Review exercise 1

1 The inequality can be solved as follows:

$$\frac{2}{x-2} < \frac{1}{x+1}$$

$$\frac{2}{x-2} - \frac{1}{x+1} < 0$$

$$\frac{2x+2-x+2}{(x-2)(x+1)} < 0$$

$$\frac{x+4}{(x-2)(x+1)} < 0$$

The inequality is satisfied when numerator and denominator do not have the same sign. The numerator is positive for $x > -4$, while the denominator is positive for $x > 2$ or $x < -1$
Therefore, the inequality holds for $x < -4$ or for $-1 < x < 2$

2

$$\frac{x^2}{x-2} > 2x$$

$$\frac{x^2}{x-2} - 2x > 0$$

$$\frac{x^2 - 2x(x-2)}{x-2} > 0$$

$$\frac{4x - x^2}{x-2} > 0$$

$$\frac{x(4-x)}{x-2} > 0$$

You collect the terms together on one side of the inequality, write the expression as a single fraction and factorise the result as far as possible.

Considering $f(x) = \frac{x(4-x)}{x-2}$,

the critical values are $x = 0$, 2 and 4

You find the critical values by solving the numerator equal to zero and the denominator equal to zero. In this case the numerator = 0, gives $x = 0$, 4 and the denominator gives $x = 2$

	$x < 0$	$0 < x < 2$	$2 < x < 4$	$4 < x$
Sign of f(x)	+	–	+	–

For example if $4 < x$, then $\frac{x(4-x)}{x-2} = \frac{\text{positive} \times \text{negative}}{\text{positive}}$, which is negative.

The solution of $\frac{x^2}{x-2} > 2x$ is

$$\{x : x < 0\} \cup \{x : 2 < x < 4\}$$

3 $\dfrac{x^2-12}{x}>1$

Multiply both sides by x^2

x cannot be zero as $\dfrac{x^2-12}{x}$ would be undefined, so x^2 is positive and you can multiply both sides of an inequality by a positive number or expression without changing the inequality. You could **not** multiply both sides of the inequality by x as x could be positive or negative.

$\dfrac{x^2-12}{\cancel{x}}\times x^{\cancel{2}}>x^2$

$x(x^2-12)-x^2>0$

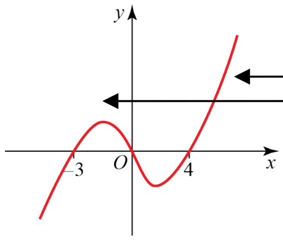
$x^3-12x-x^2>0$

$x(x^2-x-12)>0$

$x(x-4)(x+3)>0$

Sketching $y=x(x-4)(x+3)$

The graph of $y=x(x-4)(x+3)$ crosses the y axis at $x=-3$, 0 and 4

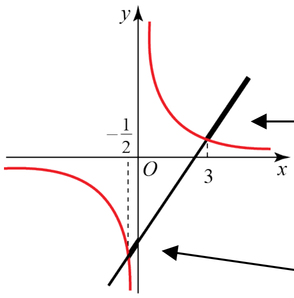


You can see from the sketch that the graph is above the x-axis for $-3<x<0$ and $x>4$
You can then just write down this answer.

If you preferred, you could solve this question using the method illustrated in the solutions to questions 2 and 3 above.

The solution of $\dfrac{x^2-12}{x}>1$ is $\{x:-3<x<0\}\cup\{x:x>4\}$

4



Both $y=2x-5$ and $y=\dfrac{3}{x}$ are straightforward graphs to sketch and so this is a suitable question for a graphical method. The question, however, specifies no method and so you can use any method which gives an exact

After sketching the two graphs, $2x-5>\dfrac{3}{x}$ is the set of values of x for which the line is above the curve. These parts of the line have been drawn thickly on the sketch.

$2x-5=\dfrac{3}{x}$

$x(2x-5)=3$

$2x^2-5x-3=0$

$(2x+1)(x-3)=0$

$x=-\frac{1}{2},3$

You need to find the x-coordinates of the points where the line and curve meet to find two end points of the intervals. The other end point ($x=0$) can be seen by inspecting the sketch.

The solution to $2x-5>\dfrac{3}{x}$ is $\{x:-\frac{1}{2}<x<0\}\cup\{x:x>3\}$

5 $$\frac{x+k}{x+4k} > \frac{k}{x}$$

$$\frac{x+k}{x+4k} - \frac{k}{x} > 0$$

$$\frac{(x+k)x - k(x+4k)}{(x+4k)x} > 0$$

$$\frac{x^2 - 4k^2}{(x+4k)x} > 0$$

$$\frac{(x+2k)(x-2k)}{(x+4k)x} > 0$$

Considering $\text{f}(x) = \frac{(x+2k)(x-2k)}{(x+4k)x}$,

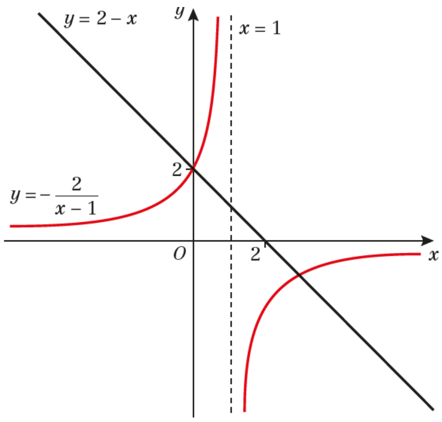
the critical values are $x = -4k, -2k, 0$ and $2k$

For example, when k is positive, in the interval $0 < x < 2k$, $\frac{(x+2k)(x-2k)}{(x+4k)x} = \frac{\text{positive} \times \text{negative}}{\text{positive} \times \text{positive}}$, which is

	$x < -4k$	$-4k < x < -2k$	$-2k < x < 0$	$0 < x < 2k$	$2k < x$
Sign of f(x)	+	–	+	–	+

The solution of $\frac{x+k}{x+4k} > \frac{k}{x}$ is $\{x : x < -4k\} \cup \{x : -2k < x < 0\} \cup \{x : x < 0\} > 2k$

6 a



b The points of intersection are those whose x-coordinate satisfies the equation $2 - x = -\frac{2}{x-1}$

We solve this:

$(2-x)(x-1) = -2$

$(x-2)(x-1) = 2$

$x^2 - 3x + 2 = 2$

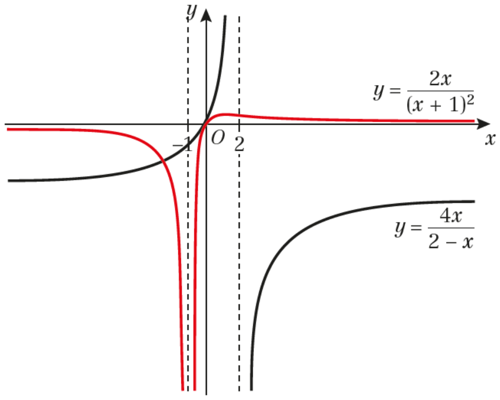
$x^2 - 3x = 0$

which is solved by $x = 0$ and $x = 3$

Both solutions are acceptable. Therefore, the points of intersection are (0, 2) and $(3, -1)$

c It is clear from the graph that the solution to the inequality is $x < 0$ or $1 < x < 3$

7 a



b The points of intersection are those whose x-coordinate satisfies the equation $\frac{4x}{2-x} = \frac{2x}{(x+1)^2}$

We solve this:

$4x(x+1)^2 = 2x(2-x)$

$4x(x^2+2x+1) = 4x-2x^2$

$4x^3+8x^2+4x = 4x-2x^2$

$4x^3+10x^2 = 0$

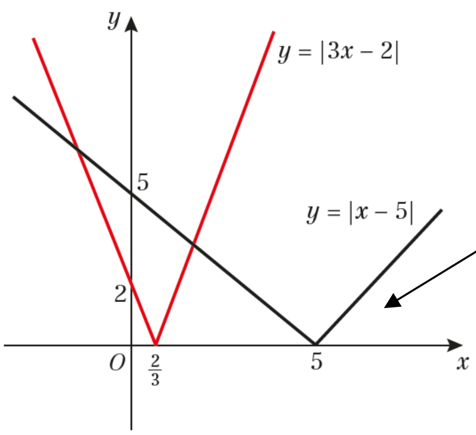
$x^2(2x+5) = 0$

Which is solved by $x = 0$ and $x = -\frac{5}{2}$

These are both acceptable solutions. Therefore, the points of intersection are $(0, 0)$ and $\left(-\frac{5}{2}, -\frac{20}{9}\right)$

c It is clear from the graph that the set of the solutions to the inequality is $\{x : x \leqslant -\frac{5}{2}\} \cup \{x : x = 0\} \cup \{x : x > 2\}$

8 a



You should mark the coordinates of the points where the graphs meet the axes.

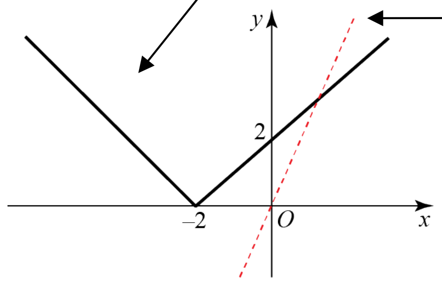
Inequalities which contain both an expression in x with a modulus sign and an expression in x without a modulus sign, are usually best answered by drawing a sketch. In this case, you have been instructed to draw the sketch first. The continuous line is the graph of $y = |x+2|$

You should mark the coordinates of the points where the graph cuts the axis.

b The points of intersection are those whose x-coordinate satisfies the equation $|x-5| = |3x-2|$
For $x > 5$ or $x < \frac{2}{3}$, this is equivalent to $x - 5 = 3x - 2$, which is solved by $x = -\frac{3}{2}$, which is acceptable. For $\frac{2}{3} < x < 5$, it is equivalent to $5 - x = 3x - 2$, which is solved by $x = \frac{7}{4}$, which is also acceptable. Therefore, there are two points of intersection: $\left(-\frac{3}{2}, \frac{13}{2}\right)$ and $\left(\frac{7}{4}, \frac{13}{4}\right)$

8 c It is clear from the graph that the set of the solutions to the inequality is $\left\{x : x < -\frac{3}{2}\right\} \cup \left\{x : x > \frac{7}{4}\right\}$

9 a



Inequalities which contain both an expression in x with a modulus sign and an expression in x without a modulus sign, are usually best answered by drawing a sketch. In this case, you have been instructed to draw the sketch first. The continuous line is the graph of $y = |x+2|$

You should mark the coordinates of the points where the graph cuts the axis.

