e^{i\frac{\pi}{2}},\ \sqrt{8}e^{i\frac{3\pi}{2}}$$

The two solutions are

$$w = \sqrt{8}e^{i\frac{\pi}{2}} = \sqrt{8}\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) = 2\sqrt{2}i$$

and

$$w = \sqrt{8}e^{i\frac{3\pi}{2}} = \sqrt{8}\left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right) = (-2\sqrt{2})i,$$

as required.

Using $\cos\frac{\pi}{2} = \cos\frac{3\pi}{2} = 0$, $\sin\frac{\pi}{2} = 1$ and $\sin\frac{3\pi}{2} = -1$.

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Exercise A, Question 34

Question:

The transformation from the z-plane to the w-plane is given by

$$w = \frac{2z - 1}{z - 2}.$$

Show that the circle $|z| = 1$ is mapped onto the circle $|w| = 1$.

Solution:

$$w = \frac{2z - 1}{z - 2} \Rightarrow wz - 2w = 2z - 1$$

$$wz - 2z = 2w - 1 \Rightarrow z(w - 2) = 2w - 1$$

$$z = \frac{2w - 1}{w - 2}$$

$$|z| = 1 \Rightarrow \left|\frac{2w + 1}{w - 2}\right| = 1$$

You know that $|z| = 1$ and you are trying to find out about w. So it is a good idea to change the subject of the formula to z. You can then put the modulus of the right hand side of the new formula, which contains w, equal to 1.

$$|2w - 1| = |w - 2|$$

Let $w = u + \mathrm{i}v$

$$|2(u + \mathrm{i}v) - 1| = |u + \mathrm{i}v - 2|$$

$$|(2u - 1) + \mathrm{i}2v| = |(u - 2) + \mathrm{i}v|$$

$$|(2u - 1) + \mathrm{i}2v|^2 = |(u - 2) + \mathrm{i}v|^2$$

It is not easy to interpret this locus geometrically and so it is sensible to transform the problem into algebra, using the rule that if $z = x + \mathrm{i}y$, then $|z|^2 = x^2 + y^2$.

$$(2u - 1)^2 + 4v^2 = (u - 2)^2 + v^2$$

$$4u^2 - 4u + 1 + 4v^2 = u^2 - 4u + 4 + v^2$$

$$3u^2 + 3v^2 = 3 \Rightarrow u^2 + v^2 = 1$$

This is a circle centre O, radius 1 and has the equation $|w| = 1$ in the Argand plane.

Hence, the circle $|z| = 1$ is mapped onto the circle $|w| = 1$, as required.

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Exercise A, Question 35

Question:

a Solve the equation

$$z^5 = 4 + 4\mathrm{i}$$

giving your answers in the form $z = r\,\mathrm{e}^{\mathrm{i}k\pi}$, where r is the modulus of z and k is a rational number such that $0 \leqslant k \leqslant 2$.

b Show on an Argand diagram the points representing your solutions.

Solution:

a Let $4 + 4\text{i} = r(\cos\theta + \text{i}\sin\theta) = r\cos\theta + \text{i}r\sin\theta$

Equating real parts

$$4 = r\cos\theta \quad ①$$

Equating imaginary parts

$$4 = r\sin\theta \quad ②$$

Dividing ② by ①

$$\tan\theta = 1 \Rightarrow \theta = \frac{\pi}{4}$$

Substituting $\theta = \frac{\pi}{4}$ into ①

$$4 = r\cos\frac{\pi}{4} \Rightarrow 4 = r \times \frac{1}{\sqrt{2}} \Rightarrow r = 4\sqrt{2}$$

Hence

$$4 + 4\text{i} = 4\sqrt{2}\left(\cos\frac{\pi}{4} + \text{i}\sin\frac{\pi}{4}\right)$$

$$= 2^{\frac{5}{2}}\text{e}^{\text{i}\frac{\pi}{4}}$$

$$z^5 = 2^{\frac{5}{2}}\text{e}^{\text{i}\frac{\pi}{4}},\ 2^{\frac{5}{2}}\text{e}^{\text{i}\frac{9\pi}{4}},\ 2^{\frac{5}{2}}\text{e}^{\text{i}\frac{17\pi}{4}},\ 2^{\frac{5}{2}}\text{e}^{\text{i}\frac{25\pi}{4}},\ 2^{\frac{5}{2}}\text{e}^{\text{i}\frac{33\pi}{4}}$$

$$z = 2^{\frac{1}{2}}\text{e}^{\text{i}\frac{\pi}{20}},\ 2^{\frac{1}{2}}\text{e}^{\text{i}\frac{9\pi}{20}},\ 2^{\frac{1}{2}}\text{e}^{\text{i}\frac{17\pi}{20}},\ 2^{\frac{1}{2}}\text{e}^{\text{i}\frac{25\pi}{20}},\ 2^{\frac{1}{2}}\text{e}^{\text{i}\frac{33\pi}{20}}$$

This is the required form with $r = \sqrt{2}$ and

$k = \frac{1}{20}, \frac{9}{20}, \frac{17}{20}, \frac{25}{20}\left(= \frac{5}{4}\right), \frac{33}{20}.$

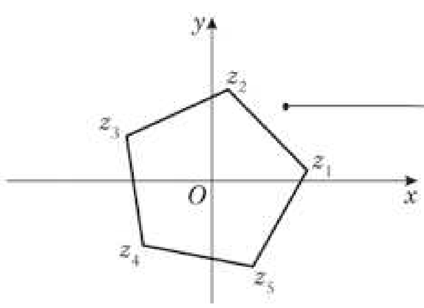
Finding the roots of a complex number is usually easier if you obtain the number in the form $r\text{e}^{\text{i}\theta}$. As you will use Euler's relation, the first step towards this is to get the complex number into the form $r(\cos\theta + \text{i}\sin\theta)$.

To take the fifth root, write $4\sqrt{2} = 2^{\frac{5}{2}}$.

For example, if $z^5 = 2^{\frac{5}{2}}\text{e}^{\text{i}\frac{9\pi}{4}}$ then

$z = \left(2^{\frac{5}{2}}\text{e}^{\text{i}\frac{9\pi}{4}}\right)^{\frac{1}{5}} = 2^{\frac{5}{2}\times\frac{1}{5}}\text{e}^{\text{i}\frac{9\pi}{4}\times\frac{1}{5}} = 2^{\frac{1}{2}}\text{e}^{\text{i}\frac{9\pi}{20}}.$

b



The points representing the 5 roots are the vertices of a regular pentagon.

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Exercise A, Question 36

Question:

The point P represents the complex number z in an Argand diagram. Given that

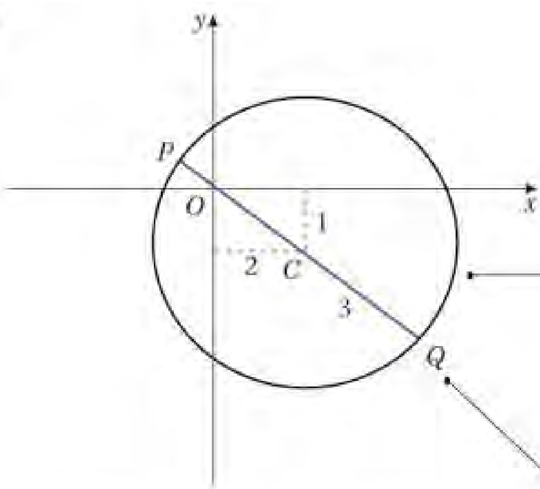
$$|z - 2 + \mathrm{i}| = 3,$$

a sketch the locus of P in an Argand diagram,

b find the exact values of the maximum and minimum of $|z|$.

Solution:

a



The locus of $|z - a| = k$, where a is a complex number and k is a real number, is a circle with radius k and centre the point representing a. Rewriting the relation in the question as $|z - (2 - \mathrm{i})| = 3$, this locus is a circle of radius 3 with centre $(2, -1)$.

b $OC^2 = 1^2 + 2^2 = 5 \Rightarrow OC = \sqrt{5}$

$OQ = OC + CQ = \sqrt{5} + 3$

Hence the maximum value of $|z|$ is $3 + \sqrt{5}$.

$OP = CP - CO = 3 - \sqrt{5}$

Hence the minimum value of $|z|$ is $3 - \sqrt{5}$.

$|z|$ is the distance of the point representing z from the origin. The point on the circle furthest from O is marked by Q on the diagram and the point closest to O by P. The distances of Q and P from O represent the maximum and minimum values of $|z|$ respectively.

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Exercise A, Question 37

Question:

The transformation T from the z-plane to the w-plane is given by

$$w = \frac{1}{z-2},\ z \neq 2,$$

where $z = x + \mathrm{i}y$ and $w = u + \mathrm{i}v$.

Show that under T the straight line with equation

$$2x + y = 5$$

is transformed to a circle in the w-plane with centre $\left(1, -\frac{1}{2}\right)$ and radius $\frac{\sqrt{5}}{2}$.

Solution:

$$w = \frac{1}{z-2}$$

$$z - 2 = \frac{1}{w}$$

$$x + \mathrm{i}y - 2 = \frac{1}{u + \mathrm{i}v}$$

$$x - 2 + \mathrm{i}y = \frac{1}{u+\mathrm{i}v} \times \frac{u - \mathrm{i}v}{u - \mathrm{i}v} = \frac{u}{u^2+v^2} - \frac{\mathrm{i}v}{u^2+v^2}$$

Multiplying the numerator and denominator by the conjugate complex of the denominator.

Equating real parts

$$x - 2 = \frac{u}{u^2+v^2} \Rightarrow x = 2 + \frac{u}{u^2+v^2}$$

Equating imaginary parts

$$y = -\frac{v}{u^2+v^2}$$

Hence $\quad 2x + y = 5$

maps to $\quad 2\left(2 + \frac{u}{u^2+v^2}\right) - \frac{v}{u^2+v^2} = 5$

This is the equation of the curve in the w-plane. The rest of the solution is showing that this is the equation of a circle, using the method of completing the square.

$$\frac{2u}{u^2+v^2} - \frac{v}{u^2+v^2} = 1$$

$$2u - v = u^2 + v^2$$

$$u^2 - 2u + v^2 + v = 0$$

$$u^2 - 2u + 1 + v^2 + v + \tfrac{1}{4} = \tfrac{5}{4}$$

$$(u-1)^2 + \left(v + \tfrac{1}{2}\right)^2 = \left(\tfrac{1}{2}\sqrt{5}\right)^2$$

This is a circle in the w-plane with centre $\left(1, -\frac{1}{2}\right)$

and radius $\frac{1}{2}\sqrt{5}$, as required.

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Exercise A, Question 38

Question:

a Use de Moivre's theorem to show that $\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$.

b Hence find 3 distinct solutions of the equation $16x^5 - 20x^3 + 5x + 1 = 0$, giving your answers to 3 decimal places where appropriate.

Solution:

a By de Moivre's theorem

$$\cos 5\theta + i\sin 5\theta = (\cos\theta + i\sin\theta)^5 = (c + is)^5\text{, say}$$
$$= c^5 + 5c^4 is + 10c^3 i^2 s^2 + 10c^2 i^3 s^3 + 5c i^4 s^4 + i^5 s^5$$
$$= c^5 + i5c^4 s - 10c^3 s^2 - i10c^2 s^3 + 5cs^4 - is^5$$

It is sensible to abbreviate $\cos\theta$ and $\sin\theta$ as c and s respectively when you have as many powers of $\cos\theta$ and $\sin\theta$ to write out as you have in this question.

Use the binomial expansion.

Use $i^2 = -1$, $i^3 = -i$, $i^4 = 1$ and $i^5 = i \times i^4 = i \times 1 = i$.

Equating real parts

$$\cos 5\theta = c^5 - 10c^3 s^2 + 5cs^4$$

Using $\cos^2\theta + \sin^2\theta = 1$

$$\cos 5\theta = c^5 - 10c^3(1 - c^2) + 5c(1 - c^2)^2$$
$$= c^5 - 10c^3 + 10c^5 + 5c - 10c^3 + 5c^5$$
$$= 16c^5 - 20c^3 + 5c$$
$$= 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta\text{, as required}$$

b Substitute $x = \cos\theta$ into $16x^5 - 20x^3 + 5x + 1 = 0$

$$16\cos^5\theta - 20\cos^3\theta + 5\cos\theta + 1 = 0$$
$$16\cos^5\theta - 20\cos^3\theta + 5\cos\theta = -1$$

Using the result of part **a**

$$\cos 5\theta = -1$$
$$5\theta = \pi, 3\pi, 5\pi$$
$$\theta = \frac{\pi}{5}, \frac{3\pi}{5}, \frac{5\pi}{5}$$
$$x = \cos\theta = \cos\frac{\pi}{5}, \cos\frac{3\pi}{5}, \cos\pi$$
$$= 0.809, -0.309, -1$$

Additional solutions are found by increasing the angles in steps of 2π. You are asked for 3 answers, so you need 3 angles at this stage.

The two approximate answers are given to 3 decimal places, as the question specified; the remaining answer -1 is exact.

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Exercise A, Question 39

Question:

a Use de Moivre's theorem to show that $\sin^5\theta = \frac{1}{16}(\sin 5\theta - 5\sin 3\theta + 10\sin\theta)$.

b Hence, or otherwise, show that

$$\int_0^{\frac{\pi}{2}} \sin^5\theta \, d\theta = \frac{8}{15}.$$

Solution:

a $\sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}$

Let $z = e^{i\theta}$

Putting $z = e^{i\theta}$ shortens the working.

then $\sin\theta = \dfrac{z - z^{-1}}{2i}$

Use Pascal's triangle to remember the coefficients in $(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$.

$$\sin^5\theta = \left(\frac{z - z^{-1}}{2i}\right)^5$$

$$= \frac{1}{(2i)^5}(z^5 - 5z^4 \times z^{-1} + 10z^3 \times z^{-2} - 10z^2 \times z^{-3} + 5z \times z^{-4} - z^{-5})$$

$$= \frac{1}{32i}(z^5 - 5z^3 + 10z - 10z^{-1} + 5z^{-3} - z^{-5})$$

$$= \frac{1}{16}\left(\frac{z^5 - z^{-5}}{2i} - \frac{5(z^3 - z^{-3})}{2i} + \frac{10(z - z^{-1})}{2i}\right)$$

The general relation is

$$\sin n\theta = \frac{e^{in\theta} - e^{-in\theta}}{2i} = \frac{z^n - z^{-n}}{2i}$$

$$= \frac{1}{16}(\sin 5\theta - 5\sin 3\theta + 10\sin\theta), \text{ as required}$$

b $\displaystyle\int_0^{\frac{\pi}{2}} \sin^5\theta \, d\theta = \frac{1}{16}\int_0^{\frac{\pi}{2}} (\sin 5\theta - 5\sin 3\theta + 10\sin\theta)\, d\theta$

Each term on the right hand side of the identity shown in part **a** can be integrated using the formula $\displaystyle\int \sin n\theta \, d\theta = -\frac{\cos n\theta}{n}$.

$$= \frac{1}{16}\left[-\frac{1}{5}\cos 5\theta + \frac{5}{3}\cos 3\theta - 10\cos\theta\right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{16}\left(0 - \left(-\frac{1}{5} + \frac{5}{3} - 10\right)\right)$$

$$= \frac{1}{16} \times \frac{128}{15} = \frac{8}{15}, \text{ as required}$$

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Exercise A, Question 40

Question:

The transformation from the z-plane to the w-plane is given by

$$w = \frac{z - \mathrm{i}}{z}.$$

a Show that under this transformation the line $\operatorname{Im} z = \frac{1}{2}$ is mapped to the circle with equation $|w| = 1$.

b Hence, or otherwise, find, in the form $w = \frac{az + b}{cz + d}$, where a, b, c and $d \in \mathbb{C}$, the transformation that maps the line $\operatorname{Im} z = \frac{1}{2}$ to the circle, centre $(3 - \mathrm{i})$ and radius 2.

Solution:

a $z = x + \frac{1}{2}\text{i}$

The real part of a complex number on $\text{Im}\, z = \frac{1}{2}$ can have any real value, which you can represent by the symbol x, but the imaginary part must be $\frac{1}{2}$.

$$w = \frac{z - \text{i}}{z}$$

$$zw = z - \text{i} \Rightarrow z - wz = \text{i}$$

$$z = \frac{\text{i}}{1 - w}$$

Let $w = u + \text{i}v$

$$x + \tfrac{1}{2}\text{i} = \frac{\text{i}}{1 - u - \text{i}v}$$

Multiplying the numerator and denominator by $1 - u + \text{i}v$

Multiply the numerator and the denominator of the right hand side by the conjugate complex of $1 - u - \text{i}v$ which is $1 - u + \text{i}v$.

$$x + \tfrac{1}{2}\text{i} = \frac{\text{i}(1 - u + \text{i}v)}{(1 - u)^2 + v^2},$$

$$= \frac{-v}{(1 - u)^2 + v^2} + \frac{1 - u}{(1 - u)^2 + v^2}\text{i}$$

Equating imaginary parts

$$\frac{1}{2} = \frac{1 - u}{u^2 - 2u + 1 + v^2}$$

You are aiming at $|w| = 1$. If $w = u + \text{i}v$, this is the equivalent to $u^2 + v^2 = 1$. So that is the expression you are looking for.

$$u^2 - 2u + 1 + v^2 = 2 - 2u$$

$$u^2 + v^2 = 1$$

$u^2 + v^2 = 1$ is a circle centre O, radius 1.

Hence the line, $\text{Im}\, z = \frac{1}{2}$ is mapped onto the circle with equation $|w| = 1$.

b The transformation $w' = \frac{z - \text{i}}{z}$ maps the line $\text{Im}\, z = \frac{1}{2}$ onto the circle with centre O and radius 1.

The first transformation is the transformation in part **a**.

The transformation $w'' = 2w'$ maps the circle with centre O and radius 1 onto the circle with centre O and radius 2.

The transformation $z \mapsto kz$ increases the radius of the circle by a factor of k. This transformation is an enlargement, factor k, centre of enlargement O.

The transformation $w = w'' + 3 - \text{i}$ maps the circle with centre O and radius 2 onto the circle with centre $3 - \text{i}$ and radius 2.

The transformation $z \mapsto z + a$ maps a circle centre O to a circle centre a. This transformation is a translation.

Combining the transformations

$$w = 2\left(\frac{z - \text{i}}{z}\right) + 3 - \text{i}$$

$$= \frac{2z - 2\text{i} + 3z - \text{i}z}{z}$$

$$= \frac{(5 - \text{i})z - 2\text{i}}{z}$$

Solutionbank FP2
Edexcel AS and A Level Modular Mathematics

Exercise A, Question 41

Question:

a Solve the equation

$$z^3 = 32 + 32\sqrt{3}\mathrm{i},$$

giving your answers in the form $r\,\mathrm{e}^{\mathrm{i}\theta}$, where $r > 0$, $-\pi < \theta \leqslant \pi$.

b Show that your solutions satisfy the equation

$$z^9 + 2^k = 0,$$

for an integer k, the value of which should be stated.

Solution:

a Let $32 + 32\sqrt{3}\text{i} = r(\cos\theta + \text{i}\sin\theta) = r\cos\theta + \text{i}\,r\sin\theta$

Equating real parts

$$32 = r\cos\theta \qquad ①$$

Equating imaginary parts

$$32\sqrt{3} = r\sin\theta \qquad ②$$

Dividing ② by ①

$$\tan\theta = \sqrt{3} \Rightarrow \theta = \frac{\pi}{3}$$

Substituting $\theta = \frac{\pi}{3}$ into ①

$$32 = r\cos\frac{\pi}{3} \Rightarrow 32 = r \times \frac{1}{2} \Rightarrow r = 64$$

Hence

$$32 + 32\sqrt{3}\text{i} = 64\left(\cos\frac{\pi}{3} + \text{i}\sin\frac{\pi}{3}\right)$$

$$= 64\text{e}^{\text{i}\frac{\pi}{3}}$$

$$z^3 = 64\text{e}^{\text{i}\frac{\pi}{3}},\ 64\text{e}^{\text{i}\frac{7\pi}{3}},\ 64\text{e}^{\text{i}\frac{-5\pi}{3}}$$

$$z = 4\text{e}^{\text{i}\frac{\pi}{9}},\ 4\text{e}^{\text{i}\frac{7\pi}{9}},\ 4\text{e}^{-\text{i}\frac{5\pi}{9}}$$

The solutions are $r\text{e}^{\text{i}\theta}$ where $r = 4$ and

$$\theta = -\frac{5\pi}{9}, \frac{\pi}{9}, \frac{7\pi}{9}$$

Finding the roots of a complex number is usually easier if you obtain the number in the form $r\text{e}^{\text{i}\theta}$. As you will use Euler's relation, the first step towards this is to get the complex number into the form $r(\cos\theta + \text{i}\sin\theta)$.

Additional solutions are found by increasing or decreasing the arguments in steps of 2π. You are asked for 3 answers, so you need 3 arguments. Had you increased the argument $\frac{7\pi}{3}$ by 2π to $\frac{13\pi}{3}$, this would have given a correct solution to the equation but it would lead to $\theta = \frac{13\pi}{9}$, which does not satisfy the condition $\theta \leqslant \pi$ in the question. So the third argument has to be found by subtracting 2π from $\frac{\pi}{3}$.

b $$z = 4\text{e}^{\text{i}\frac{\pi}{9}},\ 4\text{e}^{\text{i}\frac{7\pi}{9}},\ 4\text{e}^{-\text{i}\frac{-5\pi}{9}}$$

$$z^9 = \left(4\text{e}^{\text{i}\frac{\pi}{9}}\right)^9, \left(4\text{e}^{\text{i}\frac{7\pi}{9}}\right)^9, \left(4\text{e}^{-\text{i}\frac{-5\pi}{9}}\right)^9$$

$$= 4^9\text{e}^{\text{i}\pi},\ 4^9\text{e}^{\text{i}7\pi},\ 4^9\text{e}^{-\text{i}5\pi}$$

$\text{e}^{\text{i}\pi} = \cos\pi + \text{i}\sin\pi = -1 + 0 = -1$. Similarly for the arguments 7π and -5π.

The value of all three of these expressions is $-4^9 = -2^{18}$

Hence the solutions satisfy $z^9 + 2^k = 0$, where $k = 18$.

Solutionbank FP2
Edexcel AS and A Level Modular Mathematics

Exercise A, Question 42

Question:

a Use de Moivre's theorem to show that $\sin 5\theta = \sin\theta\,(16\cos^4\theta - 12\cos^2\theta + 1)$.

b Hence, or otherwise, solve, for $0 \leqslant \theta < \pi$,

$$\sin 5\theta + \cos\theta \sin 2\theta = 0.$$

Solution:

a By de Moivre's theorem

$\cos 5\theta + \mathrm{i}\sin 5\theta = (\cos\theta + \mathrm{i}\sin\theta)^5 = (c + \mathrm{i}s)^5$, say

$= c^5 + 5c^4\mathrm{i}s + 10c^3\mathrm{i}^2s^2 + 10c^2\mathrm{i}^3s^3 + 5c\mathrm{i}^4s^4 + \mathrm{i}^5s^5$

$= c^5 + \mathrm{i}5c^4s - 10c^3s^2 - \mathrm{i}10c^2s^3 + 5cs^4 - \mathrm{i}s^5$

Equating imaginary parts

$$\begin{aligned}\sin 5\theta &= 5c^4s - 10c^2s^3 + s^5\\ &= s(5c^4 - 10c^2s^2 + s^4)\\ &= s\left(5c^4 - 10c^2(1-c^2) + (1-c^2)^2\right)\\ &= s(5c^4 - 10c^2 + 10c^4 + 1 - 2c^2 + c^4)\\ &= s(16c^4 - 12c^2 + 1)\\ &= \sin\theta(16\cos^4\theta - 12\cos^2\theta + 1), \text{ as required}\end{aligned}$$

Repeatedly using the identity $\cos^2\theta + \sin^2\theta = 1$, which in this context is $s^2 = 1 - c^2$.

b

$$\begin{aligned}\sin 5\theta + \cos\theta\sin 2\theta &= 0\\ \sin\theta(16\cos^4\theta - 12\cos^2\theta + 1) + 2\sin\theta\cos^2\theta &= 0\\ \sin\theta(16\cos^4\theta - 10\cos^2\theta + 1) &= 0\\ \sin\theta(2\cos^2\theta - 1)(8\cos^2\theta - 1) &= 0\end{aligned}$$

Using the identity proved in part **a** and the identity $\sin 2\theta = 2\sin\theta\cos\theta$.

Hence $\sin\theta = 0$, $\cos^2\theta = \frac{1}{2}$, $\cos^2\theta = \frac{1}{8}$

$\sin\theta = 0 \Rightarrow \theta = 0$

$\cos^2\theta = \frac{1}{2} \Rightarrow \cos\theta = \pm\frac{1}{\sqrt{2}}$

$\cos\theta = \frac{1}{\sqrt{2}} \Rightarrow \theta = \frac{\pi}{4}$

$\cos\theta = -\frac{1}{\sqrt{2}} \Rightarrow \theta = \frac{3\pi}{4}$

You must consider the negative as well as the positive square roots.

$\cos^2\theta = \frac{1}{8} \Rightarrow \cos\theta = \pm\frac{1}{2\sqrt{2}}$

$\cos\theta = \frac{1}{2\sqrt{2}} \Rightarrow \theta = 1.209$ (3 d.p.)

$\cos\theta = -\frac{1}{2\sqrt{2}} \Rightarrow \theta = 1.932$ (3 d.p.)

The question has specified no accuracy and any sensible accuracy would be accepted for the approximate answers.

The solutions of the equation are

$0, \frac{\pi}{4}, \frac{3\pi}{4}$, 1.209 (3 d.p.) and 1.932 (3 d.p.).

Solutionbank FP2
Edexcel AS and A Level Modular Mathematics

Exercise A, Question 43

Question:

a Given that $z = \cos\theta + \mathrm{i}\sin\theta$, show that $z^n + z^{-n} = 2\cos n\theta$.

b Express $\cos^6\theta$ in terms of cosines of multiples of θ.

c Hence show that

$$\int_0^{\frac{\pi}{2}} \cos^6\theta \, \mathrm{d}\theta = \frac{5\pi}{32}.$$

Solution:

a $\quad z = \cos\theta + \mathrm{i}\sin\theta$

Using de Moivre's theorem

$$z^n = (\cos\theta + \mathrm{i}\sin\theta)^n = \cos n\theta + \mathrm{i}\sin n\theta \quad ①$$

From ①

$$z^{-n} = \frac{1}{z^n} = \frac{1}{\cos n\theta + \mathrm{i}\sin n\theta}$$

$$= \frac{1}{\cos n\theta + \mathrm{i}\sin n\theta} \times \frac{\cos n\theta - \mathrm{i}\sin n\theta}{\cos n\theta - \mathrm{i}\sin n\theta}$$

Multiply the numerator and denominator by $\cos n\theta - \mathrm{i}\sin n\theta$, the conjugate complex number of $\cos n\theta + \mathrm{i}\sin n\theta$.

$$= \frac{\cos n\theta - \mathrm{i}\sin n\theta}{\cos^2 n\theta + \sin^2 n\theta} = \cos n\theta - \mathrm{i}\sin n\theta \quad ②$$

Use $\cos^2 n\theta + \sin^2 n\theta = 1$.

$$z^n + z^{-n} = \cos n\theta + \mathrm{i}\sin n\theta + \cos n\theta - \mathrm{i}\sin n\theta$$

$$= 2\cos n\theta, \text{ as required.}$$

b $\cos\theta = \dfrac{\mathrm{e}^{\mathrm{i}\theta} + \mathrm{e}^{-\mathrm{i}\theta}}{2} = \dfrac{z + z^{-1}}{2}$

$$\cos^6\theta = \left(\frac{z + z^{-1}}{2}\right)^6$$

$$= \tfrac{1}{64}(z^6 + 6z^5z^{-1} + 15z^4z^{-2} + 20z^3z^{-3} + 15z^2z^{-4} + 6z^1z^{-5} + z^{-6})$$

Pair the terms as shown.

$$= \tfrac{1}{64}(z^6 + 6z^4 + 15z^2 + 20 + 15z^{-2} + 6z^{-4} + z^{-6})$$

$$= \tfrac{1}{32}\left(\frac{z^6 + z^{-6}}{2} + \frac{6(z^4 + z^{-4})}{2} + \frac{15(z^2 + z^{-2})}{2} + \frac{20}{2}\right)$$

You use $\dfrac{z^n + z^{-n}}{2} = \cos n\theta$ with $n = 6$, 4 and 2.

$$= \tfrac{1}{32}(\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10)$$

c $\displaystyle\int_0^{\frac{\pi}{2}} \cos^6\theta\,\mathrm{d}\theta = \tfrac{1}{32}\int_0^{\frac{\pi}{2}} (\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10)\,\mathrm{d}\theta$

$$= \tfrac{1}{32} - \left[\tfrac{1}{6}\sin 6\theta + \tfrac{6}{4}\sin 4\theta + \tfrac{15}{2}\sin 2\theta + 10\,\theta\right]_0^{\frac{\pi}{2}}$$

With the exception of 10θ all of these terms have value 0 at both the upper and the lower limit.

$$= \tfrac{1}{32} \times 10 \times \frac{\pi}{2} = \frac{5\pi}{32}, \text{ as required}$$

Solutionbank FP2
Edexcel AS and A Level Modular Mathematics

Exercise A, Question 44

Question:

a Prove that

$$(z^n - e^{i\theta})(z^n - e^{-i\theta}) = z^{2n} - 2z^n \cos\theta + 1.$$

b Hence, or otherwise, find the roots of the equation

$$z^6 - z^3\sqrt{2} + 1 = 0,$$

in the form $\cos\alpha + i\sin\alpha$, where $-\pi < \alpha \leqslant \pi$.

Solution:

a $(z^n - e^{i\theta})(z^n - e^{-i\theta}) = z^{2n} - z^n e^{-i\theta} - z^n e^{i\theta} + e^{i\theta}e^{-i\theta}$

$$= z^{2n} - 2z^n\left(\frac{e^{i\theta} + e^{-i\theta}}{2}\right) + 1$$

$$= z^{2n} - 2z^n\cos\theta + 1, \text{ as required}$$

The specification requires you to be familiar with

$\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$ and

$\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$.

b Using the result of part **a** with $n = 3$ and $\theta = \frac{\pi}{4}$,

$$z^6 - 2z^3\cos\frac{\pi}{4} + 1 = \left(z^3 - e^{i\frac{\pi}{4}}\right)\left(z^3 - e^{-i\frac{\pi}{4}}\right)$$

As $\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$, $2z^3\cos\frac{\pi}{4} = z^3\sqrt{2}$

$$z^6 - z^3\sqrt{2} + 1 = \left(z^3 - e^{i\frac{\pi}{4}}\right)\left(z^3 - e^{-i\frac{\pi}{4}}\right) = 0$$

$$z^3 = e^{i\frac{\pi}{4}}, e^{-i\frac{\pi}{4}}$$

$$z^3 = e^{i\frac{\pi}{4}}, e^{i\frac{9\pi}{4}}, e^{-i\frac{7\pi}{4}} \Rightarrow z = e^{i\frac{\pi}{12}}, e^{i\frac{9\pi}{12}}, e^{-i\frac{7\pi}{12}}$$

$$z^3 = e^{-i\frac{\pi}{4}}, e^{i\frac{7\pi}{4}}, e^{-i\frac{9\pi}{4}} \Rightarrow z = e^{-i\frac{\pi}{12}}, e^{i\frac{7\pi}{12}}, e^{-i\frac{9\pi}{12}}$$

The solutions of $z^6 - z^3\sqrt{2} + 1 = 0$ are $\cos\alpha + i\sin\alpha$,

where $\alpha = -\frac{3\pi}{4}, -\frac{7\pi}{12}, -\frac{\pi}{12}, \frac{\pi}{12}, \frac{7\pi}{12}, \frac{3\pi}{4}$

Each of these two expressions give rise to 3 distinct expressions when multiples of 2π are added and subtracted from the arguments. The original equation is of degree 6 and there will, usually, be 6 distinct answers.

