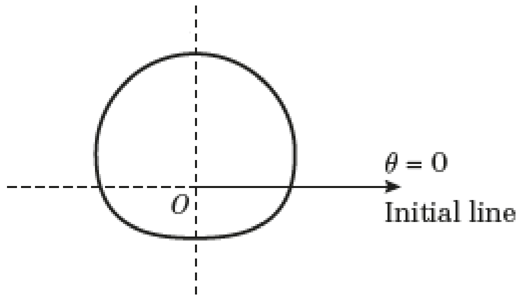
Chapter review 8

1



$$r = a\left(1+\frac{1}{2}\sin\theta\right)$$

$$\text{Area} = \frac{1}{2}a^2\int_0^{2\pi}\left(1+\frac{1}{2}\sin\theta\right)^2 \mathrm{d}\theta$$

$$= \frac{a^2}{2}\int_0^{2\pi}\left(1+\sin\theta+\frac{1}{4}\sin^2\theta\right)\mathrm{d}\theta$$

$$= \frac{a^2}{2}\int_0^{2\pi}\left(\frac{9}{8}+\sin\theta-\frac{\cos 2\theta}{8}\right)\mathrm{d}\theta$$

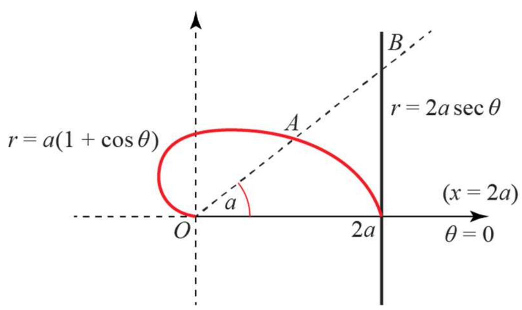
Use $\cos 2\theta = 1 - 2\sin^2\theta$.

$$= \frac{a^2}{2}\left[\frac{9}{8}\theta-\cos\theta-\frac{\sin 2\theta}{16}\right]_0^{2\pi}$$

$$= \frac{a^2}{2}\left[\left(\frac{9\pi}{4}-1-0\right)-(0-1-0)\right]$$

$$= \frac{9\pi a^2}{8}$$

2 a,b



c

$$OB = 2a\sec\alpha$$

$$OA = a(1+\cos\alpha)$$

$$2OA = OB \Rightarrow 1+\cos\alpha = \sec\alpha$$

$$\cos^2\alpha + \cos\alpha - 1 = 0$$

$$\cos\alpha = \frac{-1\pm\sqrt{1+4}}{2}$$

$\therefore$ α is acute.

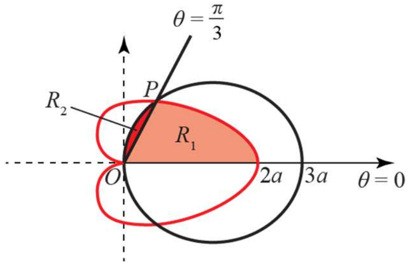
$$\cos\alpha = \frac{\sqrt{5}-1}{2}$$

3 First find P:

$1+\cos\theta=3\cos\theta$

$1=2\cos\theta$

$\Rightarrow \quad \theta=\arccos\frac{1}{2}=\frac{\pi}{3}$



By symmetry the required area $=2(R_1+R_2)$

$$R_1=\frac{1}{2}\int_0^{\frac{\pi}{3}}(1+\cos\theta)^2\,d\theta=\frac{1}{2}\int_0^{\frac{\pi}{3}}(1+2\cos\theta+\cos^2\theta)\,d\theta$$

$$R_1=\frac{1}{2}\int_0^{\frac{\pi}{3}}\left(\frac{3}{2}+2\cos\theta+\frac{\cos 2\theta}{2}\right)d\theta$$

Use $\cos 2\theta=2\cos^2\theta-1$.

$$=\frac{1}{2}\left[\frac{3}{2}\theta+2\sin\theta+\frac{1}{4}\sin 2\theta\right]_0^{\frac{\pi}{3}}$$

$$=\frac{1}{2}\left[\left(\frac{\pi}{2}+2\sin\frac{\pi}{3}+\frac{1}{4}\sin\frac{2\pi}{3}\right)-(0)\right]$$

$$=\frac{1}{2}\left[\frac{\pi}{2}+\sqrt{3}+\frac{\sqrt{3}}{8}\right]=\frac{\pi}{4}+\frac{9\sqrt{3}}{16}$$