You should now add the graph $y = 2x$ to your sketch. This has been done with a dotted line. You find the solution to the inequality by identifying the values of x where the dotted line is above the continuous line.

b The intersection occurs when $x > -2$

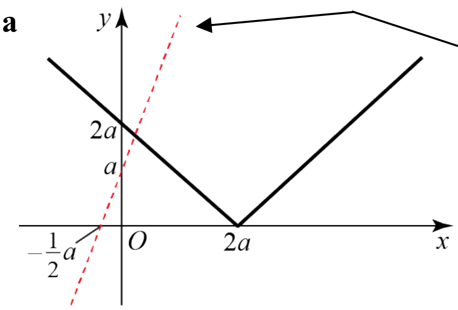
When $-2, |x+2| = x+2$

When f(x) is positive, $|f(x)| = f(x)$

$$2x = x + 2$$
$$x = 2$$

The solution of $2x > |x+2|$ is $x > 2$

10 a



The dotted line is added to the sketch in part **a** to help you to solve part **b**. The dotted line is the graph of $y = 2x + a$ and the solution to the Inequality in part **b** is found by identifying where the continuous line, which corresponds to $|x-2a|$, is above the dotted line, which corresponds to $2x + a$

b The intersection occurs when $x < 2a$

When $x < 2a, |x-2a| = 2a - x$

If f(x) is negative, then $|f(x)| = -f(x)$

$$2a - x = 2x + a$$
$$-3x = -a \Rightarrow x = \tfrac{1}{3}a$$

The solution of $|x-2a| > 2x + a$ is $x < \frac{1}{3}a$

11 We have two cases, depending on the sign of $\frac{x}{x-3}$

If $x>3$ or $x<0$, then it is positive and the inequality becomes $\frac{x}{x-3}<8-x$, which can be solved as follows:

$\frac{x}{x-3}+x-8<0$

$\frac{x+x^2-3x-8x+24}{x-3}<0$

$\frac{x^2-10x+24}{x-3}<0$

$\frac{(x-6)(x-4)}{x-3}<0$

This holds when numerator and denominator do not have the same sign; the numerator is positive for $x>6$ or $x<4$, while the denominator is positive for $x>3$

Therefore, this inequality is solved for $x<3$ or for $4<x<6$; we were looking for solutions with $x>3$ or $x<0$, therefore the set satisfying the first case is $\{x: x<0\}\cup\{x: 4<x<6\}$

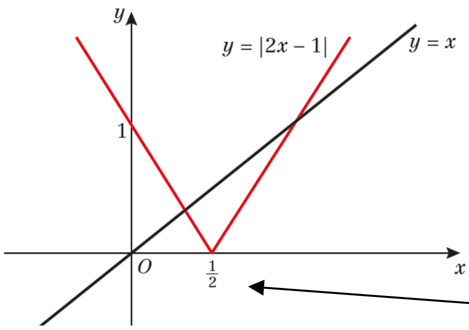
In the second case, we have $0<x<3$

Here, the inequality becomes $\frac{x}{3-x}<8-x$, which leads to $\frac{x^2-12x+24}{x-3}<0$

This is solved by $x<6-2\sqrt{3}$ or $x>3$

Clearly only the first one of these solutions is acceptable. Therefore, the set of the solutions is $\{x: 4<x<6\}\cup\{x: x<6-2\sqrt{3}\}$

12 a



At $x=0, y=|-1|=1$

At $x=\frac{1}{2}, y=0$

b There are two points of intersection.

At the right hand point of intersection,

$x>\frac{1}{2}\Rightarrow|2x-1|=2x-1$

$2x-1=x\Rightarrow x=1$

At the left hand point of intersection,

$x<\frac{1}{2}\Rightarrow|2x-1|=1-2x$

$1-2x=x\Rightarrow x=\frac{1}{3}$

If $f(x)>0$, then $|f(x)|=f(x)$

If $f(x)<0$, then $|f(x)|=-f(x)$

The points of intersection of the two graphs are

$\left(\frac{1}{3},\frac{1}{3}\right)$ and (1, 1)

You need to give both the x-coordinates and the y-coordinates.

c The solution of $|2x-1| > x$ is $\{x : x < \frac{1}{3}\} \cup \{x : x > 1\}$

13
$$|x-3| > 2|x+1|$$
$$(x-3)^2 > 4(x+1)^2$$
$$x^2 - 6x + 9 > 4x^2 + 8x + 4$$
$$0 > 3x^2 + 14x - 5$$
$$(x+5)(3x-1) < 0$$

As both $|x-3|$ and $2|x+1|$ are positive you can square both sides of the inequality without changing the direction of the inequality sign. If a and b are both positive, it is true that $a > b \Rightarrow a^2 > b^2$
You cannot make this step if either or both of a and b are negative.

Considering $f(x) = (x+5)(3x-1)$, the critical values are $x = -5$ and $\frac{1}{3}$

Alternatively you can draw a sketch of $y = (x + 5)(3x - 1)$ and identify the region where the curve is below the y-axis.

	$x < -5$	$-5 < x < \frac{1}{3}$	$\frac{1}{3} < x$
Sign of f(x)	+	–	+

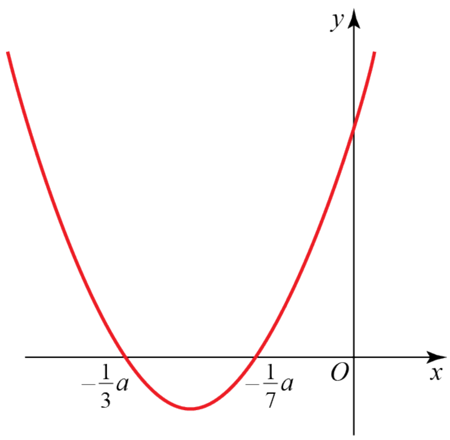
The solution of $|x-3| > 2|x+1|$ is
$\{x : -5 < x < \frac{1}{3}\}$

14
$$|5x+a| \leqslant |2x|$$
$$(5x+a)^2 \leqslant (2x)^2$$
$$25x^2 + 10ax + a^2 \leqslant 4x^2$$
$$21x^2 + 10ax + a^2 \leqslant 0$$
$$(3x+a)(7x+a) \leqslant 0$$

As a is positive, both $|5x+a|$ and $|2x|$ are positive and you can square both sides of the inequality.

Sketching $y = (3x+a)(7x+a)$

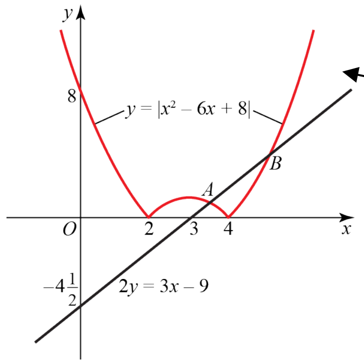
The graph is a parabola intersecting the x-axis at $x = -\frac{1}{3}a$ and $x = -\frac{1}{7}a$



A common error here is not to realise that, for a positive $a, -\frac{1}{3}a$ is a smaller number than $-\frac{1}{7}a$
It is very easy to get the inequality the wrong way round.

The solution of $|5x+a| \leqslant |2x|$ is $-\frac{1}{3}a \leqslant x \leqslant -\frac{1}{7}a$

15 a



As $x^2-6x+8=(x-2)(x-4)$ the curve meets the x-axis at $x = 2$ and $x = 4$
The sketching of the graphs of modulus functions is in Chapter 5 of book C3

The curve meets the x-axis at (2, 0) and (4, 0)
The line meets the x-axis at (3, 0)

15 b To find the coordinates of A. The x-coordinate of A is in the interval $2 < x < 4$

In this interval $x^2 - 6x + 8$ is negative and, hence,

$$|x^2-6x+8| = -x^2+6x-8$$

If $f(x) < 0$, then $|f(x)| = -f(x)$

$$-x^2+6x-8=\frac{3x-9}{2}$$
$$-2x^2+12x-16=3x-9$$
$$2x^2-9x+7=0$$
$$(2x-7)(x-1)=0$$
$$x=\tfrac{7}{2},\ \not{1}$$

As the x-coordinate of A is in the interval $2 < x < 4$, the solutions $x = 1$ must be rejected.

$$y=\frac{3\times\frac{7}{2}-9}{2}=\frac{3}{4}$$

The coordinates of A are $\left(\tfrac{7}{2},\tfrac{3}{4}\right)$,

To find the coordinates of B. The x-coordinate of B is in the interval $x > 4$

In this interval x^2-6x+8 is positive and, hence,

$$|x^2-6x+8| = x^2-6x-8$$

If $f(x) > 0$, then $|f(x)| = f(x)$

$$x^2+6x+8=\frac{3x-9}{2}$$
$$2x^2-12x+16=3x-9$$
$$2x^2-15x+25=0$$
$$(x-5)(2x-5)=0$$
$$x=5,\ \not{\tfrac{5}{2}}$$
$$y=\frac{3\times5-9}{2}=3$$

As the x-coordinate of B is in the interval $x > 4$, the solution $x=2\frac{1}{2}$ must be rejected.

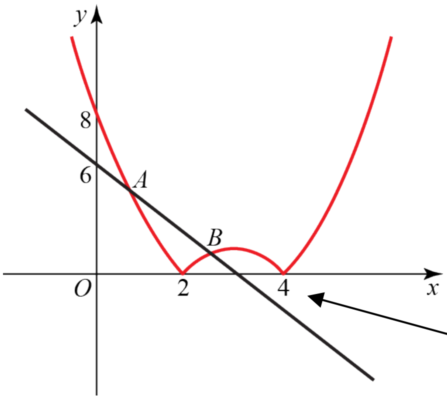
The coordinates of B are (5, 3)

The solution of $2|x^2-6x+8|>3x-9$ is

$x<\tfrac{7}{2}, x>5$

You solve the inequality by inspecting the graphs. You look for the values of x where the curve is above the line.

16 a



You should mark the coordinates of the points where the graphs meet the axes.

16 b Let the points where the graphs intersect be A and B

For A, $(x-2)(x-4)$ is positive

$$(x-2)(x-4)=6-2x$$
$$x^2-6x+8=6-2x$$
$$x^2-4x=-2$$
$$x^2-4x+4=2$$
$$(x-2)^2=2$$
$$x=2-\sqrt{2}$$

The quadratic equations have been solved by completing the square. You could use the formula for solving a quadratic but the conditions of the question require exact solutions and you should not use decimals.

The quadratic equation has another solution $2+\sqrt{2}$ but the diagram shows that the x-coordinate of A is less than 2, so this solution is rejected.

For B, $(x-2)(x-4)$ is negative

$$-(x-2)(x-4)=6-2x$$
$$-x^2+6x-8=6-2x$$
$$x^2-8x=-14$$
$$x^2-8x+16=2$$
$$(x-4)^2=2$$
$$x=4-\sqrt{2}$$

The quadratic equation has another solution $2+\sqrt{2}$ but the diagram shows that the x-coordinate of B is less than 4, so this solution is rejected.

The values of x for which $|(x-2)(x-4)|=6-2x$ are $2-\sqrt{2}$ and $4-\sqrt{2}$

c The solution of $|(x-2)(x-4)|<6-2x$ is $2-\sqrt{2}<x<4-\sqrt{2}$

You look for the values of x where the curve is below the line.

17 a For $x > -2$, $x + 2$ is positive and the equation is

$$\frac{x^2-1}{x+2} = 3(1-x)$$

$$x^2 - 1 = 3(1-x)(x+2) = -3x^2 - 3x + 6$$

$$4x^2 + 3x - 7 = (4x+7)(x-1) = 0$$

$$x = -\tfrac{7}{4}, 1$$

As both of these answers are greater than –2 both are valid.

For $x < -2$, $x + 2$ is negative and the equation is

$$\frac{x^2-1}{-(x+2)} = 3(1-x)$$

$$x^2 - 1 = -3(1-x)(x+2) = 3x^2 + 3x - 6$$

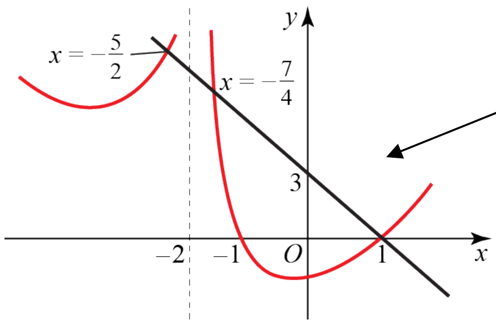
$$2x^2 + 3x - 5 = (2x+5)(x-1) = 0$$

$$x = -\tfrac{5}{2}, \cancel{1}$$

As 1 is not less than –2 the answer 1 should be 'rejected' here. However, the earlier working has already shown 1 to be a correct solution.

The solutions are $-\frac{5}{2}, -\frac{7}{4}$ and 1

17 b



To complete the question, you add the graph of y = 3(1 – x) to the graph which has already been drawn for you. You know the x-coordinates of the points of intersection from part **a**.

The solution of $\dfrac{x^2-1}{|x+2|} < 3(1-x)$ is

$$\{x : x < -\tfrac{5}{2}\} \cup \{x : -\tfrac{7}{4} < x < 1\}$$

You look for the values of x on the graph where the curve is below the line.