Solutionbank FP2
Edexcel AS and A Level Modular Mathematics

Exercise A, Question 45

Question:

The transformation

$$w = \frac{z + 2}{z + i},$$

where $z \neq i$, $w \neq i$, maps the complex number $z = x + iy$ onto the complex number $w = u + iv$.

a Show that, if the point representing w lies on the real axis, the point representing z lies on a straight line.

b Show further that, if the point representing w lies on the imaginary axis, the point representing z lies on the circle

$$\left|z + 1 + \tfrac{1}{2}i\right| = \frac{\sqrt{5}}{2}.$$

Solution:

a On the real axis, $w = u$

If the point lies on the real axis in the w-plane, the imaginary part of the associated complex number is zero. So $w = u + 0\text{i} = u$.

$$w = u = \frac{z+2}{z+\text{i}}$$

$$uz + u\text{i} = z + 2 \Rightarrow uz - z = 2 - u\text{i} \Rightarrow z = \frac{2 - u\text{i}}{u - 1}$$

Hence

$$z = x + \text{i}y = \frac{2}{u-1} - \frac{u\text{i}}{u-1}$$

Equating real and imaginary parts

$$x = \frac{2}{u-1} \qquad ①$$

$$y = -\frac{u}{u-1} \qquad ②$$

After equating real and imaginary parts, you obtain x and y in terms of the parameter u. Eliminating u gives the Cartesian equation of the locus of the point in the z-plane.

From ① $xu - x = 2$ ③
Dividing ② by ①

$$\frac{y}{x} = -\tfrac{1}{2}u \Rightarrow u = -\frac{2y}{x}$$

Substituting for u in ③

$$x \times -\frac{2y}{x} - x = 2$$

$$-2y - x = 2 \Rightarrow x + 2y + 2 = 0$$

This is the equation of a straight line in the z-plane, as required.

b On the imaginary axis, $w = \text{i}v$

If the point lies on the imaginary axis in the z-plane, then the real part of the associated complex number is zero. So $w = 0 + \text{i}v = \text{i}v$.

$$w = \text{i}v = \frac{z+2}{z+\text{i}}$$

$$\text{i}vz - v = z + 2 \Rightarrow \text{i}vz - z = v + 2 \Rightarrow z = \frac{v+2}{-1+\text{i}v}$$

$$z = \frac{v+2}{-1+\text{i}v} \times \frac{-1-\text{i}v}{-1-\text{i}v} = \frac{-(v+2) - v(v+2)\text{i}}{v^2+1}$$

$$z = x + \text{i}y = -\frac{v+2}{v^2+1} - \frac{v(v+2)\text{i}}{v^2+1}$$

Equating real and imaginary parts

$$x = -\frac{v+2}{v^2+1} \quad ①$$

$$y = -\frac{v(v+2)\mathrm{i}}{v^2+1} \quad ②$$

As in part **a**, after equating real and imaginary parts, you obtain x and y in terms of a parameter; in this case v. Eliminating v gives the Cartesian equation of the locus of the point in the z-plane.

From ① $xv^2 + x = -v - 2$ ③

Dividing ② by ①

$$\frac{y}{x} = v$$

Substituting for v in ③

$$x \times \frac{y^2}{x^2} + x = -\frac{y}{x} - 2$$

Multiplying by x

$$y^2 + x^2 = -y - 2x$$

$$x^2 + 2x + y^2 + y = 0$$

$$x^2 + 2x + 1 + y^2 + y + \tfrac{1}{4} = \tfrac{5}{4}$$

$$(x+1)^2 + \left(y + \tfrac{1}{2}\right)^2 = \left(\tfrac{1}{2}\sqrt{5}\right)^2$$

Completing squares gives you the centre and radius of the circle.

This is the Cartesian equation of a circle with centre $\left(-1, -\frac{1}{2}\right)$ and radius $= \frac{1}{2}\sqrt{5}$.

In the z-plane, z lies on the circle $\left|z + 1 + \frac{1}{2}\mathrm{i}\right| = \frac{1}{2}\sqrt{5}$, as required.

The locus of $|z - a| = k$, where a is a complex number and k is a real number, is a circle with radius k and centre the point representing a. As you know the centre and the radius, you can write down the locus of z without further working.

Solutionbank FP2
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Exercise A, Question 46

Question:

A complex number z is represented by the point P in the Argand diagram.
Given that

$$|z - 3\text{i}| = 3,$$

a sketch the locus of P.

b Find the complex number z which satisfies both $|z - 3\text{i}| = 3$ and $\arg(z - 3\text{i}) = \frac{3\pi}{4}$.

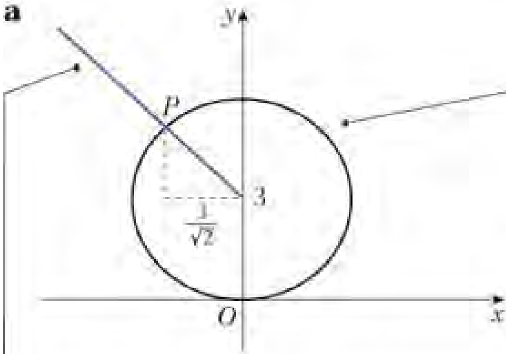
The transformation T from the z-plane to the w-plane is given by

$$w = \frac{2\text{i}}{z}.$$

c Show that T maps $|z - 3\text{i}| = 3$ to a line in the w-plane, and give the Cartesian equation of this line.

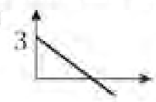
Solution:

a



The locus of P is the circle with centre (0, 3) and radius 3. The coordinates (0, 3) represent the complex number 3i in the Argand diagram.

The half-line representing $\arg(z-3\text{i})=\frac{3\pi}{4}$ has been added to the diagram. This starts at (0, 3) and makes an angle of $\frac{3\pi}{4}$ with the positive x-direction. It is a common error to turn this half into a full line. The half line

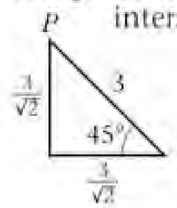


has a different equation, $\arg(z-3\text{i})=-\frac{\pi}{4}$.

b From the diagram, z is the intersection of the circle and the half line marked P in the diagram.

$$z=-\frac{3}{\sqrt{2}}+\text{i}\left(3+\frac{3}{\sqrt{2}}\right)$$

The geometry of the point of intersection is shown here. The coordinates of P can then be just written down.



c The circle $|z-3\text{i}|=3$ has Cartesian equation

$$x^2+(y-3)^2=9$$

$$x^2+y^2-6y+9=9$$

$$x^2+y^2=6y \quad ①$$

If $z=x+\text{i}y$,

$$w=\frac{2\text{i}}{z}=\frac{2\text{i}}{x+\text{i}y}=\frac{2\text{i}}{x+\text{i}y}\times\frac{x-\text{i}y}{x-\text{i}y}$$

$$=\frac{2y+\text{i}2x}{x^2+y^2}$$

Multiplying the numerator and the denominator by the conjugate complex of $x+\text{i}y$ which is $x-\text{i}y$. $(x+\text{i}y)(x-\text{i}y)=x^2+y^2$.

$$u+\text{i}v=\frac{2y}{x^2+y^2}+\text{i}\frac{2x}{x^2+y^2}$$

From ① above, $x^2+y^2=6y$

Hence $u+\text{i}v=\frac{2y}{6y}+\text{i}\frac{2x}{6y}=\frac{1}{3}+\text{i}\frac{x}{3y}$

Equating real parts

$$u=\tfrac{1}{3}$$

A 'simple' equation like $u=\frac{1}{3}$ is quite difficult to recognise in this context. This is the equation of the straight line parallel to the v (imaginary) axis in the w-plane.

The circle maps to the straight line with equation $u=\frac{1}{3}$ in the w-plane.

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Exercise A, Question 47

Question:

The point P on the Argand diagram represents the complex number z.

a Given that $|z| = 1$, sketch the locus of P.

The point Q is the image of P under the transformation

$$w = \frac{1}{z-1}.$$

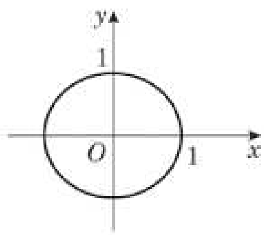
b Given that $z = e^{i\theta}$, $0 < \theta < 2\pi$, show that

$$w = -\tfrac{1}{2} - \tfrac{1}{2}i \cot \tfrac{1}{2}\theta.$$

c Make a separate sketch of the locus Q.

Solution:

a



b If $z = e^{i\theta}$

$$w = \frac{1}{z-1} = \frac{1}{e^{i\theta}-1}$$

$$= \frac{1}{\cos\theta + i\sin\theta - 1} = \frac{1}{\cos\theta - 1 + i\sin\theta}$$

Using Euler's relation $e^{i\theta} = \cos\theta + i\sin\theta$.

$$= \frac{1}{\cos\theta - 1 + i\sin\theta} \times \frac{\cos\theta - 1 - i\sin\theta}{\cos\theta - 1 - i\sin\theta}$$

Multiply the numerator and denominator by $\cos\theta - 1 - i\sin\theta$, the conjugate complex of $\cos\theta - 1 + i\sin\theta$.

$$= \frac{\cos\theta - 1 - i\sin\theta}{(\cos\theta - 1)^2 + \sin^2\theta}$$

$$= \frac{\cos\theta - 1 - i\sin\theta}{\cos^2\theta - 2\cos\theta + 1 + \sin^2\theta}$$

$$= \frac{\cos\theta - 1}{2 - 2\cos\theta} - \frac{i\sin\theta}{2 - 2\cos\theta}$$

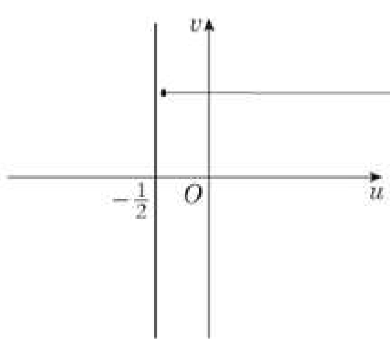
Using $\sin\theta = 2\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta$.

$$= \frac{\cos\theta - 1}{2(1-\cos\theta)} - i\frac{2\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta}{4\sin^2\frac{1}{2}\theta}$$

Using $\cos\theta = 1 - 2\sin^2\frac{1}{2}\theta$.

$= -\frac{1}{2} - \frac{1}{2}i\cot\frac{1}{2}\theta$, as required

c



$w = u + iv = -\frac{1}{2} - \frac{1}{2}i\cot\frac{1}{2}\theta$. Equating real parts gives $u = -\frac{1}{2}$. This is the equation of a straight line parallel to the v (imaginary) axis.

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Exercise A, Question 48

Question:

In an Argand diagram the point P represents the complex number z.

Given that $\arg\left(\frac{z-2i}{z+2}\right)=\frac{\pi}{2}$,

a sketch the locus of P,

b deduce the value of $|z+1-i|$.

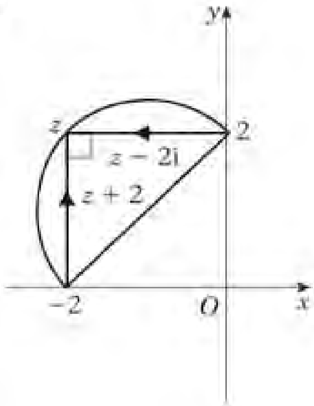
The transformation T from the z-plane to the w-plane is defined by

$$w=\frac{2(1+i)}{z+2},\ z\neq 2.$$

c Show that the locus of P in the z-plane is mapped to part of a straight line in the w-plane, and show this in an Argand diagram.

Solution:

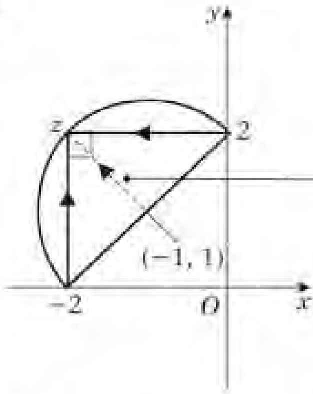
a



$\arg\left(\frac{z-2i}{z+2}\right) = \arg(z-2i) - \arg(z+2) = \frac{\pi}{2}$.

The angles which the vectors make with the positive x-axis differ by a right angle. As drawn here, the difference is $\pi - \frac{\pi}{2} = \frac{\pi}{2}$. The locus of the points, where the difference is a right angle, is a semi-circle, with the line joining -2 on the real axis to 2 on the imaginary axis as diameter.
It is a common error to complete the circle. The lower right hand completion of the circle has equation $\arg\left(\frac{z-2i}{z+2}\right) = -\frac{\pi}{2}$.

b



The dotted line represents the complex number $z+1-i = z-(-1+i)$. The length of this vector is the radius of the circle.

The diameter of the circle is given by

$$d^2 = 2^2 + 2^2 = 8$$

$$|z+1-i| = \frac{\sqrt{8}}{2} = \sqrt{2}$$

c

$$w = \frac{2(1+i)}{z+2}$$

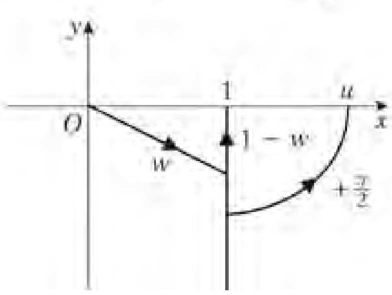
$$z = \frac{2(1+i)}{w} - 2$$

You find the transformation of $\arg\left(\frac{z-2i}{z+2}\right) = \frac{\pi}{2}$ under T by making z 'the subject of the transformation' and using this to substitute for z in the expression $\frac{z-2i}{z+2}$.

$$\frac{z-2i}{z+2} = \frac{\frac{2(1+i)}{w} - 2 - 2i}{\frac{2(1+i)}{w} - 2 + 2} = \frac{2(1+i) - 2(1+i)w}{2(1+i)} = 1 - w$$

Hence the transformation of $\arg\left(\frac{z-2i}{z+2}\right) = \frac{\pi}{2}$

under T is $\arg(1-w) = \frac{\pi}{2}$. This is a half-line as shown in the following diagram.



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Edexcel AS and A Level Modular Mathematics

Exercise A, Question 49

Question:

The transformation T from the complex z-plane to the complex w-plane is given by

$$w = \frac{z+1}{z+\mathrm{i}}, \; z \neq \mathrm{i}.$$

a Show that T maps points on the half-line $\arg z = \frac{\pi}{4}$ in the z-plane into points on the circle $|w| = 1$ in the w-plane.

b Find the image under T in the w-plane of the circle $|z| = 1$ in the z-plane.

c Sketch on separate diagrams the circle $|z| = 1$ in the z-plane and its image under T in the w-plane.

d Mark on your sketches the point P, where $z = \mathrm{i}$, and its image Q under T in the w-plane.

Solution:

a If $z = x + \mathrm{i}y$, then $\arg z = \frac{\pi}{4} \Rightarrow \frac{y}{x} = 1$

Let $x = y = \lambda$

$$w = \frac{\lambda + \lambda\mathrm{i} + 1}{\lambda + \lambda\mathrm{i} + \mathrm{i}} = \frac{(\lambda + 1) + \lambda\mathrm{i}}{\lambda + (\lambda + 1)\mathrm{i}}$$

$$|w| = \left|\frac{(\lambda + 1) + \lambda\mathrm{i}}{\lambda + (\lambda + 1)\mathrm{i}}\right| = \frac{|(\lambda + 1) + \lambda\mathrm{i}|}{|\lambda + (\lambda + 1)\mathrm{i}|}$$

For all complex numbers a and b, $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$

$$= \frac{((\lambda + 1)^2 + \lambda^2)^{\frac{1}{2}}}{(\lambda^2 + (\lambda + 1)^2)^{\frac{1}{2}}} = 1$$

Hence the points on $\arg z = \frac{\pi}{4}$ map, under T, onto points on the circle $|w| = 1$.

As $\lambda \neq 0$, the image would only be part of this circle but the wording of the question does not require you to be more specific. You are only required to show that the image points are points on the circle; not all of the points on the circle. (The image is, in fact, just the lower right quadrant of the circle.)

b

$$wz + w\mathrm{i} = z + 1$$

$$wz - z = 1 - \mathrm{i}w$$

$$z = \frac{1 - \mathrm{i}w}{w - 1}$$

$$|z| = \frac{|1 - \mathrm{i}w|}{|w - 1|} = 1$$

Hence $|1 - \mathrm{i}w| = |w - 1|$

This is the image under T of $|z| = 1$ but it is difficult to interpret and part **c** would be difficult without some further working.

$|1 - \mathrm{i}w| = |-\mathrm{i}(w + \mathrm{i})| = |-\mathrm{i}||w + \mathrm{i}| = 1 \times |w + \mathrm{i}| = |w + \mathrm{i}|$

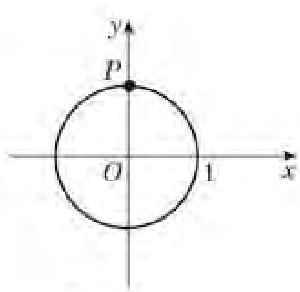
The image of $|z| = 1$ in the z-plane is

$$|w + \mathrm{i}| = |w - 1|$$

in the w-plane.

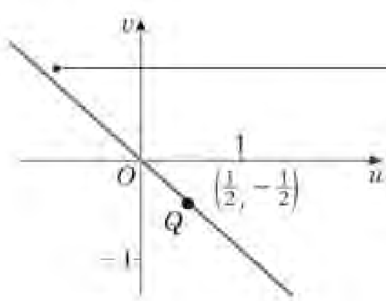
This is the locus of points equidistant from the points in the Argand plane representing $-\mathrm{i}$ and one. That is the perpendicular bisector of $(0, -1)$ and $(1, 0)$.

c d



z

$$z = \mathrm{i} \Rightarrow w = \frac{1 + \mathrm{i}}{2\mathrm{i}} = \frac{\mathrm{i} + 1}{2\mathrm{i}} = \tfrac{1}{2} - \tfrac{1}{2}\mathrm{i}$$



w

The perpendiclar bisector of $(0, -1)$ and $(1, 0)$ is the line $v = -u$.

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Exercise A, Question 50

Question:

a Find the roots z_1 and z_2 of the equation

$$z^2 - 2iz - 2 = 0.$$

The transformation

$$w = \frac{az + b}{z + d},\ z \neq -d,$$

where a, b and d are complex numbers, maps the complex number z onto the complex number w. Given that z_1 and z_2 are invariant under this transformation and that $z = 0$ maps to $w = i$,

b find the values of a, b and d.

Using your values of a, b and d,

c show that $|z| = 2\left|\frac{w - i}{w}\right|$.

d Hence, or otherwise, find the radius and centre of the circle described by w when z moves on the unit circle $|z| = 1$.

Solution:

a $z^2 - 2iz = 2$

$z^2 - 2iz + i^2 = 2 + i^2$

$(z - i)^2 = 1$

$z - i = \pm 1$

$z = 1 + i, -1 + i$

You can use any method to solve this quadratic. Completing the square is an efficient method in this case.

b For an invariant point $w = z$

$$z = \frac{az + b}{z + d}$$

An invariant point is a point unchanged by the mapping. So w and z are the same point and the expression can be transformed into a quadratic.

$z^2 + dz = az + b$

$z^2 + (d - a)z - b = 0$

This must be the same equation as that in part **a**, which is

$z^2 - 2iz - 2 = 0$

The complex numbers $1 + i$ and $-1 + i$ must be the roots of both this quadratic equation and the quadratic equation in part **a**. So, the two equations must be the same and equating the coefficients of x and the constant coefficients gives a simple relation between a and d and the value of b.

Hence, equating coefficients,

$d - a = -2i$ and $b = 2$

$z = 0, w = i$

$$i = \frac{b}{d} \Rightarrow d = \frac{b}{i} = \frac{ib}{i^2} = -ib$$

$d = -2i$ and $a = 0$

$a = 0, b = 2, d = -2i$

c

$$w = \frac{2}{z - 2\text{i}}$$

$$zw - 2\text{i}\,w = 2 \Rightarrow z = \frac{2 + 2\text{i}\,w}{w}$$

$$z = \frac{2\text{i}(w - \text{i})}{w}$$

$$|z| = |2\text{i}|\left|\frac{(w - \text{i})}{w}\right|$$

For all complex numbers a and b, $|ab| = |a||b|$.

As $|2\text{i}| = 2$.

$$|z| = 2\left|\frac{w - \text{i}}{w}\right|, \text{ as required}$$

d

$$|z| = 1 \Rightarrow 2\left|\frac{w - \text{i}}{w}\right| = 1$$

$$2|w - \text{i}| = |w|$$

$$4|w - \text{i}|^2 = |w|^2$$

Let $w = u + \text{i}v$

$$4|u + \text{i}(v - 1)|^2 = |u + \text{i}v|^2$$

$$4(u^2 + (v - 1)^2) = u^2 + v^2$$

$$4u^2 + 4v^2 - 8v + 4 = u^2 + v^2$$

$$3u^2 + 3v^2 - 8v + 4 = 0$$

$$u^2 + v^2 - \tfrac{8}{3}v = -\tfrac{4}{3}$$

$$u^2 + v^2 - \tfrac{8}{3}v + \tfrac{16}{9} = -\tfrac{4}{3} + \tfrac{16}{9}$$

$$u^2 + \left(v - \tfrac{4}{3}\right)^2 = \tfrac{4}{9} = \left(\tfrac{2}{3}\right)^2$$

Completing the square gives the centre and radius of the circle.

The image is a circle, centre $\left(0, \tfrac{4}{3}\right)$, radius $\tfrac{2}{3}$.

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Exercise A, Question 1

Question:

Find, in the form $y = \mathrm{f}(x)$, the general solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{4}{x}y = 6x - 5, \quad x > 0.$$

Solution:

The integrating factor is

$$\mathrm{e}^{\int \frac{4}{x}\,\mathrm{d}x} = \mathrm{e}^{4\ln x} = \mathrm{e}^{\ln x^4} = x^4$$

If the differential equation has the form $\frac{\mathrm{d}y}{\mathrm{d}x} + Py = Q$, the integrating factor is $\mathrm{e}^{\int P\,\mathrm{d}x}$.

For any function $\mathrm{f}(x)$, $\mathrm{e}^{\ln \mathrm{f}(x)} = \mathrm{f}(x)$.

Multiply the equation throughout by x^4

$$x^4\frac{\mathrm{d}y}{\mathrm{d}x} + 4x^3y = 6x^5 - 5x^4$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(x^4y) = 6x^5 - 5x^4$$

$$x^4y = \int(6x^5 - 5x^4)\,\mathrm{d}x = x^6 - x^5 + C$$

$$y = x^2 - x + \frac{C}{x^4}$$

It is important that you remember to add the constant of integration. When you divide by x^4, the constant becomes a function of x and its omission would be a significant error.

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Exercise A, Question 2

Question:

Solve the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{y}{x} = x^2, \quad x > 0,$$

giving your answer for y in terms of x.

Solution:

The interating factor is

$$e^{\int -\frac{1}{x}\mathrm{d}x} = e^{-\ln x} = e^{\ln \frac{1}{x}} = \frac{1}{x}$$

For all n, $n \ln x = \ln x^n$, so for $n = -1$, $-\ln x = \ln x^{-1} = \ln \frac{1}{x}$.

Multiply the equation throughout by $\frac{1}{x}$

$$\frac{1}{x}\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{y}{x^2} = x$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y}{x}\right) = x$$

The product rule for differentiating, in this case $\frac{\mathrm{d}}{\mathrm{d}x}\left(y \times \frac{1}{x}\right) = \frac{\mathrm{d}y}{\mathrm{d}x} \times \frac{1}{x} + y \times -\frac{1}{x^2}$, enables you to write the differential equation as an exact equation, where one side is the exact derivative of a product and the other side can be integrated with respect to x.

$$\frac{y}{x} = \frac{x^2}{2} + C$$

$$y = \frac{x^3}{2} + Cx$$

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Exercise A, Question 3

Question:

Find the general solution of the differential equation

$$(x+1)\frac{dy}{dx}+2y=\frac{1}{x}, \quad x>0,$$

giving your answer in the form $y = f(x)$.

Solution:

$$(x+1)\frac{dy}{dx}+2y=\frac{1}{x}$$

$$\frac{dy}{dx}+\frac{2}{x+1}y=\frac{1}{x(x+1)}$$

If the equation is in the form $R\frac{dy}{dx}+Sy=T$, you must begin by dividing throughout by R, in this case $(x+1)$, before finding the integrating factor.

The integrating factor is

$$e^{\int\frac{2}{x+1}dx}=e^{2\ln(x+1)}=e^{\ln(x+1)^2}=(x+1)^2$$

Multiply throughout by $(x+1)^2$

$$(x+1)^2\frac{dy}{dx}+2(x+1)y=\frac{x+1}{x}$$

To integrate $\frac{x+1}{x}$, write $\frac{x+1}{x}=\frac{x}{x}+\frac{1}{x}=1+\frac{1}{x}$.

$$\frac{d}{dx}\left((x+1)^2y\right)=1+\frac{1}{x}$$

$$(x+1)^2y=\int\left(1+\frac{1}{x}\right)dx=x+\ln x+C$$

$$y=\frac{x+\ln x+C}{(x+1)^2}$$

You divide throughout by $(x+1)^2$ to obtain the equation in the form $y = f(x)$. This is required by the wording of the question.

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Exercise A, Question 4

Question:

Obtain the solution of

$$\frac{dy}{dx} + y\tan x = e^{2x}\cos x,\ 0 \leqslant x < \frac{\pi}{2},$$

for which $y = 2$ at $x = 0$, giving your answer in the form $y = f(x)$.

Solution:

The integrating factor is $e^{\int \tan x\, dx}$

$$\int \tan x\, dx = \int \frac{\sin x}{\cos x} dx = -\ln\cos x = \ln\frac{1}{\cos x} = \ln\sec x$$

Hence

$$e^{\int \tan x\, dx} = e^{\ln\sec x} = \sec x$$

In C4 you learnt that $\int \frac{f'(x)}{f(x)} dx = \ln f(x)$. As $-\sin x$ is the derivative of $\cos x$, $\int \frac{-\sin x}{\cos x} dx = \ln\cos x$.

Multiply the differential equation throughout by $\sec x$

$$\sec x\frac{dy}{dx} + y\sec x\tan x = e^{2x}\sec x\cos x = e^{2x}$$

$\sec x\cos x = \frac{1}{\cos x} \times \cos x = 1$

$$\frac{d}{dx}(y\sec x) = e^{2x}$$

$$y\sec x = \int e^{2x} dx = \frac{e^{2x}}{2} + C$$

Multiply throughout by $\cos x$

$$y = \left(\frac{e^{2x}}{2} + C\right)\cos x$$

$y = 2$ at $x = 0$

$$2 = \tfrac{1}{2} + C \Rightarrow C = \tfrac{3}{2}$$

The condition $y = 2$ at $x = 0$ enables you to evaluate the constant of integration and find the particular solution of the differential equation for these values.

$$y = \tfrac{1}{2}(e^{2x} + 3)\cos x$$

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Exercise A, Question 5

Question:

Find the general solution of the differential equation

$$\frac{dy}{dx} + 2y \cot 2x = \sin x, \quad 0 < x < \frac{\pi}{2},$$

giving your answer in the form $y = f(x)$.

Solution:

The integrating factor is $e^{\int 2 \cot 2x \, dx}$

$$\int 2 \cot 2x \, dx = \int \frac{2 \cos 2x}{\sin 2x} dx = \ln \sin 2x$$

Hence

$$e^{\int 2 \cot 2x \, dx} = e^{\ln \sin 2x} = \sin 2x$$

Multiply the differential equation throughout by $\sin 2x$

$$\sin 2x \frac{dy}{dx} + 2y \cos 2x = \sin x \sin 2x$$

Using the identity $\sin 2x = 2 \sin x \cos x$.

$$\frac{d}{dx}(y \sin 2x) = 2 \sin^2 x \cos x$$

$$y \sin 2x = \frac{2 \sin^3 x}{3} + C$$

As $\frac{d}{dx}(\sin^3 x) = 3 \sin^2 x \cos x$, then $\int \sin^2 x \cos x \, dx = \frac{\sin^3 x}{3}$. It saves time to find integrals of this type by inspection. However, you can use the substitution $\sin x = s$ if you find inspection difficult.

$$y = \frac{2 \sin^3 x}{3 \sin 2x} + \frac{C}{\sin 2x}$$

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Exercise A, Question 6

Question:

Solve the differential equation

$$(1 + x)\frac{dy}{dx} - xy = xe^{-x},$$

given that $y = 1$ at $x = 0$.

Solution:

$$(1+x)\frac{dy}{dx} - xy = xe^{-x}$$

$$\frac{dy}{dx} - \frac{xy}{1+x} = \frac{xe^{-x}}{1+x} \quad ①$$

If the equation is in the form $R\frac{dy}{dx} + Sy = T$, you must begin by dividing throughout by R, in this case $(1+x)$ before finding the integrating factor.

The integrating factor is $e^{\int -\frac{x}{1+x}dx}$

$$\frac{x}{1+x} = \frac{1+x-1}{1+x} = 1 - \frac{1}{1+x}$$

To integrate an expression in which the degree of the numerator is greater or equal to the degree of the denominator, you must transform the expression into one with a proper fraction. This can be done using partial fractions, long division or, as here, using decomposition.

Hence

$$\int \frac{x}{1+x}dx = x - \ln(1+x)$$

and the integrating factor is

$$e^{-x+\ln(1+x)} = e^{-x}e^{\ln(1+x)} = e^{-x}(1+x)$$

Multiplying ① throughout by $(1+x)e^{-x}$

$$(1+x)e^{-x}\frac{dy}{dx} - xe^{-x}y = xe^{-2x}$$

$$\frac{d}{dx}\left(y(1+x)e^{-x}\right) = xe^{-2x}$$

$$y(1+x)e^{-x} = \int xe^{-2x}dx$$

You integrate xe^{-2x} using integration by parts.

$$= -\frac{xe^{-2x}}{2} + \int \frac{e^{-2x}}{2}dx = -\frac{xe^{-2x}}{2} - \frac{e^{-2x}}{4} + C$$

$$\frac{e^{-2x}}{e^{-x}} = e^{-2x-(-x)} = e^{-2x+x} = e^{-x}$$

$$y = -\frac{xe^{-2x}}{2(1+x)e^{-x}} - \frac{e^{-2x}}{4(1+x)e^{-x}} + \frac{C}{(1+x)e^{-x}}$$

$$= -\frac{xe^{-x}}{2(1+x)} - \frac{e^{-x}}{4(1+x)} + \frac{Ce^{x}}{(1+x)}$$

$y = 1$ at $x = 0$

$$1 = 0 - \tfrac{1}{4} + C \Rightarrow C = \tfrac{5}{4}$$

$$y = \frac{5e^{x}}{4(1+x)} - \frac{xe^{-x}}{2(1+x)} - \frac{e^{-x}}{4(1+x)}$$

This expression could be put over a common denominator but, other than requiring that y is expressed in terms of x, the question asks for no particular form and this is an acceptable answer.