$$R_2=\frac{9}{2}\int_{\frac{\pi}{3}}^{\frac{\pi}{2}}\cos^2\theta\,d\theta=\frac{9}{4}\int_{\frac{\pi}{3}}^{\frac{\pi}{2}}(1+\cos 2\theta)\,d\theta$$

$$=\frac{9}{4}\left[\theta+\frac{1}{2}\sin 2\theta\right]_{\frac{\pi}{3}}^{\frac{\pi}{2}}=\frac{9}{4}\left[\left(\frac{\pi}{2}+0\right)-\left(\frac{\pi}{3}+\frac{\sqrt{3}}{4}\right)\right]$$

$$=\frac{3\pi}{8}-\frac{9\sqrt{3}}{16}$$

$$\therefore \text{Area required}=2\left(\frac{3\pi}{8}+\frac{\pi}{4}\right)=\frac{5\pi}{4}$$

4 $r^2 = a^2 \sin 2\theta$ (must have $\sin 2\theta \geqslant 0$)

$r = a\sqrt{\sin 2\theta}$

$x = r\cos\theta = a\cos\theta\sqrt{\sin 2\theta}$

$\frac{dx}{d\theta} = 0 \Rightarrow 0 = -\sin\theta\sqrt{\sin 2\theta} + \frac{1}{2}\cos\theta\frac{1}{\sqrt{\sin 2\theta}}2\cos 2\theta$

i.e. $0 = -\sin\theta \times \sin 2\theta + \cos\theta\cos 2\theta$

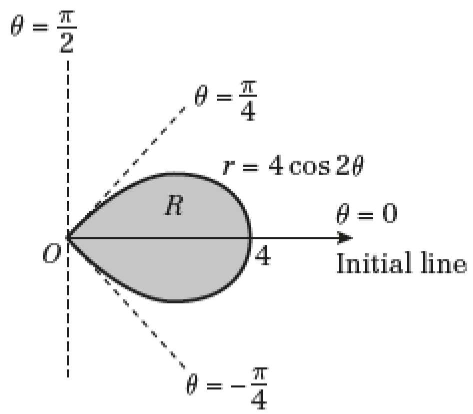
i.e. $0 = \cos 3\theta$

$\therefore$ $3\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{7\pi}{2}$

$\therefore$ $\theta = \frac{\pi}{6}, \frac{\pi}{2}, \frac{7\pi}{6}$

So $\left(a\sqrt{\frac{\sqrt{3}}{2}}, \frac{\pi}{6}\right), \left(a\sqrt{\frac{\sqrt{3}}{2}}, \frac{7\pi}{6}\right)$ and $\left(0, \frac{\pi}{2}\right)$

5 a



b $\text{Area} = 2 \times \frac{1}{2}\int_0^{\frac{\pi}{4}} 16\cos^2 2\theta \, d\theta$

$= \int_0^{\frac{\pi}{4}} (8 + 8\cos 4\theta) d\theta$

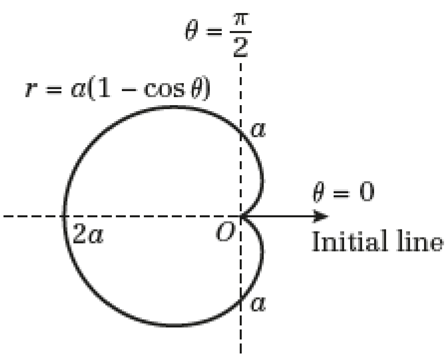
$= [8\theta + 2\sin 4\theta]_0^{\frac{\pi}{4}}$

$= 2\pi + 0 - 0$

$= 2\pi$

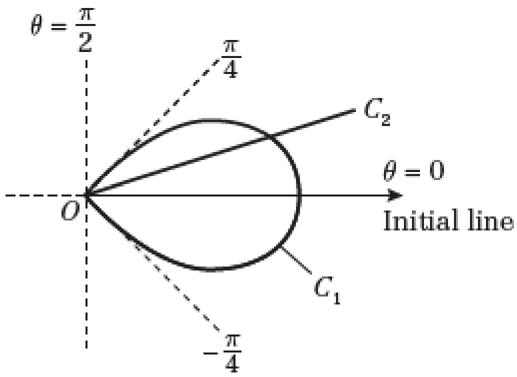
$2\cos^2\theta = 1 + \cos 2\theta.$

6



Max r is $2a$ at point $(2a, \pi)$

7 a



b $$\text{Area} = \frac{1}{2}\int_{\frac{\pi}{12}}^{\frac{\pi}{4}} 4\cos^2 2\theta$$