18 $$\frac{2}{(r+1)(r+2)} = \frac{A}{r+1} + \frac{B}{r+2}$$

$$2 = A(r+2) + B(r+1)$$

$$2 = A = -B$$

Let $f(r) = \dfrac{1}{r+1}$

$$\sum_{r=1}^{n} \frac{2}{(r+1)(r+2)} = 2\sum_{r=1}^{n}\left(\frac{1}{r+1} - \frac{1}{r+2}\right)$$

$$= 2(f(1) - f(n+1))$$

$$= 2\left(\frac{1}{2} - \frac{1}{n+2}\right)$$

$$= 1 - \frac{2}{n+2} = \frac{n}{n+2}$$

19 $\frac{2}{(r+1)(r+3)} = \frac{A}{r+1} + \frac{B}{r+3}$

$2 = A(r+3) + B(r+1)$

$2 = 2A = -2B$

Let $f(r) = \frac{1}{r+1}$

$$\sum_{r=1}^{n} \frac{2}{(r+1)(r+3)} = \sum_{r=1}^{n}\left(\frac{1}{r+1} - \frac{1}{r+3}\right)$$
$$= f(1) + f(2) - f(n+1) - f(n+2)$$
$$= \frac{1}{2} + \frac{1}{3} - \frac{1}{n+2} - \frac{1}{n+3}$$
$$= \frac{5(n+2)(n+3) - 6(n+3) - 6(n+2)}{6(n+2)(n+3)}$$
$$= \frac{n(5n+13)}{6(n+2)(n+3)}$$

Hence $a = 5, b = 13$ and $c = 6$

20 a $\text{LHS} = \frac{r+1}{r+2} - \frac{r}{r+1}$
$$= \frac{(r+1)^2 - r(r+2)}{(r+1)(r+2)}$$
$$= \frac{r^2 + 2r + 1 - r^2 - 2r}{(r+1)(r+2)}$$
$$= \frac{1}{(r+1)(r+2)}$$
$= \text{RHS}$, as required

To show that an algebraic identity is true, you should start from one side of the identity, here the left hand side (LHS), and use algebra to show that it is equal to the other side of the identity, here the right hand side (RHS).

You use the identity that you proved in part **a** to break up each term in the summation into two parts.

This is the LHS of the identity with $r = 1$.

This is the LHS of the identity with $r = 2$.

This is the LHS of the identity with $r = 3$.

b $\displaystyle\sum_{r=1}^{n}\frac{1}{(r+1)(r+2)}=\sum_{r=1}^{n}\left(\frac{r+1}{r+2}-\frac{r}{r+1}\right)$

$$=\frac{2}{3}-\frac{1}{2}$$
$$+\frac{3}{4}-\frac{2}{3}$$
$$+\frac{4}{5}-\frac{3}{4}$$
$$\vdots$$
$$+\frac{n}{n+1}-\frac{n-1}{n}$$
$$+\frac{n+1}{n+2}-\frac{n}{n+1}$$
$$=\frac{n+1}{n+2}-\frac{1}{2}$$
$$=\frac{2(n+1)-(n+2)}{2(n+2)}=\frac{2n+2-n-2}{2(n+2)}$$
$$=\frac{n}{2(n+2)}$$

This is the LHS of the identity with $r = n - 1$.
$$\frac{r+1}{r+2}-\frac{r}{r+1}=\frac{n-1+1}{n-1+2}-\frac{n-1}{n-1+1}$$
$$=\frac{n}{n+1}-\frac{n-1}{n}$$

This is the LHS of the identity with $r = n$.

The only terms which have not cancelled one another out are the $-\frac{1}{2}$ in the first line of the summation and the $\frac{n+1}{n+2}$ in the last line.

21 a Let $\frac{2}{(x+1)(x+2)(x+3)} = \frac{A}{x+1} + \frac{B}{x+2} + \frac{C}{x+3}$

Multiplying throughout by $(x+1)(x+2)(x+3)$

$2 = A(x+2)(x+3) + B(x+1)(x+3) + C(x+1)(x+2)$

When -1 is substituted for x then both $B(x+1)(x+3)$ and $C(x+1)(x+2)$ become

Substitute $x = -1$

$2 = A \times 1 \times 2 \Rightarrow A = 1$

Substitute $x = -2$

$2 = B \times -1 \times 1 \Rightarrow B = -2$

Substitute $x = -3$

$2 = C \times -2 \times -1 \Rightarrow C = 1$

Hence

$$f(x) = \frac{1}{x+1} - \frac{2}{x+2} + \frac{1}{x+3}$$

You use the partial fractions in part **a** to break up each term in the summation into three parts.

b Using the result in part **a** with $x = r$

$$\sum_{r=1}^{n} f(r) = \frac{1}{r+1} - \frac{2}{r+2} + \frac{1}{r+3}$$

$$= \frac{1}{2} - \frac{2}{3} + \frac{1}{4}$$
$$+ \frac{1}{3} - \frac{2}{4} + \frac{1}{5}$$
$$+ \frac{1}{4} - \frac{2}{5} + \frac{1}{6}$$
$$\vdots$$
$$+ \frac{1}{n-1} - \frac{2}{n} + \frac{1}{n+1}$$
$$+ \frac{1}{n} - \frac{2}{n+1} + \frac{1}{n+2}$$
$$+ \frac{1}{n+1} - \frac{2}{n+2} + \frac{1}{n+3}$$

Three terms at the beginning of the summation and three terms at the end have not been cancelled out.

$$= \frac{1}{2} - \frac{2}{3} + \frac{1}{3} + \frac{1}{n+2} - \frac{2}{n+2} + \frac{1}{n+3}$$
$$= \frac{1}{6} - \frac{1}{n+2} + \frac{1}{n+3}$$

This question asks for no particular form of the answer. You should collect together like terms but, otherwise, the expression can be left as it is. You do not have to express your answer as a single fraction unless the question asks you to do this.

22 a $\frac{1}{(r-1)^2} - \frac{1}{r^2} = \frac{r^2 - (r-1)^2}{r^2(r-1)^2}$

$= \frac{r^2 - (r^2 - 2r + 1)}{r^2(r-1)^2}$

$= \frac{2r-1}{r^2(r-1)^2}$

Methods for simplifying algebraic fractions can be found in Chapter 1 of book C3.

b $\sum_{r=2}^{n} \frac{2r-1}{r^2(r-1)^2} = \sum_{r=2}^{n}\left(\frac{1}{(r-1)^2} - \frac{1}{r^2}\right)$

$= \frac{1}{1^2} - \frac{1}{2^2}$

$+ \frac{1}{2^2} - \frac{1}{3^2}$

$+ \frac{1}{3^2} - \frac{1}{4^2}$

$\vdots$

$+ \frac{1}{(n-2)^2} - \frac{1}{(n-1)^2}$

$+ \frac{1}{(n-1)^2} - \frac{1}{n^2}$

$= \frac{1}{1^2} - \frac{1}{n^2} = 1 - \frac{1}{n^2}$, as required

This summation starts from $r = 2$ and not from the more common $r = 1$. It could not start from $r = 1$ as $\frac{1}{(r-1)^2}$ is not defined for that value.

In this summation all of the terms cancel out with one another except for one term at the beginning and one term at the end.

23 a

$\frac{4}{r(r+2)} = \frac{A}{r} + \frac{B}{r+2}$

$4 = A(r+2) + Br$

$4 = 2A = -2B$

Let $f(r) = \frac{1}{r}$

$\sum_{r=1}^{n} \frac{4}{r(r+2)} = 2\sum_{r=1}^{n}\left(\frac{1}{r} - \frac{1}{r+2}\right)$

$= 2(f(1) + f(2) - f(n+1) - f(n+2))$

$= 2\left(1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2}\right)$

$= \frac{3(n+1)(n+2) - 2(n+2) - 2(n+1)}{(n+1)(n+2)}$

$= \frac{n(3n+5)}{(n+1)(n+2)}$

Hence $a = 3$ and $b = 5$

23 b $\sum_{r=50}^{100} \frac{4}{r(r+2)} = \sum_{r=1}^{100} \frac{4}{r(r+2)} - \sum_{r=1}^{49} \frac{4}{r(r+2)}$

$= \frac{100\times305}{101\times102} - \frac{49\times152}{50\times51}$

$= 2.960\ 590\ldots - 2.920\ 784$

$= 0.0398$ (4 d.p.)

$\sum_{r=50}^{100} f(r) = \sum_{r=1}^{100} f(r) - \sum_{r=1}^{49} f(r)$
You find the sum from the 50th to the 100th term by subtracting the sum from the first to the 49th term from the sum from the first to the 100th term.
It is a common error to subtract one term too many, in this case the 50th term. The sum you are finding starts with the 50th term. You must not subtract it from the series – you have to leave it in the series.

24 a $4r^2 - 1 = (2r-1)(2r+1)$

Let

$\frac{2}{4r^2-1} = \frac{2}{(2r-1)(2r+1)} = \frac{A}{2r-1} + \frac{B}{2r+1}$

Multiply throughout by $(2r-1)(2r+1)$

$2 = A(2r+1) + B(2r-1)$

Substitute $r = \frac{1}{2}$

$2 = 2A \Rightarrow A = 1$

Substitute $r = -\frac{1}{2}$

$2 = -2B \Rightarrow B = -1$

Hence

$\frac{2}{4r^2-1} = \frac{1}{2r-1} - \frac{1}{2r+1}$

$\sum_{r=1}^{n} \frac{2}{4r^2-1} = \sum_{r=1}^{n} \left(\frac{1}{2r-1} - \frac{1}{2r+1}\right)$

$= \frac{1}{1} - \frac{1}{3}$

$+ \frac{1}{3} - \frac{1}{5}$

$+ \frac{1}{5} - \frac{1}{7}$

$\vdots$

$+ \frac{1}{2n-3} - \frac{1}{2n-1}$

$+ \frac{1}{2n-1} - \frac{1}{2n+1}$

$= 1 - \frac{1}{2n+1}$, as required

This question gives you the option to choose your own method (the question has 'or otherwise') and, as you are given the answer, you could, if you preferred, use the method of mathematical induction.
If the method of differences is used, you begin by factorising $4r^2 - 1$, using the difference of two squares, and then express $\frac{2}{(2r-1)(2r+1)}$ in partial fractions.

With $r = 1$,
$\frac{1}{2r-1} - \frac{1}{2r+1} = \frac{1}{2\times1-1} - \frac{1}{2\times1+1} = \frac{1}{1} - \frac{1}{3}$

With $r = n - 1$,
$\frac{1}{2r-1} - \frac{1}{2r+1} = \frac{1}{2\times(n-1)-1} - \frac{1}{2\times(n-1)}$
$= \frac{1}{2n-2-1} - \frac{1}{2n-2+1} = \frac{1}{2n-3} - \frac{1}{2n-1}$

The only terms which are not cancelled out in the summation are the $\frac{1}{1}$ at the beginning and the $-\frac{1}{2n+1}$ at the end.

24 b $\sum_{r=11}^{n}\frac{2}{4r^2-1}=\sum_{r=1}^{20}\frac{2}{4r^2-1}-\sum_{r=1}^{10}\frac{2}{4r^2-1}$

$=\left(1-\frac{1}{41}-1+\frac{1}{21}\right)$

$=-\frac{1}{41}+\frac{1}{21}=\frac{-21+41}{41\times 21}$

$=\frac{20}{861}$

You find the sum from the 11th to the 20th term by subtracting the sum from the first to the 10th term from the sum from the first to the 20th term.

The conditions of the question require an exact answer, so you must not use decimals.

25 a Using the binomial expansion

$(2r+1)^3=8r^3+12r^2+6r+1$ **(1)**

$(2r-1)^3=8r^3-12r^2+6r-1$ **(2)**

Subtracting **(2)** from **(1)**

$(2r+1)^3-(2r-1)^3=24r^2+2$ **(3)**

$A=24,\ B=2$

Subtracting the two expansions gives an expression in r^2. This enables you to sum r^2 using the method of differences.

b Using identity **(3)** in part **a**

$\sum_{r=1}^{n}(24r^2+2)=\sum_{r=1}^{n}\left((2r+1)^3-(2r-1)^3\right)$

$24\sum_{r=1}^{n}r^2+\sum_{r=1}^{n}2=\sum_{r=1}^{n}\left((2r+1)^3-(2r-1)^3\right)$

$\sum_{r=1}^{n}2=\underbrace{2+2+2+\ldots+2}_{n\text{ times}}=2n$

It is a common error to write $\sum_{r=1}^{n}2=2$.