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Exercise A, Question 7

Question:

a By using the substitution $y = \frac{1}{2}(u - x)$, or otherwise, find the general solution of the differential equation

$$\frac{dy}{dx} = x + 2y.$$

Given that $y = 2$ at $x = 0$,

b express y in terms of x.

Solution:

a $y = \frac{1}{2}u - \frac{1}{2}x$

Differentiate throughout with respect to x.

$$\frac{dy}{dx} = \frac{1}{2}\frac{du}{dx} - \frac{1}{2}$$

$$\frac{dy}{dx} = x + 2y$$

$y = \frac{1}{2}(u - x) \Rightarrow 2y = u - x$

transforms to

$$\frac{1}{2}\frac{du}{dx} - \frac{1}{2} = x + u - x = u$$

$$\frac{du}{dx} - 1 = 2u$$

$$\frac{du}{dx} = 2u + 1$$

This is a separable equation. You learnt how to solve separable equations in C4.

$$\int \frac{1}{2u + 1}\,du = \int 1\,dx$$

Separating the variables.

$$\frac{1}{2}\ln(2u + 1) = x + A$$

$$\ln(2u + 1) = 2x + B$$

Twice one arbitrary constant A is another arbitrary constant, $B = 2A$.

$$e^{\ln(2u+1)} = e^{2x+B} = e^B e^{2x} = C\,e^{2x}$$

e to an arbitrary constant is another arbitrary constant. Here $C = e^B$.

$$2u + 1 = 4y + 2x + 1 = C\,e^{2x}$$

$$y = \frac{C\,e^{2x} - 2x - 1}{4}$$

This is the general solution of the original differential equation.

b $y = 2$ at $x = 0$

$$2 = \frac{C - 1}{4} \Rightarrow 8 = C - 1 \Rightarrow C = 9$$

$$y = \frac{9e^{2x} - 2x - 1}{4}$$

This is the particular solution of the original differential equation for which $y = 2$ at $x = 0$.

Solutionbank FP2

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Exercise A, Question 8

Question:

a Find the general solution of the differential equation

$$t\frac{dv}{dt} - v = t, \quad t > 0$$

and hence show that the solution can be written in the form $v = t(\ln t + c)$, where c is an arbitrary constant.

b This differential equation is used to model the motion of a particle which has speed $v\,\text{m s}^{-1}$ at time t seconds. When $t = 2$ the speed of the particle is $3\,\text{m s}^{-1}$. Find, to 3 significant figures, the speed of the particle when $t = 4$.

Solution:

a $t\frac{dv}{dt} - v = t$

Divide throughout by t

$$\frac{dv}{dt} - \frac{v}{t} = 1 \qquad ①$$

The integrating factor is

$$e^{\int -\frac{1}{t}dt} = e^{-\ln t} = e^{\ln \frac{1}{t}} = \frac{1}{t}$$

Multiply ① throughout by $\frac{1}{t}$

$$\frac{1}{t}\frac{dv}{dt} - \frac{v}{t^2} = \frac{1}{t}$$

$$\frac{d}{dt}\left(\frac{v}{t}\right) = \frac{1}{t}$$

The product rule for differentiating, in this case $\frac{d}{dt}(v \times t^{-1}) = \frac{dv}{dt} \times t^{-1} + v \times (-1)t^{-2}$, enables you to write the differential equation as an exact equation, where one side is the exact derivative of a product and the other side can be integrated with respect to t.

$$\frac{v}{t} = \int \frac{1}{t}dt = \ln t + c$$

$v = t(\ln t + c)$, as required

b $v = 3$ when $t = 2$

$$3 = 2(\ln 2 + c) = 2\ln 2 + 2c \Rightarrow c = 1.5 - \ln 2$$

$$v = t(\ln t + 1.5 - \ln 2)$$

When $t = 4$

$$v = 4(\ln 4 + 1.5 - \ln 2)$$

Use your calculator to evaluate this expression.

$$\approx 8.77$$

The speed of the particle when $t = 4$ is $8.77\,\text{m s}^{-1}$ (3 s.f.).

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Exercise A, Question 9

Question:

a Use the substitution $y = vx$ to transform the equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{(4x + y)(x + y)}{x^2} \quad x > 0, \qquad ①$$

into the equation

$$x\frac{\mathrm{d}v}{\mathrm{d}x} = (2 + v)^2. \qquad ②$$

b Solve the differential equation ② to find v in terms of x.

c Hence show that

$$y = -2x - \frac{x}{\ln x + c}, \text{ where } c \text{ is an}$$

arbitrary constant, is a general solution of differential equation ①.

Solution:

a $y = vx$

$$\frac{dy}{dx} = x\frac{dv}{dx} + v$$

Differentiating vx as a product,
$\frac{d}{dx}(vx) = \frac{dv}{dx}x + v\frac{d}{dx}(x) = x\frac{dv}{dx} + v$,
as $\frac{d}{dx}(x) = 1$.

Substituting $y = vx$ and $\frac{dy}{dx} = x\frac{dv}{dx} + v$ into equation ① in the question

$$x\frac{dv}{dx} + v = \frac{(4x + vx)(x + vx)}{x^2}$$

$$= \frac{x^2(4 + v)(1 + v)}{x^2} = (4 + v)(1 + v) = 4 + 5v + v^2$$

$x\frac{dv}{dx} = 4 + 4v + v^2 = (2 + v)^2$, as required.

This is a separable equation and the first step in its solution is to separate the variables, by collecting together the terms in v and dv on one side of the equation and the terms in x and dx on the other side of the equation.

b $\int \frac{1}{(2 + v)^2}\,dv = \int \frac{1}{x}\,dx$

$$-\frac{1}{2 + v} = \ln x + c$$

$\int (2 + v)^{-2}\,dv = \frac{(2 + v)^{-1}}{-1} = -\frac{1}{2 + v}$

$$2 + v = -\frac{1}{\ln x + c}$$

$$v = -2 - \frac{1}{\ln x + c}$$

c $y = vx \Rightarrow v = \frac{y}{x}$

Substituting $v = \frac{y}{x}$ into the answer to part **b**

$$\frac{y}{x} = -2 - \frac{1}{\ln x + c}$$

$y = -2x - \frac{x}{\ln x + c}$, as required.

Multiply throughout by x to obtain the printed answer.

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Exercise A, Question 10

Question:

a Using the substitution $t = x^2$, or otherwise, find

$$\int x^3 \, e^{-x^2} \, dx.$$

b Find the general solution of the differential equation

$$x\frac{dy}{dx} + 3y = xe^{-x^2}.$$

Solution:

a $t = x^2 \Rightarrow \frac{dt}{dx} = 2x \Rightarrow x\frac{dx}{dt} = \frac{1}{2}$

$$\int x^3 e^{-x^2} dx = \int x^2 e^{-x^2}\left(x\frac{dx}{dt}\right) dt$$

$$= \int t e^{-t}\left(\tfrac{1}{2}\right) dt = \tfrac{1}{2}\int t e^{-t} dt$$

$$= -\frac{t e^{-t}}{2} + \int \frac{e^{-t}}{2} dt$$

$$= -\frac{t e^{-t}}{2} - \frac{e^{-t}}{2} + C$$

The first part of this question is integration by substitution and could have been set on a C4 paper. Its purpose here is to help you with part **b**. Realising this helps you to check your work. When you come to the integration in part **b**, it should turn out to be the integration you have already carried out in part **a**. If it was not, you would need to check your work carefully.

Returning to the original variable

$$\int x^3 e^{-x^2} dx = -\frac{x^2 e^{-x^2}}{2} - \frac{e^{-x^2}}{2} + C$$

b $x\frac{dy}{dx} + 3y = x e^{-x^2}$

$$\frac{dy}{dx} + \frac{3}{x}y = e^{-x^2} \quad ①$$

Divide throughout by x.

The integrating factor is

$$e^{\int \frac{3}{x} dx} = e^{3\ln x} = e^{\ln x^3} = x^3$$

Multiply ① throughout by x^3

$$x^3\frac{dy}{dx} + 3x^2 y = x^3 e^{-x^2}$$

$$\frac{d}{dx}(yx^3) = x^3 e^{-x^2}$$

This is an exact equation, where one side is the exact derivative of a product and the other side is the expression you have already integrated in part **a**.

$$yx^3 = \int x^3 e^{-x^2} dx$$

$$= -\frac{x^2 e^{-x^2}}{2} - \frac{e^{-x^2}}{2} + C$$

$$y = -\frac{e^{-x^2}}{2x} - \frac{e^{-x^2}}{2x^3} + \frac{C}{x^3}$$

Solutionbank FP2
Edexcel AS and A Level Modular Mathematics

Exercise A, Question 11

Question:

a Find the general solution of the differential equation

$$\cos x \frac{dy}{dx} + (\sin x)y = \cos^3 x.$$

b Show that, for $0 \leqslant x \leqslant 2\pi$, there are two points on the x-axis through which all the solution curves for this differential equation pass.

c Sketch the graph, $0 \leqslant x \leqslant 2\pi$, of the particular solution for which $y = 0$ at $x = 0$.

Solution:

a Dividing throughout by $\cos x$

$$\frac{dy}{dx} + \frac{\sin x}{\cos x} y = \cos^2 x \quad ①$$

$$\int \frac{\sin x}{\cos x} dx = -\int \frac{-\sin x}{\cos x} dx = -\ln \cos x = \ln \frac{1}{\cos x} = \ln \sec x$$

In C4 you learnt that $\int \frac{f'(x)}{f(x)} dx = \ln f(x)$. As $-\sin x$ is the derivative of $\cos x$, $-\int \frac{-\sin x}{\cos x} dx = -\ln \cos x$.

As $\ln 1 = 0$, $-\ln \cos x = \ln 1 - \ln \cos x = \ln \frac{1}{\cos x}$, using the log law $\ln a - \ln b = \ln \frac{a}{b}$.

Hence the integrating factor is $e^{\ln \sec x} = \sec x$

Multiply ① by $\sec x$

$$\sec x \frac{dy}{dx} + \sec x \frac{\sin x}{\cos x} y = \cos^2 x \sec x$$

$$\sec x \frac{dy}{dx} + (\sec x \tan x) y = \cos x$$

$$\frac{d}{dx}(y \sec x) = \cos x$$

$$y \sec x = \int \cos x \, dx = \sin x + C$$

Multiplying throughout by $\cos x$

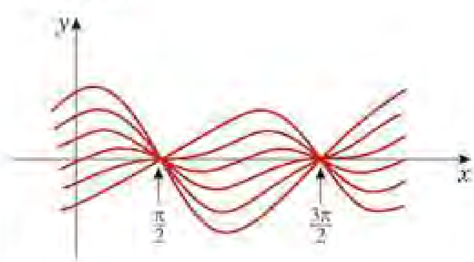
$$y = \sin x \cos x + C \cos x$$

b Where $\cos x = 0$ and $0 \leqslant x \leqslant 2\pi$

$$x = \frac{\pi}{2}, \frac{3\pi}{2}$$

The points $\left(\frac{\pi}{2}, 0\right)$ and $\left(\frac{3\pi}{2}, 0\right)$ lie on all of the solution curves of the differential equation.

In general, for a given value of x, different values of c give different values of y. However, if $\cos x = 0$, the c will have no effect and y will be zero for any value of c.



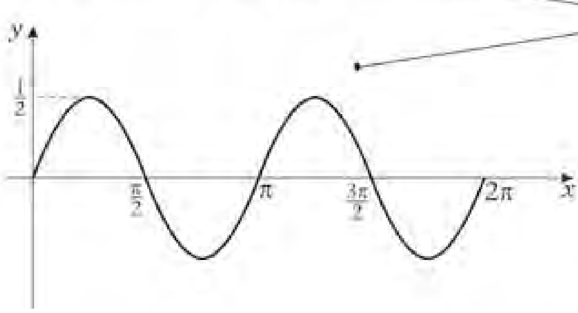
c $y = \sin x \cos x + C \cos x$

At $x = 0$, $y = 0$

$$0 = 0 + C \Rightarrow C = 0$$

$$y = \sin x \cos x = \tfrac{1}{2} \sin 2x$$

Using the identity $\sin 2x = 2 \sin x \cos x$. $\sin 2x$ is a function with period π. So the curve makes two complete oscillations in the interval $0 \leqslant x \leqslant 2\pi$



Solutionbank FP2
Edexcel AS and A Level Modular Mathematics

Exercise A, Question 12

Question:

a Find the general solution of the differential equation

$$\frac{dy}{dx} + 2y = x.$$

Given that $y = 1$ at $x = 0$,

b find the exact values of the coordinates of the minimum point of the particular solution curve,

c draw a sketch of the particular solution curve.

Solution:

a The integrating factor is

$$e^{\int 2\,dx} = e^{2x}$$

Multiplying the differential equation throughout by e^{2x}

$$e^{2x}\frac{dy}{dx} + 2e^{2x}y = x\,e^{2x}$$

$$\frac{d}{dx}(y\,e^{2x}) = x\,e^{2x}$$

$$y\,e^{2x} = \int x\,e^{2x}\,dx$$

Integrate by parts.

$$= \frac{x\,e^{2x}}{2} - \int \frac{e^{2x}}{2}\,dx = \frac{x\,e^{2x}}{2} - \frac{e^{2x}}{4} + C$$

$$y = \frac{x}{2} - \frac{1}{4} + C\,e^{-2x}$$

b $y = 1$ at $x = 0$

$$1 = 0 - \tfrac{1}{4} + C \Rightarrow C = \tfrac{5}{4}$$

$$y = \frac{x}{2} - \frac{1}{4} + \frac{5\,e^{-2x}}{4}$$

This is the particular solution of the differential equation for $y = 1$ at $x = 0$. You are asked to sketch this in part **c**.

For a minimum $\frac{dy}{dx} = 0$

$$\frac{dy}{dx} = \frac{1}{2} - \frac{5\,e^{-2x}}{2} = 0 \Rightarrow 5\,e^{-2x} = 1 \Rightarrow e^{2x} = 5$$

$$\ln e^{2x} = \ln 5 \Rightarrow 2x = \ln 5$$

$$x = \tfrac{1}{2}\ln 5$$

At the minimum, the differential equation reduces to

$$2y = x$$

The differential equation is $\frac{dy}{dx} + 2y = x$. At the minimum, $\frac{dy}{dx} = 0$ and so $2y = x$. If you did not see this you could, of course, substitute $x = \frac{1}{2}\ln 5$ into the particular solution and find y. This would take longer but would gain full marks.

Hence

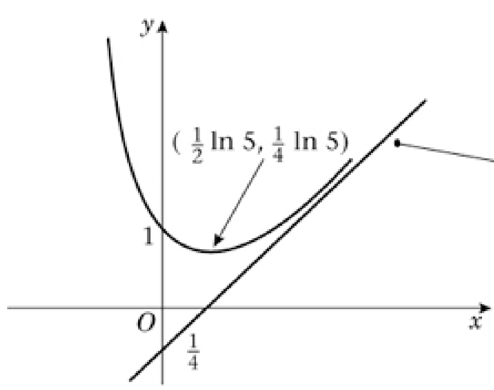
$$y = \tfrac{1}{2}x = \tfrac{1}{4}\ln 5$$

$$\frac{d^2y}{dx^2} = 5\,e^{-2x} > 0 \text{ for any real } x$$

This confirms the point is a minimum.

The coordinates of the minimum are $\left(\frac{1}{2}\ln 5, \frac{1}{4}\ln 5\right)$.

c



As x increases, $e^{-2x} \to 0$ and so $\frac{x}{2} - \frac{1}{4} + \frac{5\,e^{-2x}}{4} \to \frac{x}{2} - \frac{1}{4}$. This means that $y = \frac{x}{2} - \frac{1}{4}$ is an asymptote of the curve. This has been drawn on the graph. It is not essential to do this, but if you recognise that this line is an asymptote, it helps you to draw the correct shape of the curve.

Solutionbank FP2

Edexcel AS and A Level Modular Mathematics

Exercise A, Question 13

Question:

During an industrial process, the mass of salt, S kg, dissolved in a liquid t minutes after the process begins is modelled by the differential equation

$$\frac{dS}{dt} + \frac{2S}{120 - t} = \frac{1}{4}, \quad 0 \leqslant t < 120.$$

Given that $S = 6$ when $t = 0$,

a find S in terms of t,

b calculate the maximum mass of salt that the model predicts will be dissolved in the liquid at any one time during the process.

Solution:

a $\int \frac{2}{120-t}\,dt = -2\ln(120-t) = \ln(120-t)^{-2} = \ln\frac{1}{(120-t)^2}$

Using the log law $n\log a = \log a^n$, with $n = -2$.

Hence the integrating factor is

$$e^{\int \frac{2}{120-t}dt} = e^{\ln\frac{1}{(120-t)^2}} = \frac{1}{(120-t)^2}$$

Multiply the equation throughout by $\frac{1}{(120-t)^2}$

$$\frac{1}{(120-t)^2}\frac{dS}{dt} + \frac{2}{(120-t)^3}S = \frac{1}{4(120-t)^2}$$

$$\frac{d}{dt}\left(\frac{S}{(120-t)^2}\right) = \frac{1}{4}(120-t)^{-2}$$

$\frac{d}{dt}\left(S(120-t)^{-2}\right)$

$= \frac{dS}{dt}(120-t)^{-2} - S\times(-2)(120-t)^{-3}$

$= \frac{1}{(120-t)^2}\frac{dS}{dt} + \frac{2}{(120-t)^3}S$

This product enables you to write the differential equation as a complete equation.

Integrating both sides with respect to t

$$\frac{S}{(120-t)^2} = \frac{1}{4}\int(120-t)^{-2}dt = -\frac{1}{4}\,\frac{(120-t)^{-1}}{-1} + C$$

$$\frac{S}{(120-t)^2} = \frac{1}{4(120-t)} + C$$

Multiply this equation by $(120-t)^2$.

$$S = \frac{120-t}{4} + C(120-t)^2$$

Remember to multiply the C by $(120-t)^2$. It is a common error to obtain C instead of $C(120-t)^2$ at this stage.

$S = 6$ when $t = 0$

$$6 = 30 + C\times 120^2 \Rightarrow C = -\frac{24}{120^2} = -\frac{1}{600}$$

$$S = \frac{120-t}{4} - \frac{(120-t)^2}{600}$$

b For a maximum value

$$\frac{dS}{dt} = -\frac{1}{4} + \frac{2(120-t)}{600} = 0$$

$$240 - 2t = 150 \Rightarrow t = \frac{240-150}{2} = 45$$

$$\frac{d^2S}{dt^2} = -\frac{1}{300} < 0 \Rightarrow \text{maximum}$$

Maximum value is given by

$$S = \frac{120-45}{4} - \frac{(120-45)^2}{600} = \frac{75}{4} - \frac{75}{8} = \frac{75}{8} = 9\frac{3}{8}$$

The maximum mass of salt predicted is $9\frac{3}{8}$ kg.

Solutionbank FP2
Edexcel AS and A Level Modular Mathematics

Exercise A, Question 14

Question:

A fertilized egg initially contains an embryo of mass m_0 together with a mass $100m_0$ of nutrient, all of which is available as food for the embryo. At time t the embryo has mass m and the mass of nutrient which has been consumed is $5(m - m_0)$.

a Show that, when three-quarters of the nutrient has been consumed, $m = 16m_0$.

The rate of increase of the mass of the embryo is a constant μ multiplied by the product of the mass of the embryo and the mass of the remaining nutrient.

b Show that $\frac{dm}{dt} = 5\,\mu m\,(21m_0 - m)$.

The egg hatches at time T, when three-quarters of the nutrient has been consumed.

c Show that $105\,\mu m_0 T = \ln 64$.

Solution:

a Three quarters of the nutrient is $\frac{3}{4} \times 100m_0 = 75m_0$

At time t, the nutrient consumed is $5(m - m_0)$

Hence

$$5(m - m_0) = 75m_0$$

$$5m - 5m_0 = 75m_0 \Rightarrow 5m = 80m_0$$

$$m = \frac{80m_0}{5} = 16m_0, \text{ as required}$$

b Rate of increase of mass = $\mu \times$ mass $\times$ nutrient remaining

The nutrient remaining is the nutrient consumed, $5(m - m_0)$, subtracted from the original nutrient $100m_0$.

$$\frac{dm}{dt} = \mu \times m \times [100m_0 - 5(m - m_0)]$$

$$\frac{dm}{dt} = \mu m\,(100m_0 - 5m + 5m_0)$$

$$= \mu m\,(105m_0 - 5m)$$

$$= 5\,\mu m\,(21m_0 - m), \text{ as required}$$

c $\frac{dm}{dt} = 5\,\mu m\,(21m_0 - m)$

This is a separable equation.

$$\int 5\mu\, dt = \int \frac{1}{m(21m_0 - m)}\, dm$$

Separating the variables.

Let $\frac{1}{m(21m_0 - m)} = \frac{A}{m} + \frac{B}{21m_0 - m}$

To integrate the right hand side of this equation, you must break the expression up into partial fractions using one of the methods you learnt in C4.

Multiplying throughout by $m(21m_0 - m)$

$$1 = A(21m_0 - m) + Bm$$

Let $m = 0$

$$1 = A \times 21m_0 \Rightarrow A = \frac{1}{21m_0}$$

Let $m = 21m_0$

$$1 = B \times 21m_0 \Rightarrow B = \frac{1}{21m_0}$$

Hence

$$5\mu t = \frac{1}{21m_0}\int \left(\frac{1}{m} + \frac{1}{21m_0 - m}\right) dm$$

$$105\mu m_0 t = \int \left(\frac{1}{m} + \frac{1}{21m_0 - m}\right) dm = \ln m - \ln(21m_0 - m) + C$$

When $t = 0$, $m = m_0$

Initially, the mass of the embryo is m_0. This enables you to find the particular solution of the differential equation. The initial conditions are often known in scientific applications of mathematics.

$$0 = \ln m_0 - \ln 20m_0 + C$$

$$C = \ln 20m_0 - \ln m_0 = \ln \frac{20m_0}{m_0} = \ln 20$$

$$105\mu m_0 t = \ln m - \ln(21m_0 - m) + \ln 20 = \ln\left(\frac{20m}{21m_0 - m}\right)$$

Combining the logarithms at this stage simplifies the next stage of the calculation. The form of the simplification is $\ln a - \ln b + \ln c = \ln \frac{ac}{b}$

From part **a**, when $t = T$, $m = 16m_0$

$$105\mu m_0 T = \ln\left(\frac{20 \times 16m_0}{21m_0 - 16m_0}\right) = \ln\left(\frac{320m_0}{5m_0}\right)$$

$$= \ln 64, \text{ as required}$$

Solutionbank FP2
Edexcel AS and A Level Modular Mathematics

Exercise A, Question 15

Question:

a Show that the substitution $y = vx$ transforms the differential equation

$$\frac{dy}{dx} = \frac{3x - 4y}{4x + 3y} \qquad ①$$

into the differential equation

$$x\frac{dv}{dx} = -\frac{3v^2 + 8v - 3}{3v + 4} \qquad ②$$

b By solving differential equation ②, find the general solution of differential equation ②.

c Given that $y = 7$ at $x = 1$, show that the particular solution of differential equation ① can be written as

$$(3y - x)(y + 3x) = 200.$$

Solution:

a $y = vx$

$$\frac{dy}{dx} = x\frac{dx}{dx} + v$$

Differentiating vx as a product,
$\frac{d}{dx}(vx) = \frac{dv}{dx}x + v\frac{d}{dx}(x) = x\frac{dv}{dx} + v,$
as $\frac{d}{dx}(x) = 1.$

Substitute $y = vx$ and $\frac{dy}{dx} = x\frac{dv}{dx} + v$ into equation ① in the question

$$x\frac{dv}{dx} + v = \frac{3x - 4vx}{4x + 3vx} = \frac{\cancel{x}(3 - 4v)}{\cancel{x}(4 + 3v)}$$

$$x\frac{dv}{dx} = \frac{3 - 4v}{4 + 3v} - v = \frac{3 - 4v - 4v - 3v^2}{4 + 3v} = \frac{3 - 8v - 3v^2}{4 + 3v}$$

$$x\frac{dv}{dx} = -\frac{3v^2 + 8v - 3}{3v + 4}, \text{ as required.}$$

This is a separable equation and in part **b** you solve it by collecting together the terms in v and dv on one side of the equation and the terms in x and dx on the other side.

b $\int \frac{3v + 4}{3v^2 + 8v - 3}\,dv = \frac{1}{2}\int \frac{6v + 8}{3v^2 + 8v - 3}\,dv = -\int \frac{1}{x}\,dx$

$$\tfrac{1}{2}\ln(3v^2 + 8v - 3) = -\ln x + A$$

$$\ln(3v^2 + 8v - 3) = -2\ln x + B$$

$$= \ln\frac{1}{x^2} + \ln C = \ln\frac{C}{x^2}$$

$\int \frac{f'(x)}{f(x)}\,dx = \ln f(x)$ is a standard formula you should know. As $6v + 8$ is the derivative of $3v^2 + 8v - 3$,
$\int \frac{6v + 8}{3v^2 + 8v - 3}\,dv = \ln(3v^2 + 8v - 3).$

An arbitrary constant B can be written as the logarithm of another arbitrary constant $\ln C$.

Hence

$$3v^2 + 8v - 3 = \frac{C}{x^2}$$

c $y = xv \Rightarrow v = \frac{y}{x}$

Substituting into the answer to part **b**

$$\frac{3y^2}{x^2} + \frac{8y}{x} - 3 = \frac{C}{x^2}$$

$$3y^2 + 8yx - 3x^2 = C$$

Multiply each term in the equation by x^2.

$y = 7$ at $x = 1$

$$3 \times 49 + 56 - 3 = C \Rightarrow C = 200$$

Factorising the left hand side of the equation

$$(3y - x)(y + 3x) = 200, \text{ as required.}$$

Solutionbank FP2
Edexcel AS and A Level Modular Mathematics

Exercise A, Question 16

Question:

a Use the substitution $u = y^{-2}$ to transform the differential equation

$$\frac{dy}{dx} + 2xy = xe^{-x^2}y^3 \quad ①$$

into the differential equation

$$\frac{du}{dx} - 4xu = -2xe^{-x^2}. \quad ②$$

b Find the general solution of differential equation ②.

c Hence obtain the solution of differential equation ① for which $y = 1$ at $x = 0$.

Solution:

a $u = y^{-2}$

$$\frac{du}{dx} = -2 \times y^{-3} \times \frac{dy}{dx}$$

Differentiate both sides implicitly with respect to x.

Hence

$$\frac{dy}{dx} = -\frac{y^3}{2}\frac{du}{dx}$$

You transform this equation, making $\frac{dy}{dx}$ the subject of the formula as you need to substitute for $\frac{dy}{dx}$ in ①.

Substituting in equation ① in the question

$$-\frac{y^3}{2}\frac{du}{dx} + 2xy = x\,e^{-x^2}y^3$$

Divide by y^3

$$-\frac{1}{2}\frac{du}{dx} + \frac{2x}{y^2} = x\,e^{-x^2}$$

As $u = \frac{1}{y^2}$

$$-\frac{1}{2}\frac{du}{dx} + 2xu = x\,e^{-x^2}$$

Multiply by (-2)

$$\frac{du}{dx} - 4xu = -2x\,e^{-x^2}\text{, as required}$$

b The integrating factor of ② is

$$e^{\int -4x\,dx} = e^{-2x^2}$$

Multiplying ② throughout by e^{-2x^2}

$$e^{-2x^2}\frac{du}{dx} - 4xu\,e^{-2x^2} = -2x\,e^{-x^2} \times e^{-2x^2} = -2x\,e^{-3x^2}$$

$$\frac{d}{dx}(u\,e^{-2x^2}) = -2x\,e^{-3x^2}$$

$$u\,e^{-2x^2} = -2\int x\,e^{-3x^2}\,dx = \tfrac{1}{3}e^{-3x^2} + C$$

This integration can be carried out by inspection. As $\frac{d}{dx}(e^{-3x^2}) = -6x\,e^{-3x^2}$, then $\int x\,e^{-3x^2}\,dx = -\frac{1}{6}e^{-3x^2}$.

Multiplying throughout by e^{2x^2}

$$u = \tfrac{1}{3}e^{-x^2} + C\,e^{2x^2}$$

c As $u = \frac{1}{y^2}$

$$\frac{1}{y^2} = \tfrac{1}{3}e^{-x^2} + C\,e^{2x^2}$$

$y = 1$ at $x = 0$

$$1 = \tfrac{1}{3} + C \Rightarrow C = \tfrac{2}{3}$$

$$\frac{1}{y^2} = \frac{1}{3}e^{-x^2} + \frac{2}{3}e^{2x^2}$$

As no form of the answer has been specified in the question, this is an acceptable answer for the particular solution of ①.

Solutionbank FP2
Edexcel AS and A Level Modular Mathematics

Exercise A, Question 17

Question:

Given that θ satisfies the differential equation

$$\frac{d^2\theta}{dt^2} + 4\frac{d\theta}{dt} + 5\theta = 0$$

and that, when $t = 0$, $\theta = 3$ and $\frac{d\theta}{dt} = -6$, express θ in terms of t.

Solution:

The auxiliary equation is

$$m^2 + 4m + 5 = 0$$
$$m^2 + 4m + 4 = -1$$
$$(m + 2)^2 = -1$$
$$m = -2 \pm i$$

The general solution is

$$\theta = e^{-2t}(A\cos t + B\sin t)$$

If the solutions to the auxiliary equation are $\alpha \pm i\beta$, you may quote the result that the general solution of the differential equation is $e^{\alpha t}(A\cos\beta t + B\sin\beta t)$.

$t = 0$, $\theta = 3$

$$3 = A$$

$$\frac{d\theta}{dt} = -2e^{-2t}(A\cos t + B\sin t) + e^{-2t}(-A\sin t + B\cos t)$$

$t = 0$, $\frac{d\theta}{dt} = -6$

$$-6 = -2A + B$$

Using $\sin 0 = 0$ and $\cos 0 = 1$.

$$B = 2A - 6 = 0$$

As $A = 3$

The particular solution is

$$\theta = 3e^{-2t}\cos t$$

Solutionbank FP2

Edexcel AS and A Level Modular Mathematics

Exercise A, Question 18

Question:

Given that $3x \sin 2x$ is a particular integral of the differential equation

$$\frac{d^2y}{dx^2} + 4y = k \cos 2x,$$

where k is a constant,

a calculate the value of k,

b find the particular solution of the differential equation for which at $x = 0$, $y = 2$, and for which at $x = \frac{\pi}{4}$, $y = \frac{\pi}{2}$.

Solution:

a $y = 3x\sin 2x \Rightarrow \frac{dy}{dx} = 3\sin 2x + 6x\cos 2x$

$\frac{d^2y}{dx^2} = 6\cos 2x + 6\cos 2x - 12x\sin 2x$

$= 12\cos 2x - 12x\sin 2x$

Use the product rule for differentiating.

Substituting into the differential equation

$12\cos 2x - \cancel{12x\sin 2x} + \cancel{12x\sin 2x} = k\cos 2x$

Hence

$k = 12$

b The auxiliary equation is

$m^2 + 4 = 0$

$m = \pm 2\text{i}$

The complementary function is given by

$y = A\cos 2x + B\sin 2x$

If the solutions to the auxiliary equation are $m = \pm\alpha\text{i}$, you may quote the result that the complementary function is $A\cos\alpha x + B\sin\alpha x$.

From **a**, the general solution is

$y = A\cos 2x + B\sin 2x + 3x\sin 2x$

Part **a** of the question gives you that $3x\sin 2x$ is a particular integral of the differential equation and general solution = complementary function + particular integral.