$$= \int_{\frac{\pi}{12}}^{\frac{\pi}{4}} (1+\cos 4\theta)\,\mathrm{d}\theta$$

$\cos 4\theta = 2\cos^2 2\theta - 1$

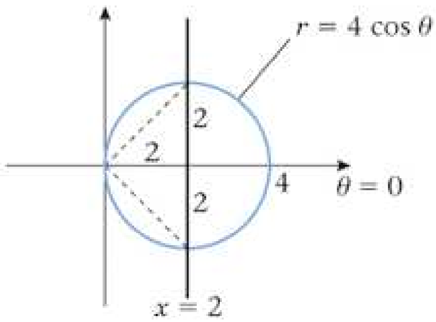
$$= \left[\theta + \frac{1}{4}\sin 4\theta\right]_{\frac{\pi}{12}}^{\frac{\pi}{4}}$$

$$= \left(\frac{\pi}{4} + 0\right) - \left(\frac{\pi}{12} + \frac{1}{4}\sin\frac{\pi}{3}\right)$$

$$= \frac{\pi}{6} - \frac{1}{4}\times\frac{\sqrt{3}}{2}$$

$$= \frac{\pi}{6} - \frac{\sqrt{3}}{8}$$

8 a $$r = 2\sec\theta$$
$$r\cos\theta = 2$$
$$x = 2$$



b $x = 2$ is a diameter

$$r = \sqrt{2^2 + 2^2} = 2\sqrt{2}$$

So polar coordinates are

$$\left(2\sqrt{2}, \frac{\pi}{4}\right) \quad \left(2\sqrt{2}, -\frac{\pi}{4}\right)$$

9 a $$a(1+\cos\theta) = 3a\cos\theta$$
$$1 = 2\cos\theta$$
$$\cos\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}$$

So P is $\left(\frac{3}{2}a, \frac{\pi}{3}\right)$

9 b Area $= \frac{a^2}{2}\int_0^{\frac{\pi}{3}}\left(\frac{3}{2}+2\cos\theta+\frac{\cos 2\theta}{2}\right)d\theta+\frac{9}{2}a^2\int_{\frac{\pi}{3}}^{\frac{\pi}{2}}\cos^2\theta\,d\theta$

$$= \frac{a^2}{2}\left[\frac{3}{2}\theta+2\sin\theta+\frac{1}{4}\sin 2\theta\right]_0^{\frac{\pi}{3}}+\frac{9}{4}a^2\left[\theta+\frac{1}{2}\sin 2\theta\right]_{\frac{\pi}{3}}^{\frac{\pi}{2}}$$

$$= \frac{a^2}{2}\left[\frac{\pi}{2}+\sqrt{3}+\frac{\sqrt{3}}{8}\right]+\frac{9}{4}a^2\left[\frac{\pi}{6}-\frac{\sqrt{3}}{4}\right]$$

$$= \frac{5\pi}{8}a^2$$

10 a

$$r^2 = \sec 2\theta$$

$$r^2\cos 2\theta = 1$$

$$r^2(2\cos^2\theta-1) = 1$$

$$2r^2\cos^2\theta = 1+r^2$$

$$2x^2 = 1+x^2+y^2$$

$$\therefore \quad y^2 = x^2-1$$

b

$$r^2 = \operatorname{cosec}2\theta$$

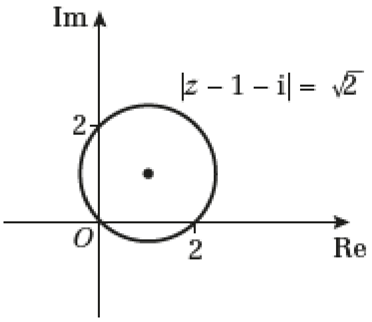
$$\Rightarrow \quad r^2\sin 2\theta = 1$$

$$\Rightarrow 2r\sin\theta\, r\cos\theta = 1$$

$$\Rightarrow \quad 2xy = 1$$

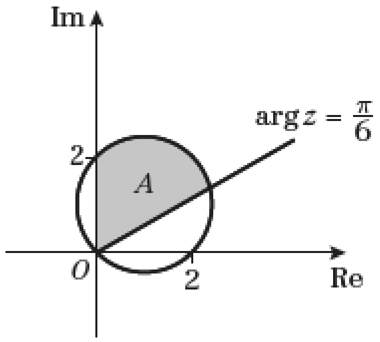
$$y = \frac{1}{2x}$$

11 a $|z-1-i| = \sqrt{2}$ is a circle centred at $(1,1)$ with radius $\sqrt{2}$.



b The Cartesian equation of a circle centred at $(1,1)$ with radius $\sqrt{2}$ is $(x-1)^2+(y-1)^2 = 2$.
Converting this to polar coordinates gives $(r\cos\theta-1)^2+(r\sin\theta-1)^2 = 2$ which simplifies to $r = 2\cos\theta+2\sin\theta$ when $r \neq 0$.