$24\sum_{r=1}^{n}r^2+2n=\cancel{3^3}-1^3$

$+\cancel{5^3}+\cancel{3^3}$

$+\cancel{7^3}-\cancel{5^3}$

$\vdots$

$+\cancel{(2n-1)^3}-\cancel{(2n-3)^3}$

$+(2n+1)^3-\cancel{(2n-1)^3}$

The expression is $(2r+1)^3-(2r-1)^3$ with $n-1$ substituted for r.

$(2(n-1)+1)^3-(2(n-1)-1)^3$

$=(2n-1)^3-(2n-3)^3$

$24\sum_{r=1}^{n}r^2+2n=(2n+1)^3-1$

Summing gives you an equation in $\sum r^2$, which you solve. You then factorise the result to give the answer in the form required by the question.

$24\sum_{r=1}^{n}r^2=8n^2+12n^2+6n+1-1-2n$

$=8n^3+12n^2+4n=4n(2n^2+3n+1)$

$=4n(n+1)(2n+1)$

$\sum_{r=1}^{n}r^2=\frac{4n(n+1)(2n+1)}{24}=\frac{1}{6}n(n+1)(2n+1)$, as required.

25 c $(3r-1)^2 = 9r^2 - 6r + 1$

Hence

$$\sum_{r=1}^{40}(3r-1)^2 = 9\sum_{r=1}^{40}r^2 - 6\sum_{r=1}^{40}r + \sum_{r=1}^{40}1$$

In the formula proved in part **b**, you replace the n by 40.

Using the result in part **b**.

$$9\sum_{r=1}^{40}r^2 = 9\times\tfrac{1}{6}\times 40\times 41\times 81 = 199\ 260$$

Using the standard result $\sum_{r=1}^{n} r = \frac{n(n+1)}{2}$,

$$6\sum_{r=1}^{40} r = 6\times\frac{40\times 41}{2} = 4920$$

$$\sum_{r=1}^{40}1 = 40$$

$\sum_{r=1}^{40}1 = 40$ is 40 ones added together which is, of course, 40.

Combining these results

$$\sum_{r=1}^{40}(3r-1)^2 = 199\ 260 - 4920 + 40 = 194\ 380$$

26 $\frac{1}{r(r+1)(r+2)} = \frac{A}{r} + \frac{B}{r+1} + \frac{C}{r+2}$

$1 = A(r+1)(r+2) + Br(r+2) + Cr(r+1)$

$r = 0 : 1 = 2A$

$r = 1 : 1 = -B$

$r = 2 : 1 = 2C$

Let $\mathrm{f}(r) = \frac{1}{r}$

$$\sum_{r=1}^{2n}\frac{1}{r(r+1)(r+2)} = \frac{1}{2}\sum_{r=1}^{2n}\left(\frac{1}{r} - \frac{1}{r+1} + \frac{1}{r+2} - \frac{1}{r+1}\right)$$

$$= \frac{1}{2}(f(1) - f(2n+1) + f(2n+2) - f(2))$$

$$= \frac{1}{2}\left(1 - \frac{1}{2} - \frac{1}{2n+1} + \frac{1}{2n+2}\right)$$

$$= \frac{1}{4}\left(1 - \frac{2}{2n+1} + \frac{1}{n+1}\right)$$

$$= \frac{1}{4}\left(\frac{(n+1)(2n+1) - 2(n+1) + (2n+1)}{(n+1)(2n+1)}\right)$$

$$= \frac{1}{4}\frac{2n^2+3n}{(n+1)(2n+1)} = \frac{n(2n+3)}{4(n+1)(2n+1)}$$

Hence $a = 2, b = 3$ and $c = 4$

27 a $\text{RHS} = r - 1 + \frac{1}{r} - \frac{1}{r+1}$

$= \frac{(r-1)r(r+1) + (r+1) - r}{r(r+1)}$

$= \frac{r(r^2-1)+1}{r(r+1)}$

$= \frac{r^3 - r + 1}{r(r+1)} = \text{LHS, as required}$

To show that an algebraic identity is true, you should start from one side of the identity, here the right hand side (RHS), and use algebra to show that it is equal to the other side of the identity, here the left hand side (LHS).

b Using the result in part **a**

This summation is broken up into 3 separate summations. Only the third of these uses the method of differences.

$$\sum_{r=1}^{n} \frac{r^3 - r + 1}{r(r+1)} = \sum_{r=1}^{n} r - \sum_{r=1}^{n} 1 + \sum_{r=1}^{n} \left(\frac{1}{r} - \frac{1}{r+1}\right)$$

$$\sum_{r=1}^{n} r = \frac{n(n+1)}{2}$$

$$\sum_{r=1}^{n} 1 = n$$

$$\sum_{r=1}^{n} \left(\frac{1}{r} - \frac{1}{r+1}\right) = \frac{1}{1} - \frac{1}{2}$$

$$+ \frac{1}{2} - \frac{1}{3}$$

$$+ \frac{1}{3} - \frac{1}{4}$$

$$\vdots$$

$$+ \frac{1}{n-1} + \frac{1}{n}$$

$$+ \frac{1}{n} - \frac{1}{n+1}$$

$$= 1 + \frac{1}{n+1}$$

In the summation, using the method of differences, all of the terms cancel out with one another except for one term at the beginning and one term at the end.

Combining the three summations

$$\sum_{r=1}^{n} \frac{r^3 - r + 1}{r(r+1)} = \frac{n(n+1)}{2} - n + 1 - \frac{1}{n+1}$$

$$= \frac{n(n+1)^2 - 2n(n+1) + 2(n+1) - 2}{2(n+1)}$$

$$= \frac{n^3 + 2n^2 + n - 2n^2 - 2n + 2n + 2 - 2}{2(n+1)}$$

$$= \frac{n^3 + n}{2(n+1)} = \frac{n(n^2+1)}{2(n+1)}$$

To complete the question, you put the results of the three summations over a common denominator and simplify the resulting expression as far as possible.

28 $\frac{2r+3}{r(r+1)} = \frac{A}{r} + \frac{B}{r+1}$

$2r+3 = A(r+1) + Br$

$3 = A, 1 = -B$

$\frac{2r+3}{r(r+1)}\frac{1}{3^r} = \frac{1}{3^r}\left(\frac{3}{r} - \frac{1}{r+1}\right)$

$= \frac{1}{3^{r-1}}\frac{1}{r} - \frac{1}{3^r}\frac{1}{r+1}$

Let $f(r) = \frac{1}{3^{r-1}}\frac{1}{r}$

$\sum_{r=1}^{n}\frac{2r+3}{r(r+1)}\frac{1}{3^r} = \sum_{r=1}^{n}\left(f(r) - f(r+1)\right)$

$= (f(1) - f(n+1))$

$= 1 - \frac{1}{3^n(n+1)}$

29 Using Euler's solution $e^{i\theta} = \cos\theta + i\sin\theta$,

$\cos 2x + i\sin 2x = e^{i2x}$

$\cos 9x - i\sin 9x = \cos(-9x) + i\sin(-9x) = e^{i(-9x)}$

For any angle, $\theta, \cos\theta = \cos(-\theta)$ and $-\sin\theta = \sin(-\theta)$ You will find these relations useful when finding the arguments of

Hence

$\frac{\cos 2x + i\sin 2x}{\cos 9x - i\sin 9x} = \frac{e^{i2x}}{e^{i(-9x)}} = e^{i(2x+9x)} = e^{i11x}$

Manipulating the arguments in $e^{i\theta}$ you use the ordinary laws of indices.

$= \cos 11x + i\sin 11x$

This is the required form with $n = 11$.

30 a By de Moivre's theorem

It is sensible to abbreviate $\cos\theta$ and $\sin\theta$ as c and s respectively when you have as many powers of $\cos\theta$ and $\sin\theta$ to write out as you have

$\cos 5\theta + i\sin 5\theta = (\cos\theta + i\sin\theta)^5 = (c + is)^5$, say

$= c^5 + 5c^4 is + 10c^3 i^2 s^2 + 10c^2 i^3 s^3 + 5ci^4 s^4 + i^5 s^5$

$= c^5 + i5c^4 s - 10c^3 s^3 - i10c^2 s^3 + 5cs^4 - is^5$

Use the binomial expansion.

Use $i^2 = -1, i^3 = -i, i^4 = 1$

Equating real parts

$\cos 5\theta = c^5 - 10c^3 s^2 + 5cs^4$

Using $\cos^2\theta + \sin^2\theta = 1$

$\cos 5\theta = c^5 - 10c^3(1 - c^2) + 5c(1 - c^2)^2$

$= c^5 - 10c^3 + 10c^5 + 5c - 10c^3 + 5c^5$

$= 16c^5 - 20c^3 + 5c$

$= 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$, as required

30 b Substitute $x = \cos\theta$ into $16x^5 - 20x^3 + 5x + 1 = 0$

$$16\cos^5\theta - 20\cos^3\theta + 5\cos\theta + 1 = 0$$

$$16\cos^5\theta - 20\cos^3\theta + 5\cos\theta = -1$$

Using the result of part **a**

$$\cos 5\theta = -1$$

$$5\theta = \pi,\ 3\pi,\ 5\pi$$

$$\theta = \frac{\pi}{5},\ \frac{3\pi}{5},\ \frac{5\pi}{5}$$

$$x = \cos\theta = \cos\frac{\pi}{5},\ \cos\frac{3\pi}{5},\ \cos\pi$$

$$= 0.809,\ -0.309,\ -1$$

Additional solutions are found by increasing the angles in steps of 2π. You are asked for 3 answers, so you need 3 angles at this stage.

The two approximate answers are given to 3 decimal places, as the question specified; the remaining answer -1 is exact.

31 a By de Moivre's theorem

$$\cos 5\theta + i\sin 5\theta = (\cos\theta + i\sin\theta)^5 = (c + is)^5, \text{ say}$$

$$= c^5 + 5c^4 is + 10c^3 i^2 s^2 + 10c^2 i^2 s^3 + 5ci^3 s^4 + i^5 s^5$$

$$= c^5 + i5c^4 s - 10c^3 s^2 - i10c^2 s^3 + 5cs^4 - is^5$$

Equating imaginary parts

$$\sin 5\theta = 5c^4 s - 10c^2 s^3 + s^5$$

$$= s(5c^4 - 10c^2 s^2 + s^4)$$

$$= s(5c^4 - 10c^2(1 - c^2) + (1 - c^2)^2)$$

$$= s(5c^4 - 10c^2 + 10c^4 + 1 - 2c^2 + c^4)$$

$$= s(16c^4 - 12c^2 + 1)$$

$$= \sin\theta(16\cos^4\theta - 12\cos^2\theta + 1), \text{ as required}$$

Repeatedly using the identity $\cos^2\theta + \sin^2\theta = 1$, which in this context is $s^2 = 1 - c^2$.

31 b

$$\sin 5\theta + \cos\theta \sin 2\theta = 0$$

$$\sin\theta(16\cos^4\theta - 12\cos^2\theta + 1) + 2\sin\theta\cos^2\theta = 0$$

Using the identity proved in part **a** and the identity $\sin 2\theta = 2\sin\theta\cos\theta$.

$$\sin\theta(16\cos^4\theta - 10\cos^2\theta + 1) = 0$$

$$\sin\theta(2\cos^2\theta - 1)(8\cos^2\theta - 1) = 0$$

Hence $\sin\theta = 0,\ \cos^2\theta = \frac{1}{2},\ \cos^2\theta = \frac{1}{8}$

$$\sin\theta = 0 \Rightarrow \theta = 0$$

$$\cos^2\theta = \frac{1}{2} \Rightarrow \cos\theta = \pm\frac{1}{\sqrt{2}}$$

$$\cos\theta = \frac{1}{\sqrt{2}} \Rightarrow \theta = \frac{\pi}{4}$$

$$\cos\theta = -\frac{1}{\sqrt{2}} \Rightarrow \theta = \frac{3\pi}{4}$$

You must consider the negative as well as the positive square roots.

$$\cos^2\theta = \frac{1}{8} \Rightarrow \cos\theta = \pm\frac{1}{2\sqrt{2}}$$

$$\cos\theta = \frac{1}{2\sqrt{2}} \Rightarrow \theta = 1.209 \text{ (3 d.p.)}$$

$$\cos\theta = -\frac{1}{2\sqrt{2}} \Rightarrow \theta = 1.932 \text{ (3 d.p.)}$$

The question has specified no accuracy and any sensible accuracy would be accepted for the approximate answers.