$x = 0, y = 2$

$2 = A$

$x = \frac{\pi}{4}, y = \frac{\pi}{2}$

$\frac{\pi}{2} = A\cos\frac{\pi}{2} + B\sin\frac{\pi}{2} + 3 \times \frac{\pi}{4}\sin\frac{\pi}{2}$

$\frac{\pi}{2} = B + \frac{3\pi}{4} \Rightarrow B = -\frac{\pi}{4}$

Use $\cos\frac{\pi}{2} = 0$ and $\sin\frac{\pi}{2} = 1$.

The particular solution is

$y = 2\cos 2x - \frac{\pi}{4}\sin 2x + 3x\sin 2x$

Solutionbank FP2
Edexcel AS and A Level Modular Mathematics

Exercise A, Question 19

Question:

Given that $a + bx$ is a particular integral of the differential equation

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 16 + 4x,$$

a find the values of the constants a and b.

b Find the particular solution of this differential equation for which $y = 8$ and $\frac{dy}{dx} = 9$ at $x = 0$.

Solution:

a $y = a + bx \Rightarrow \dfrac{dy}{dx} = b$ and $\dfrac{d^2y}{dx^2} = 0$

Substituting into the differential equation

$$0 - 4b + 4a + 4bx = 16 + 4x$$

Equating the coefficients of x

$$4b = 4 \Rightarrow b = 1$$

Equating the constant coefficients

$$-4b + 4a = 16$$

$$-4 + 4a = 16 \Rightarrow a = 5$$

Use $b = 1$.

$$a = 5,\ b = 1$$

b The auxiliary equation is

$$m^2 - 4m + 4 = 0$$

$$(m - 2)^2 = 0$$

$$m = 2, \text{ repeated}$$

The complementary function is given by

$$y = e^{2x}(A + Bx)$$

If the auxiliary equation has a repeated root α, then the complementary function is $e^{\alpha x}(A + Bx)$. You can quote this result.

The general solution is

$$y = e^{2x}(A + Bx) + 5 + x$$

general solution = complementary function + particular integral.

$y = 8, x = 0$

$$8 = A + 5 \Rightarrow A = 3$$

$$\frac{dy}{dx} = 2e^{2x}(A + Bx) + Be^{2x} + 1$$

$\dfrac{dy}{dx} = 9, x = 0$

$$9 = 2A + B + 1 \Rightarrow B = 8 - 2A = 2$$

Use $A = 3$

The particular solution is

$$y = e^{2x}(3 + 2x) + 5 + x$$

Solutionbank FP2
Edexcel AS and A Level Modular Mathematics

Exercise A, Question 20

Question:

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 5y = 65 \sin 2x, \quad x > 0.$$

a Find the general solution of the differential equation.

b Show that for large values of x this general solution may be approximated by a sine function and find this sine function.

Solution:

a The auxiliary equation is

$$\begin{aligned} m^2 + 4m + 5 &= 0 \\ m^2 + 4m + 4 &= -1 \\ (m+2)^2 &= -1 \\ m &= -2 \pm \mathrm{i} \end{aligned}$$

The complementary function is given by

$y = \mathrm{e}^{-2x}(A\cos x + B\sin x)$

For a particular integral, let $y = p\cos 2x + q\sin 2x$

If the right hand side of the second order differential equation is a sine or cosine function, then you should try a particular integral of the form $p\cos\omega x + q\sin\omega x$, with an appropriate ω. Here $\omega = 2$.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -2p\sin 2x + 2q\cos 2x$$

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -4p\cos 2x - 4q\sin 2x$$

Substituting into the differential equation

$$-4p\cos 2x - 4q\sin 2x - 8p\sin 2x + 8q\cos 2x + 5p\cos 2x + 5q\sin 2x = 65\sin 2x$$

$$(-4p + 8q + 5p)\cos 2x + (-4q - 8p + 5q)\sin 2x = 65\sin 2x$$

Equating the coefficients of $\cos 2x$ and $\sin 2x$

The coefficients of $\cos 2x$ and $\sin 2x$ can be equated separately. The coefficient of $\cos 2x$ on the right hand side of this equation is zero.

$\cos 2x$: $\quad -4p + 8q + 5p = 0 \Rightarrow p + 8q = 0 \quad ①$

$\sin 2x$: $\quad -4q - 8p + 5q = 65 \Rightarrow -8p + q = 65 \quad ②$

$8p + 64q = 0 \quad ③$

$65q = 65 \Rightarrow q = 1$

Multiply ① by 8 and add the result to ②.

Substitute $q = 1$ into ①

$p + 8 = 0 \Rightarrow p = -8$

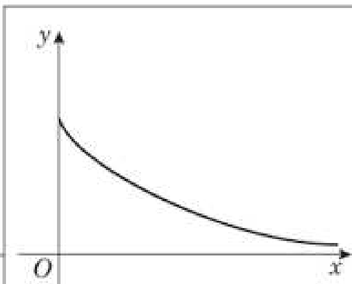
A particular integral is $-8\cos 2x + \sin 2x$

The general solution is

$y = \mathrm{e}^{-2x}(A\cos x + B\sin x) + \sin 2x - 8\cos 2x$

b As $x \to \infty$, $\mathrm{e}^{-2x} \to 0$ and, hence,

$y \to \sin 2x - 8\cos 2x$



The graph of e^{-2x} against x has this shape. As x becomes larger e^{-2x} is close to zero, so $\mathrm{e}^{-2x}(A\cos x + B\sin x)$ is also small.

Let

$$\begin{aligned} \sin 2x - 8\cos 2x &= R\sin(2x - \alpha) \\ &= R\sin 2x\cos\alpha - R\cos 2x\sin\alpha \end{aligned}$$

Equating the coefficients of $\cos 2x$ and $\sin 2x$

$1 = R\cos\alpha \ldots \quad ④$

$8 = R\sin\alpha \ldots \quad ⑤$

$R^2\cos^2\alpha + R^2\sin^2\alpha = 1^2 + 8^2 = 65$

$R^2 = 65 \Rightarrow R = \sqrt{65}$

Add ④ squared to ⑤ squared and use the identity $\cos^2\alpha + \sin^2\alpha = 1$.

$\dfrac{R\sin\alpha}{R\cos\alpha} = \dfrac{8}{1} \Rightarrow \tan\alpha = 8$

Divide ⑤ by ④.

Hence, for large x, y can be approximated by the sine function $\sqrt{65}\sin(2x - \alpha)$, where $\tan\alpha = 8$ ($\alpha \approx 82.9°$)

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Exercise A, Question 21

Question:

a Find the general solution of the differential equation

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 2y = 2e^{-t}.$$

b Find the particular solution of this differential equation for which $y = 1$ and $\frac{dy}{dt} = 1$ at $t = 0$.

Solution:

a The auxiliary equation is

$$m^2 + 2m + 2 = 0$$
$$m^2 + 2m + 1 = -1$$
$$(m + 1)^2 = -1$$
$$m = -1 \pm i$$

The complementary function is

$$y = e^{-t}(A\cos t + B\sin t)$$

Try a particular integral $y = k\,e^{-t}$

If the right hand side of the differential equation is $\lambda\,e^{at+b}$, where λ is any constant, then a possible form of the particular integral is $k\,e^{at+b}$.

$$\frac{dy}{dt} = -k\,e^{-t},\ \frac{d^2y}{dt^2} = k\,e^{-t}$$

Substituting into the differential equation

$$k\,e^{-t} - 2k\,e^{-t} + 2k\,e^{-t} = 2\,e^{-t}$$

Divide throughout by e^{-t}.

$$k - 2k + 2k = 2 \Rightarrow k = 2$$

A particular integral is $2\,e^{-t}$

The general solution is

$$y = e^{-t}(A\cos t + B\sin t) + 2\,e^{-t}$$

b $y = 1,\ t = 0$

$$1 = A + 2 \Rightarrow A = -1$$

Substitute the boundary condition $y = 1,\ t = 0$ into the general solution gives you an equation for one arbitrary constant.

$$\frac{dy}{dt} = -e^{-t}(A\cos t + B\sin t) + e^{-t}(-A\sin t + B\cos t) - 2\,e^{-t}$$

Use the product rule for differentiating.

$$\frac{dy}{dt} = 1,\ t = 0$$

$$1 = -A + B - 2 \Rightarrow B = 3 + A = 2$$

As $A = -1$.

The particular solution is

$$y = e^{-t}(2\sin t - \cos t) + 2\,e^{-t}$$

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Exercise A, Question 22

Question:

a Find the general solution of the differential equation

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 5x = 0$$

b Given that $x = 1$ and $\frac{dx}{dt} = 1$ at $t = 0$, find the particular solution of the differential equation, giving your answer in the form $x = f(t)$.

c Sketch the curve with equation $x = f(t)$, $0 \leqslant t \leqslant \pi$, showing the coordinates, as multiples of π, of the points where the curve cuts the t-axis.

Solution:

a The auxiliary equation is

$m^2 + 2m + 5 = 0$

$m^2 + 2m + 1 = -4$

$(m + 1)^2 = -4$

$m = -1 \pm 2i$

You may use any appropriate method to solve the quadratic. Completing the square works efficiently when the coefficient of m is even.

The general solution is

$x = e^{-t}(A\cos 2t + B\sin 2t)$

b $x = 1,\ t = 0$

$1 = A$

$\frac{dx}{dt} = -e^{-t}(A\cos 2t + B\sin 2t) + 2e^{-t}(-A\sin 2t + B\cos 2t)$

Use the product rule for differentiation.

$\frac{dx}{dt} = 1,\ t = 0$

$1 = -A + 2B \Rightarrow 2B = A + 1 = 2 \Rightarrow B = 1$

The particular solution is

$x = e^{-t}(\cos 2t + \sin 2t)$

Both A and B are 1.

c The curve crosses the t-axis where

$e^{-t}(\cos 2t + \sin 2t) = 0$

e^{-t} can never be zero.

$\cos 2t + \sin 2t = 0$

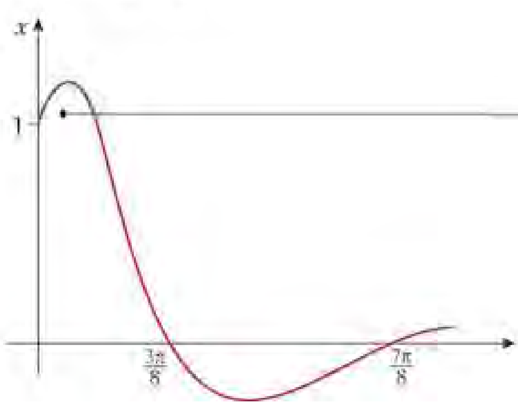
$\sin 2t = -\cos 2t$

$\tan 2t = -1$

Divide both sides by $\cos 2t$ and use the identity $\tan\theta = \frac{\sin\theta}{\cos\theta}$.

$2t = \frac{3\pi}{4}, \frac{7\pi}{4}$

$t = \frac{3\pi}{8}, \frac{7\pi}{8}$



The boundary conditions give you that at $t = 0$, $x = 1$ and the curve has a positive gradient. The curve must then turn down and cross the axis at the two points where $t = \frac{3\pi}{8}$ and $\frac{7\pi}{8}$.

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Exercise A, Question 23

Question:

a Find the general solution of the differential equation

$$2\frac{d^2y}{dt^2} + 7\frac{dy}{dt} + 3y = 3t^2 + 11t$$

b Find the particular solution of this differential equation for which $y = 1$ and $\frac{dy}{dt} = 1$ when $t = 0$.

c For this particular solution, calculate the value of y when $t = 1$.

Solution:

a The auxiliary equation is

$$2m^2 + 7m + 3 = 0$$
$$(2m + 1)(m + 3) = 0$$
$$m = -\tfrac{1}{2}, -3$$

The complementary function is given by

$$y = A\,e^{-\frac{1}{2}t} + B\,e^{-3t}$$

If the auxiliary equation has two real solutions α and β, the complementary function is $y = A\,e^{\alpha t} + B\,e^{\beta t}$. You can quote this result.

For a particular integral, try $y = at^2 + bt + c$

If the right hand side of the differential equation is a polynomial of degree n, then you can try a particular integral of the same degree. Here the right hand side is a quadratic, so you try the general quadratic $at^2 + bt + c$.

$$\frac{dy}{dt} = 2at + b,\ \frac{d^2y}{dt^2} = 2a$$

Substitute into the differential equation

$$4a + 14at + 7b + 3at^2 + 3bt + 3c = 3t^2 + 11t$$
$$3at^2 + (14a + 3b)t + 4a + 7b + 3c = 3t^2 + 11t$$

Equating the coefficients of t^2

$$3a = 3 \Rightarrow a = 1$$

Equating the coefficients of t

Use $a = 1$.

$$14a + 3b = 11 \Rightarrow 3b = 11 - 14a = -3 \Rightarrow b = -1$$

Equating the constant coefficients

Use $a = 1$ and $b = -1$.

$$4a + 7b + 3c = 0 \Rightarrow 3c = -4a - 7b = 3 \Rightarrow c = 1$$

A particular integral is $t^2 - t + 1$.

The general solution is $y = A\,e^{-\frac{1}{2}t} + B\,e^{-3t} + t^2 - t + 1$.

b $y = 1,\ t = 0$

$$1 = A + B + 1 \Rightarrow A + B = 0 \qquad ①$$

$$\frac{dy}{dt} = -\tfrac{1}{2}A\,e^{-\frac{1}{2}t} - 3B\,e^{-3t} + 2t - 1$$

Differentiate the general solution in part **a** with respect to t.

$\frac{dy}{dt} = 1,\ t = 0$

$$1 = -\tfrac{1}{2}A - 3B - 1 \Rightarrow \tfrac{1}{2}A + 3B = -2 \qquad ②$$
$$A + 6B = -4 \qquad ③$$
$$5B = -4 \Rightarrow B = -\tfrac{4}{5}$$

Multiply ② by 2 and then subtract ① from ③.

Substituting $B = -\frac{4}{5}$ into ①

$$A - \tfrac{4}{5} = 0 \Rightarrow A = \tfrac{4}{5}$$

The particular solution is $y = \frac{4}{5}\left(e^{-\frac{1}{2}t} - e^{-3t}\right) + t^2 - t + 1$.

c When $t = 1$, $y = \frac{4}{5}\left(e^{-\frac{1}{2}} - e^{-3}\right) + 1 = 1.45$ (3 s.f.)

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Exercise A, Question 24

Question:

a Find the value of λ for which $\lambda x \cos 3x$ is a particular integral of the differential equation

$$\frac{d^2y}{dx^2} + 9y = -12 \sin 3x$$

b Hence find the general solution of this differential equation.

The particular solution of the differential equation for which $y = 1$ and $\frac{dy}{dx} = 2$ at $x = 0$, is $y = g(x)$.

c Find $g(x)$.

d Sketch the graph of $y = g(x)$, $0 \leqslant x \leqslant \pi$.

Solution:

a Let $y = \lambda x \cos 3x$

$$\frac{dy}{dx} = \lambda \cos 3x - 3\lambda x \sin 3x$$

$$\frac{d^2y}{dx^2} = -3\lambda \sin 3x - 3\lambda \sin 3x - 9\lambda x \cos 3x$$

$$= -6\lambda \sin 3x - 9\lambda x \cos 3x$$

Use the product rule for differentiation

$$\frac{d}{dx}(x \sin 3x) = \frac{d}{dx}(x) \sin 3x + x \frac{d}{dx}(\sin 3x)$$
$$= \sin 3x + 3x \cos 3x$$

Substituting into the differential equation

$$-6\lambda \sin 3x - \cancel{9\lambda x \cos 3x} + \cancel{9\lambda x \cos 3x} = -12 \sin 3x$$

Hence

$$\lambda = 2$$

b The auxiliary equation is

$$m^2 + 9 = 0 \Rightarrow m^2 = -9$$
$$m = \pm 3i$$

The complementary function is given by

$$y = A \cos 3x + B \sin 3x$$

The general solution is

$$y = A \cos 3x + B \sin 3x + 2x \cos 3x$$

Part **a** shows that $2x \cos 3x$ is a particular integral of the differential equation and general solution = complementary function + particular integral

c $y = 1, x = 0$

$$1 = A$$

$$\frac{dy}{dx} = -3A \sin 3x + 3B \cos 3x + 2 \cos 3x - 6x \sin 3x$$

Differentiate the general solution in part **b** with respect to x.

$$\frac{dy}{dx} = 2, x = 0$$

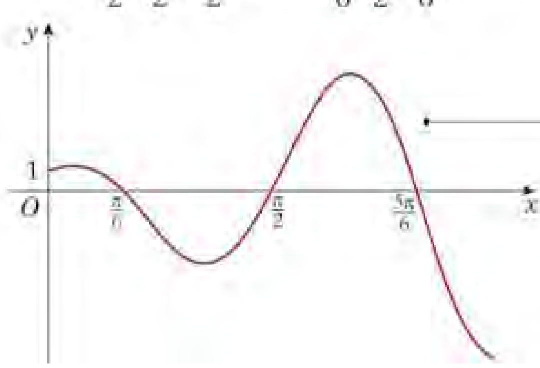
$$2 = 3B + 2 \Rightarrow B = 0$$

The particular solution is

$$y = \cos 3x + 2x \cos 3x = (1 + 2x) \cos 3x$$

d For $x > 0$, the curve crosses the x-axis at $\cos 3x = 0$

$$3x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2} \Rightarrow x = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$$



The boundary conditions give you that at $x = 0$, $y = 1$ and the curve has a positive gradient. The curve must then turn down and cross the axis at the three points where $x = \frac{\pi}{6}, \frac{\pi}{2}$ and $\frac{5\pi}{6}$.

The $(1 + 2x)$ factor in the general solution means that the size of the oscillations increases as x increases.

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Exercise A, Question 25

Question:

$$\frac{d^2y}{dt^2} - 6\frac{dy}{dt} + 9y = 4e^{3t},\ t \geqslant 0$$

a Show that Kt^2e^{3t} is a particular integral of the differential equation, where K is a constant to be found.

b Find the general solution of the differential equation.

Given that a particular solution satisfies

$y = 3$ and $\frac{dy}{dt} = 1$ when $t = 0$,

c find this solution.

Another particular solution which satisfies

$y = 3$ and $\frac{dy}{dt} = 1$ when $t = 0$, has equation $y = (1 - 3t + 2t^2)e^{3t}$

d For this particular solution, draw a sketch graph of y against t, showing where the graph crosses the t-axis. Determine also the coordinates of the minimum point on the sketch graph.

Solution:

a If $y = Kt^2 e^{3t}$

$$\frac{dy}{dt} = 2Kt\,e^{3t} + 3Kt^2\,e^{3t}$$

$$\frac{d^2y}{dt^2} = 2K\,e^{3t} + 6Kt\,e^{3t} + 6Kt\,e^{3t} + 9Kt^2\,e^{3t}$$

$$= 2K\,e^{3t} + 12Kt\,e^{3t} + 9Kt^2\,e^{3t}$$

Substituting into the differential equation

$$2K\,e^{3t} + \cancel{12Kt\,e^{3t}} + \cancel{9Kt^2\,e^{3t}} - \cancel{12Kt\,e^{3t}} - \cancel{18Kt^2\,e^{3t}} + \cancel{9Kt^2\,e^{3t}} = 4\,e^{3t}$$

e^{3t} cannot be zero, so you can divide throughout by e^{3t}.

Hence

$$2K = 4 \Rightarrow K = 2$$

$2t^2\,e^{3t}$ is a particular integral of the differential equation.

b The auxiliary equation is

$$m^2 - 6m + 9 = 0$$

$$(m - 3)^2 = 0$$

$$m = 3, \text{ repeated}$$

The complementary function is given by

$$y = e^{3t}(A + Bt)$$

If the auxiliary equation has a repeated root α, then the complementary function is $e^{\alpha t}(A + Bt)$. You can quote this result.

The general solution is

$$y = e^{3t}(A + Bt) + 2t^2\,e^{3t} = (A + Bt + 2t^2)\,e^{3t}$$

c $y = 3,\ t = 0$

$$3 = A$$

$$\frac{dy}{dt} = (B + 4t)\,e^{3t} + 3(A + Bt + 2t^2)\,e^{3t}$$

$$\frac{dy}{dt} = 1,\ t = 0$$

$$1 = B + 3A \Rightarrow B = 1 - 3A \Rightarrow B = -8$$

As $A = 3$.

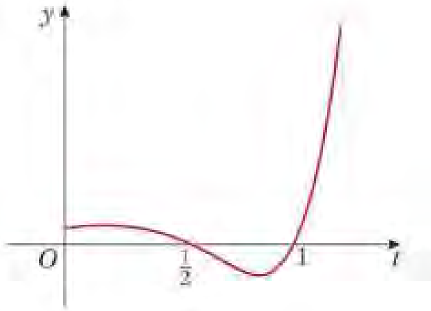
The particular solution is

$$y = (3 - 8t + 2t^2)\,e^{3t}$$

d This particular solution crosses the t-axis where

$1 - 3t + 2t^2 = (1 - 2t)(1 - t) = 0$

$t = \frac{1}{2}, 1$



For a minimum $\frac{dy}{dt} = 0$

$(-3 + 4t)\,e^{3t} + (1 - 3t + 2t^2)3\,e^{3t} = 0$

e^{3t} cannot be zero, so you can divide throughout by e^{3t}.

$-3 + 4t + 3 - 9t + 6t^2 = 0$

$6t^2 - 5t = t\,(6t - 5) = 0 \Rightarrow t = 0, \frac{5}{6}$

From the digram $t = \frac{5}{6}$ gives the minimum

It is clear from the diagram that there is a minimum point between $t = \frac{1}{2}$ and $t = 1$. You do not have to consider the second derivative to show that it is a minimum.

At $t = \frac{5}{6}$

$y = \left(1 - 3 \times \frac{5}{6} + 2 \times \left(\frac{5}{6}\right)^2\right) e^{3 \times \frac{5}{6}} = -\frac{1}{9}\,e^{\frac{5}{2}}$

The coordinates of the minumum point are

$\left(\frac{5}{6}, -\frac{1}{9}\,e^{\frac{5}{2}}\right)$.

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Exercise A, Question 26

Question:

a Find the general solution of the differential equation

$$2\frac{d^2x}{dt^2} + 5\frac{dx}{dt} + 2x = 2t + 9$$

b Find the particular solution of this differential equation for which $x = 3$ and $\frac{dx}{dt} = -1$ when $t = 0$.

The particular solution in part **b** is used to model the motion of the particle P on the x-axis. At time t seconds ($t \geqslant 0$), P is x metres from the origin O.

c Show that the minimum distance between O and P is $\frac{1}{2}(5 + \ln 2)$ m and justify that the distance is a minimum.

Solution:

a The auxiliary equation is

$$2m^2 + 5m + 2 = 0$$

$$(2m + 1)(m + 2) = 0$$

$$m = -\tfrac{1}{2}, -2$$

The complementary function is given by

$$x = A\,e^{-\frac{1}{2}t} + B\,e^{-2t}$$

If the auxiliary equation has two real solutions α and β, the complementary function is $x = A\,e^{\alpha t} + B\,e^{\beta t}$. You can quote this result.

For a particular integral, try $x = pt + q$

If the right hand side of the differential equation is a polynomial of degree n, then you can try a particular integral of the same degree. Here the right hand side is linear, so you try the general linear function $pt + q$.

$$\frac{dx}{dt} = p, \frac{d^2x}{dt^2} = 0$$

Substituting into the differential equation

$$0 + 5p + 2pt + 2q = 2t + 9$$

Equating the coefficients of t

$$2p = 2 \Rightarrow p = 1$$

Equating the constant coefficients

$$5p + 2q = 9 \Rightarrow q = \frac{9 - 5p}{2} \Rightarrow q = 2$$

A particular integral is $t + 2$

The general solution is

$$x = A\,e^{-\frac{1}{2}t} + B\,e^{-2t} + t + 2$$

b $x = 3,\ t = 0$

$3 = A + B + 2 \Rightarrow A + B = 1 \qquad ①$

$\frac{dx}{dt} = -\frac{1}{2}A\,e^{-\frac{1}{2}t} - 2B\,e^{-2t} + 1$

Differentiating the general solution in part **a**.

$\frac{dx}{dt} = -1,\ t = 0$

$-1 = -\frac{1}{2}A - 2B + 1 \Rightarrow \frac{1}{2}A + 2B = 2 \qquad ②$

$A + 4B = 4 \qquad ③$

$3B = 3 \Rightarrow B = 1$

Multiplying ② by 2 and subtracting ① from ③.

Substituting $B = 1$ into ①

$A + 1 = 1 \Rightarrow A = 0$

The particular solution is

$x = e^{-2t} + t + 2$

c For a minimum

$\frac{dx}{dt} = -2\,e^{-2t} + 1 = 0$

$e^{-2t} = \frac{1}{2}$

You take logarithms of both sides of this equation and use $e^{\ln f(x)} = f(x)$.

$-2t = \ln\frac{1}{2} = -\ln 2$

$\ln\frac{1}{2} = \ln 1 - \ln 2 = -\ln 2$, as $\ln 1 = 0$.

$t = \frac{1}{2}\ln 2$

$\frac{d^2x}{dt^2} = 4\,e^{-2t} > 0$, for any real t

Hence the stationary value is a minimum value

When $t = \frac{1}{2}\ln 2$

$e^{-\ln 2} = e^{\ln 1 - \ln 2} = e^{\ln\frac{1}{2}} = \frac{1}{2}$

$x = e^{-\ln 2} + \frac{1}{2}\ln 2 + 2 = \frac{1}{2} + \frac{1}{2}\ln 2 + 2 = \frac{5}{2} + \frac{1}{2}\ln 2$

The minimum distance is $\frac{1}{2}(5 + \ln 2)$ m, as required.

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Exercise A, Question 27

Question:

Given that $x = At^2 e^{-t}$ satisfies the differential equation

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + x = e^{-t},$$

a find the value of A.

b Hence find the solution of the differential equation for which $x = 1$ and $\frac{dx}{dt} = 0$ at $t = 0$.

c Use your solution to prove that for $t \geqslant 0$, $x \leqslant 1$.

Solution:

a If $x = At^2 e^{-t}$

$$\frac{dx}{dt} = 2At\,e^{-t} - At^2\,e^{-t}$$

$$\frac{d^2x}{dt^2} = 2A\,e^{-t} - 2At\,e^{-t} - 2At\,e^{-t} + At^2\,e^{-t}$$

$$= 2A\,e^{-t} - 4At\,e^{-t} + At^2\,e^{-t}$$

Substituting into the differential equation

$$2A\,e^{-t} - \cancel{4At\,e^{-t}} + \cancel{At^2\,e^{-t}} + \cancel{4At\,e^{-t}} - \cancel{2At^2\,e^{-t}} + \cancel{At^2\,e^{-t}} = e^{-t}$$

e^{-t} cannot be zero, so you can divide throughout by e^{-t}.

Hence

$$2A = 1 \Rightarrow A = \tfrac{1}{2}$$

b The auxiliary equation is

$$m^2 + 2m + 1 = (m + 1)^2 = 0$$

$$m = -1, \text{ repeated}$$

The complementary function is given by

$$x = e^{-t}(A + Bt)$$

If the auxiliary equation has a repeated root α, then the complementary function is $e^{\alpha t}(A + Bt)$. You can quote this result.

The general solution is

$$x = e^{-t}(A + Bt) + \tfrac{1}{2}t^2\,e^{-t} = (A + Bt + \tfrac{1}{2}t^2)e^{-t}$$

From part **a**, $\frac{1}{2}t^2\,e^{-t}$ is a particular integral of the differential equation.

$x = 1, t = 0$

$$1 = A$$

$$\frac{dx}{dt} = (B + t)\,e^{-t} - (A + Bt + \tfrac{1}{2}t^2)e^{-t}$$

$\frac{dx}{dt} = 0, t = 0$

$$0 = B - A \Rightarrow B = A = 1$$

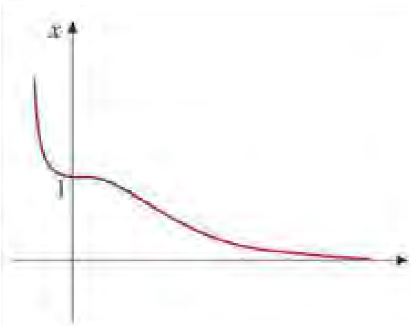
The particular solution is

$$x = \left(1 + t + \tfrac{1}{2}t^2\right)e^{-t}$$

c $\frac{dx}{dt} = (1 + t)\,e^{-t} - \left(1 + t + \tfrac{1}{2}t^2\right)e^{-t}$

$$= -\tfrac{1}{2}t^2\,e^{-t} \leqslant 0, \text{ for all real } t.$$

When $t = 0$, $x = 1$ and x has a negative gradient for all positive t, x is a decreasing function of t. Hence, for $t \geqslant 0$, $x \leqslant 1$, as required.



The graph of x against t, shows the curve crossing the x-axis at $x = 1$ and then decreasing. For all positive t, x is less than 1.

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Exercise A, Question 28

Question:

Given that $y = kx$ is a particular solution of the differential equation

$$\frac{d^2y}{dx^2} + y = 3x,$$

a find the value of the constant k.

b Find the most general solution of this differential equation for which $y = 0$ at $x = 0$.

c Prove that all curves given by this solution pass through the point $(\pi, 3\pi)$ and that they all have equal gradients when $x = \frac{\pi}{2}$.

d Find the particular solution of the differential equation for which $y = 0$ at $x = 0$ and at $x = \frac{\pi}{2}$.

e Show that a minimum value of the solution in part **d** is

$$3 \arccos\left(\frac{2}{\pi}\right) - \frac{3}{2}\sqrt{(\pi^2 - 4)}$$

Solution:

a $y = kx \Rightarrow \frac{dy}{dx} = k \Rightarrow \frac{d^2y}{dx^2} = 0$

Substituting into $\frac{d^2y}{dx^2} + y = 3x$

$$0 + kx = 3x$$
$$k = 3$$

b The auxiliary equation is

$$m^2 + 1 = 0 \Rightarrow m = \pm i$$

The complementary function is given by

$$y = A\sin x + B\cos x$$

and the general solution is

$$y = A\sin x + B\cos x + 3x$$

$y = 0, x = 0$

$$0 = B + 0 \Rightarrow B = 0$$

The most general solution is

$$y = A\sin x + 3x$$

In part **b**, only one condition is given, so only one of the arbitrary constants can be found. The solution is a family of functions, some of which are illustrated in the diagram below.

c At $x = \pi$

$$y = A\sin\pi + 3\pi = 3\pi$$

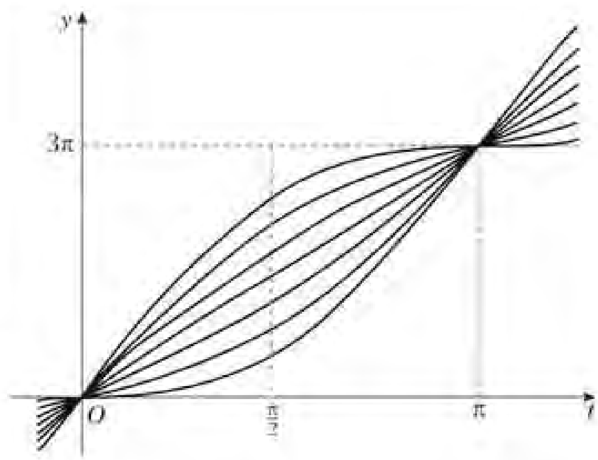
This is independent of the value of A. Hence, all curves given by the solution in part **a** pass through $(\pi, 3\pi)$.

$$\frac{dy}{dx} = A\cos x + 3$$

At $x = \frac{\pi}{2}$

$$\frac{dy}{dx} = A\cos\frac{\pi}{2} + 3 = 3$$

This is independent of the value of A. Hence, all curves given by the solution in part **a** have an equal gradient of 3 at $x = \frac{\pi}{2}$.