11 c The set of points $A = \left\{ z : \frac{\pi}{6} \leqslant \arg z \leqslant \frac{\pi}{2} \right\} \cap \left\{ z : |z - 1 - i| \leqslant \sqrt{2} \right\}$ is the green sector of the circle. It represents the intersection of all possible $\arg z$ such that $\frac{\pi}{6} \leqslant \arg z \leqslant \frac{\pi}{2}$ and the red circle which represents all z such that $|z - 1 - i| \leqslant \sqrt{2}$.



d In order to find the area of the region bounded between the lines $\theta = \frac{\pi}{6}$, $\theta = \frac{\pi}{2}$ and the arc A, we calculate

$$Area = \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (2\cos\theta + 2\sin\theta)^2 \, d\theta$$

$$= 2\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (\cos\theta + \sin\theta)^2 \, d\theta$$

$$= 2\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (1 + 2\cos\theta\sin\theta) \, d\theta$$

$$= 2\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (1 + \sin 2\theta) d\theta$$

$$= 2\left[\theta - \frac{\cos 2\theta}{2}\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$$

$$= 2\left(\frac{\pi}{3} + \frac{3}{4}\right)$$

$$\approx 3.59 (3 \text{ s.f.})$$

12 In order to find the area of the shaded region, we must find the area of the sector bounded by the curve and the line OA, then subtract the area of the triangle OAB. The value of θ at the point A can be found by solving $r = 4\cos 2\theta = 2$, leading to $\theta = \frac{\pi}{6}$.

We now find the area of the sector bounded by the curve and the line OA.

$$\begin{aligned} A_{sector} &= \frac{1}{2}\int_0^{\frac{\pi}{6}} (4\cos 2\theta)^2 \,\mathrm{d}\theta \\ &= 8\int_0^{\frac{\pi}{6}} (\cos^2 2\theta)\,\mathrm{d}\theta \\ &= 4\int_0^{\frac{\pi}{6}} (\cos 4\theta + 1)\,\mathrm{d}\theta \\ &= 4\left[\left(\tfrac{1}{4}\sin 4\theta + \theta\right)\right]_0^{\frac{\pi}{6}} \\ &= \frac{\sqrt{3}}{2} + \frac{2\pi}{3} \\ &\approx 2.9604 \end{aligned}$$

Now we find the area of the triangle OAB by using the formula

$$\begin{aligned} Area_{OAB} &= \frac{1}{2} \times Base \times Height \\ &= \frac{1}{2}|x_A||y_A| \end{aligned}$$

where

x_A is the x coordinate of A and

x_B is the y coordinate of A.

$$\begin{aligned} Area_{OAB} &= \frac{1}{2}|x_A||y_A| \\ &= \frac{1}{2}|r\cos\theta||r\sin\theta| \\ &= \frac{1}{2}|4\cos 2\theta\cos\theta||4\cos 2\theta\sin\theta| \\ &= 8|\cos\frac{\pi}{3}\cos\frac{\pi}{6}||\cos\frac{\pi}{3}\sin\frac{\pi}{6}| \\ &= \frac{\sqrt{3}}{2} \\ &\approx 0.86603 \end{aligned}$$

Thus, the area of the shaded region is found to be

$$\begin{aligned} A &= Area_{sector} - Area_{OAB} \\ &\approx 2.9604 - 0.86603 \\ &\approx 2.09 \text{ (3 s.f.)} \end{aligned}$$

13 First we need to find the point for which the tangent to the curve is perpendicular to the initial line. We form an expression for x and differentiate with respect to θ.

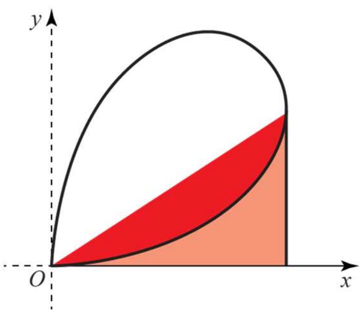
$$x = r\cos\theta$$
$$= 4\sin 2\theta\cos\theta$$
$$\frac{dx}{d\theta} = 8\cos 2\theta\cos\theta - 4\sin 2\theta\sin\theta$$
$$= 8(2\cos^2\theta - 1)\cos\theta - 8\cos\theta\sin^2\theta$$
$$= 24\cos^3\theta - 16\cos\theta$$
$$= 8\cos\theta(3\cos^2\theta - 2).$$

We now solve equal to 0 in order to find our required θ values. We choose to neglect the solutions arising from the $\cos\theta = 0$ factor, since a tangent at the origin is not what we are looking for, even though it is perpendicular to the initial line.