The solutions of the equation are

$0,\ \frac{\pi}{4},\ \frac{3\pi}{4},\ 1.209$ (3 d.p.) and 1.932 (3 d.p.).

32 a $\sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}$

Let $z = e^{i\theta}$

Putting $z = e^{i\theta}$ shortens the working.

then $\sin\theta = \dfrac{z - z^{-1}}{2i}$

$$\sin^5\theta = \left(\frac{z - z^{-1}}{2i}\right)^5$$

Use Pascal's triangle to remember the coefficients in $(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$.

$$= \frac{1}{(2i)^5}(z^5 - 5z^4 \times z^{-1} + 10z^3 \times z^{-2} - 10z^2 \times z^{-3} + 5z \times z^{-4} - z^{-5})$$

$$= \frac{1}{32i}(z^5 - 5z^3 + 10z - 10z^{-1} + 5z^{-3} - z^{-5})$$

$$= \frac{1}{16}\left(\frac{z^5 - z^{-5}}{2i} - \frac{5(z^3 - z^{-3})}{2i} + \frac{10(z - z^{-1})}{2i}\right)$$

The general relation is $\sin n\theta = \dfrac{e^{in\theta} - e^{-in\theta}}{2i} = \dfrac{z^n - z^{-n}}{2i}$

$$= \frac{1}{16}(\sin 5\theta - 5\sin 3\theta + 10\sin\theta), \text{ as required}$$

b $$\int_0^{\frac{\pi}{2}} \sin^5\theta \, d\theta = \frac{1}{16}\int_0^{\frac{\pi}{2}} (\sin 5\theta - 5\sin 3\theta + 10\sin\theta)\, d\theta$$

Each term on the right hand side of the identity shown in part **a** can be integrated using the formula $\int \sin n\theta \, d\theta = -\dfrac{\cos n\theta}{n}$.

$$= \frac{1}{16}\left[-\frac{1}{5}\cos 5\theta + \frac{5}{3}\cos 3\theta - 10\cos\theta\right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{16}\left(0 - \left(-\frac{1}{5} + \frac{5}{3} - 10\right)\right)$$

$$= \frac{1}{16} \times \frac{128}{15} = \frac{8}{15}, \text{ as required}$$

33 a $z = \cos\theta + i\sin\theta$

Using de Moivre's theorem

$z^n = (\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$ **(1)**

From **(1)**

$$z^{-n} = \frac{1}{z^n} = \frac{1}{\cos n\theta + i\sin n\theta}$$

$$= \frac{1}{\cos n\theta + i\sin n\theta} \times \frac{\cos n\theta - i\sin n\theta}{\cos n\theta - i\sin n\theta}$$

Multiply the numerator and denominator by $\cos n\theta - i\sin n\theta$, the conjugate complex number of $\cos n\theta + i\sin n\theta$.

$$= \frac{\cos n\theta - i\sin n\theta}{\cos^2 n\theta + \sin^2 n\theta} = \cos n\theta - i\sin n\theta \quad \textbf{(2)}$$

Use $\cos^2 n\theta + \sin^2 n\theta = 1$.

$$z^n + z^{-n} = \cos n\theta + i\sin n\theta + \cos n\theta - i\sin n\theta$$
$$= 2\cos n\theta, \text{ as required.}$$

33 b $\cos\theta = \dfrac{e^{i\theta} + e^{i\theta}}{2} = \dfrac{z + z^{-1}}{2}$

$$\cos^6\theta = \left(\frac{z+z^{-1}}{2}\right)^6$$

$$= \frac{1}{64}(z^6 + 6z^5z^{-1} + 15z^4z^{-2} + 20z^3z^{-3} + 15z^2z^{-4} + 6z^1z^{-5} + z^{-6})$$

Pair the terms as shown.

$$= \frac{1}{64}(z^6 + 6z^4 + 15z^2 + 20 + 15z^{-2} + 6z^{-4} + z^{-6})$$

$$= \frac{1}{32}\left(\frac{z^6+z^{-6}}{2} + \frac{6(z^4+z^{-4})}{2} + \frac{15(z^2+z^{-2})}{2} + \frac{20}{2}\right)$$

$$= \frac{1}{32}(\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10)$$

You use $\dfrac{z^n + z^{-n}}{2} = \cos n\theta$ with $n = 6,\ 4$ and 2.

c $\displaystyle\int_0^{\frac{\pi}{2}} \cos^6\theta \, d\theta = \frac{1}{32}\int_0^{\frac{\pi}{2}} (\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10)\, d\theta$

$$= \frac{1}{32} - \left[\frac{1}{6}\sin 6\theta + \frac{6}{4}\sin 4\theta + \frac{15}{2}\sin 2\theta + 10\theta\right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{32} \times 10 \times \frac{\pi}{2} = \frac{5\pi}{32}, \text{ as required}$$

With the exception of 10θ all of these terms have value 0 at both the upper and the lower limit.

34 a Let $4+4\text{i}=r(\cos\theta+\text{i}\sin\theta)=r\cos\theta+\text{i}r\sin\theta$

Equating real parts

$4=r\cos\theta$ **(1)**

Equating imaginary parts

$4=r\sin\theta$ **(2)**

Dividing **(2)** by **(1)**

$\tan\theta=1\Rightarrow\theta=\frac{\pi}{4}$

Substituting $\theta=\frac{\pi}{4}$ into **(1)**

$4=r\cos\frac{\pi}{4}\Rightarrow 4=r\times\frac{1}{\sqrt{2}}\Rightarrow r=4\sqrt{2}$

Hence

$4+4\text{i}=4\sqrt{2}\left(\cos\frac{\pi}{4}+\text{i}\sin\frac{\pi}{4}\right)$

$=2^{\frac{5}{2}}\text{e}^{\text{i}\frac{\pi}{4}}$

$z^5=2^{\frac{5}{2}}\text{e}^{\text{i}\frac{\pi}{4}},\ 2^{\frac{5}{2}}\text{e}^{\text{i}\frac{9\pi}{4}},2^{\frac{5}{2}}\text{e}^{\text{i}\frac{17\pi}{4}},\ 2^{\frac{5}{2}}\text{e}^{\text{i}\frac{25\pi}{4}},\ 2^{\frac{5}{2}}\text{e}^{\text{i}\frac{33\pi}{4}}$

$z=2^{\frac{1}{2}}\text{e}^{\text{i}\frac{\pi}{20}},\ 2^{\frac{1}{2}}\text{e}^{\text{i}\frac{9\pi}{20}},\ 2^{\frac{1}{2}}\text{e}^{\text{i}\frac{17\pi}{20}},\ 2^{\frac{1}{2}}\text{e}^{\text{i}\frac{25\pi}{20}},\ 2^{\frac{1}{2}}\text{e}^{\text{i}\frac{33\pi}{20}}$

This is the required form with $r=\sqrt{2}$ and

$k=\frac{1}{20},\frac{9}{20},\frac{17}{20},\frac{25}{20}\left(=\frac{5}{4}\right),\frac{33}{20}.$

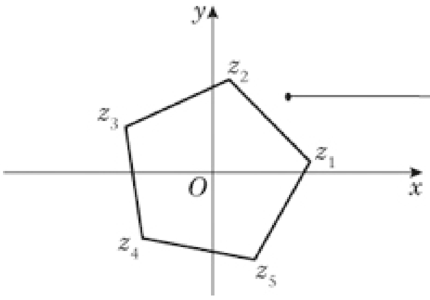
Finding the roots of a complex number is usually easier if you obtain the number in the form $r\text{e}^{\text{i}\theta}$. As you will use Euler's relation, the first step towards this is to get the complex number into the form $r(\cos\theta+\text{i}\sin\theta)$.

To take the fifth root, write $4\sqrt{2}=2^{\frac{5}{2}}$.

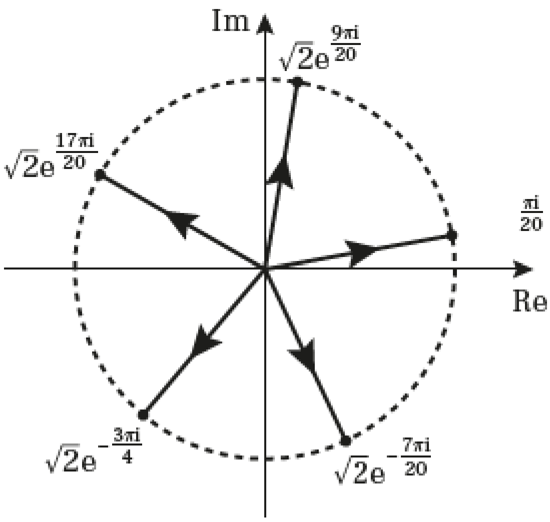
For example, if $z^5=2^{\frac{5}{2}}\text{e}^{\text{i}\frac{9\pi}{4}}$ then

$z=\left(2^{\frac{5}{2}}\text{e}^{\text{i}\frac{9\pi}{4}}\right)^{\frac{1}{5}}=2^{\frac{5}{2}\times\frac{1}{5}}\text{e}^{\text{i}\frac{9\pi}{4}\times\frac{1}{5}}=2^{\frac{1}{2}}\text{e}^{\text{i}\frac{9\pi}{20}}.$

34 b



The points representing the 5 roots are the vertices of a regular pentagon.



35 a Let $32+32\sqrt{3}\mathrm{i}=r(\cos\theta+\mathrm{i}\sin\theta)=r\cos\theta+\mathrm{i}r\sin\theta$

Equating real parts

$32=r\cos\theta$ **(1)**

Equating imaginary parts

$32\sqrt{3}=r\sin\theta$ **(2)**

Dividing **(2)** by **(1)**

$\tan\theta=\sqrt{3}\Rightarrow\theta=\dfrac{\pi}{3}$

Substituting $\theta=\dfrac{\pi}{3}$ into **(1)**

$32=r\cos\dfrac{\pi}{3}\Rightarrow 32=r\times\dfrac{1}{2}\Rightarrow r=64$

Hence

$32+32\sqrt{3}\mathrm{i}=64\left(\cos\dfrac{\pi}{3}+\mathrm{i}\sin\dfrac{\pi}{3}\right)$

$=64\mathrm{e}^{\mathrm{i}\frac{\pi}{3}}$

Finding the roots of a complex number is usually easier if you obtain the number in the form $r\mathrm{e}^{\mathrm{i}\theta}$. As you will use Euler's relation, the first step towards this is to get the complex number into the form $r(\cos\theta+\mathrm{i}\sin\theta)$.

$z^3=64\mathrm{e}^{\mathrm{i}\frac{\pi}{3}},\ 64\mathrm{e}^{\mathrm{i}\frac{7\pi}{3}},\ 64\mathrm{e}^{\mathrm{i}\frac{-5\pi}{3}}$

$z=4\mathrm{e}^{\mathrm{i}\frac{\pi}{9}},\ 4\mathrm{e}^{\mathrm{i}\frac{7\pi}{9}},\ 4\mathrm{e}^{-\mathrm{i}\frac{5\pi}{9}}$

The solutions are $r\mathrm{e}^{\mathrm{i}\theta}$ where $r=4$ and

$\theta=-\dfrac{5\pi}{9},\dfrac{\pi}{9},\dfrac{7\pi}{9}$

Additional solutions are found by increasing or decreasing the arguments in steps of 2π. You area asked for 3 answers, so you need 3 arguments. Had you increased the argument $\dfrac{7\pi}{3}$ by 2π to $\dfrac{13\pi}{3}$, this would have given a correct solution to the equation but it would lead to $\theta=\dfrac{13\pi}{9}$, which does not satisfy the condition $\theta\leq\pi$ in the question. So the third argument has to be found by subtracting 2π from $\dfrac{\pi}{3}$.

b $z=4\mathrm{e}^{\mathrm{i}\frac{\pi}{9}},\ 4\mathrm{e}^{\mathrm{i}\frac{7\pi}{9}},\ 4\mathrm{e}^{-\mathrm{i}\frac{5\pi}{9}}$

$z^9=\left(4\mathrm{e}^{\mathrm{i}\frac{\pi}{9}}\right)^9,\left(4\mathrm{e}^{\mathrm{i}\frac{7\pi}{9}}\right)^9,\left(4\mathrm{e}^{-\mathrm{i}\frac{5\pi}{9}}\right)^9$

$=4^9\mathrm{e}^{\mathrm{i}\pi},\ 4^9\mathrm{e}^{\mathrm{i}7\pi},\ 4^9\mathrm{e}^{-\mathrm{i}5\pi}$

$\mathrm{e}^{\mathrm{i}\pi}=\cos\pi+\mathrm{i}\sin\pi=-1+0=-1$. Similarly for the arguments 7π and -5π.