As is illustrated by this diagram, the family of curves $y = A\sin x + 3x$ all go through $(0, 0)$ and $(\pi, 3\pi)$. The tangent to the curves at $x = \frac{\pi}{2}$ are all parallel to each other.

d $y = 0, x = \frac{\pi}{2}$

Substituting into $y = A\sin x + 3x$

$$0 = A\sin\frac{\pi}{2} + \frac{3\pi}{2} = A + \frac{3\pi}{2} \Rightarrow A = -\frac{3\pi}{2}$$

The particular solution is

$$y = 3x - \frac{3\pi}{2}\sin x$$

e For a minimum

$$\frac{dy}{dx} = 3 - \frac{3\pi}{2}\cos x = 0$$

$$\cos x = \frac{2}{\pi} \Rightarrow x = \arccos\left(\frac{2}{\pi}\right)$$

$$\frac{d^2y}{dx^2} = \frac{3\pi}{2}\sin x$$

In the interval $0 \leqslant x \leqslant \frac{\pi}{2}$,

$$\frac{d^2y}{dx^2} > 0 \Rightarrow \text{minimum}$$

$\cos x = \frac{2}{\pi}$ has an infinite number of solutions. This shows that the solution in the first quadrant gives a minimum as $\sin x$ is positive in that quadrant.

$$\sin^2 x = 1 - \cos^2 x = 1 - \frac{4}{\pi^2} = \frac{\pi^2 - 4}{\pi^2}$$

In the interval $0 \leqslant x \leqslant \frac{\pi}{2}$

$$\sin x = +\left(\frac{\pi^2 - 4}{\pi^2}\right)^{\frac{1}{2}} = \frac{\sqrt{\pi^2 - 4}}{\pi}$$

$$y = 3\arccos\left(\frac{2}{\pi}\right) - \frac{3\pi}{2} \times \frac{\sqrt{\pi^2 - 4}}{\pi}$$

$$= 3\arccos\left(\frac{2}{\pi}\right) - \frac{3}{2}\sqrt{\pi^2 - 4}\text{, as required.}$$

Solutionbank FP2
Edexcel AS and A Level Modular Mathematics

Exercise A, Question 29

Question:

a Show that the transformation $y = xv$ transforms the equation

$$x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + (2 + 9x^2)y = x^5, \qquad ①$$

into the equation

$$\frac{d^2v}{dx^2} + 9v = x^2. \qquad ②$$

b Solve the differential equation ② to find v as a function of x.

c Hence state the general solution of the differential equation ①.

Solution:

a $y = xv$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

Use the product rule for differentiation
$\frac{d}{dx}(xv) = \frac{d}{dx}(x) \times v + x \times \frac{dv}{dx} = 1 \times v + x\frac{dv}{dx}$

$$\frac{d^2y}{dx^2} = \frac{dv}{dx} + \frac{dv}{dx} + x\frac{d^2v}{dx^2} = 2\frac{dv}{dx} + x\frac{d^2v}{dx^2}$$

Substituting for $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ into ①

$$x^2\left(x\frac{d^2v}{dx^2} + 2\frac{dv}{dx}\right) - 2x\left(v + x\frac{dv}{dx}\right) + (2 + 9x^2)vx = x^5$$

$$x^3\frac{d^2v}{dx^2} + \cancel{2x^2\frac{dv}{dx}} - \cancel{2xv} - \cancel{2x^2\frac{dv}{dx}} + \cancel{2xv} + 9x^3v = x^5$$

$$x^3\frac{d^2v}{dx^2} + 9x^3v = x^5$$

Divide by x^3.

$$\frac{d^2v}{dx^2} + 9v = x^2, \text{ as required}$$

b The auxiliary equation is

$$m^2 + 9 = 0 \Rightarrow m^2 = -9$$
$$m = \pm 3\text{i}$$

The complementary function is given by

$$v = A\cos 3x + B\sin 3x$$

For a particular integral, try $v = px^2 + qx + r$

$$\frac{dv}{dx} = 2px + q,\ \frac{d^2v}{dx^2} = 2p$$

If the right hand side of the differential equation is a polynomial of degree n, then you can try a particular integral of the same degree. Here the right hand side is a quadratic x^2, so you try a general quadratic $px^2 + qx + r$.

Substituting into ②

$$2p + 9px^2 + 9qx + 9r = x^2$$

Equating coefficients of x^2

$$9p = 1 \Rightarrow p = \tfrac{1}{9}$$

Equating coefficients of x

$$9q = 0 \Rightarrow q = 0$$

Equating constant coefficients

As $p = \frac{1}{9}$.

$$2p + 9r = 0 \Rightarrow 9r = -2p = -\tfrac{2}{9} \Rightarrow r = -\tfrac{2}{81}$$

A particular integral is $\frac{1}{9}x^2 - \frac{2}{81}$

A general solution of ② is

$$v = A\cos 3x + B\sin 3x + \tfrac{1}{9}x^2 - \tfrac{2}{81}$$

$y = vx \Rightarrow v = \dfrac{y}{x}$.

c $\dfrac{y}{x} = A\cos 3x + B\sin 3x + \frac{1}{9}x^2 - \frac{2}{81}$

The question does not ask for a particular form of the answer in part **c**, so this would be an acceptable answer.

$y = Ax\cos 3x + Bx\sin 3x + \frac{1}{9}x^3 - \frac{2}{81}x$

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Edexcel AS and A Level Modular Mathematics

Exercise A, Question 30

Question:

Given that $x = t^{\frac{1}{2}}$, $x > 0$, $t > 0$, and that y is a function of x,

a find $\frac{dy}{dx}$ in terms of $\frac{dy}{dt}$ and t.

Assuming that $\frac{d^2y}{dx^2} = 4t\frac{d^2y}{dt^2} + 2\frac{dy}{dt}$,

b show that the substitution $x = t^{\frac{1}{2}}$, transforms the differential equation

$$\frac{d^2y}{dx^2} + \left(6x - \frac{1}{x}\right)\frac{dy}{dx} - 16x^2y = 4x^2e^{2x^2} \qquad ①$$

into the differential equation

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} - 4y = e^{2t}$$

c Hence find the general solution of ① giving y in terms of x.

Solution:

a $x = t^{\frac{1}{2}} \Rightarrow \frac{dx}{dt} = \frac{1}{2}t^{-\frac{1}{2}} = \frac{1}{2t^{\frac{1}{2}}}$

$\frac{dt}{dx} = \frac{1}{\frac{1}{2t^{\frac{1}{2}}}} = 2t^{\frac{1}{2}}$

Use $\frac{dt}{dx} = \frac{1}{\frac{dx}{dt}}$.

$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{dy}{dt} \times 2t^{\frac{1}{2}} = 2t^{\frac{1}{2}}\frac{dy}{dt}$

You obtain an expression for $\frac{dy}{dx}$ using the chain rule.

b Substituting $x = t^{\frac{1}{2}}$, the result of part **a** and the given $\frac{d^2y}{dx^2} = 4t\frac{d^2y}{dt^2} + 2\frac{dy}{dt}$ into ①

$\left(6t^{\frac{1}{2}} - \frac{1}{t^{\frac{1}{2}}}\right)2t^{\frac{1}{2}} = 6t^{\frac{1}{2}} \times 2t^{\frac{1}{2}} - \frac{2t^{\frac{1}{2}}}{t^{\frac{1}{2}}} = 12t - 2$

$4t\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + \left(6t^{\frac{1}{2}} - \frac{1}{t^{\frac{1}{2}}}\right)2t^{\frac{1}{2}}\frac{dy}{dt} - 16ty = 4te^{2t}$

$4t\frac{d^2y}{dt^2} + \cancel{2\frac{dy}{dt}} + 12t\frac{dy}{dt} - \cancel{2\frac{dy}{dt}} - 16ty = 4te^{2t}$

$4t\frac{d^2y}{dt^2} + 12t\frac{dy}{dt} - 16ty = 4te^{2t}$

Divide throughout by $4t$.

$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} - 4y = e^{2t}$, as required

c The auxiliary equation is

$$m^2 + 3m - 4 = (m - 1)(m + 4) = 0$$

$$m = 1, -4$$

The complementary function is

$$y = Ae^t + Be^{-4t}$$

For a particular integral try, $y = ke^{2t}$

If the right hand side of the equation is $e^{\alpha t}$, you can try $ke^{\alpha t}$ as a particular integral. This will work unless α is a solution of the auxiliary equation.

$$\frac{dy}{dt} = 2ke^{2t}, \frac{d^2y}{dt^2} = 4ke^{2t}$$

Substituting into $\frac{d^2y}{dt^2} + 3\frac{dy}{dt} - 4y = e^{2t}$

$$\cancel{4ke^{2t}} + 6ke^{2t} - \cancel{4ke^{2t}} = e^{2t}$$

As e^{2t} cannot be zero, you can divide throughout by e^{2t}.

$$6k = 1 \Rightarrow k = \frac{1}{6}$$

A particular integral is $\frac{1}{6}e^{2t}$

The general solution of the differential equation in y and t is

$$y = Ae^t + Be^{-4t} + \frac{1}{6}e^{2t}$$

$x = t^{\frac{1}{2}} \Rightarrow t = x^2$

The general solution of ① is

$$y = Ae^{x^2} + Be^{-4x^2} + \frac{1}{6}e^{2x^2}$$

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Exercise A, Question 31

Question:

A scientist is modelling the amount of a chemical in the human bloodstream. The amount x of the chemical, measured in mg l^{-1}, at time t hours satisfies the differential equation

$$2x\frac{d^2x}{dt^2} - 6\left(\frac{dx}{dt}\right)^2 = x^2 - 3x^4, \quad x > 0.$$

a Show that the substitution $y = \frac{1}{x^2}$ transforms this differential equation into

$$\frac{d^2y}{dt^2} + y = 3. \qquad ①$$

b Find the general solution of differential equation ①.

Given that at time $t = 0$, $x = \frac{1}{2}$ and $\frac{dx}{dt} = 0$,

c find an expression for x in terms of t,

d write down the maximum value of x as t varies.

Solution:

a $y = x^{-2}$

Differentiating implicitly with respect to t

$$\frac{dy}{dt} = -2x^{-3}\frac{dx}{dt}$$

Use $\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$.

Differentiating again implicitly with respect to t.

$$\frac{d^2y}{dt^2} = 6x^{-4}\left(\frac{dx}{dt}\right)^2 - 2x^{-3}\frac{d^2x}{dt^2} \quad ②$$

This expression is closely related to the left hand side of the original differential equation in the question. This suggests to you that if you divide the original equation by $-x^4$, then the left hand side can just be replaced by $\frac{d^2x}{dt^2}$

Dividing the differential equation given in the question by $-x^4$, it becomes

$$-2x^{-3}\frac{d^2x}{dt^2} + 6x^{-4}\left(\frac{dx}{dt}\right)^2 = -x^{-2} + 3$$

Using equation ② and $y = x^{-2}$

$$\frac{d^2y}{dt^2} = -y + 3$$

$$\frac{d^2y}{dt^2} + y = 3, \text{ as required}$$

b The auxiliary equation is

$$m^2 + 1 = 0 \Rightarrow m = \pm i$$

The complementary function is given by

$$y = A\cos t + B\sin t$$

By inspection, a particular integral of ① is 3

As $\frac{d^2}{dt^2}(3) = 0$, $y = 3$ satisfies $\frac{d^2y}{dt^2} + y = 3$, by inspection and you need not write down any working.

The general solution of ② is

$$y = A\cos t + B\sin t + 3$$

c The general solution of the differential equation in x and t is

$$\frac{1}{x^2} = A\cos t + B\sin t + 3 \quad ③$$

When $t = 0$, $x = \frac{1}{2}$

$$4 = A + 3 \Rightarrow A = 1$$

Differentiating ③ implicitly with respect to t

$$-\frac{2}{x^3}\frac{dx}{dt} = -A\sin t + B\cos t$$

Use the chain rule $\frac{d}{dt}(x^{-2}) = \frac{d}{dx}(x^{-2}) \times \frac{dx}{dt} = -2x^{-3}\frac{dx}{dt}$.

When $t = 0$, $x = \frac{1}{2}$ and $\frac{dx}{dt} = 0$

$$0 = B$$

The particular solution is

$$\frac{1}{x^2} = \cos t + 3$$

As $x > 0$, $t > 0$

$$x = \frac{1}{\sqrt{(\cos t + 3)}}$$

As x and t are both positive, the negative square root need not be considered.

d The maximum value of x is

$$x = \frac{1}{\sqrt{(-1 + 3)}} = \frac{1}{\sqrt{2}}$$

The maximum value of this fraction is when the denominator has its least value. The smallest possible value of $\cos t$ is -1. So you can write down the maximum value without using calculus.

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Exercise A, Question 32

Question:

Given that $x = \ln t$, $t > 0$, and that y is a function of x,

a find $\frac{dy}{dx}$ in terms of $\frac{dy}{dt}$ and t,

b show that $\frac{d^2y}{dx^2} = t^2\frac{d^2y}{dt^2} + t\frac{dy}{dt}$.

c Show that the substitution $x = \ln t$ transforms the differential equation

$$\frac{d^2y}{dx^2} - (1 - 6e^x)\frac{dy}{dx} + 10y\,e^{2x} = 5e^{2x}\sin 2e^x \qquad ①$$

into the differential equation

$$\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 10y = 5\sin 2t \qquad ②$$

d Hence find the general solution of ①, giving your answer in the form $y = f(x)$.

Solution:

a $x = \ln t \Rightarrow \dfrac{dx}{dt} = \dfrac{1}{t} \Rightarrow \dfrac{dt}{dx} = t$

$\dfrac{dt}{dx} = \dfrac{1}{\frac{dx}{dt}}$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{dy}{dt} \times t$$

$$\frac{dy}{dx} = t\frac{dy}{dt}$$

b $\dfrac{d^2y}{dx^2} = \dfrac{d}{dx}\left(\dfrac{dy}{dx}\right) = \dfrac{dt}{dx} \times \dfrac{d}{dt}\left(\dfrac{dy}{dx}\right)$

It is a common error to proceed from $\dfrac{dy}{dx} = t\dfrac{dy}{dt}$ to $\dfrac{d^2y}{dx^2} = \dfrac{dy}{dt} + t\dfrac{d^2y}{dt^2}$. This is incorrect because the left hand side has been differentiated with respect to x and the right hand side with respect to t. The version of the chain rule given here must be used.

$$= t\frac{d}{dt}\left(t\frac{dy}{dt}\right)$$

$$= t\left(\frac{dy}{dt} + t\frac{d^2y}{dt^2}\right)$$

$$= t^2\frac{d^2y}{dt^2} + t\frac{dy}{dt}, \text{ as required}$$

c Substituting $x = \ln t$, $\dfrac{dy}{dx} = t\dfrac{dy}{dt}$ and

$\dfrac{d^2y}{dx^2} = t^2\dfrac{d^2y}{dt^2} + t\dfrac{dy}{dt}$ into ①

$e^{2\ln t} = e^{\ln t^2} = t^2$, using the log rule $n \ln a = \ln a^n$ and $e^{\ln f(t)} = f(t)$.

$$t^2\frac{d^2y}{dt^2} + t\frac{dy}{dt} - (1 - 6t)t\frac{dy}{dt} + 10yt^2 = 5t^2 \sin 2t$$

$$t^2\frac{d^2y}{dt^2} + \cancel{t\frac{dy}{dt}} - \cancel{t\frac{dy}{dt}} + 6t^2\frac{dy}{dt} + 10yt^2 = 5t^2 \sin 2t$$

After cancelling, divide throughout by t^2.

$$\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 10y = 5 \sin 2t, \text{ as required}$$

d The auxiliary equation of ① is

$$m^2 + 6m + 10 = 0$$
$$m^2 + 6m + 9 = -1$$
$$(m + 3)^2 = -1$$
$$m + 3 = \pm i$$
$$m = -3 \pm i$$

The complementary function is given by

$$y = e^{-3t}(A\cos t + B\sin t)$$

For a particular integral try $y = p\sin 2t + q\cos 2t$

If the right hand side of the second order differential equation is a $k\sin nt$ or $k\cos nt$ function, then you should try a particular integral of the form $p\cos nt + q\sin nt$.

$$\frac{dy}{dt} = 2p\cos 2t - 2q\sin 2t$$

$$\frac{d^2y}{dt^2} = -4p\sin 2t - 4q\cos 2t$$

Substituting into ②

$$-4p\sin 2t - 4q\cos 2t + 12p\cos 2t - 12q\sin 2t + 10p\sin 2t + 10q\cos 2t = 5\sin 2t$$
$$(-4p - 12q + 10p)\sin 2t + (-4q + 12p + 10q)\cos 2t = 5\sin 2t$$
$$(6p - 12q)\sin 2t + (12p + 6q)\cos 2t = 5\sin 2t$$

Equating the coefficients of $\sin 2t$

$$6p - 12q = 5 \quad ③$$
$$12p + 6q = 0 \quad ④$$

You can solve the simultaneous equations by any appropriate method.

From ④ $p = -\frac{6}{12}q = -\frac{1}{2}q$

Substitute into ③

$$-3q - 12q = -15q = 5 \Rightarrow q = -\tfrac{1}{3}$$

Hence $p = -\frac{1}{2}q = -\frac{1}{2} \times -\frac{1}{3} = \frac{1}{6}$

The general solution of ② is

$$y = e^{-3t}(A\cos t + B\sin t) + \tfrac{1}{6}\sin 2t - \tfrac{1}{3}\cos 2t$$

$x = \ln t \Rightarrow t = e^x$

The general solution of ① is

$$y = e^{-3e^x}(A\cos(e^x) + B\sin(e^x)) + \tfrac{1}{6}\sin(2e^x) - \tfrac{1}{3}\cos(2e^x)$$

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Exercise A, Question 33

Question:

Given that x is so small that terms in x^3 and higher powers of x may be neglected, show that

$$11\sin x - 6\cos x + 5 = A + Bx + Cx^2,$$

stating the values of the constants A, B and C.

Solution:

a $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots$

$= 1 - \frac{x^2}{2}$, neglecting terms in x^3 and higher powers

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$

The series of $\cos x$ and $\sin x$ are both given in the formulae book and may be quoted without proof, unless the question specifically asks for a proof.

$= x$, neglecting terms in x^3 and higher powers

$$11\sin x - 6\cos x + 5 = 11x - 6\left(1 - \frac{x^2}{2}\right) + 5$$
$$= 11x - 6 + 3x^2 + 5$$
$$= -1 + 11x + 3x^2$$

You substitute the abbreviated series into the expression and collect together terms.

$A = -1$, $B = 11$, $C = 3$

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Exercise A, Question 34

Question:

Show that for $x > 1$,

$$\ln(x^2 - x + 1) + \ln(x + 1) - 3\ln x = \frac{1}{x^3} - \frac{1}{2x^6} + \ldots + \frac{(-1)^{n-1}}{nx^{3n}} + \ldots .$$

Solution:

a LHS $= \ln(x^2 - x + 1) + \ln(x + 1) - 3\ln x$

$= \ln[(x^2 - x + 1)(x + 1)] - \ln x^3$

$= \ln\left(\frac{x^3 + 1}{x^3}\right) = \ln\left(1 + \frac{1}{x^3}\right)$

You collect together the three terms of the left hand side (LHS) of the expression into a single logarithm using all three log rules;

$\log x + \log y = \log xy$

$\log x - \log y = \log\left(\frac{x}{y}\right)$,

and $n \log x = \log x^n$.

$(x^2 - x + 1)(x + 1) = x^3 + x^2 - x^2 - x + x + 1$
$= x^3 + 1$

Substituting $\frac{1}{x^3}$ for x and n for r in the series

$$\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \ldots + \frac{(-1)^{r+1}x^r}{r} + \ldots$$

This series is given in the formulae booklet. It is valid for $-1 < x \leqslant 1$ and, if $x > 1$, then $0 < \frac{1}{x^3} < 1$ so the series is valid for this question.

LHS $= \frac{1}{x^3} - \frac{1}{2x^6} + \ldots + \frac{(-1)^{n-1}}{nx^{3n}} + \ldots$, as required

$(-1)^{n+1} = (-1)^{n-1}$. If n is odd, both sides are 1. If n is even, both sides are -1.

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Exercise A, Question 35

Question:

Given that x is so small that terms in x^4 and higher powers of x may be neglected, find the values of the constants A, B, C and D for which

$$e^{-2x} \cos 5x = A + Bx + Cx^2 + Dx^3.$$

Solution:

a $e^{-2x} = 1 + (-2x) + \frac{(-2x)^2}{2!} + \frac{(-2x)^3}{3!} + \ldots$

$= 1 - 2x + 2x^2 - \frac{4}{3}x^3 + \ldots$

Substituting $-2x$ for x in the formula $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$ and ignoring terms in x^4 and higher powers.

$\cos 5x = 1 - \frac{(5x)^2}{2!} + \ldots$

$= 1 - \frac{25}{2}x^2 + \ldots$

Substituting $5x$ for x in the formula $\cos x = 1 - \frac{x^2}{2!} + \ldots$ and ignoring terms in x^4 and higher powers.

$e^{-2x} \cos 5x = \left(1 - 2x + 2x^2 - \frac{4}{3}x^3 + \ldots\right)\left(1 - \frac{25}{2}x^2 + \ldots\right)$

$= 1 - \frac{25}{2}x^2 - 2x + 25x^3 + 2x^2 - \frac{4}{3}x^3 + \ldots$

When multiplying out the brackets, you discard terms in x^4 and higher powers. For example, multiplying $2x^2$ by $-\frac{25}{2}x^2$ gives $-25x^4$ and you just ignore his term.

$= 1 - 2x + \left(-\frac{25}{2} + 2\right)x^2 + \left(25 - \frac{4}{3}\right)x^3 + \ldots$

$= 1 - 2x - \frac{21}{2}x^2 + \frac{71}{3}x^3 + \ldots$

$A = 1, B = -2, C = -\frac{21}{2}, D = \frac{71}{3}$

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Exercise A, Question 36

Question:

a Find the first four terms of the expansion, in ascending powers of x, of

$(2x+3)^{-1}$, $|x| < \frac{2}{3}$.

b Hence, or otherwise, find the first four non-zero terms of the expansion, in ascending powers of x, of

$\frac{\sin 2x}{3+2x}$, $|x| < \frac{2}{3}$.

Solution:

a $(2x+3)^{-1} = 3^{-1}\left(1+\frac{2x}{3}\right)^{-1}$

$= \frac{1}{3}\left(1 - \frac{2x}{3} + \frac{(-1)(-2)}{2.1}\left(\frac{2x}{3}\right)^2 + \frac{(-1)(-2)(-3)}{3.2.1}\left(\frac{2x}{3}\right)^3 + \dots\right)$

$= \frac{1}{3}\left(1 - \frac{2}{3}x + \frac{4}{9}x^2 - \frac{8}{27}x^3 + \dots\right)$

$= \frac{1}{3} - \frac{2}{9}x + \frac{4}{27}x^2 - \frac{8}{81}x^3 + \dots$

Part **a** is a binomial series with a rational index. This is in the C3 specification. The FP2 specification prerequisites states 'A knowledge of the specifications for C1, C2, C3, C4 and FP1, their prerequisites, preambles and associated formulae is assumed and may be tested.' In part **b**, this series is combined with a series in the FP2 specification.

b $\frac{\sin 2x}{3+2x} = \sin 2x(3+2x)^{-1}$

$= \left(2x - \frac{(2x)^3}{3!} + \dots\right)\left(\frac{1}{3} - \frac{2}{9}x + \frac{4}{27}x^2 - \frac{8}{81}x^3 + \dots\right)$

$= \left(2x - \frac{4}{3}x^3 + \dots\right)\left(\frac{1}{3} - \frac{2}{9}x + \frac{4}{27}x^2 - \frac{8}{81}x^3 + \dots\right)$

$= \frac{2}{3}x - \frac{4}{9}x^2 + \frac{8}{27}x^3 - \frac{16}{81}x^4 - \frac{4}{9}x^3 + \frac{8}{27}x^4 + \dots$

When multiplying out the brackets, you discard terms in x^4 and higher powers. For example, multiplying $-\frac{4}{3}x^3$ by $\frac{4}{27}x^2$ gives $-\frac{16}{81}x^5$ and you ignore this term.

$= \frac{2}{3}x - \frac{4}{9}x^2 + \left(\frac{8}{27} - \frac{4}{9}\right)x^3 + \left(\frac{8}{27} - \frac{16}{81}\right)x^4 + \dots$

$= \frac{2}{3}x - \frac{4}{9}x^2 - \frac{4}{27}x^3 + \frac{8}{81}x^4 + \dots$

Solutionbank FP2
Edexcel AS and A Level Modular Mathematics

Exercise A, Question 37

Question:

a By using the power series expansion for $\cos x$ and the power series expansion for $\ln(1+x)$, find the series expansion for $\ln(\cos x)$ in ascending powers of x up to and including the term in x^4.

b Hence, or otherwise, obtain the first two non-zero terms in the series expansion for $\ln(\sec x)$ in ascending powers of x.

Solution:

a $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots$

$$= 1 + \left(-\frac{x^2}{2} + \frac{x^4}{24}\right) \quad ①,$$

neglecting terms above x^4

The expression $-\frac{x^2}{2!} + \frac{x^4}{4!}$ is used to replace the x in the standard series for $\ln(1+x)$.

$$\ln(1+x) = x - \frac{x^2}{2} + \ldots$$

Using the expansion ①

$$\ln(\cos x) = \ln\left(1 + \left(-\frac{x^2}{2} + \frac{x^4}{24}\right)\right)$$

$$= \left(-\frac{x^2}{2} + \frac{x^4}{24}\right) - \frac{1}{2}\left(-\frac{x^2}{2} + \frac{x^4}{24}\right)^2 + \ldots$$

$$= -\frac{x^2}{2} + \frac{x^4}{24} - \frac{x^4}{8} + \ldots$$

$$= -\frac{x^2}{2} - \frac{x^4}{12} - \ldots$$

$-\frac{1}{2}\left(-\frac{x^2}{2} + \frac{x^4}{24}\right)^2 = -\frac{x^4}{8} + \frac{x^6}{48} - \frac{x^8}{1152}$ but, as the expansion is only required up to the term in x^4, you only need the first of the three terms.

b $\ln(\sec x) = \ln\left(\frac{1}{\cos x}\right) = \ln 1 - \ln \cos x$

$$= -\ln \cos x$$

Using the log rule $\log\left(\frac{a}{b}\right) = \log a - \log b$ and the fact that $\ln 1 = 0$.

Using the result to part **a**

$$\ln(\sec x) = -\left(-\frac{x^2}{2} - \frac{x^4}{12} - \ldots\right) = \frac{x^2}{2} + \frac{x^4}{12} + \ldots$$

Solutionbank FP2
Edexcel AS and A Level Modular Mathematics

Exercise A, Question 38

Question:

a Find the Taylor expansion of $\cos 2x$ in ascending powers of $\left(x - \frac{\pi}{4}\right)$ up to and including the term in $\left(x - \frac{\pi}{4}\right)^5$.

b Use your answer to part **a** to obtain an estimate of $\cos 2$, giving your answer to 6 decimal places.

Solution:

a Let $f(x) = \cos 2x$ $\qquad f\left(\frac{\pi}{4}\right) = \cos\frac{\pi}{2} = 0$

$f'(x) = -2\sin 2x$ $\qquad f'\left(\frac{\pi}{4}\right) = -2\sin\frac{\pi}{2} = -2$

$f''(x) = -4\cos 2x$ $\qquad f''\left(\frac{\pi}{4}\right) = -4\cos\frac{\pi}{2} = 0$

$f'''(x) = 8\sin 2x$ $\qquad f'''\left(\frac{\pi}{4}\right) = 8\sin\frac{\pi}{2} = 8$

$f^{(iv)}(x) = 16\cos 2x$ $\qquad f^{(iv)}\left(\frac{\pi}{4}\right) = 16\cos\frac{\pi}{2} = 0$

$f^{(v)}(x) = -32\sin 2x$ $\qquad f^{(v)}\left(\frac{\pi}{4}\right) = -32\sin\frac{\pi}{2} = -32$

Taylor's and Maclaurin's series need repeated differentiation and substitution. You need to display these in a systematic form, both to help you substitute correctly and to show your working clearly so that the examiner can award you marks.

$f^{(iv)}(x)$ and $f^{(v)}(x)$ are symbols which can be used for the fourth and fifth derivatives of $f(x)$ respectively.

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \frac{(x-a)^4}{4!}f^{(iv)}(a) + \frac{(x-a)^5}{5!}f^{(v)}(a) + \ldots$$

This is the appropriate form of Taylor's series for this question. It is given in the formula booklet.

Substituting $f(x) = \cos 2x$ and $a = \frac{\pi}{4}$

$$\cos 2x = \left(x - \frac{\pi}{4}\right) \times (-2) + \frac{\left(x-\frac{\pi}{4}\right)^3}{6} \times 8 + \frac{\left(x-\frac{\pi}{4}\right)^5}{120} \times (-32) + \ldots$$

$$= -2\left(x - \frac{\pi}{4}\right) + \frac{4}{3}\left(x - \frac{\pi}{4}\right)^3 - \frac{4}{15}\left(x - \frac{\pi}{4}\right)^5 + \ldots$$

All of the even derivatives are zero at $x = \frac{\pi}{4}$.

b Let $x = 1$, then $x - \frac{\pi}{4} = 0.2146\ldots$

Substituting into the result of part **a**

$$\cos 2 = -2(0.2146\ldots) + \frac{4}{3}(0.2146\ldots)^3 - \frac{4}{15}(0.2146\ldots)^5 + \ldots$$

$$\approx -0.416147 \text{ (6 d.p.)}$$

Work out $x - \frac{\pi}{4}$ on your calculator and then use the ANS button to complete the calculation.

This is a very accurate estimate and is correct to 6 decimal places.

Solutionbank FP2
Edexcel AS and A Level Modular Mathematics

Exercise A, Question 39

Question:

a Find the Taylor expansion of $\ln(\sin x)$ in ascending powers of $\left(x - \frac{\pi}{6}\right)$ up to and including the term in $\left(x - \frac{\pi}{6}\right)^3$.

b Use your answer to part **a** to obtain an estimate of $\ln(\sin 0.5)$, giving your answer to 6 decimal places.

Solution:

a Let $f(x) = \ln(\sin x)$ $\quad f\left(\frac{\pi}{6}\right) = \ln\frac{1}{2} = -\ln 2$

$f'(x) = \frac{\cos x}{\sin x} = \cot x$ $\quad f'\left(\frac{\pi}{6}\right) = \cot\frac{\pi}{6} = \sqrt{3}$

$f''(x) = -\operatorname{cosec}^2 x$ $\quad f''\left(\frac{\pi}{6}\right) = -4$

$\operatorname{cosec}\frac{\pi}{6} = \frac{1}{\sin\frac{\pi}{6}} = \frac{1}{\frac{1}{2}} = 2$

$f'''(x) = 2\operatorname{cosec}^2 x \cot x$ $\quad f'''\left(\frac{\pi}{6}\right) = 2 \times 2^2 \times \sqrt{3} = 8\sqrt{3}$

Using the chain rule,

$\frac{d}{dx}(-\operatorname{cosec}^2 x) = -2\operatorname{cosec} x \frac{d}{dx}(\operatorname{cosec} x)$

$= -2\operatorname{cosec} x \times -\operatorname{cosec} x \cot x$

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \ldots$$

This is the appropriate form of Taylor's series for this question. It is given in the formula booklet.