So, $3\cos^2\theta - 2 = 0$ gives $\cos\theta = \pm\sqrt{\frac{2}{3}}$ and we choose to neglect the negative solution since $0 \leqslant \theta \leqslant \frac{\pi}{2}$.

Thus our tangent perpendicular to the initial line occurs at $\theta = \theta_A = \arccos\left(\sqrt{\frac{2}{3}}\right)$.

To find the area of the region, we will need to find the area of the sector that lies between $0 \leqslant \theta \leqslant \theta_A$ as shown in the diagram (red region).



$$A_{sector} = \frac{1}{2}\int_0^{\theta_A} (4\sin 2\theta)^2 d\theta$$
$$= 8\int_0^{\theta_A} (\sin^2 2\theta) d\theta$$
$$= 4\int_0^{\theta_A} (1 - \cos 4\theta) d\theta$$
$$= [4\theta - \sin 4\theta]_0^{\theta_A}$$
$$= 4\theta_A - \sin 4\theta_A$$
$$= 4\arccos\left(\sqrt{\frac{2}{3}}\right) - \sin\left(4\arccos\left(\sqrt{\frac{2}{3}}\right)\right)$$
$$\approx 1.8334.$$

13 (continued)

Now we find the area of the right-angle triangle bounded by the horizontal axis, the tangent and the line $\theta = \theta_A$.

Using the formula

$$\begin{aligned}A_{tri} &= \frac{1}{2} \times Base \times Height \\ &= \frac{1}{2}|x||y| \\ &= \frac{1}{2}r^2|\cos\theta||\sin\theta| \\ &= 8(\sin^2 2\theta)|\cos\theta||\sin\theta|\end{aligned}$$

and substituting in $\theta = \theta_A$, we find that $A_{tri} = \frac{64\sqrt{2}}{27}$.

So our shaded region is

$$\begin{aligned}A &= A_{tri} - A_{sector} \\ &= \frac{64\sqrt{2}}{27} - 1.8334 \\ &\approx 1.52 \text{ (2 d.p.)}\end{aligned}$$

Challenge

First we find expressions for x and y in terms of θ.

$x = r\cos\theta = \sqrt{2}\theta\cos\theta$

$y = r\sin\theta = \sqrt{2}\theta\sin\theta.$

Now differentiating with respect to θ we obtain

$\frac{dx}{d\theta} = \sqrt{2}\cos\theta - \sqrt{2}\theta\sin\theta$

$\frac{dy}{d\theta} = \sqrt{2}\sin\theta + \sqrt{2}\theta\cos\theta$

$\frac{dy}{dx} = \frac{\sin\theta + \theta\cos\theta}{\cos\theta - \theta\sin\theta}.$

So at $\theta = \frac{\pi}{4}$, $\sin\theta = \cos\theta = \frac{\sqrt{2}}{2}$

$\therefore \frac{dy}{dx} = \frac{\sqrt{2} + \frac{\pi}{4}\sqrt{2}}{\sqrt{2} - \frac{\pi}{4}\sqrt{2}}$

$= \frac{4+\pi}{4-\pi}.$

If we use the linear formula $y = mx + c$, we can find a value for c by substituting in values for x and y at $\theta = \frac{\pi}{4}$.

At $\theta = \frac{\pi}{4}$,

$x = \sqrt{2}\theta\cos\theta = \frac{\pi}{4}$

$y = \sqrt{2}\theta\sin\theta = \frac{\pi}{4}.$

So,

$y = mx + c$

$y = \left(\frac{4+\pi}{4-\pi}\right)x + c$

$\frac{\pi}{4} = \left(\frac{4+\pi}{4-\pi}\right)\frac{\pi}{4} + c$

$c = \frac{\pi}{4}\left(1 - \frac{4+\pi}{4-\pi}\right)$

$c = \frac{\pi}{2}\left(\frac{\pi}{\pi-4}\right)$

Now we can substitute into the linear equation to obtain

$y = \left(\frac{4+\pi}{4-\pi}\right)x + \frac{\pi}{2}\left(\frac{\pi}{\pi-4}\right)$

$2(\pi-4)y + 2(\pi+4)x = \pi^2.$