The value of all three of these expressions is $-4^9=-2^{18}$

Hence the solutions satisfy $z^9+2^k=0$, where $k=18$.

36 $\mathrm{i}=\cos\frac{\pi}{2}+\mathrm{i}\sin\frac{\pi}{2}=\mathrm{e}^{\mathrm{i}\frac{\pi}{2}}$

The modulus of the complex number i is 1 and its argument is $\frac{\pi}{2}$. So $\mathrm{i}=1\mathrm{e}^{\mathrm{i}\frac{\pi}{2}}$.

$z^5=\mathrm{e}^{\mathrm{i}\frac{\pi}{2}},\ \mathrm{e}^{\mathrm{i}\frac{5\pi}{2}},\ \mathrm{e}^{\mathrm{i}\frac{9\pi}{2}},\ \mathrm{e}^{\mathrm{i}\frac{13\pi}{2}},\ \mathrm{e}^{\mathrm{i}\frac{17\pi}{2}},$

Additional solutions are found by increasing the arguments in steps of 2π. As the equation is of degree 5, there are exactly 5 distinct answers.

$z=\mathrm{e}^{\mathrm{i}\frac{\pi}{10}},\ \mathrm{e}^{\mathrm{i}\frac{5\pi}{10}},\ \mathrm{e}^{\mathrm{i}\frac{9\pi}{10}},\ \mathrm{e}^{\mathrm{i}\frac{13\pi}{10}},\ \mathrm{e}^{\mathrm{i}\frac{17\pi}{10}}$

For example, if $z^5=\mathrm{e}^{\mathrm{i}\frac{9\pi}{2}}$ then $z=\left(\mathrm{e}^{\mathrm{i}\frac{9\pi}{2}}\right)^{\frac{1}{5}}=\left(\mathrm{e}^{\mathrm{i}\frac{9\pi}{10}}\right)$.

Hence

$z=\cos\theta+\mathrm{i}\sin\theta$, where

$\theta=\frac{\pi}{10},\ \frac{5\pi}{10}\left(=\frac{\pi}{2}\right),\ \frac{9\pi}{10},\ \frac{13\pi}{10},\ \frac{17\pi}{10}$

37 a $z^5=16+16i\sqrt{3}=32\left(\cos\left(\frac{\pi}{3}+2k\pi\right)+i\sin\left(\frac{\pi}{3}+2k\pi\right)\right)$

as $\sqrt{16^2+\left(16\sqrt{3}\right)^2}=32$, $\arctan\frac{16\sqrt{3}}{16}=\frac{\pi}{3}$

$z=r(\cos\theta+i\sin\theta)$

$z^5=r^5(\cos5\theta+i\sin5\theta)$

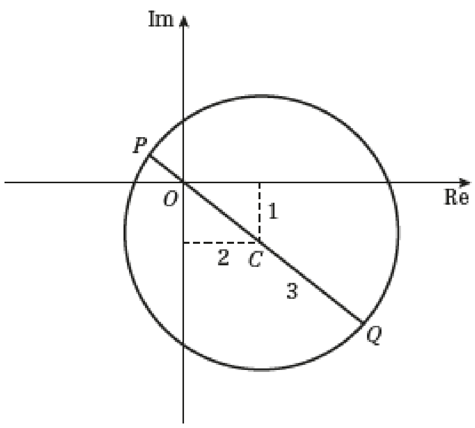
$r^5=32, r=2$

$\theta=\frac{\pi}{15}+\frac{2k\pi}{5}=\frac{\pi}{15},\frac{7\pi}{15},\frac{13\pi}{15},-\frac{\pi}{3},-\frac{11\pi}{15}$

$z=2e^{\frac{i\pi}{15}},2e^{\frac{7i\pi}{15}},2e^{\frac{13i\pi}{15}},2e^{-\frac{i\pi}{3}},2e^{-\frac{11i\pi}{15}}$

b The polygon is a regular pentagon.

38 a $|z-2+\mathrm{i}|=3$ is the circle with centre (2, −1) and radius 3

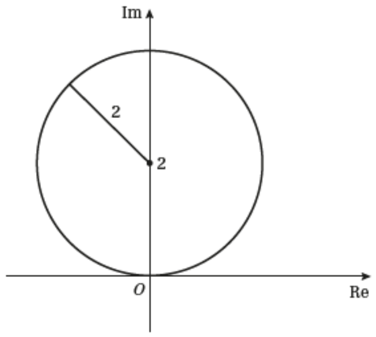


b $|OC|=\sqrt{1^2+2^2}=\sqrt{5}$

Therefore,

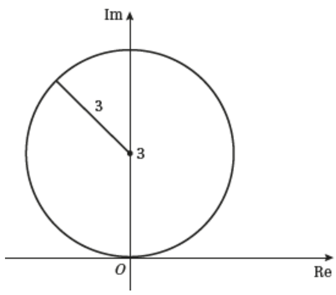
Min $|z|=3-\sqrt{5}$ and Max $|z|=3+\sqrt{5}$

39 a $|z-2i|=2$ is the circle with centre (0, 2) and radius 2

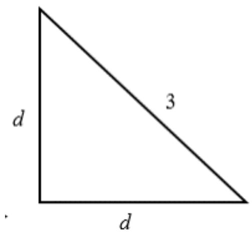


b $|z|_{max}=4$

40 a $|z-3i|=3$ is the circle with centre (0, 3) and radius 3



b $\arg(z-3i)=\dfrac{3\pi}{4}$ is the half-line originating at (0, 3) at $\dfrac{3\pi}{4}$ to the positive x-axis

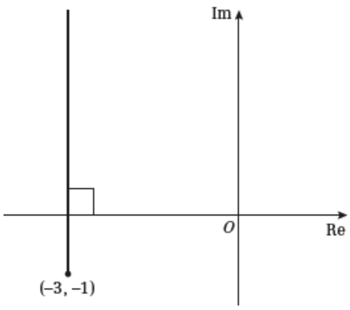


$2d^2=9$

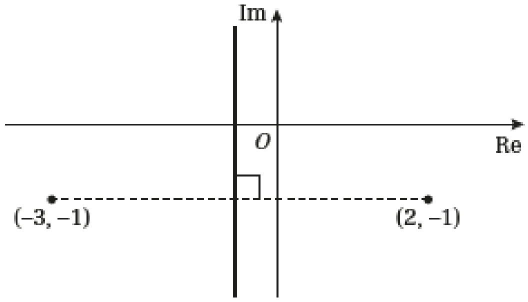
$d=\dfrac{3\sqrt{2}}{2}$

Therefore $z=-\dfrac{3\sqrt{2}}{2}+\left(3+\dfrac{3\sqrt{2}}{2}\right)i$

41 $\arg(z+3+i)=\dfrac{\pi}{2}$ is the half-line originating at (−3, −1) at $\dfrac{\pi}{2}$ to the positive x-axis



42 a $|z+3+\mathrm{i}|=|z-2+\mathrm{i}|$



b $|z|_{\min}=\dfrac{1}{2}$

c $\arg z=-\dfrac{3\pi}{4}$ is the half-line originating at (0, 0) at $-\dfrac{3\pi}{4}$ to the positive x-axis

$\arg z=-\dfrac{3\pi}{4}$ is part of the line $y=x$

Substituting $x=-\dfrac{1}{2}$ into $y=x$ gives $y=-\dfrac{1}{2}$

Therefore $z=-\dfrac{1}{2}-\dfrac{1}{2}\mathrm{i}$

43 a The locus forms a major arc since $\theta = \frac{\pi}{4} < \frac{\pi}{2}$

b Let $z = x + \text{i}y$

Then $\frac{z+\text{i}}{z-\text{i}} = \frac{x+\text{i}(y+1)}{x+\text{i}(y-1)} = \frac{(x+\text{i}(y+1))(x-\text{i}(y-1))}{x^2+(y-1)^2} = \frac{x^2+y^2-1+2\text{i}x}{x^2+(y-1)^2}$

$= \frac{x^2+y^2-1}{x^2+(y-1)^2} + \text{i}\left(\frac{2x}{x^2+(y-1)^2}\right)$

Now $\arg\left(\frac{z+\text{i}}{z-\text{i}}\right) = \frac{\pi}{4}$, so $\tan\left(\arg\left(\frac{z+\text{i}}{z-\text{i}}\right)\right) = \tan\frac{\pi}{4} = 1$

Therefore $\dfrac{\frac{2x}{x^2+(y-1)^2}}{\frac{x^2+y^2-1}{x^2+(y-1)^2}} = 1$

$\frac{2x}{x^2+(y-1)^2} = \frac{x^2+y^2-1}{x^2+(y-1)^2}$

$x^2+y^2-1 = 2x$

$x^2-2x+y^2-1 = 0$

$(x-1)^2-1+y^2-1 = 0$

$(x-1)^2+y^2 = 2$

Hence the centre is at $(1, 0)$

44 a A is represented by the complex number $-1+3\text{i}$.

b The radius of the circle is given by $|\overrightarrow{XA}| = |-1-2\text{i}-(-1+3\text{i})| = |-5\text{i}| = 5$

So the area of the sector is $\frac{1}{2}r^2\theta = \frac{25\pi}{12}$

Substituting $r = 5$ gives $\frac{25\theta}{2} = \frac{25\pi}{12}$

Therefore $\theta = \frac{\pi}{6}$

Now $\overrightarrow{XB} = -5\sin\frac{\pi}{6} + 5\cos\frac{\pi}{6}\text{i} = -\frac{5}{2} + \frac{5\sqrt{3}}{2}\text{i}$

Therefore $b = \overrightarrow{OB} = -1-2\text{i} - \frac{5}{2} + \frac{5\sqrt{3}}{2}\text{i}$

$= -\frac{7}{2} + \frac{5\sqrt{3}-4}{2}\text{i}$

45 Let $z = x + \mathrm{i}y$

$|z-1| = \sqrt{2}|z-\mathrm{i}|$

$|(x-1)+\mathrm{i}y| = \sqrt{2}|x+(y-1)\mathrm{i}|$

Squaring the modulus gives

$|x-1+\mathrm{i}y|^2 = 2|x+(y-1)\mathrm{i}|^2$

$(x-1)^2 + y^2 = 2\left(x^2+(y-1)^2\right)$

$x^2 - 2x + 1 + y^2 = 2x^2 + 2y^2 - 4y + 2$

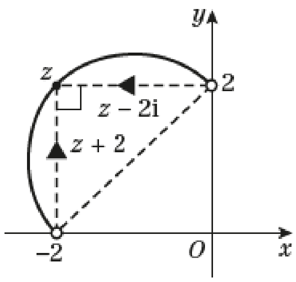
$x^2 + 2x + y^2 - 4y + 1 = 0$

$(x+1)^2 - 1 + (y-2)^2 - 4 + 1 = 0$

$(x+1)^2 + (y-2)^2 = 4$

Hence the circle has radius 2 and centre $(-1, 2)$

46 a

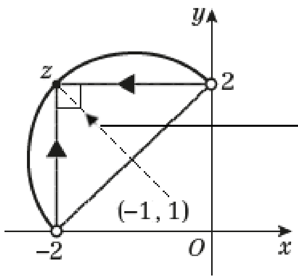


$\arg\left(\frac{z-2\mathrm{i}}{z+2}\right) = \arg(z-2\mathrm{i}) - \arg(z+2) = \frac{\pi}{2}$.

The angles which the vectors make with the positive x-axis differ by a right angle. As drawn here, the difference is $\pi - \frac{\pi}{2} = \frac{\pi}{2}$. The locus of the points, where the difference is a right angle, is a semi-circle, with the line joining –2 on the real axis to 2 on the imaginary axis as diameter.