Substituting $f(x) = \ln(\sin x)$ and $a = \frac{\pi}{6}$

$$\ln(\sin x) = -\ln 2 + \left(x - \frac{\pi}{6}\right) \times \sqrt{3} + \frac{1}{2}\left(x - \frac{\pi}{6}\right)^2 \times (-4) + \frac{1}{6}\left(x - \frac{\pi}{6}\right)^3 \times 8\sqrt{3} + \ldots$$

$$= -\ln 2 + \sqrt{3}\left(x - \frac{\pi}{6}\right) - 2\left(x - \frac{\pi}{6}\right)^2 + \frac{4\sqrt{3}}{3}\left(x - \frac{\pi}{6}\right)^3 + \ldots$$

b Let $x = 0.5$, then $x - \frac{\pi}{6} = -0.023\,598\,7\ldots$

Work out $x - \frac{\pi}{4}$ on your calculator and then use the ANS button to complete the calculation.

Substituting into the result of part **a**

$$\ln(\sin 0.5) = -\ln 2 + \sqrt{3}(-0.023\,598\ldots) - 2(-0.023\,598\ldots)^2 + \frac{4\sqrt{3}}{3}(-0.023\,598\ldots)^3 + \ldots$$

$$\approx -0.735166 \text{ (6 d.p.)}$$

Solutionbank FP2
Edexcel AS and A Level Modular Mathematics

Exercise A, Question 40

Question:

Given that $y = \tan x$,

a find $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ and $\frac{d^3y}{dx^3}$.

b Find the Taylor series expansion of $\tan x$ in ascending powers of $\left(x - \frac{\pi}{4}\right)$ up to and including the term in $\left(x - \frac{\pi}{4}\right)^3$.

c Hence show that

$$\tan \frac{3\pi}{10} \approx 1 + \frac{\pi}{10} + \frac{\pi^2}{200} + \frac{\pi^3}{3000}.$$

Solution:

a $y = \tan x$

$\frac{dy}{dx} = \sec^2 x$

$\frac{d^2y}{dx^2} = 2\sec x \frac{d}{dx}(\sec x) = 2\sec x \times \sec x \tan x$

Using the chain rule for differentiation.

$= 2\sec^2 x \tan x$

$\frac{d^3y}{dx^3} = \tan x \frac{d}{dx}(2\sec^2 x) + 2\sec^2 x \frac{d}{dx}(\tan x)$

Using the product rule for differentiation $\frac{d}{dx}(uv) = v\frac{du}{dx} + u\frac{dv}{dx}$ with $u = 2\sec^2 x$ and $v = \tan x$.

$= 4\sec^2 x \tan^2 x + 2\sec^4 x$

b Let $y = f(x) = \tan x$

$f\left(\frac{\pi}{4}\right) = \tan\frac{\pi}{4} = 1$

Using the results in part **a**

$f'\left(\frac{\pi}{4}\right) = \sec^2\frac{\pi}{4} = (\sqrt{2})^2 = 2$

$f''(x) = 2\sec^2\frac{\pi}{4}\tan\frac{\pi}{4} = 2 \times (\sqrt{2})^2 \times 1 = 4$

$\sec\frac{\pi}{4} = \sqrt{2}$ and $\tan\frac{\pi}{4} = 1$

$f'''\left(\frac{\pi}{4}\right) = 4\sec^2\frac{\pi}{4}\tan^2\frac{\pi}{4} + 2\sec^4\frac{\pi}{4}$

$= 4(\sqrt{2})^2 \times 1^2 + 2(\sqrt{2})^4 = 8 + 8 = 16$

$f(x) = f(a) + (x - a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \dots$

This is the first four terms of Taylor's series.

Substituting $f(x) = \tan x$ and $x = \frac{\pi}{4}$

You are expanding $\tan x$ about the point $x = \frac{\pi}{4}$, using Taylor's series.

$\tan x = 1 + \left(x - \frac{\pi}{4}\right) \times 2 + \frac{1}{2}\left(x - \frac{\pi}{4}\right)^2 \times 4 + \frac{1}{6}\left(x - \frac{\pi}{4}\right)^3 \times 16 + \dots$

$= 1 + 2\left(x - \frac{\pi}{4}\right) + 2\left(x - \frac{\pi}{4}\right)^2 + \frac{8}{3}\left(x - \frac{\pi}{4}\right)^3 + \dots$

c Let $x = \frac{3\pi}{10}$, then $x - \frac{\pi}{4} = \frac{3\pi}{10} - \frac{\pi}{4} = \frac{\pi}{20}$

Substituting into the result in part **b**

$\tan\frac{3\pi}{10} = 1 + 2\left(\frac{\pi}{20}\right) + 2\left(\frac{\pi}{20}\right)^2 + \frac{8}{3}\left(\frac{\pi}{20}\right)^3 + \dots$

$\approx 1 + \frac{\pi}{10} + \frac{\pi^2}{200} + \frac{\pi^3}{3000}$, as required

Solutionbank FP2
Edexcel AS and A Level Modular Mathematics

Exercise A, Question 41

Question:

$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + 2y = 0$$

At $x = 0$, $y = 2$ and $\frac{dy}{dx} = -1$.

a Find the value of $\frac{d^3y}{dx^3}$ at $x = 0$.

b Express y as a series in ascending powers of x, up to and including the term in x^3.

Solution:

a $(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + 2y = 0$ ①

Differentiate ① throughout with respect to x

$$-2x\frac{d^2y}{dx^2} + (1-x^2)\frac{d^3y}{dx^3} - \frac{dy}{dx} - x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 0 \quad ②$$

Substituting $x = 0$, $y = 2$ and $\frac{dy}{dx} = -1$ into ②

$$0 + \frac{d^3y}{dx^3} + 1 - 0 - 2 = 0$$

At $x = 0$, $\frac{d^3y}{dx^3} = 1$

Using the product rule for differentiation

$\frac{d}{dx}(uv) = v\frac{du}{dx} + u\frac{dv}{dx}$ with

$u = 1 - x^2$ and $v = \frac{d^2y}{dx^2}$,

$\frac{d}{dx}\left((1-x^2)\frac{d^2y}{dx^2}\right)$

$= \frac{d^2y}{dx^2}\frac{d}{dx}(1-x^2) + (1-x^2)\frac{d}{dx}\left(\frac{d^2y}{dx^2}\right)$

$= \frac{d^2y}{dx^2} \times -2x + (1-x^2)\frac{d^3y}{dx^3}$

b Let $y = f(x)$

From the data in the question

$f(0) = 2$, $f'(0) = -1$

At $x = 0$, ① above becomes

$f''(0) + 2 \times 2 = 0 \Rightarrow f''(0) = -4$

And the result to part **a** becomes

$f'''(0) = 1$

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \ldots$$

The formula for Maclaurin's series is given in the formulae booklet. For this question, you need the terms up to and including the term in x^3.

$$y = 2 + x \times (-1) + \frac{x^2}{2} \times (-4) + \frac{x^3}{6} \times 1 + \ldots$$

$$= 2 - x - 2x^2 + \frac{1}{6}x^3 + \ldots$$

Solutionbank FP2
Edexcel AS and A Level Modular Mathematics

Exercise A, Question 42

Question:

$(1 + 2x)\dfrac{dy}{dx} = x + 4y^2.$

a Show that

$$(1 + 2x)\frac{d^2y}{dx^2} = 1 + 2(4y - 1)\frac{dy}{dx} \qquad ①$$

b Differentiate equation ① with respect to x to obtain an equation involving

$\dfrac{d^3y}{dx^3}, \dfrac{d^2y}{dx^2}, \dfrac{dy}{dx}, x$ and y.

Given that $y = \frac{1}{2}$ at $x = 0$,

c find a series solution for y, in ascending powers of x, up to and including the term in x^3.

Solution:

a $(1+2x)\dfrac{dy}{dx} = x + 4y^2 \quad \star$

You need to differentiate $4y^2$ implicitly with respect to x.

$$\frac{d}{dx}(4y^2) = \frac{dy}{dx} \times \frac{d}{dy}(4y^2) = 8y\frac{dy}{dx}.$$

Differentiate $\star$ throughout with respect to x

$$2\frac{dy}{dx} + (1+2x)\frac{d^2y}{dx^2} = 1 + 8y\frac{dy}{dx}$$

$$(1+2x)\frac{d^2y}{dx^2} = 1 + 8y\frac{dy}{dx} - 2\frac{dy}{dx}$$

$$= 1 + 2(4y-1)\frac{dy}{dx} \quad \text{① as required.}$$

b Differentiate ① throughout with respect to x

$$2\frac{d^2y}{dx^2} + (1+2x)\frac{d^3y}{dx^3} = 8\left(\frac{dy}{dx}\right)^2 + 2(4y-1)\frac{d^2y}{dx^2} \ldots \quad ②$$

When using the product rule for differentiation

$\dfrac{d}{dx}(uv) = v\dfrac{du}{dx} + u\dfrac{dv}{dx}$ with

$u = 2(4y-1)$ and $v = \dfrac{dy}{dx}$,

$2(4y-1)$ must be differentiated implicitly with respect to x. So

$$\frac{d}{dx}\left(2(4y-1)\frac{dy}{dx}\right)$$

$$= 8\frac{dy}{dx} \times \frac{dy}{dx} + 2(4y-1)\frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$= 8\left(\frac{dy}{dx}\right)^2 + 2(4y-1)\frac{d^2y}{dx^2}.$$

c Let $y = f(x)$

From the data in the question

$$f(0) = \frac{1}{2}$$

At $x = 0$, $y = \frac{1}{2}$, $\star$ becomes

$$f'(0) = 4 \times \left(\frac{1}{2}\right)^2 = 1$$

At $x = 0$, $y = \frac{1}{2}$, $\frac{dy}{dx} = 1$, ① becomes

$$f''(0) = 1 + 2\left(4 \times \frac{1}{2} - 1\right) \times 1 = 3$$

At $x = 0$, $y = \frac{1}{2}$, $\frac{dy}{dx} = 1$, $\frac{d^2y}{dx^2} = 3$, ② becomes

$$2 \times 3 + f'''(0) = 8 \times 1^2 + 2\left(4 \times \frac{1}{2} - 1\right) \times 3$$

$$6 + f'''(0) = 8 + 6 \Rightarrow f'''(0) = 8$$

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \ldots$$

The formula for Maclaurin's series is given in the formulae booklet. For this question, you need the terms up to and including the term in x^3.

$$y = \frac{1}{2} + x \times 1 + \frac{x^2}{2} \times 3 + \frac{x^3}{6} \times 8 + \ldots$$

$$= \frac{1}{2} + x + \frac{3}{2}x^2 + \frac{4}{3}x^3 + \ldots$$

Solutionbank FP2
Edexcel AS and A Level Modular Mathematics

Exercise A, Question 43

Question:

$$\frac{dy}{dx} = y^2 + xy + x,\ y = 1 \text{ at } x = 0$$

a Use the Taylor series method to find y as a series in ascending powers of x, up to and including the term in x^3.

b Use your series to find y at $x = 0.1$, giving your answer to 2 decimal places.

Solution:

a Let $y = f(x)$

From the data in the question

$f(0) = 1$

$$\frac{dy}{dx} = y^2 + xy + x \quad ①$$

y^2 has to be differentiated implicitly by x. So

$$\frac{d}{dx}(y^2) = \frac{dy}{dx} \times \frac{d}{dy}(y^2) = \frac{dy}{dx} \times 2y$$

At $x = 0$, $y = 1$, ① becomes

$f'(0) = 1^2 + 0 + 0 = 1$

Differentiate ① throughout by x

$$\frac{d^2y}{dx^2} = 2y\frac{dy}{dx} + y + x\frac{dy}{dx} + 1 \quad ②$$

Using the product rule for differentiation $\frac{d}{dx}(uv) = v\frac{du}{dx} + u\frac{dv}{dx}$ with $u = x$ and $v = y$,

$$\frac{d}{dx}(xy) = y\frac{dx}{dx} + x\frac{dy}{dx} = y \times 1 + x\frac{dy}{dx}.$$

At $x = 0$, $y = 1$, $\frac{dy}{dx} = 1$, ② becomes

$f''(0) = 2 \times 1 \times 1 + 1 + 0 + 1 = 4$

Differentiate ② throughout by x

$$\frac{d^3y}{dx^3} = 2\left(\frac{dy}{dx}\right)^2 + 2y\frac{d^2y}{dx^2} + \frac{dy}{dx} + \frac{dy}{dx} + x\frac{d^2y}{dx^2} \quad ③$$

Using the product rule for differentiation $\frac{d}{dx}(uv) = v\frac{du}{dx} + u\frac{dv}{dx}$ with $u = 2y$ and $v = \frac{dy}{dx}$,

$$\frac{d}{dx}\left(2y\frac{dy}{dx}\right) = \frac{dy}{dx}\frac{d}{dx}(2y) + 2y\frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$= \frac{dy}{dx} \times 2\frac{dy}{dx} + 2y\frac{d^2y}{dx^2} = 2\left(\frac{dy}{dx}\right)^2 + 2y\frac{d^2y}{dx^2}.$$

At $x = 0$, $y = 1$, $\frac{dy}{dx} = 1$, $\frac{d^2y}{dx^2} = 4$, ③ becomes

$f'''(0) = 2 \times 1^2 + 2 \times 1 \times 4 + 1 + 1 + 0 = 12$

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \dots$$

$$y = 1 + x \times 1 + \frac{x^2}{2} \times 4 + \frac{x^3}{6} \times 12 + \dots$$

$$= 1 + x + 2x^2 + 2x^3 + \dots$$

b At 0.1,

$y = 1 + 0.1 + 2(0.1)^2 + 2(0.1)^3 + \dots$

$\approx 1 + 0.1 + 0.02 + 0.002 = 1.122$

$y \approx 1.12$ (2 d.p.)

Solutionbank FP2
Edexcel AS and A Level Modular Mathematics

Exercise A, Question 44

Question:

$$y\frac{dy}{dx}=\frac{x+3}{y+1}$$

Given that $y = 1.5$ at $x = 0$,

a Use the Taylor series method to find the series solution for y, in ascending powers of x, up to and including the term in x^3.

b Use your result to **a** to estimate, to 3 decimal places, the value of y at $x = 0.1$.

Solution:

a Rearranging the differential equation in the question

$$(y^2 + y)\frac{dy}{dx} = x + 3 \qquad ①$$

The right hand side of the equation in the question would be hard to repeatedly differentiate as a quotient, so multiply both sides by $y + 1$.

Let $y = f(x)$

From the data in the question

$$f(0) = 1.5$$

At $x = 0$, $y = 1.5$, ① becomes

$$(1.5^2 + 1.5)\,f'(0) = 0 + 3 \Rightarrow f'(0) = \frac{3}{3.75} = 0.8$$

Differentiate ① throughout by x

$$(2y+1)\left(\frac{dy}{dx}\right)^2 + (y^2 + y)\frac{d^2y}{dx^2} = 1 \qquad ②$$

At $x = 0$, $y = 1.5$, $\frac{dy}{dx} = 0.8$, ② becomes

$$4 \times 0.8^2 + (1.5^2 + 1.5)\,f''(0) = 1$$

$$f''(0) = \frac{1 - 4 \times 0.8^2}{3.75} = -0.416$$

Differentiate ② throughout by x

Differentiating $\left(\frac{dy}{dx}\right)^2$ by x, using the chain rule

$$\frac{d}{dx}\left(\left(\frac{dy}{dx}\right)^2\right) = 2\frac{dy}{dx} \times \frac{d}{dx}\left(\frac{dy}{dx}\right) = 2\frac{dy}{dx} \times \frac{d^2y}{dx^2}.$$

$$2\left(\frac{dy}{dx}\right)^3 + (2y+1)\,2 \times \frac{dy}{dx} \times \frac{d^2y}{dx^2} + (2y+1)\frac{dy}{dx} \times \frac{d^2y}{dx^2} + (y^2+y)\frac{d^3y}{dx^3} = 0.$$

$$2\left(\frac{dy}{dx}\right)^3 + 3(2y+1)\frac{dy}{dx}\frac{d^2y}{dx^2} + (y^2+y)\frac{d^3y}{dx^3} = 0 \qquad ③$$

At $x = 0$, $y = 1.5$, $\frac{dy}{dx} = 0.8$, $\frac{d^2y}{dx^2} = -0.416$, ② becomes

$$2 \times 0.8^3 + 3 \times 4 \times 0.8 \times -0.416 + (1.5^2 + 1.5)\,f'''(0) = 0$$

$$1.204 - 3.9936 + 3.75\,f'''(0) = 0$$

$$f'''(0) = \frac{3.9936 - 1.024}{3.75} = 0.79189\dot{3}$$

This is a recurring decimal. There is an exact fraction $\frac{7424}{9375}$.

$$f(x) = f(0) + x\,f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \ldots$$

$$y = 1.5 + x \times 0.8 + \frac{x^2}{2} \times -0.416 + \frac{x^3}{6} \times 0.79189\dot{3} + \ldots$$

$$= 1.5 + 0.8x - 0.208x^2 + 0.13198\dot{2}x^3 + \ldots$$

b At 0.1,

$$y = 1.5 + 0.08 - 0.00208 + 0.000\,131\,98 \ldots$$

$$\approx 1.578 \text{ (3 d.p.)}$$

The fourth term is small and this justifies you using the truncated series to make the approximation.

Solutionbank FP2
Edexcel AS and A Level Modular Mathematics

Exercise A, Question 45

Question:

$$y\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2+y=0$$

a Find an expression for $\frac{d^3y}{dx^3}$.

Given that $y = 1$ and $\frac{dy}{dx} = 1$ at $x = 0$,

b find the series solution for y, in ascending powers of x, up to and including the term in x^3.

c Comment on whether it would be sensible to use your series solution to give estimates for y at $x = 0.2$ and at $x = 50$.

Solution:

a $y\dfrac{d^2y}{dx^2} + \left(\dfrac{dy}{dx}\right)^2 + y = 0$ ①

Differentiate ① throughout with respect to x

$$\frac{dy}{dx} \times \frac{d^2y}{dx^2} + y\frac{d^3y}{dx^3} + 2\frac{dy}{dx} \times \frac{d^2y}{dx^2} + \frac{dy}{dx} = 0$$

$$y\frac{d^3y}{dx^3} = -3\frac{dy}{dx}\frac{d^2y}{dx^2} - \frac{dy}{dx} = -\frac{dy}{dx}\left(3\frac{d^2y}{dx^2} + 1\right)$$

$$\frac{d^3y}{dx^3} = -\frac{1}{y}\frac{dy}{dx}\left(3\frac{d^2y}{dx^2} + 1\right) \quad ②$$

Using the product rule for differentiation

$$\frac{d}{dx}(uv) = v\frac{du}{dx} + u\frac{dv}{dx}$$

with $u = y$ and $v = \dfrac{d^2y}{dx^2}$,

$$\frac{d}{dx}\left(y\frac{d^2y}{dx^2}\right) = \frac{d^2y}{dx^2} \times \frac{dy}{dx} + y \times \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right)$$
$$= \frac{dy}{dx} \times \frac{d^2y}{dx^2} + y\frac{d^3y}{dx^3}$$

The wording of the question requires you to make $\dfrac{d^3y}{dx^3}$ the subject of the formula. There are many possible alternative forms for the answer.

b Let $y = f(x)$

From the data in the question

$f(0) = 1,\ f'(0) = 1$

At $x = 0$, $y = 1$, $\dfrac{dy}{dx} = 1$, ① becomes

$1 \times f''(0) + 1^2 + 1 = 0 \Rightarrow f''(0) = -2$

At $x = 0$, $y = 1$, $\dfrac{dy}{dx} = 1$, $\dfrac{d^2y}{dx^2} = -2$, ② becomes

$f'''(0) = -\dfrac{1}{1} \times 1(3 \times -2 + 1) = -1(-6 + 1) = 5$

$f(x) = f(0) + x\,f'(0) + \dfrac{x^2}{2!}f''(0) + \dfrac{x^3}{3!}f'''(0) + \ldots$

$y = 1 + x \times 1 + \dfrac{x^2}{2} \times -2 + \dfrac{x^3}{6} \times 5 + \ldots$

$= 1 + x - x^2 + \dfrac{5}{6}x^3 + \ldots$

The formula for Maclaurin's series is given in the formulae booklet. For this question, you need the terms up to and including the term in x^3.

c The series expansion up to and including the term in x^3 can be used to estimate y if x is small. So it would be sensible to use it at $x = 0.2$ but not at $x = 50$.

Solutionbank FP2
Edexcel AS and A Level Modular Mathematics

Exercise A, Question 46

Question:

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 3y^2 = 6, \text{ with } y = 1 \text{ and } \frac{dy}{dx} = 0 \text{ at } x = 0.$$

a Use the Taylor series method to obtain y as a series of ascending powers of x, up to and including the term in x^4.

b Hence find the approximate value for y when $x = 0.2$.

Solution:

a $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 3y^2 = 6$ ①

$3y^2$ has to be differentiated implicitly with respect to x. So $\frac{d}{dx}(3y^2) = \frac{dy}{dx} \times \frac{d}{dy}(3y^2) = \frac{dy}{dx} \times 6y$

Let $y = f(x)$

From the data in the question

$f(0) = 1, f'(0) = 0$

At $x = 0, y = 1, \frac{dy}{dx} = 0$, ① becomes

$f''(0) - 4 \times 0 + 3 \times 1^2 = 6 \Rightarrow f''(0) = 3$

Differentiate ① throughout with respect to x

$\frac{d^3y}{dx^3} - 4\frac{d^2y}{dx^2} + 6y\frac{dy}{dx} = 0$ ②

At $x = 0, y = 1, \frac{dy}{dx} = 0, \frac{d^2y}{dx^2} = 3$, ② becomes

$f'''(0) - 4 \times 3 + 6 \times 1 \times 0 = 0 \Rightarrow f'''(0) = 12$

Differentiate ② throughout with respect to x

Using the product rule for differentiation $\frac{d}{dx}(uv) = v\frac{du}{dx} + u\frac{dv}{dx}$ with $u = 6y$ and $v = \frac{dy}{dx}$,

$$\frac{d}{dx}\left(6y\frac{dy}{dx}\right) = \frac{dy}{dx}\frac{d}{dy}(6y) + 6y\frac{d}{dx}\left(\frac{dy}{dx}\right) = 6\left(\frac{dy}{dx}\right)^2 + 6y\frac{d^2y}{dx^2}.$$

$\frac{d^4y}{dx^4} - 4\frac{d^3y}{dx^3} + 6\left(\frac{dy}{dx}\right)^2 + 6y\frac{d^2y}{dx^2} = 0$ ③

At $x = 0, y = 1, \frac{dy}{dx} = 0, \frac{d^2y}{dx^2} = 3, \frac{d^3y}{dx^3} = 12,$

③ becomes

$f^{(iv)}(0) - 4 \times 12 + 6 \times 0^2 + 6 \times 1 \times 3 = 0$

$f^{(iv)}(0) = 48 - 18 = 30$

$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{(iv)}(0) + \ldots$

The formula for Maclaurin's series is given in the formulae booklet. For this question, you need the terms up to and including the term in x^4.

$y = 1 + x \times 0 + \frac{x^2}{2} \times 3 + \frac{x^3}{6} \times 12 + \frac{x^4}{24} \times 30 + \ldots$

$= 1 + \frac{3}{2}x^2 + 2x^3 + \frac{5}{4}x^4 + \ldots$

b At $x = 0.2$

$y = 1 + 0.06 + 0.016 + 0.002 + \ldots \approx 1.078$

$y \approx 1.08$ (2 d.p.)

Solutionbank FP2

Edexcel AS and A Level Modular Mathematics

Exercise A, Question 47

Question:

Given that

$$f(x) = \ln(1 + \cos 2x), \quad 0 \leqslant x < \frac{\pi}{2},$$

Show that

a $f'(x) = -2 \tan x$

b $f''''(x) = -[f'''(x)\, f'(x) + (f''(x))^2]$.

c Use Maclaurin's theorem to find the expansion of $f(x)$, in ascending powers of x, up to and including the term in x^4.

Solution:

a Let $u = 1 + \cos 2x$, then $\mathrm{f}(x) = \ln u$

$$\frac{\mathrm{d}u}{\mathrm{d}x} = -2\sin 2x$$

$$\mathrm{f}'(x) = \mathrm{f}'(u)\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{u}\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{1+\cos 2x} \times -2\sin 2x$$

$$= \frac{-4\sin x\cos x}{2\cos^2 x}$$

Using the identities $\sin 2x = 2\sin x\cos x$ and $\cos 2x = 2\cos^2 x - 1$.

$$= \frac{-2\sin x}{\cos x} = -2\tan x, \text{ as required}$$

b $\mathrm{f}''(x) = -2\sec^2 x$

$\mathrm{f}'''(x) = -4\sec^2 x\tan x$

$\mathrm{f}''''(x) = -8\sec x \,.\, \sec x\tan x \,.\, \tan x - 4\sec^2 x \,.\, \sec^2 x$

$= -8\sec^2 x\tan^2 x - 4\sec^4 x$

$= -[-4\sec^2 x\tan x \times -2\tan x + (-2\sec^2 x)^2]$

$= -[\mathrm{f}'''(x)\,\mathrm{f}'(x) + (\mathrm{f}''(x))^2]$, as required

$\mathrm{f}''''(x)$ is a symbol used for the fourth derivative of $\mathrm{f}(x)$ with respect to x. The symbol $\mathrm{f}^{(\mathrm{iv})}(x)$ is also used for the fourth derivative.

You use the product rule for differentiation $\frac{\mathrm{d}}{\mathrm{d}x}(uv) = v\frac{\mathrm{d}u}{\mathrm{d}x} + u\frac{\mathrm{d}v}{\mathrm{d}x}$ with $u = -4\sec^2 x$ and $v = \tan x$.

You also use the chain rule $\frac{\mathrm{d}}{\mathrm{d}x}(\sec^2 x) = 2\sec x\frac{\mathrm{d}}{\mathrm{d}x}(\sec x)$
$= 2\sec x \times \sec x\tan x$.

c $\mathrm{f}(0) = \ln(1 + \cos 0) = \ln 2$

$\mathrm{f}'(0) = -2\tan 0 = 0$

$\mathrm{f}''(0) = -2\sec^2 0 = -2$

$\mathrm{f}'''(0) = -4\sec^2 0\tan 0 = 0$

$\mathrm{f}''''(0) = -[\mathrm{f}'''(0)\,\mathrm{f}'(0) + (\mathrm{f}''(0))^2]$

Using the result for part **b**.

$= -[0 \times 0 + (-2)^2] = -4$

$$\mathrm{f}(x) = \mathrm{f}(0) + x\,\mathrm{f}'(0) + \frac{x^2}{2!}\mathrm{f}''(0) + \frac{x^3}{3!}\mathrm{f}'''(0) + \frac{x^4}{4!}\mathrm{f}^{(\mathrm{iv})}(0) + \ldots$$

The formula for Maclaurin's series is given in the formulae booklet. For this question, you need the terms up to and including the term in x^4.

$$= \ln 2 + x \times 0 + \frac{x^2}{2} \times -2 + \frac{x^3}{6} \times 0 + \frac{x^4}{24} \times -4 + \ldots$$

$$= \ln 2 - x^2 - \frac{1}{6}x^4 + \ldots$$

Solutionbank FP2
Edexcel AS and A Level Modular Mathematics

Exercise A, Question 48

Question:

a Use the Taylor series method to obtain a solution in a series of ascending powers of x, up to and including the term in x^4, of the differential equation

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = e^{x^2},$$

given that $y = 1$ and $\frac{dy}{dx} = 1$ at $x = 0$.

b Working to a least 4 decimal places, use the series obtained in part **a** to obtain the value of y at **i** $x = 0.1$, **ii** $x = 0.2$.

c By differentiating the series obtained for y, obtain estimates for $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at $x = 0.1$.

Solution:

a $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = e^{x^2}$ ①

Let $y = f(x)$

From the data in the question

$f(0) = 1,\ f'(0) = 1$

At $x = 0,\ y = 1,\ \frac{dy}{dx} = 1$, ① becomes

$f''(0) - 3 \times 1 + 2 \times 1 = e^0 = 1$

$f''(0) = 1 + 3 - 2 = 2$

Differentiate ① throughout with respect to x

$\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 2x\,e^{x^2}$ ②

$\frac{d}{dx}(e^{x^2}) = \frac{d}{dx}(x^2) \times e^{x^2} = 2x\,e^{x^2}$

At $x = 0,\ y = 1,\ \frac{dy}{dx} = 1,\ \frac{d^2y}{dx^2} = 2$, ② becomes

$f'''(0) - 3 \times 2 + 2 \times 1 = 0$

$f'''(0) = 6 - 2 = 4$

Differentiate ② throughout with respect to x

$\frac{d^4y}{dx^4} - 3\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} = 2\,e^{x^2} + 4x^2\,e^{x^2}$ ③

$\frac{d}{dx}(2x\,e^{x^2}) = e^{x^2}\frac{d}{dx}(2x) + 2x\frac{d}{dx}(e^{x^2})$

At $x = 0,\ y = 1,\ \frac{dy}{dx} = 1,\ \frac{d^2y}{dx^2} = 2,\ \frac{d^3y}{dx^3} = 4$,

③ becomes

$f^{(iv)}(0) - 3 \times 4 + 2 \times 2 = 2 + 0 \Rightarrow f^{(iv)}(0) = 10$

$f(x) = f(0) + x\,f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \frac{x^4}{4!}f^{(iv)}(0) + \ldots$

$y = 1 + x \times 1 + \frac{x^2}{2} \times 2 + \frac{x^3}{6} \times 4 + \frac{x^4}{24} \times 10 + \ldots$

$= 1 + x + x^2 + \frac{2}{3}x^3 + \frac{5}{12}x^4 + \ldots$

b i At $x = 0.1$

$$y = 1 + 0.1 + 0.01 + 0.000\,666\ldots + 0.000\,041\ldots$$
$$\approx 1.110\,708 = 1.1107 \text{ (4 d.p.)}$$

ii At $x = 0.2$

$$y = 1 + 0.2 + 0.04 + 0.005\,333\ldots + 0.000\,666\ldots$$
$$\approx 1.2460 \text{ (4 d.p.)}$$

As x gets larger, the approximation gets less accurate, so the answer to **ii** will be less accurate than the answer to **i**. In this case the value at 0.1 is accurate to 4 decimal paces. The approximation at 0.2 is a very good one but the accurate answer, 1.246 064…, is 1.2641 to 4 decimal places.

c $y = 1 + x + x^2 + \frac{2}{3}x^3 + \frac{5}{12}x^4 + \ldots$

Differentiating term by term

$$\frac{dy}{dx} = 1 + 2x + 2x^2 + \frac{5}{3}x^3 + \ldots$$

At $x = 0.1$

$$\frac{dy}{dx} = 1 + 0.2 + 0.02 + 0.001\,666\ldots$$
$$\approx 1.222 \text{ (3 d.p.)}$$

$$\frac{d^2y}{dx^2} = 2 + 4x + 5x^2 + \ldots$$

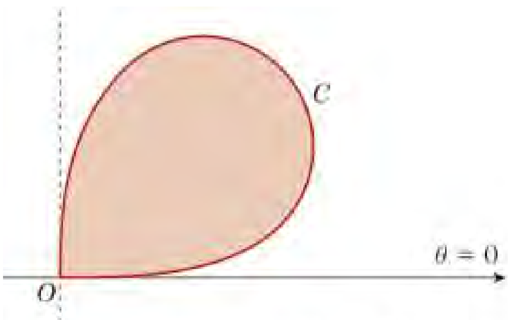
At $x = 0.1$

$$\frac{d^2y}{dx^2} = 2 + 0.4 + 0.05 + \ldots$$
$$\approx 2.45 \text{ (2 d.p.)}$$

Solutionbank FP2
Edexcel AS and A Level Modular Mathematics

Exercise A, Question 49

Question:



The figure shows a sketch of the curve C with polar equation

$$r^2 = a^2 \sin 2\theta,\ 0 \leqslant \theta \leqslant \frac{\pi}{2},$$

where a is a constant.

Find the area of the shaded region enclosed by C.

Solution:

$$A = \frac{1}{2}\int_0^{\frac{\pi}{2}} r^2 \, d\theta$$

You need to know the formula for the area of polar curves $A = \frac{1}{2}\int r^2 \, d\theta$. In this question, the diagram shows that the limits are 0 and $\frac{\pi}{2}$.

$$\frac{1}{2}\int r^2 \, d\theta = \frac{1}{2}\int a^2 \sin 2\theta \, d\theta$$

$$= \frac{a^2}{2}\left[-\frac{\cos 2\theta}{2}\right]$$

$$A = \frac{a^2}{4}\left[-\cos 2\theta\right]_0^{\frac{\pi}{2}} = \frac{a^2}{4}\left[1 - (-1)\right]$$

$\cos\left(2 \times \frac{\pi}{2}\right) = \cos \pi = -1$ and $\cos 0 = 1$.

$$= \frac{1}{2}a^2$$

Solutionbank FP2
Edexcel AS and A Level Modular Mathematics

Exercise A, Question 50

Question:

Relative to the origin O as pole and initial line $\theta = 0$, find an equation in polar coordinate form for

a a circle, centre O and radius 2,

b a line perpendicular to the initial line and passing through the point with polar coordinates $(3, 0)$.

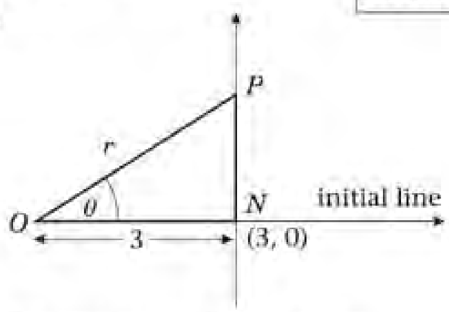
c a straight line through the points with polar coordinates $(4, 0)$ and $\left(4, \frac{\pi}{3}\right)$.

Solution:

a $r = 2$

You can just write the answer to part **a** down. The equation $r = k$ is the equation of a circle centre O and radius k, for any positive k.

b



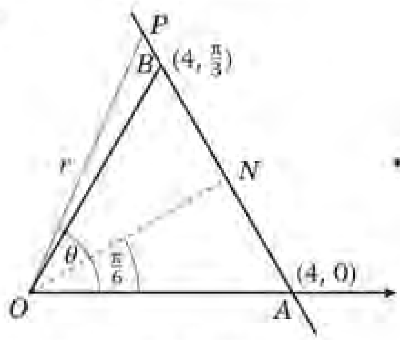
For any point P on the line

$\frac{3}{r} = \cos\theta$

If the point (3, 0) is labelled N, trigonometry on the right-angled triangle ONP gives the polar equation of the line.

$r = \frac{3}{\cos\theta} = 3\sec\theta$

c



In this diagram, the point (4, 0) is labelled A, the point $\left(4, \frac{\pi}{3}\right)$ is labelled B and the foot of the perpendicular from O to AB is labelled N. The triangle OAB is equilateral and $\angle AON = \frac{1}{2} \times 60° = 30° = \frac{\pi}{6}$ radians.

In the triangle ONA

$\frac{ON}{OA} = \frac{ON}{4} = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$

$ON = 2\sqrt{3}$

In the triangle ONP,

$\frac{ON}{OP} = \cos\left(\theta - \frac{\pi}{6}\right)$

$\frac{2\sqrt{3}}{r} = \cos\left(\theta - \frac{\pi}{6}\right)$

$r = 2\sqrt{3}\sec\left(\theta - \frac{\pi}{6}\right)$

This relation is true for any point P on the line and, as $OP = r$ this gives you the polar equation of the line.

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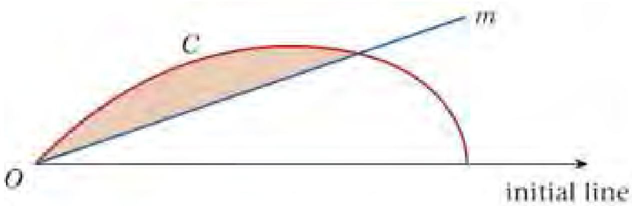
Edexcel AS and A Level Modular Mathematics

Exercise A, Question 51

Question:



The figure shows a curve C with polar equation $r = 4a\cos 2\theta$, $0 \leqslant \theta \leqslant \frac{\pi}{4}$, and a line m with polar equation $\theta = \frac{\pi}{8}$. The shaded region, shown in the figure, is bounded by C and m. Use calculus to show that the area of the shaded region is $\frac{1}{2}a^2(\pi - 2)$.

Solution:

$$A = \frac{1}{2}\int_{\frac{\pi}{8}}^{\frac{\pi}{4}} r^2\,d\theta$$

The lower limit, $\frac{\pi}{8}$, is given by the polar equation of m. The upper limit, $\frac{\pi}{4}$, can be identified from the domain of definition, $0 \leqslant \theta \leqslant \frac{\pi}{4}$ given in the question and the diagram.

$$\frac{1}{2}\int r^2\,d\theta = \frac{1}{2}\int 16a^2\cos^2 2\theta\,d\theta$$

$$= 8a^2\int \cos^2 2\theta\,d\theta$$

$$= 8a^2\int\left(\frac{1}{2} + \frac{1}{2}\cos 4\theta\right)d\theta$$

Using $\cos 2A = 2\cos^2 A - 1$ with $A = 2\theta$.

$$= 4a^2\left[\theta + \frac{\sin 4\theta}{4}\right]$$

$$A = 4a^2\left[\theta + \frac{\sin 4\theta}{4}\right]_{\frac{\pi}{8}}^{\frac{\pi}{4}}$$

$$= 4a^2\left[\left(\frac{\pi}{4} - \frac{\pi}{8}\right) + \left(0 - \frac{1}{4}\right)\right]$$

$\sin\left(4 \times \frac{\pi}{4}\right) = \sin\pi = 0$ and $\sin\left(4 \times \frac{\pi}{8}\right) = \sin\frac{\pi}{2} = 1$.

$$= 4a^2\left[\frac{\pi}{8} - \frac{1}{4}\right]$$

$$= \frac{1}{2}a^2(\pi - 2)$$

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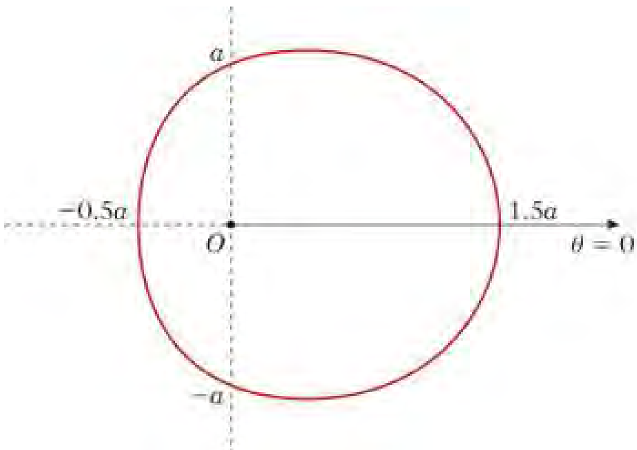
Edexcel AS and A Level Modular Mathematics

Exercise A, Question 52

Question:



The curve shown in the figure has polar equation

$r = a\left(1 + \frac{1}{2}\cos\theta\right), \quad a > 0, \quad 0 < \theta \leq 2\pi.$

Determine the area enclosed by the curve, giving your answer in terms of a and π.

Solution:

$$A = 2 \times \frac{1}{2}\int_0^{\pi} r^2 \, d\theta$$

The method used here is to find twice the area above the initial line.

$$= \int_0^{\pi} a^2\left(1 + \frac{1}{2}\cos\theta\right)^2 d\theta$$

$$= a^2\int_0^{\pi}\left(1 + \cos\theta + \frac{1}{4}\cos^2\theta\right) d\theta$$

Use $\cos 2\theta = 2\cos^2\theta - 1$.

$$= a^2\int_0^{\pi}\left(1 + \cos\theta + \frac{1}{8}\cos 2\theta + \frac{1}{8}\right) d\theta$$

$$= a^2\int_0^{\pi}\left(\frac{9}{8} + \cos\theta + \frac{1}{8}\cos 2\theta\right) d\theta$$

$$= a^2\left[\frac{9}{8}\theta + \sin\theta + \frac{\sin 2\theta}{16}\right]_0^{\pi}$$

As $\sin\pi = \sin 2\pi = 0$ and $\sin 0 = 0$, all of the terms are zero at both the lower and the upper limit except for $\frac{9}{8}\theta$, which has a non-zero value at π.

$$= a^2 \times \frac{9}{8}\pi = \frac{9}{8}\pi a^2$$

Solutionbank FP2
Edexcel AS and A Level Modular Mathematics

Exercise A, Question 53

Question:

a Sketch the curve with polar equation

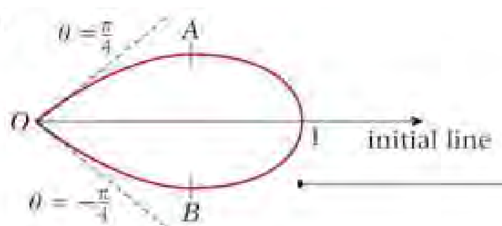
$$r = \cos 2\theta, \ -\frac{\pi}{4} \leqslant \theta \leqslant \frac{\pi}{4}$$

At the distinct points A and B on this curve, the tangents to the curve are parallel to the initial line, $\theta = 0$.

b Determine the polar coordinates of A and B, giving your answers to 3 significant figures.

Solution:

a



At $\theta = -\frac{\pi}{4}$, $r = 0$. As θ increases, r increases until $\theta = 0$. For $\theta = 0$, $\cos 2\theta$ has its greatest value of 1. After that, as θ increases, r decreases to 0 at $\theta = \frac{\pi}{4}$.

b $y = r\sin\theta = \cos 2\theta \sin\theta$

$$\frac{dy}{d\theta} = -2\sin 2\theta \sin\theta + \cos 2\theta \cos\theta = 0$$

$$-4\sin\theta\cos\theta\sin\theta + (1 - 2\sin^2\theta)\cos\theta = 0$$

$$\cos\theta(-4\sin^2\theta + 1 - 2\sin^2\theta) = 0$$

At A and B, $\cos\theta \neq 0$

$$6\sin^2\theta = 1$$

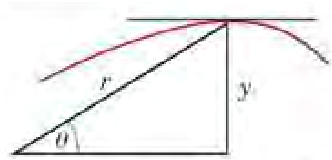
$$\sin\theta = \pm\frac{1}{\sqrt{6}}$$

$$\theta = \pm 0.420\,534\ldots$$

$$r = \cos 2\theta = 1 - 2\sin^2\theta = 1 - \tfrac{2}{6} = \tfrac{2}{3}$$

To 3 significant figures, the polar coordinates of A and B are

$(0.667, 0.421)$ and $(0.667, -0.421)$.



Where the tangent at a point is parallel to the initial line, the distance y from the point to the initial line has a stationary value. The diagram above shows that $y = r\sin\theta$. You find the polar coordinates θ of the points by finding the values of θ for which $r\sin\theta$ has a maximum or minimum value.

r has an exact value but the question specifically asks for 3 significant figures. Unless the questions specifies otherwise, in polar coordinates, you should always give the value of the angle in radians.

Solutionbank FP2
Edexcel AS and A Level Modular Mathematics

Exercise A, Question 54

Question:

a Sketch the curve with polar equation

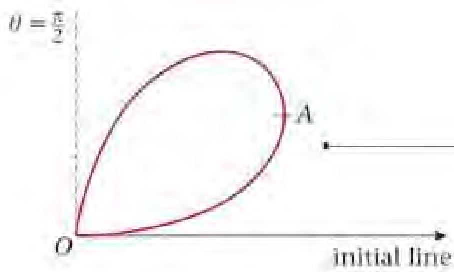
$r = \sin 2\theta,\ 0 \leqslant \theta \leqslant \frac{\pi}{2}$

At the point A, where A is distinct from O, on this curve, the tangent to the curve is parallel to $\theta = \frac{\pi}{2}$.

b Determine the polar coordinates of the point A, giving your answer to 3 significant figures.

Solution:

a



At $\theta = 0$, $r = 0$. As θ increases, r increases until $\theta = \frac{\pi}{4}$. For $\theta = \frac{\pi}{4}$, $\sin 2\theta$ has its greatest value of 1. After that, as θ increases, r decreases to $\sin\left(2 \times \frac{\pi}{2}\right) = \sin \pi = 0$ at $\theta = \frac{\pi}{2}$.

b $x = r\cos\theta = \sin 2\theta \cos\theta$

$\frac{dx}{d\theta} = 2\cos 2\theta\cos\theta - \sin 2\theta \sin\theta$

$= 2(2\cos^2\theta - 1)\cos\theta - 2\sin\theta\cos\theta\sin\theta$

$= 2(2\cos^2\theta - 1)\cos\theta - 2\sin^2\theta\cos\theta$

$= 4\cos^3\theta - 2\cos\theta - 2(1 - \cos^2\theta)\cos\theta$

$= 6\cos^3\theta - 4\cos\theta = 0$

$2\cos\theta(3\cos^2\theta - 2) = 0$

At A, $\cos\theta \neq 0$

$\cos^2\theta = \frac{2}{3}$

$\cos\theta = \left(\frac{2}{3}\right)^{\frac{1}{2}}$, for $0 \leqslant \theta \leqslant \frac{\pi}{2}$

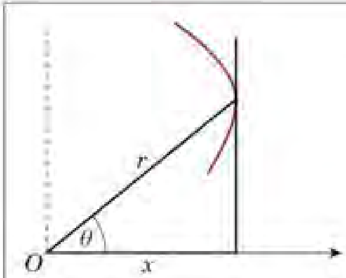
$\theta = 0.615\,479\ldots$

By calculator

$r = \sin 2\theta = 0.942\,809\ldots$

To 3 significant figures, the coordinates of A are

$(0.943, 0.615)$



Where the tangent at a point is parallel to $\theta = \frac{\pi}{2}$ (which is the same as being perpendicular to the initial line), the distance x from the half line $\theta = \frac{\pi}{2}$ has a stationary value. The diagram above shows that $x = r\cos\theta$. You find the polar coordinates θ of such points by finding the values of θ for which $r\cos\theta$ has a maximum or minimum value.

Solutionbank FP2
Edexcel AS and A Level Modular Mathematics

Exercise A, Question 55

Question:

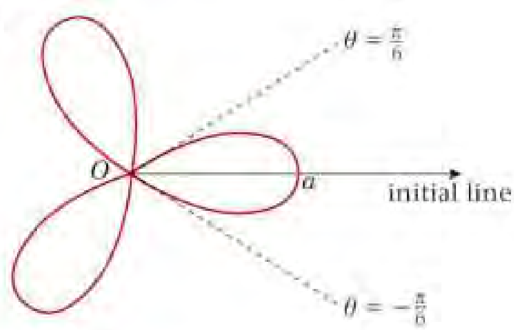
a Sketch the curve with polar equation

$r = a\cos 3\theta, \quad 0 \leq \theta < 2\pi$

b Find the area enclosed by one loop of this curve.

Solution:

a



At $\theta = -\frac{\pi}{6}$, $r = 0$. As θ increases, r increases until $\theta = 0$. For $\theta = 0$, $a\cos 6\theta$ has its greatest value of a. Then, as θ increases, r decreases to 0 at $\theta = \frac{\pi}{6}$. Between $\theta = \frac{\pi}{6}$ and $\theta = \frac{\pi}{2}$, $\cos 6\theta$ is negative and, as $r \geq 0$, the curve does not exist. The pattern repeats itself in the other intervals where the curve exists.

b $A = \frac{1}{2}\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} r^2 \, d\theta$

$$\frac{1}{2}\int a^2\cos^2 3\theta \, d\theta = \frac{a^2}{2}\int \left(\frac{1}{2}\cos 6\theta + \frac{1}{2}\right) d\theta$$

Using $\cos 2A = 2\cos^2 A - 1$ with $A = 3\theta$.

$$= \frac{a^2}{4}\left[\frac{\sin 6\theta}{6} + \theta\right]$$

$\sin\left(6 \times \frac{\pi}{6}\right) = \sin \pi = 0$

$$A = \frac{a^2}{4}\left[\frac{\sin 6\theta}{6} + \theta\right]_{-\frac{\pi}{6}}^{\frac{\pi}{6}} = \frac{a^2}{4}\left[\frac{1}{6}(0 - 0) + \frac{\pi}{6} - \left(-\frac{\pi}{6}\right)\right]$$

$$= \frac{a^2}{4} \times \frac{\pi}{3} = \frac{\pi}{12}a^2$$

Solutionbank FP2
Edexcel AS and A Level Modular Mathematics

Exercise A, Question 56

Question:

The curve C has polar equation

$$r = 6\cos\theta, \qquad -\frac{\pi}{2} \leqslant \theta < \frac{\pi}{2},$$

and the line D has polar equation

$$r = 3\sec\left(\frac{\pi}{3} - \theta\right), \qquad -\frac{\pi}{6} \leqslant \theta < \frac{5\pi}{6}$$

a Find a Cartesian equation of C and a Cartesian equation of D.

b Sketch on the same diagram the graphs of C and D, indicating where each cuts the initial line.

The graphs of C and D intersect at the points P and Q.

c Find the polar coordinates of P and Q.

Solution:

a $r = 6\cos\theta$

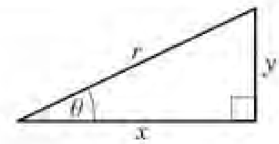
Multiplying the equation by r

$r^2 = 6r\cos\theta$

$x^2 + y^2 = 6x$

$x^2 - 6x + 9 + y^2 = 9$

$(x-3)^2 + y^2 = 9$



This diagram illustrates the relations between polar and Cartesian coordinates. The relations you need to solve the question are

$r^2 = x^2 + y^2$,

$x = r\cos\theta$ and $y = r\sin\theta$.

This is an acceptable answer but putting the equation into a form which shows that the curve is a circle, centre (3, 0) and radius 3, helps you to draw the sketch in part **b**.

$r = 3\sec\left(\frac{\pi}{3} - \theta\right)$

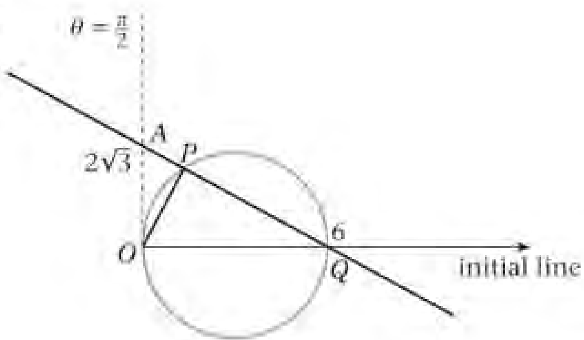
$3 = r\cos\left(\frac{\pi}{3} - \theta\right) = r\cos\frac{\pi}{3}\cos\theta + r\sin\frac{\pi}{3}\sin\theta$

$= \frac{1}{2}r\cos\theta + \frac{\sqrt{3}}{2}r\sin\theta$

$= \frac{1}{2}x + \frac{\sqrt{3}}{2}y$

$x + \sqrt{3}y = 6$

b



The initial line is the positive x-axis and the half-line $u = \frac{\pi}{2}$ is the positive y-axis. At $x = 0$, $x + \sqrt{3}y = 6$ gives $y = \frac{6}{\sqrt{3}} = 2\sqrt{3}$.

The question does not say which point is P and which is Q. You can choose which is which.

c By inspection, the polar coordinates of Q are (6, 0)

$\angle OPQ = 90°$

The angle in a semi-circle is a right angle.

In the triangle OAQ

$\tan AQO = \frac{OA}{OQ} = \frac{2\sqrt{3}}{6} = \frac{\sqrt{3}}{3} \Rightarrow \angle AQO = 30°$

In the triangle OPQ

$OP = OQ\sin PQO = 6\sin 30° = 3$

$\angle POQ = 180° - 90° - 30° = 60° = \frac{\pi}{3}$

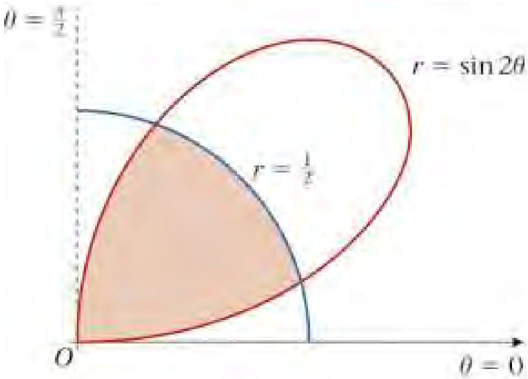
Hence the polar coordinates of P are

$(OP, \angle POQ) = \left(3, \frac{\pi}{3}\right)$

Solutionbank FP2
Edexcel AS and A Level Modular Mathematics

Exercise A, Question 57

Question:



The figure show the half lines $\theta = 0$, $\theta = \frac{\pi}{2}$ and the curves with polar equations

$r = \frac{1}{2}$, $\quad 0 \leqslant \theta \leqslant \frac{\pi}{2}$,

$r = \sin 2\theta$, $\quad 0 \leqslant \theta \leqslant \frac{\pi}{2}$

a Find the exact values of θ at the two points where the curves cross.

b Find by integration the area of the shaded region, shown in the figure, which is bounded by both curves.

Solution:

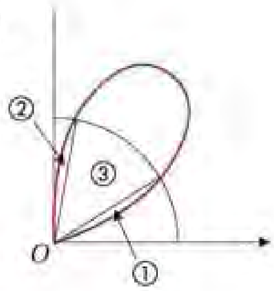
a The curves intersect at

$$\frac{1}{2} = \sin 2\theta$$

$$2\theta = \frac{\pi}{6}, \frac{5\pi}{6}$$

$$\theta = \frac{\pi}{12}, \frac{5\pi}{12}$$

b



The shaded area can be broken up into three parts. You can find the small areas labelled ① and ②, which are equal in area, by integration. The larger area is a sector of a circle and you find this using $A = \frac{1}{2}r^2\theta$, where θ is in radians.

The area of the sector ③ is given by

$$A_3 = \frac{1}{2} \times \left(\frac{1}{2}\right)^2 \times \frac{\pi}{3} = \frac{\pi}{24}$$

The radius of the sector is $\frac{1}{2}$ and the angle is $\frac{5\pi}{12} - \frac{\pi}{12} = \frac{\pi}{3}$.

The area of ① is given by

$$A_1 = \frac{1}{2}\int_0^{\frac{\pi}{12}} r^2 \,\mathrm{d}\theta$$

Using $\cos 2A = 1 - 2\sin^2 A$ with $A = 2\theta$.

$$\frac{1}{2}\int \sin^2 2\theta \,\mathrm{d}\theta = \frac{1}{2}\int\left(\frac{1}{2} - \frac{1}{2}\cos 4\theta\right)\mathrm{d}\theta$$

$$= \frac{1}{4}\left[\theta - \frac{\sin 4\theta}{4}\right]$$

$$A_1 = \frac{1}{4}\left[\theta - \frac{\sin 4\theta}{4}\right]_0^{\frac{\pi}{12}}$$

$$= \frac{1}{4}\left[\frac{\pi}{12} - 0 - \frac{1}{4}\left(\frac{\sqrt{3}}{2} - 0\right)\right]$$

$$= \frac{1}{4}\left[\frac{\pi}{12} - \frac{\sqrt{3}}{8}\right]$$

$\sin\left(4 \times \frac{\pi}{12}\right) = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$

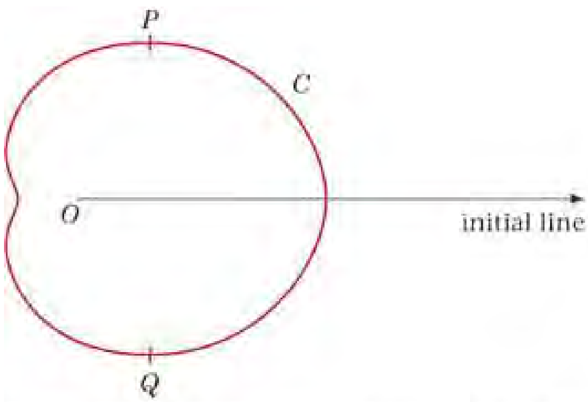
The area of the shaded region is given by

$$2 \times A_1 + A_3 = \frac{1}{2}\left[\frac{\pi}{12} - \frac{\sqrt{3}}{8}\right] + \frac{\pi}{24} = \frac{\pi}{12} - \frac{\sqrt{3}}{16}$$

Solutionbank FP2
Edexcel AS and A Level Modular Mathematics

Exercise A, Question 58

Question:



The curve C, shown in the figure, has polar equation

$r = a(3 + \sqrt{5}\cos\theta)$, $-\pi \leqslant \theta < \pi$

a Find the polar coordinates of the points P and Q where the tangents to C are parallel to the initial line.

The curve C represents the perimeter of the surface of a swimming pool. The direct distance from P to Q is 20 m.

b Calculate the value of a.

c Find the area of the surface of the pool.

Solution:

a Let $y = r\sin\theta$

Where the tangent at a point is parallel to the initial line, the distance y from the point to the initial line has a stationary value. You find the polar coordinate θ of the point by finding the value of θ for which $y = r\sin\theta$ has a stationary value.

$y = a(3 + \sqrt{5}\cos\theta)\sin\theta$

$= 3a\sin\theta + \sqrt{5}a\cos\theta\sin\theta = 3a\sin\theta + \frac{\sqrt{5}a}{2}\sin 2\theta$

$\frac{dy}{d\theta} = 3a\cos\theta + \sqrt{5}a\cos 2\theta = 0$

$3\cos\theta + \sqrt{5}(2\cos^2\theta - 1) = 0$

$2\sqrt{5}\cos^2\theta + 3\cos\theta - \sqrt{5} = 0$

$\cos\theta = \frac{-3 \pm \sqrt{(9 + 40)}}{4\sqrt{5}}$

As $|\cos\theta| \leqslant 1$, you reject the value $-\frac{10}{4\sqrt{5}} \approx -1.118$.

$= \frac{-3 + 7}{4\sqrt{5}} = \frac{1}{\sqrt{5}}$

By calculator

$\theta = \pm 1.107$ (3 d.p.)

At $\cos\theta = \frac{1}{\sqrt{5}}$

$r = a(3 + \sqrt{5}\cos\theta) = a\left(3 + \sqrt{5} \times \frac{1}{\sqrt{5}}\right) = 4a$

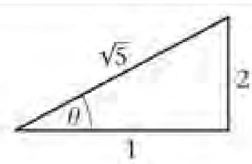
The polar coordinates are

$P{:}(4a, 1.107)$, $Q{:}(4a, -1.107)$

b $PQ = 2y = 2r\sin\theta$

$= 2 \times 4a \times \frac{2}{\sqrt{5}} = \frac{16}{\sqrt{5}}a = 20\,\text{m}$, given

$a = \frac{20\sqrt{5}}{16}\text{m} = \frac{5\sqrt{5}}{4}\text{m}$



As $1^2 + 2^2 = (\sqrt{5})^2$, the diagram illustrates that if $\cos\theta = \frac{1}{\sqrt{5}}$ then $\sin\theta = \frac{2}{\sqrt{5}}$.

c $A = 2 \times \frac{1}{2}\int_0^{\pi} r^2\,d\theta$

The method used here is to find twice the area above the initial line.

$\int a^2(3 + \sqrt{5}\cos\theta)^2\,d\theta = \int a^2(9 + 6\sqrt{5}\cos\theta + 5\cos^2\theta)\,d\theta$

$= a^2\int\left(9 + 6\sqrt{5}\cos\theta + \frac{5}{2}\cos 2\theta + \frac{5}{2}\right)d\theta$

Using $\cos 2\theta = 2\cos^2\theta - 1$.

$= a^2\int\left(\frac{23}{2} + 6\sqrt{5}\cos\theta + \frac{5}{2}\cos 2\theta\right)d\theta$

$= a^2\left[\frac{23}{2}\theta + 6\sqrt{5}\sin\theta + \frac{5}{4}\sin 2\theta\right]$

$A = a^2\left[\frac{23}{2}\theta + 6\sqrt{5}\sin\theta + \frac{5}{4}\sin 2\theta\right]_0^{\pi} = \frac{23\pi}{2}a^2$

You use the value of a you found in part **b**.

$= \frac{23\pi}{2}\left(\frac{5\sqrt{5}}{4}\right)^2\text{m}^2 = \frac{2875\pi}{32}\text{m}^2 \approx 282\,\text{m}^2$

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Edexcel AS and A Level Modular Mathematics

Exercise A, Question 59

Question:

a Sketch the curve with polar equation

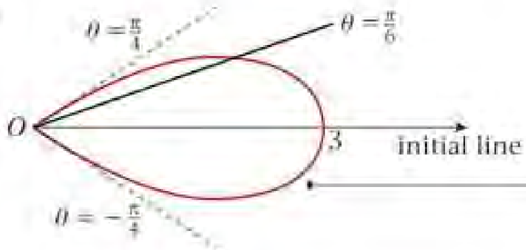
$$r = 3\cos 2\theta,\ -\frac{\pi}{4} \leq \theta < \frac{\pi}{4}$$

b Find the area of the smaller finite region enclosed between the curve and the half-line $\theta = \frac{\pi}{6}$.

c Find the exact distance between the two tangents which are parallel to the initial line.

Solution:

a



At $\theta = -\frac{\pi}{4}$, $r = 0$. As θ increases, r increases until $\theta = 0$. For $\theta = 0$, $3\cos 2\theta$ has its greatest value of 3. After that, as θ increases, r decreases to 0 at $\theta = \frac{\pi}{4}$.

b $A = \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} r^2\,d\theta$

$\frac{1}{2}\int r^2\,d\theta = \frac{1}{2}\int 9\cos^2 2\theta\,d\theta$

Using $\cos 2A = 2\cos^2 A - 1$ with $A = 2\theta$.

$= \frac{9}{2}\int\left(\frac{\cos 4\theta}{2} + \frac{1}{2}\right)d\theta = \frac{9}{4}\int(\cos 4\theta + 1)\,d\theta$

$= \frac{9}{4}\left[\frac{\sin 4\theta}{4} + \theta\right]$

$A = \frac{9}{4}\left[\frac{\sin 4\theta}{4} + \theta\right]_{\frac{\pi}{6}}^{\frac{\pi}{4}}$

$\sin\left(4 \times \frac{\pi}{6}\right) = \sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2}$

$= \frac{9}{4}\left[\frac{1}{4}\left(0 - \frac{\sqrt{3}}{2}\right) + \left(\frac{\pi}{4} - \frac{\pi}{6}\right)\right]$

$= -\frac{9\sqrt{3}}{32} + \frac{3\pi}{16} = \frac{3}{32}(2\pi - 3\sqrt{3})$

c Let $y = r\sin\theta = 3\cos 2\theta\sin\theta$

Where the tangent at a point is parallel to the initial line, the distance y from the point to the initial line has a stationary value. You find the polar coordinate θ of such a point by finding the value of θ for which $y = r\sin\theta$ has a stationary value.