It is a common error to complete the circle. The lower right hand completion of the circle has equation $\arg\left(\frac{z-2\mathrm{i}}{z+2}\right) = -\frac{\pi}{2}$.

b



The dotted line represents the complex number $z+1-\mathrm{i} = z-(-1+\mathrm{i})$. The length of this vector is the radius of the circle.

The diameter of the circle is given by $d^2 = 2^2 + 2^2 = 8$, so $d = 2\sqrt{2}$

Therefore $|z+1-\mathrm{i}| = \frac{2\sqrt{2}}{2} = \sqrt{2}$

47 a Both loci L and M are circles, hence they are similar.

47 b Computing the scale factor of enlargement amounts to computing the radii of both circles.

For L we have:

$$|z-4| = \sqrt{5}|z+2\mathrm{i}|$$
$$|(x-4)+\mathrm{i}y| = \sqrt{5}|x+(y+2)\mathrm{i}|$$
$$|(x-4)+\mathrm{i}y|^2 = 5|x+(y+2)\mathrm{i}|^2$$
$$(x-4)^2 + y^2 = 5\left(x^2+(y+2)^2\right)$$
$$x^2 - 8x + 16 + y^2 = 5x^2 + 5y^2 + 20y + 20$$
$$4x^2 + 8x + 4y^2 + 20y + 4 = 0$$
$$x^2 + 2x + y^2 + 5y + 1 = 0$$
$$(x+1)^2 - 1 + \left(y+\tfrac{5}{2}\right)^2 - \tfrac{25}{4} + 1 = 0$$

Which simplifies to

$$(x+1)^2 + \left(y+\tfrac{5}{2}\right)^2 = \tfrac{25}{4}$$

Hence the radius of L is $\frac{5}{2}$

For M we have:

$$|z-6| = \sqrt{7}|z+6\mathrm{i}|$$
$$|(x-6)+\mathrm{i}y| = \sqrt{7}|x+\mathrm{i}(y+6)|$$
$$|(x-6)+\mathrm{i}y|^2 = 7|x+(y+6)\mathrm{i}|^2$$
$$(x-6)^2 + y^2 = 7x^2 + 7(y+6)^2$$
$$x^2 - 12x + 36 + y^2 = 7x^2 + 7y^2 + 84y + 252$$
$$6x^2 + 12x + 6y^2 + 84y + 216 = 0$$
$$x^2 + 2x + y^2 + 14y + 36 = 0$$
$$(x+1)^2 - 1 + (y+7)^2 - 49 + 36 = 0$$
$$(x+1)^2 + (y+7)^2 = 14$$

Hence this circle has radius $\sqrt{14}$

So the scale factor of enlargement is $\dfrac{\sqrt{14}}{\frac{5}{2}} = \dfrac{2\sqrt{14}}{5}$

48 The locus is given by

$$\arg\left(\frac{z+1}{z}\right)=\frac{\pi}{4}$$

Substituting $z=x+\mathrm{i}y$

$$\frac{z+1}{z}=\frac{x+1+\mathrm{i}y}{x+\mathrm{i}y}=\frac{(x+1+\mathrm{i}y)(x-\mathrm{i}y)}{(x+\mathrm{i}y)(x-\mathrm{i}y)}=\frac{x^2+x+\mathrm{i}xy-\mathrm{i}xy-\mathrm{i}y+y^2}{x^2+y^2}$$

$$=\frac{x^2+x+y^2-\mathrm{i}y}{x^2+y^2}=\frac{x^2+x+y^2}{x^2+y^2}+\frac{-y}{x^2+y^2}\mathrm{i}$$

Now $\arg\left(\frac{z+1}{z}\right)=\frac{\pi}{4}$, so $\tan\left(\arg\left(\frac{z+1}{z}\right)\right)=\tan\frac{\pi}{4}=1$

So $\dfrac{\frac{-y}{x^2+y^2}}{\frac{x^2+x+y^2}{x^2+y^2}}=1$

$$-y=x^2+x+y^2$$

$$x^2+x+y^2+y=0$$

$$\left(x+\tfrac{1}{2}\right)^2-\tfrac{1}{4}+\left(y+\tfrac{1}{2}\right)^2-\tfrac{1}{4}=0$$

$$\left(x+\tfrac{1}{2}\right)^2+\left(y+\tfrac{1}{2}\right)^2=\tfrac{1}{2}$$

So the centre of the circle is $\left(-\tfrac{1}{2},-\tfrac{1}{2}\right)$ and the radius is $\frac{1}{\sqrt{2}}$

Therefore, this is the major arc of a circle.

The length of the curve required is $r(2\pi-\theta)$, where θ is the angle between the lines connecting the endpoints of the arc to the centre of the circle.

Geometrically we can see that $\tan\frac{\theta}{2}=\frac{\frac{1}{2}}{\frac{1}{2}}=1$, so $\theta=\frac{\pi}{2}$

Therefore the length of the arc is $\frac{1}{\sqrt{2}}\left(2\pi-\frac{\pi}{2}\right)=\frac{1}{\sqrt{2}}\left(\frac{3\pi}{2}\right)=\frac{3\pi}{2\sqrt{2}}$

49 We consider the locus $|z-\mathrm{i}|=\sqrt{p}|z+1|$

Squaring gives $|z-\mathrm{i}|^2=p|z+1|^2$

Substituting $z=x+\mathrm{i}y$:

$$|x+(y-1)\mathrm{i}|^2=p|(x+1)+\mathrm{i}y|^2$$

$$x^2+(y-1)^2=p\left((x+1)^2+y^2\right)$$

$$x^2+y^2-2y+1=px^2+2px+p+py^2$$

$$(p-1)x^2+2px+2y+(p-1)y^2+p-1=0$$

$$x^2+\frac{2px}{p-1}+y^2+\frac{2y}{p-1}+1=0$$

$$\left(x+\frac{p}{p-1}\right)^2-\frac{p^2}{(p-1)^2}+\left(y+\frac{1}{p-1}\right)^2-\frac{1}{(p-1)^2}+1=0$$

$$\left(x+\frac{p}{p-1}\right)^2+\left(y+\frac{1}{p-1}\right)^2=\frac{p^2}{(p-1)^2}+\frac{1}{(p-1)^2}-1$$

$$\left(x+\frac{p}{p-1}\right)^2+\left(y+\frac{1}{p-1}\right)^2=\frac{p^2+1-(p-1)^2}{(p-1)^2}$$

$$\left(x+\frac{p}{p-1}\right)^2+\left(y+\frac{1}{p-1}\right)^2=\frac{2p}{(p-1)^2}$$

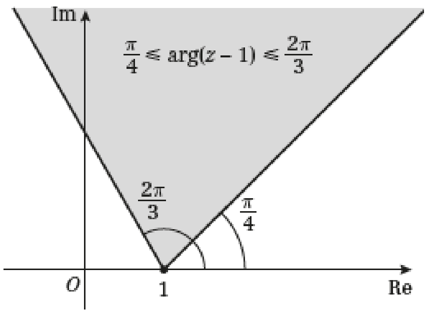
Hence the radius is $\dfrac{\sqrt{2p}}{p-1}$

For a circumference of 24π the radius is 12

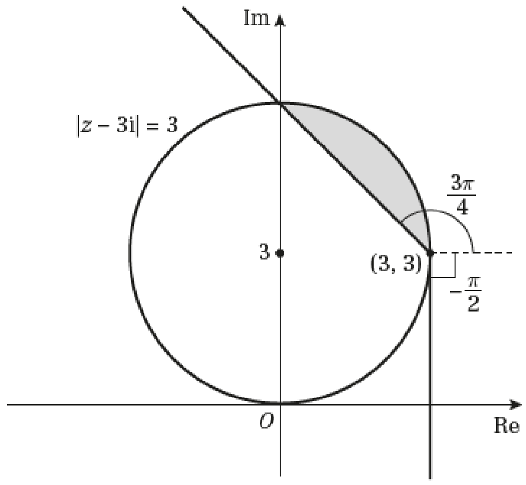
Hence $\dfrac{\sqrt{2p}}{p-1}=12$

$$2p=(12p-12)^2$$

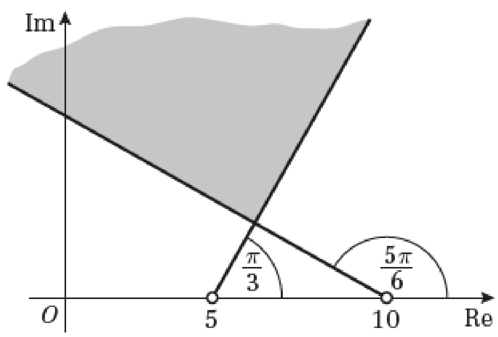
50 $\dfrac{\pi}{4}\le\arg(z-1)\le\dfrac{2\pi}{3}$



51 $\left\{z \in \mathbb{C} : -\frac{\pi}{2} < \arg(z-3-3\mathrm{i}) \leq \frac{3\pi}{4}\right\} \cap \left\{z \in \mathbb{C} : |z-3\mathrm{i}| \leq 3\right\}$



52



53 a The locus is given by the inequalities $|z-6\mathrm{i}| \leqslant 2|z-3|$ and $\mathrm{Re}(z) \leqslant k$

Taking the first inequality:

$|z-6\mathrm{i}|^2 \leqslant 4|z-3|^2$

$|x+(y-6)\mathrm{i}|^2 \leqslant 4|x-3+\mathrm{i}y|^2$

$x^2+(y-6)^2 \leqslant 4\left((x-3)^2+y^2\right)$

$x^2+y^2-12y+36 \leqslant 4x^2-24x+36+4y^2$

$0 \leqslant 3x^2-24x+3y^2+12y$

$0 \leqslant x^2-8x+y^2+4y$

$x^2-8x+y^2+4y \geqslant 0$

$(x-4)^2-16+(y+2)^2-4 \geqslant 0$

$(x-4)^2+(y+2)^2 \geqslant 20$

Hence the circle has centre $(4,-2)$

Hence, for a semi-circle, we should take $k=4$

b The area of the semicircle is $\frac{\pi r^2}{2} = \frac{\pi \times 20}{2} = 10\pi$

54 The line corresponding to $|z-p|=|z-q|$ is given by the perpendicular bisector of p and q,

that is $x=\dfrac{p+q}{2}$

The area of the triangular region is therefore $\dfrac{1}{2}\times\left(\dfrac{q-p}{2}\right)^2=x$

So $(q-p)^2=8x$

$q-p=\sqrt{8x}$

So $q=p+\sqrt{8x}$, as required

55 a The transformation defined by $w=3z+4-2\text{i}$ represents a scaling by 3, followed by a translation by the complex number $4-2\text{i}$.
The translation leaves the area of the triangle invariant.
Therefore the new area is $3^2\times 8=72$.

55 b We consider what happens to the line $\text{Im}(z)=4$ under the transformation.
Consider a point $z=x+4\text{i}$ on the line.
This is mapped to $w=3(x+4\text{i})+4-2\text{i}=3x+4+10\text{i}$.
Hence the line is mapped to the line $\text{Im}(z)=10$.

56 $w=\dfrac{2z-1}{z-2}\Rightarrow wz-2w=2z-1$

$wz-2z=2w-1\Rightarrow z(w-2)=2w-1$

$z=\dfrac{2w-1}{w-2}$

$|z|=1\Rightarrow\left|\dfrac{2w+1}{w-2}\right|=1$

$|2w-1|=|w-2|$

You know that $|z|=1$ and you are trying to find out about w. So it is a good idea to change the subject of the formula to z. You can then put the modulus of the right hand side of the new formula, which contains w, equal to 1.

Let $w=u+\text{i}v$

$|2(u+\text{i}v)-1|=|u+\text{i}v-2|$

$|(2u-1)+\text{i}2v|=|(u-2)+\text{i}v|$

$|(2u-1)+\text{i}2v|^2=|(u-2)+\text{i}v|^2$

$(2u-1)^2+4v^2=(u-2)^2+v^2$

$4u^2-4u+1+4v^2=u^2-4u+4+v^2$

$3u^2+3v^2=3\Rightarrow u^2+v^2=1$

It is not easy to interpret this locus geometrically and so it is sensible to transform the problem into algebra, using the rule that if $z=x+\text{i}y$, then $|z|^2=x^2+y^2$.

This is a circle centre O, radius 1 and has the equation $|w|=1$ in the Argand plane.

Hence, the circle $|z|=1$ is mapped onto the circle $|w|=1$, as required.

57 a $z = x + \frac{1}{2}\text{i}$

The real part of a complex number on Im $z = \frac{1}{2}$ can have any real value, which you can represent by the symbol x, but the imaginary part must be $\frac{1}{2}$.

$$w = \frac{z - \text{i}}{z}$$

$$zw = z - \text{i} \Rightarrow z - wz = \text{i}$$

$$z = \frac{\text{i}}{1 - w}$$

Let $w = u + \text{i}v$

$$x + \frac{1}{2}\text{i} = \frac{\text{i}}{1 - u - \text{i}v}$$

Multiplying the numerator and denominator by $1 - u + \text{i}v$

Multiply the numerator and the denominator of the right hand side by the conjugate complex of $1 - u - \text{i}v$ which is $1 - u + \text{i}v$.

$$x + \frac{1}{2}\text{i} = \frac{\text{i}(1 - u + \text{i}v)}{(1-u)^2 + v^2},$$

$$= \frac{-v}{(1-u)^2 + v^2} + \frac{1-u}{(1-u)^2 + v^2}\text{i}$$

Equating imaginary parts

$$\frac{1}{2} = \frac{1 - u}{u^2 - 2u + 1 + v^2}$$

You are aiming at $|w| = 1$. If $w = u + \text{i}v$, this is the equivalent to $u^2 + v^2 = 1$. So that is the expression you are looking for.

$$u^2 - 2u + 1 + v^2 = 2 - 2u$$

$$u^2 + v^2 = 1$$

$u^2 + v^2 = 1$ is a circle centre O, radius 1.

Hence the line, Im $z = \frac{1}{2}$ is mapped onto the circle with equation $|w| = 1$.

b The transformation $w' = \frac{z - \text{i}}{z}$ maps the line Im $z = \frac{1}{2}$ onto the circle with centre O and radius 1.

The first transformation is the transformation in part **a**.

The transformation $w'' = 2w'$ maps the circle with centre O and radius 1 onto the circle with centre O and radius 2.

The transformation $z \mapsto kz$ increases the radius of the circle by a factor of k. This transformation is an enlargement, factor k, centre of enlargement O.

The transformation $w = w'' + 3 - \text{i}$ maps the circle with centre O and radius 2 onto the circle with centre $3 - \text{i}$ and radius 2.

The transformation $z \mapsto z + a$ maps a circle centre O to a circle centre a. This transformation is a translation.

Combining the transformations

$$w = 2\left(\frac{z - \text{i}}{z}\right) + 3 - \text{i}$$

$$= \frac{2z - 2\text{i} + 3z - \text{i}z}{z}$$

$$= \frac{(5 - \text{i})z - 2\text{i}}{z}$$

58 a If $z = x + iy$, then $\arg z = \frac{\pi}{4} \Rightarrow \frac{y}{x} = 1$

Let $x = y = \lambda$

$$w = \frac{\lambda + \lambda i + 1}{\lambda + \lambda i + i} = \frac{(\lambda + 1) + \lambda i}{\lambda + (\lambda + 1)i}$$

$$|w| = \left|\frac{(\lambda + 1) + \lambda i}{\lambda + (\lambda + 1)i}\right| = \frac{|(\lambda + 1) + \lambda i|}{|\lambda + (\lambda + 1)i|}$$

$$= \frac{\left((\lambda + 1)^2 + \lambda^2\right)^{\frac{1}{2}}}{\left(\lambda^2 + (\lambda + 1)^2\right)^{\frac{1}{2}}} = 1$$

For all complex numbers a and b, $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$

Hence the points on $\arg z = \frac{\pi}{4}$ map, under T, onto points on the circle $|w| = 1$.

As $\lambda > 0$, the image would only be part of this circle but the wording of the question does not require you to be more specific. You are only required to show that the image points are points on the circle; not all of the points on the circle. (The image is, in fact, just the lower right quadrant of the circle.)

b $wz + wi = z + 1$

$wz - z = 1 - iw$

$$z = \frac{1 - iw}{w - 1}$$

$$|z| = \frac{|1 - iw|}{|w - 1|} = 1$$

Hence $|1 - iw| = |w - 1|$

This is the image under T of $|z| = 1$ but it is difficult to interpret and part **c** would be difficult without some further working.

$|1 - iw| = |-i(w + 1)| = |-i||w + i| = 1 \times |w + i| = |w + i|$

The image of $|z| = 1$ in the z-plane is

$|w + i| = |w - 1|$

in the w-plane.

This is the locus of points equidistant from the points in the Argand plane representing –i and one. That is the perpendicular bisector of $(0, -1)$ and $(1, 0)$.

Writing $w = u + iv$:

$|u + iv + i| = |u + iv - 1|$

$|u + (v + 1)i| = |(u - 1) + iv|$

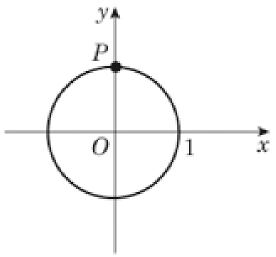
$|u + (v + 1)i|^2 = |(u - 1) + iv|^2$

$u^2 + (v + 1)^2 = (u - 1)^2 + v^2$

$u^2 + v^2 + 2v + 1 = u^2 - 2u + 1 + v^2$

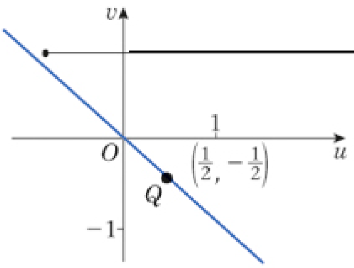
So $v = -u$

58 c, d



z

$$z = \mathrm{i} \Rightarrow w = \frac{1+\mathrm{i}}{2\mathrm{i}} = \frac{\mathrm{i}+1}{2\mathrm{i}} = \frac{1}{2} - \frac{1}{2}\mathrm{i}$$



w

The perpendicular bisector of $(0,-1)$ and $(1,0)$ is the line $v = -u$.

59 a $z = a^{-1}\mathrm{e}^{\mathrm{i}\theta}$

b Let $z = a^{-1}\mathrm{e}^{\mathrm{i}\theta}$ then we have $w = az + \dfrac{1}{z} = \mathrm{e}^{\mathrm{i}\theta} + a\mathrm{e}^{-\mathrm{i}\theta}$

$= \cos\theta + \mathrm{i}\sin\theta + a(\cos\theta - \mathrm{i}\sin\theta)$

$= (1+a)\cos\theta + (1-a)\mathrm{i}\sin\theta$

$= u + \mathrm{i}v$

Hence $u = (1+a)\cos\theta$ and $v = (1-a)\sin\theta$

$$\left(\frac{u}{1+a}\right)^2 + \left(\frac{v}{1-a}\right)^2 = 1$$

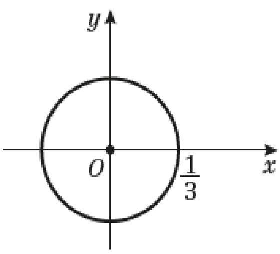
$$u^2(1-a)^2 + v^2(1+a)^2 = (1+a)^2(1-a)^2$$

$$u^2(1-a)^2 + v^2(1+a)^2 = \left[(1+a)(1-a)\right]^2$$

$$u^2(1-a)^2 + v^2(1+a)^2 = (1-a^2)^2$$

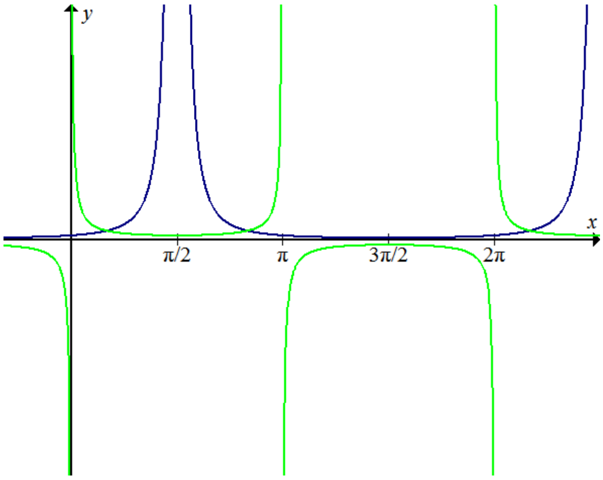
as required

c This ellipse corresponds to the case $a = 3$, hence the points on the z-plane that are transformed to the ellipse are those such that $|z| = \frac{1}{3}$



Challenge

1 $\dfrac{1}{1-\sin x} < \dfrac{1}{\sin x}$ for $0 < x < 2\pi$



Equating $y = \dfrac{1}{1-\sin x}$ and $y = \dfrac{1}{\sin x}$ gives:

$\sin x = 1 - \sin x$

$2\sin x = 1$

$\sin x = \dfrac{1}{2}$

In the range $0 < x < 2\pi$

$\sin x = \dfrac{1}{2}$ at $\dfrac{\pi}{6}$ and $\dfrac{5\pi}{6}$

Therefore:

$0 < x < \dfrac{\pi}{6}$ and $\dfrac{5\pi}{6} < x < \pi$

2 a $\omega = e^{\frac{2\pi i}{3}}, \omega^{3k} = e^{2\pi i k} = 1$

$$n = 0: \frac{1^n + \omega^n + (\omega^2)^n}{3} = \frac{1+1+1}{3} = 1$$

$$n = 3k: \frac{1^{3k} + \omega^{3k} + (\omega^2)^{3k}}{3} = \frac{1^{3k} + \omega^{3k} + (\omega^{3k})^2}{3}$$

$$= \frac{1+1+1}{3} = 1$$

$$n = 3k+1: \frac{1^{3k+1} + \omega^{3k+1} + (\omega^2)^{3k+1}}{3} = \frac{1 + \omega^{3k+1} + \omega^{6k+2}}{3}$$

$$= \frac{1 + \omega^{3k}\omega + (\omega^{3k})^2\omega^2}{3} = \frac{1+\omega+\omega^2}{3} = 0$$

$$n = 3k+2: \frac{1^{3k+1} + \omega^{3k+2} + (\omega^2)^{3k+2}}{3} = \frac{1 + \omega^{3k+2} + \omega^{6k+4}}{3}$$

$$= \frac{1 + \omega^{3k}\omega^2 + (\omega^{3k})^2\omega^3\omega}{3} = \frac{1+\omega^2+\omega}{3} = 0$$

b $\mathrm{f}(x)=\sum a_n x^n$

$\mathrm{f}(1)=\sum a_n, \mathrm{f}(\omega)=\sum a_n\omega^n, \mathrm{f}(\omega^2)=\sum a_n\left(\omega^2\right)^n$

$$\frac{\mathrm{f}(1)+\mathrm{f}(\omega)+\mathrm{f}(\omega^2)}{3}=\sum a_n\frac{1^n+\omega^n+\left(\omega^2\right)^n}{3}$$

$$=\sum a_n 1_{\{n=3k\}},$$

where $1_{\{n=3k\}}=1$ if $n=3k$ and 0 otherwise

c $\mathrm{f}(x)=(1+x)^{45}=\sum_{r=0}^{45}\binom{45}{r}x^r$

$$S=\sum_{r=0}^{45}\binom{45}{r}1_{\{r=3k\}}=\sum_{r=0}^{15}\binom{45}{3r}$$

$$=\frac{\mathrm{f}(1)+\mathrm{f}(\omega)+\mathrm{f}(\omega^2)}{3}=\frac{2^{45}+(1+\omega)^{45}+(1+\omega^2)^{45}}{3}$$

$$=\frac{2^{45}+(-\omega^2)^{45}+(-\omega)^{45}}{3}=\frac{2^{45}-(\omega^3)^{30}-(\omega^3)^{15}}{3}$$

$$=\frac{2^{45}-2}{3}\text{ as }1+\omega+\omega^2=0\text{ and }\omega^3=0$$