$\frac{dy}{d\theta} = -6\sin 2\theta\sin\theta + 3\cos 2\theta\cos\theta = 0$

$2\sin 2\theta\sin\theta = \cos 2\theta\cos\theta$

$\frac{\sin 2\theta\sin\theta}{\cos 2\theta\cos\theta} = \tan 2\theta\tan\theta = \frac{1}{2}$

$\frac{2\tan^2\theta}{1 - \tan^2\theta} = \frac{1}{2}$

Using $\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}$.

$4\tan^2\theta = 1 - \tan^2\theta$

$5\tan^2\theta = 1$

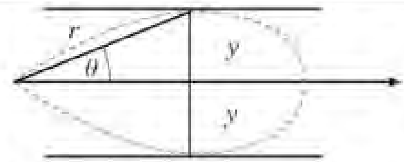
$\tan\theta = \frac{1}{\sqrt{5}}$

One value of $\tan\theta$ is sufficient to complete the question. r is not needed.

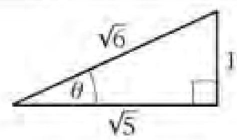
The distance between the two tangents is given by
$2y = 2r\sin\theta = 6\cos 2\theta \sin\theta = 6(2\cos^2\theta - 1)\sin\theta$

$$= 6 \times \left(2 \times \frac{5}{6} - 1\right) \times \frac{1}{\sqrt{6}} = 6 \times \frac{2}{3} \times \frac{1}{\sqrt{6}}$$

$$= \frac{2\sqrt{6}}{3}$$



This sketch shows you that the distance between the two tangents parallel to the initial line is given by $2y = 2r\sin\theta$.



As $(\sqrt{5})^2 + 1^2 = (\sqrt{6})^2$, if $\tan\theta = \frac{1}{\sqrt{5}}$, then $\sin\theta = \frac{1}{\sqrt{6}}$ and $\cos\theta = \frac{\sqrt{5}}{\sqrt{6}}$.

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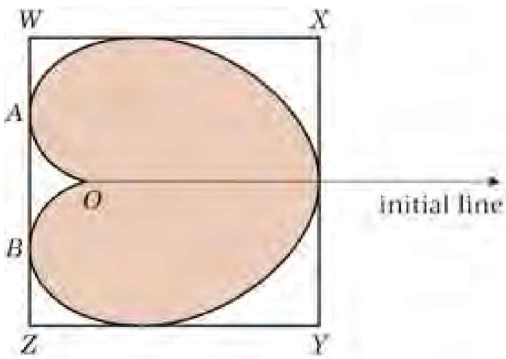
Exercise A, Question 60

Question:



The figure shows a sketch of the cardioid C with equation $r = a(1 + \cos\theta)$, $-\pi < \theta \leqslant \pi$. Also shown are the tangents to C that are parallel and perpendicular to the initial line. These tangents form a rectangle $WXYZ$.

a Find the area of the finite region, shaded in the figure, bounded by the curve C.

b Find the polar coordinates of the points A and B where WZ touches the curve C.

c Hence find the length of WX.

Given that the length of WZ is $\frac{3\sqrt{3}a}{2}$,

d find the area of the rectangle $WXYZ$.

A heart-shape is modelled by the cardioid C, where $a = 10\text{ cm}$. The heart shape is cut from the rectangular card $WXYZ$, shown the figure.

e Find a numerical value for the area of card wasted in making this heart shape.

Solution:

a $A = 2 \times \frac{1}{2}\int_0^\pi r^2\,d\theta$

The total area is twice the area above the initial line.

$$\int r^2\,d\theta = \int a^2(1+\cos\theta)^2\,d\theta = \int a^2(1+2\cos\theta+\cos^2\theta)\,d\theta$$

$$= a^2\int\left(1+2\cos\theta+\frac{1}{2}\cos 2\theta+\frac{1}{2}\right)d\theta$$

$$= a^2\int\left(\frac{3}{2}+2\cos\theta+\frac{1}{2}\cos 2\theta\right)d\theta$$

$$= a^2\left[\frac{3}{2}\theta+2\sin\theta+\frac{1}{4}\sin 2\theta\right]$$

$$A = a^2\left[\frac{3}{2}\theta+2\sin\theta+\frac{1}{4}\sin 2\theta\right]_0^\pi = \frac{3}{2}\pi a^2$$

As $\sin\pi = \sin 2\pi = 0$ and $\sin 0 = 0$, all of the terms are zero at both the lower and the upper limit except for $\frac{3}{2}\theta$, which has a non-zero value at π.

b Let $x = r\cos\theta$

$$= a(1+\cos\theta)\cos\theta = a\cos\theta + a\cos^2\theta$$

$$\frac{dx}{d\theta} = -a\sin\theta - 2a\sin\theta\cos\theta = 0$$

When the tangent at a point is perpendicular to the initial line, you find the polar coordinates θ of the points by finding any values of θ for which $r\cos\theta$ has a stationary value.

$$\sin\theta(2\cos\theta+1) = 0$$

$$\cos\theta = -\frac{1}{2}$$

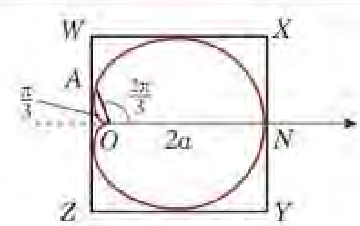
$\sin\theta = 0$ corresponds to the point where XY cuts the curve C and can be rejected as a solution to part **b**.

$$\theta = \pm\frac{2\pi}{3}$$

At A and B

$$r = a(1+\cos\theta) = a\left(1-\frac{1}{2}\right) = \frac{1}{2}a$$

$$A:\left(\frac{1}{2}a, \frac{2\pi}{3}\right),\ B:\left(\frac{1}{2}a, -\frac{2\pi}{3}\right)$$



This sketch illustrates the geometry which is used to solve parts **c** and **d**.

c $WX = AO\cos\frac{\pi}{3} + ON$

$$= \frac{1}{2}a \times \frac{1}{2} + 2a = \frac{9}{4}a$$

d Area of rectangle $WXYZ$ is given by

$$WX \times WZ = \frac{9}{4}a \times \frac{3\sqrt{3}}{2}a = \frac{27\sqrt{3}}{8}a^2$$

e The area wasted is given by

The area wasted is the answer to part **a** subtracted from the answer to part **d**.

$$\frac{27\sqrt{3}}{8}a^2 - \frac{3}{2}\pi a^2 = \left(\frac{27\sqrt{3}}{8} - \frac{3\pi}{2}\right)a^2 = \left(\frac{27\sqrt{3}}{8} - \frac{3\pi}{2}\right)10^2\,\text{cm}^2$$

$$= 113\,\text{cm}^2\ (3\text{ s.f.})$$

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Edexcel AS and A Level Modular Mathematics

Exercise A, Question 61

Question:

a Sketch, on the same diagram, the curves defined by the polar equations $r = a$ and $r = a(1 + \cos\theta)$, where a is a positive constant and $-\pi < \theta \leq \pi$.

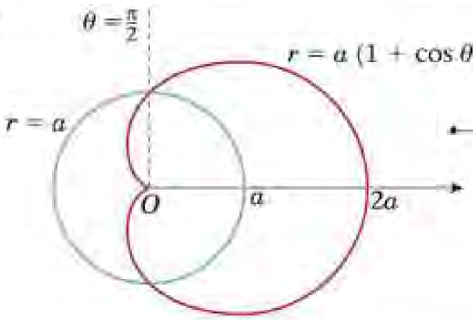
b By considering the stationary values of $r\sin\theta$, or otherwise, find equations of the tangents to the curve $r = a(1 + \cos\theta)$ which are parallel to the initial line.

c Show that the area of the region for which

$a < r < a(1 + \cos\theta)$ is $\dfrac{(\pi + 8)a^2}{4}$.

Solution:

a



$r = a(1 + \cos\theta)$ is a cardioid and $r = a$ is a circle centre O, radius a.

b Let $y = r\sin\theta = a(1 + \cos\theta)\sin\theta$

Where the tangent at a point is parallel to the initial line, the distance y from the point to the initial line has a stationary value. You find the polar coordinates θ of such points by finding the values of θ for which $y = r\sin\theta$ has stationary values.

$= a\sin\theta + a\cos\theta\sin\theta = a\sin\theta + \frac{a}{2}\sin 2\theta$

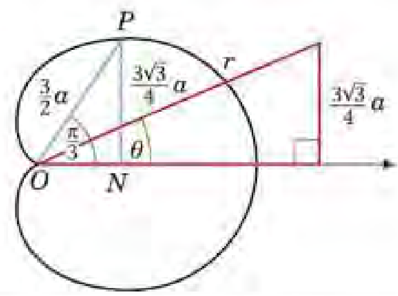
$\frac{dy}{d\theta} = a\cos\theta + a\cos 2\theta = 0$

$\cos 2\theta + \cos\theta = 2\cos^2\theta - 1 + \cos\theta = 0$

$2\cos^2\theta + \cos\theta - 1 = (2\cos\theta - 1)(\cos\theta + 1) = 0$

$\cos\theta = \frac{1}{2}$, $\cos\theta = -1$

$\theta = \pm\frac{\pi}{3}$, $\theta = \pi$



At $\theta = \frac{\pi}{3}$,

$r = a\left(1 + \cos\frac{\pi}{3}\right) = a\left(1 + \frac{1}{2}\right) = \frac{3}{2}a$

And $y = r\sin\frac{\pi}{3} = \frac{3}{2}a \times \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{4}a$

You find the distance (labelled PN in the diagram above) from the point where the tangent meets the curve to the initial line.

The polar equation of the tangent is given by

$r\sin\theta = \frac{3\sqrt{3}}{4}a$

The polar equation is found by trigonometry in the triangle marked in red on the diagram above.

$r = \frac{3\sqrt{3}a}{4}\operatorname{cosec}\theta$

Similarly at $\theta = -\frac{\pi}{3}$, the equation of the tangent is $r = -\frac{3\sqrt{3}a}{4}\operatorname{cosec}\theta$.

At $\theta = \pi$, the equation of the tangent is

$\theta = \pi$.

It is easy to overlook this case. The half-line $\theta = \pi$ does touch the cardioid at the pole.

The circle and the cardioid meet when

$$a = a(1 + \cos\theta) \Rightarrow \cos\theta = \theta$$

$$\theta = \pm\frac{\pi}{2}$$

To find the area of the cardioid between

$\theta = -\frac{\pi}{2}$ and $\theta = \frac{\pi}{2}$

$$A = 2 \times \frac{1}{2}\int_0^{\frac{\pi}{2}} r^2\,d\theta$$

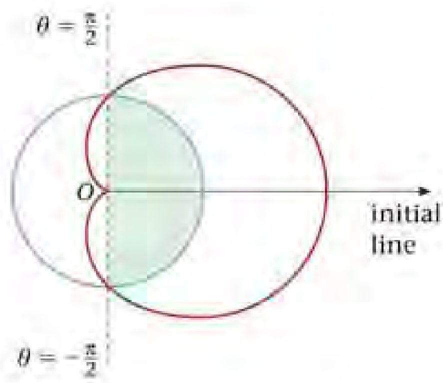
The total area is twice the area above the initial line.

$$\int r^2\,d\theta = \int a^2(1 + \cos\theta)^2\,d\theta = \int a^2(1 + 2\cos\theta + \cos^2\theta)\,d\theta$$

$$= a^2\int\left(1 + 2\cos\theta + \frac{1}{2}\cos 2\theta + \frac{1}{2}\right)d\theta$$

$$= a^2\int\left(\frac{3}{2} + 2\cos\theta + \frac{1}{2}\cos 2\theta\right)d\theta$$

$$= a^2\left[\frac{3}{2}\theta + 2\sin\theta + \frac{1}{4}\sin 2\theta\right]$$



$$A = a^2\left[\frac{3}{2}\theta + 2\sin\theta + \frac{1}{4}\sin 2\theta\right]_0^{\frac{\pi}{2}}$$

$$= a^2\left(\frac{3\pi}{4} + 2\right)$$

The required area is A less half of the circle

$$\left(\frac{3\pi}{4} + 2\right)a^2 - \frac{1}{2}\pi a^2 = \frac{1}{4}\pi a^2 + 2a^2$$

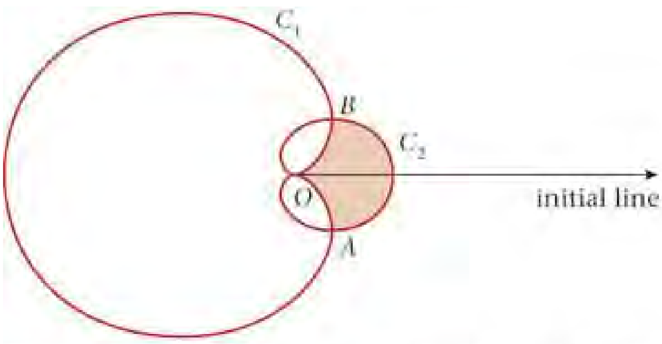
$$= \left(\frac{\pi + 8}{4}\right)a^2, \text{ as required}$$

The area you are asked to find is inside the cardioid and outside the circle. You find it by subtracting the shaded semi-circle from the area of the cardioid bounded by the half-lines $\theta = \frac{\pi}{2}$ and $\theta = -\frac{\pi}{2}$.

Solutionbank FP2
Edexcel AS and A Level Modular Mathematics

Exercise A, Question 62

Question:



The figure is a sketch of two curves C_1 and C_2 with polar equations

$$C_1: r = 3a(1 - \cos\theta), \qquad -\pi \leqslant \theta < \pi$$

and $$C_2: r = a(1 + \cos\theta), \qquad -\pi \leqslant \theta < \pi$$

The curves meet at the pole O and at the points A and B.

a Find, in terms of a, the polar coordinates of the points A and B.

b Show that the length of the line AB is $\frac{3\sqrt{3}}{2}a$.

The region inside C_2 and outside C_1 is shaded in the figure.

c Find, in terms of a, the area of this region.

A badge is designed which has the shape of the shaded region.

Given that the length of the line AB is 4.5 cm,

d calculate the area of this badge, giving your answer to 3 significant figures.

Solution:

a C_1 and C_2 intersect where

$$3\cancel{a}(1 - \cos\theta) = \cancel{a}(1 + \cos\theta)$$

$$3 - 3\cos\theta = 1 + \cos\theta$$

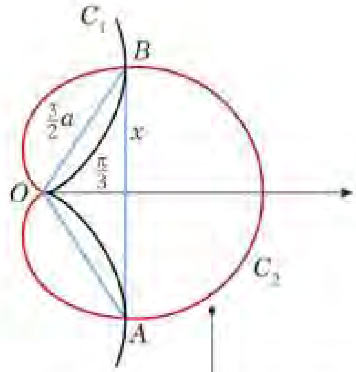
$$4\cos\theta = 2 \Rightarrow \cos\theta = \frac{1}{2}$$

$$\theta = \pm\frac{\pi}{3}$$

Where $\cos\theta = \frac{1}{2}$

$$r = a(1 + \cos\theta) = a\left(1 + \frac{1}{2}\right) = \frac{3}{2}a$$

$$A{:}\left(\frac{3}{2}a, -\frac{\pi}{3}\right),\ B{:}\left(\frac{3}{2}a, \frac{\pi}{3}\right)$$



Referring to the diagram,

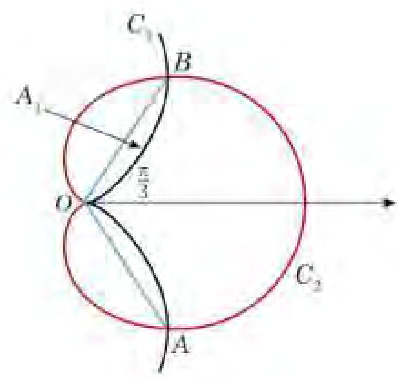
$$\frac{x}{\frac{3}{2}a} = \sin\frac{\pi}{3} \Rightarrow x = \frac{3}{2}a\sin\frac{\pi}{3}$$

and $AB = 2x$.

b $AB = 2 \times \frac{3}{2}a\sin\frac{\pi}{3} = 3a \times \frac{\sqrt{3}}{2}$

$= \frac{3\sqrt{3}}{2}a$, as required

c The area A_1 enclosed by OB and C_1 is given by



$$A_1 = \frac{1}{2}\int_0^{\frac{\pi}{3}} r^2\,d\theta$$

$$\int r^2\,d\theta = \int 9a^2(1 - \cos\theta)^2\,d\theta = \int 9a^2(1 - 2\cos\theta + \cos^2\theta)\,d\theta$$

$$= 9a^2\int\left(1 - 2\cos\theta + \frac{1}{2}\cos 2\theta + \frac{1}{2}\right)d\theta$$

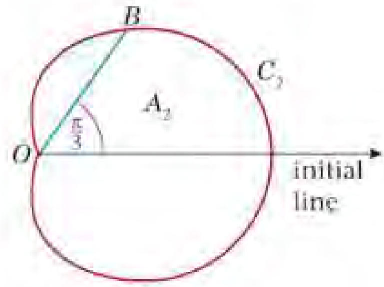
$$= 9a^2\int\left(\frac{3}{2} - 2\cos\theta + \frac{1}{2}\cos 2\theta\right)d\theta$$

$$= 9a^2\left[\frac{3}{2}\theta - 2\sin\theta + \frac{1}{4}\sin 2\theta\right]$$

$$A_1 = \frac{1}{2} \times 9a^2\left[\frac{3}{2}\theta - 2\sin\theta + \frac{1}{4}\sin 2\theta\right]_0^{\frac{\pi}{3}}$$

$$= \frac{9}{2}a^2\left[\frac{\pi}{2} - \sqrt{3} + \frac{\sqrt{3}}{8}\right] = \frac{9a^2}{16}(4\pi - 7\sqrt{3})$$

The area A_2 enclosed by the initial line, C_2 and OB is given by



$$A_2 = \frac{1}{2}\int_0^{\frac{\pi}{3}} r^2\,d\theta$$

$$\int r^2\,d\theta = \int a^2(1+\cos\theta)^2\,d\theta = a^2\int(1+2\cos\theta+\cos^2\theta)\,d\theta$$

$$= a^2\int\left(1+2\cos\theta+\frac{1}{2}\cos 2\theta+\frac{1}{2}\right)d\theta$$

$$= a^2\int\left(\frac{3}{2}+2\cos\theta+\frac{1}{2}\cos 2\theta\right)d\theta$$

$$= a^2\left[\frac{3}{2}\theta+2\sin\theta+\frac{1}{4}\sin 2\theta\right]$$

$$A_2 = \frac{1}{2}\times a^2\left[\frac{3}{2}\theta+2\sin\theta+\frac{1}{4}\sin 2\theta\right]_0^{\frac{\pi}{3}}$$

$$= \frac{a^2}{2}\left[\frac{\pi}{2}+\sqrt{3}+\frac{\sqrt{3}}{8}\right] = \frac{a^2}{16}(4\pi+9\sqrt{3})$$

The required area R is given by

$$R = 2(A_2 - A_1)$$

$$= 2\left[\frac{a^2}{16}(4\pi+9\sqrt{3}) - \frac{9a^2}{16}(4\pi-7\sqrt{3})\right]$$

$$= \frac{2a^2}{16}[4\pi+9\sqrt{3}-(36\pi-63\sqrt{3})]$$

$$= \frac{a^2}{8}[72\sqrt{3}-32\pi] = (9\sqrt{3}-4\pi)a^2$$

d $\frac{3\sqrt{3}}{2}a = 4.5\text{ cm}$

You use the result from part **b** to find a and substitute the value of a into the result of part **c**.

$a = \frac{9}{3\sqrt{3}}\text{ cm} = \sqrt{3}\text{ cm}$

The area of the badge is

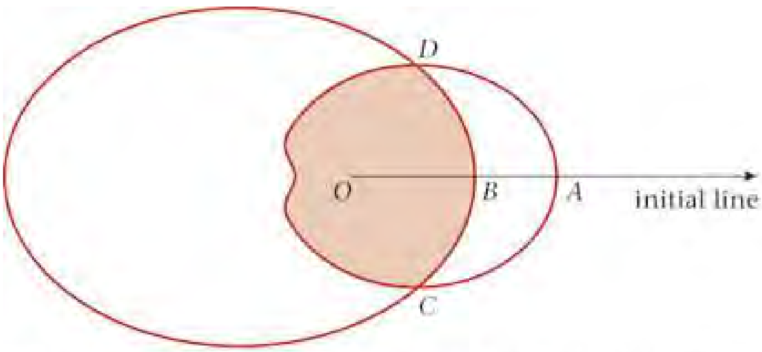
$(9\sqrt{3}-4\pi)a^2 = (9\sqrt{3}-4\pi)\times 3\text{ cm}^2$

$= 9.07\text{ cm}^2$ (3 s.f.)

Solutionbank FP2
Edexcel AS and A Level Modular Mathematics

Exercise A, Question 63

Question:



A logo is designed which consists of two overlapping closed curves.

The polar equations of these curves are

$r = a(3 + 2\cos\theta)$ and

$r = a(5 - 2\cos\theta)$ $0 \leqslant \theta < 2\pi$

The figure is a sketch (not to scale) of these two curves.

a Write down the polar coordinates of the points A and B where the curves meet the initial line.

b Find the polar coordinates of the points C and D where the two curves meet.

c Show that the area of the overlapping region, which is shaded in the figure, is

$$\frac{a^2}{3}(49\pi - 48\sqrt{3})$$

Solution:

a $A{:}(5a, 0)$, $B{:}(3a, 0)$

For A, at $\theta = 0$, $r = a(3 + 2\cos 0) = a(3 + 2) = 5a$.
For B, at $\theta = 0$, $r = a(5 - 2\cos 0) = a(5 - 2) = 3a$.

b The curves intersect where

$$\cancel{a}(3 + 2\cos\theta) = \cancel{a}(5 - 2\cos\theta)$$

$$4\cos\theta = 2 \Rightarrow \cos\theta = \frac{1}{2}$$

$$\theta = \frac{\pi}{3}, \frac{5\pi}{3}$$

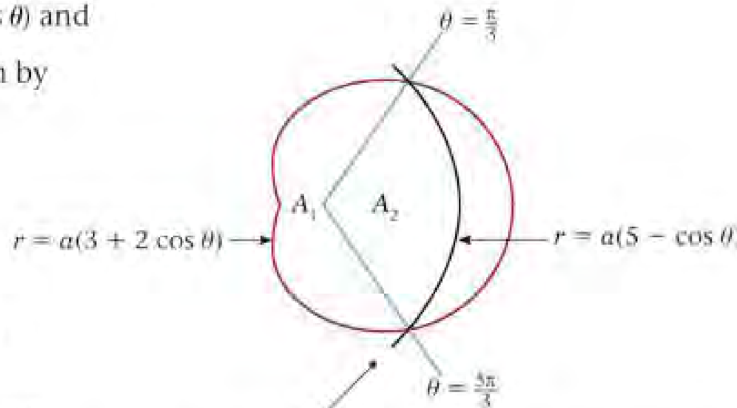
In this question $0 \leqslant \theta < 2\pi$.

Where $\cos\theta = \frac{1}{2}$

$$r = a(3 + 2\cos\theta) = a\left(3 + 2 \times \frac{1}{2}\right) = 4a$$

$$C{:}\left(4a, \frac{5\pi}{3}\right),\ D{:}\left(4a, \frac{\pi}{3}\right)$$

c The area A_1 enclosed by $r = a(3 + 2\cos\theta)$ and the half-lines $\theta = \frac{\pi}{3}$ and $\theta = \frac{5\pi}{3}$ is given by



The shaded area in the question is the sum of the two areas A_1 and A_2 shown in the diagram above. It is important that you carefully distinguish which curve is which.

$$A_1 = 2 \times \frac{1}{2}\int_{\frac{\pi}{3}}^{\pi} r^2\,d\theta$$

$$\int r^2\,d\theta = \int a^2(3 + 2\cos\theta)^2\,d\theta$$

$$= a^2\int(9 + 12\cos\theta + 4\cos^2\theta)\,d\theta$$

$$= a^2\int(9 + 12\cos\theta + 2\cos 2\theta + 2)\,d\theta$$

$$= a^2\int(11 + 12\cos\theta + 2\cos 2\theta)\,d\theta$$

$$= a^2[11\theta + 12\sin\theta + \sin 2\theta]$$

$$A_1 = a^2\Big[11\theta + 12\sin\theta + \sin 2\theta\Big]_{\frac{\pi}{3}}^{\pi}$$

$$= a^2\left[11\left(\pi - \frac{\pi}{3}\right) + 12\left(0 - \frac{\sqrt{3}}{2}\right) + \left(0 - \frac{\sqrt{3}}{2}\right)\right]$$

$$= a^2\left[\frac{22\pi}{3} - \frac{13\sqrt{3}}{2}\right]$$

The area A_2 enclosed by $r = a(5 - 2\cos\theta)$ and the half-lines $\theta = \frac{5\pi}{3}$ and $\theta = \frac{\pi}{3}$ is given by

$$A_2 = 2 \times \frac{1}{2}\int_0^{\frac{\pi}{3}} r^2 \, d\theta$$

$$\int r^2 \, d\theta = \int a^2(5 - 2\cos\theta)^2 \, d\theta = a^2\int (25 - 20\cos\theta + 4\cos^2\theta) \, d\theta$$

$$= a^2\int (25 - 20\cos\theta + 2\cos 2\theta + 2) \, d\theta$$

$$= a^2\int (27 - 20\cos\theta + 2\cos 2\theta) \, d\theta$$

$$= a^2 [27\theta - 20\sin\theta + \sin 2\theta]$$

The double angle formulae, here $\cos 2\theta = 2\cos^2\theta - 1$, are used in all questions involving the areas of cardioids.

$$A_2 = a^2\Big[27\theta - 20\sin\theta + \sin 2\theta\Big]_0^{\frac{\pi}{3}}$$

$$= a^2\left[27 \times \frac{\pi}{3} - 20 \times \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2}\right]$$

$$= a^2\left[\frac{27\pi}{3} - \frac{19\sqrt{3}}{2}\right]$$

The area of the overlapping region is given by

$$A_1 + A_2 = a^2\left(\frac{22\pi}{3} - \frac{13\sqrt{3}}{2} + \frac{27\pi}{3} - \frac{19\sqrt{3}}{2}\right)$$

$$= a^2\left(\frac{49\pi}{3} - 16\sqrt{3}\right)$$

$$= \frac{a^2}{3}(49\pi - 48\sqrt{3}), \text{ as required}$$

Solutionbank FP2
Edexcel AS and A Level Modular Mathematics

Exercise A, Question 64

Question:

The curve C has polar equation $r = 3a\cos\theta$, $-\frac{\pi}{2} \leqslant \theta < \frac{\pi}{2}$. The curve D has polar equation $r = a(1 + \cos\theta)$, $-\pi \leqslant \theta < \pi$. Given that a is positive,

a sketch, on the same diagram, the graphs of C and D, indicating where each curve cuts the initial line.

The graphs of C and D intersect at the pole O and at the points P and Q.

b Find the polar coordinates of P and Q.

c Use integration to find the exact value of the area enclosed by the curve D and the lines $\theta = 0$ and $\theta = \frac{\pi}{3}$.

The region R contains all points which lie outside D and inside C.

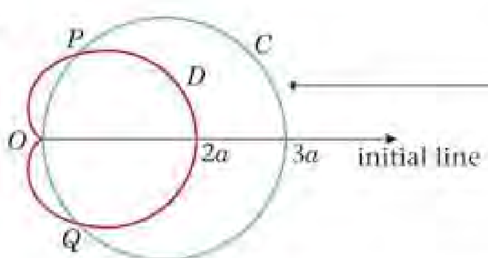
Given that the value of the smaller area enclosed by the curve C and the line $\theta = \frac{\pi}{3}$ is

$\frac{3a^2}{16}(2\pi - 3\sqrt{3})$,

d show that the area of R is πa^2.

Solution:

a



The curve C is a circle of diameter $3a$ and the curve D is a cardioid. The points of intersection of C and D have been marked on the diagram. The question does not specify which is P and which is Q. They could be interchanged. This would make no substantial difference to the solution of the question.

b The points of intersection of C and D are given by

$3\cancel{a}\cos\theta = \cancel{a}(1 + \cos\theta)$

$2\cos\theta = 1 \Rightarrow \cos\theta = \frac{1}{2}$

$\theta = \pm\frac{\pi}{3}$

In this question $-\frac{\pi}{2} \leqslant \theta < \frac{\pi}{2}$.

Where $\cos\theta = \frac{1}{2}$

$r = 3a\cos\frac{\pi}{3} = 3a \times \frac{1}{2} = \frac{3}{2}a$

$P: \left(\frac{3}{2}a, \frac{\pi}{3}\right)$, $Q: \left(\frac{3}{2}a, -\frac{\pi}{3}\right)$

c The area between D, the initial line and OP is given by

$$A_1 = \frac{1}{2}\int_0^{\frac{\pi}{3}} r^2\,\mathrm{d}\theta$$

$$\int r^2\,\mathrm{d}\theta = \int a^2(1+\cos\theta)^2\,\mathrm{d}\theta = a^2\int(1+2\cos\theta+\cos^2\theta)\,\mathrm{d}\theta$$

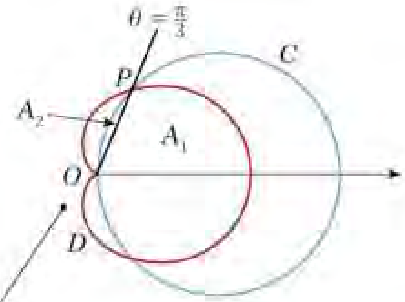
$$= a^2\int\left(1+2\cos\theta+\frac{1}{2}\cos 2\theta+\frac{1}{2}\right)\mathrm{d}\theta$$

$$= a^2\int\left(\frac{3}{2}+2\cos\theta+\frac{1}{2}\cos 2\theta\right)\mathrm{d}\theta$$

$$= a^2\left[\frac{3}{2}\theta+2\sin\theta+\frac{1}{4}\sin 2\theta\right]$$

$$A_1 = \frac{1}{2}\times a^2\left[\frac{3}{2}\theta+2\sin\theta+\frac{1}{4}\sin 2\theta\right]_0^{\frac{\pi}{3}}$$

$$= \frac{a^2}{2}\left[\frac{\pi}{2}+\sqrt{3}+\frac{\sqrt{3}}{8}\right] = \frac{a^2}{16}(4\pi+9\sqrt{3})$$



d Let the smaller area enclosed by C and the half-line $\theta = \frac{\pi}{3}$ be A_2.

$$R = \pi\left(\frac{3a}{2}\right)^2 - 2A_1 - 2A_2$$

By the symmetry of the figure, to find the area inside C but outside D, you subtract two areas A_1 and two areas A_2 from the area inside C. C is a circle of radius $\frac{3a}{2}$.

$$= \frac{9a^2\pi}{4} - \frac{2a^2}{16}(4\pi+9\sqrt{3}) - \frac{6a^2}{16}(2\pi-3\sqrt{3})$$

This is twice the area you are given in the question.

$$= \frac{9a^2\pi}{4} - \frac{\pi a^2}{2} - \cancel{\frac{9\sqrt{3}a^2}{8}} - \frac{3\pi a^2}{4} + \cancel{\frac{9\sqrt{3}a^2}{8}} = \pi a^2, \text{ as required}$$