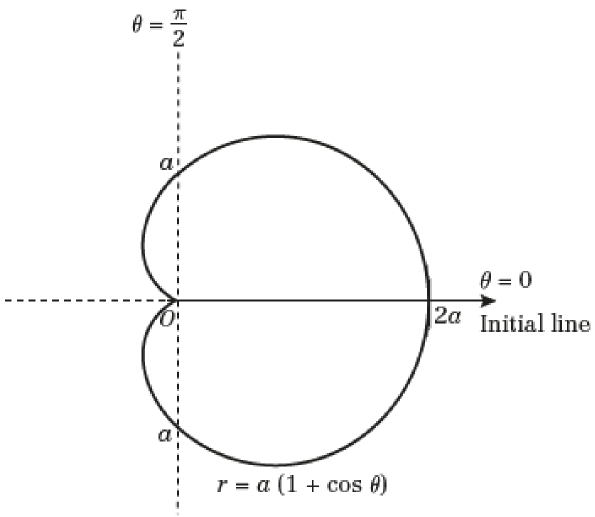
Exercise 8D



1 $r = a(1+\cos\theta)$

Require $\frac{d}{d\theta}(r\cos\theta) = 0$

i.e. $\frac{d}{d\theta}(a\cos\theta + a\cos^2\theta) = a(-\sin\theta - 2\cos\theta\sin\theta)$

So $0 = -a\sin\theta(1+2\cos\theta)$

$\sin\theta = 0 \Rightarrow \theta = 0, \pi$ (from sketch π is not allowed)

$\cos\theta = -\frac{1}{2} \Rightarrow \theta = \pm\frac{2\pi}{3} \Rightarrow r = a\left(1-\frac{1}{2}\right) = \frac{a}{2}$

$\therefore$ points are $(2a, 0)$ and $\left(\frac{a}{2}, \frac{2\pi}{3}\right), \left(\frac{a}{2}, \frac{-2\pi}{3}\right)$

2 $r = e^{2\theta}$

a $x = r\cos\theta = e^{2\theta}\cos\theta$

$\frac{dx}{d\theta} = 0 \Rightarrow 0 = 2e^{2\theta}\cos\theta - e^{2\theta}\sin\theta$

$0 = e^{2\theta}(2\cos\theta - \sin\theta)$

$\Rightarrow \tan\theta = 2$

$\therefore \theta = 1.107$ (rads)

$r = e^{2\times 1.107} = 9.1549\ldots$

So at (9.15, 1.11) the tangent is perpendicular to initial line.

b $y = r\sin\theta = e^{2\theta}\sin\theta$

$\frac{dy}{d\theta} = 0 \Rightarrow 0 = 2e^{2\theta}\sin\theta + e^{2\theta}\cos\theta$

$0 = e^{2\theta}(2\sin\theta + \cos\theta)$

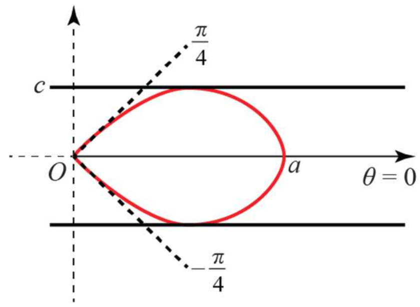
$\Rightarrow \tan\theta = -\frac{1}{2}$

$\therefore \theta = 2.6779\ldots$

$r = e^{2\times 2.6779\ldots} = 211.852\ldots$

So at (212, 2.68) the tangent is parallel to initial line.

3 a $r = a\cos 2\theta$



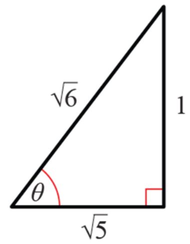
$$y = r\sin\theta = a\sin\theta\cos 2\theta$$

$$\frac{\mathrm{d}y}{\mathrm{d}\theta} = 0 \Rightarrow 0 = a[\cos\theta\cos 2\theta - 2\sin 2\theta\sin\theta]$$

$$0 = a\cos\theta[\cos 2\theta - 4\sin^2\theta]$$

$$0 = a\cos\theta[\cos^2\theta - 5\sin^2\theta]$$

$$\cos\theta = \Rightarrow \theta = \frac{\pi}{2} \quad \text{(outside range)}$$



$$\therefore \quad \tan^2\theta = \frac{1}{5} \Rightarrow \tan\theta = \pm\frac{1}{\sqrt{5}}$$

$$\theta = \pm 0.42053\ldots$$

$$r = a\left(\cos^2\theta - \sin^2\theta\right) = a\left(\frac{5}{6} - \frac{1}{6}\right) = \frac{2a}{3}$$

$$\therefore \quad \text{points are}\left(\frac{2a}{3}, \pm 0.421\right)$$

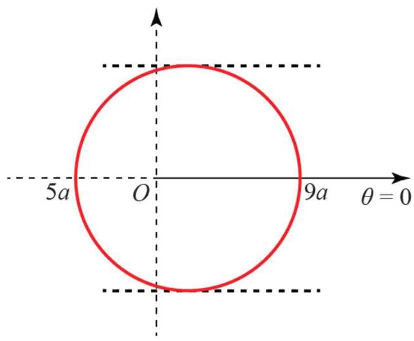
b The lines are $y = \pm c$ where $c = r\sin\,(0.42053\ldots)$

$$= \frac{2a}{3} \times \frac{1}{\sqrt{6}} = \frac{a\sqrt{6}}{9}$$

The line $y = c$ is $r\sin\theta = \dfrac{a\sqrt{6}}{9}$

$\therefore$ Tangents have equations $r = \pm\dfrac{a\sqrt{6}}{9}\operatorname{cosec}\theta$

4 $r = a\ (7 + 2\cos\theta)$



$$y = r\sin\theta = a\,(7\sin\theta + 2\cos\theta\sin\theta)$$

$$y = a\,(7\sin\theta + \sin 2\theta)$$

$$\frac{\mathrm{d}y}{\mathrm{d}\theta} = 0 \Rightarrow 0 = a(7\cos\theta + 2\cos 2\theta)$$

$$\Rightarrow \quad 0 = 4\cos^2\theta + 7\cos\theta - 2$$

$$0 = (4\cos\theta - 1)(\cos\theta + 2)$$

$$\cos\theta = \frac{1}{4}(\text{or } -2)$$

$$\Rightarrow \quad \theta = \pm 1.318\ldots$$

$$r = a\left(7 + \frac{2}{4}\right) = \frac{15}{2}a$$

$\therefore$ tangents are parallel at $\left(\dfrac{15}{2}a, \pm 1.32\right)$

5 $r = 2 + \cos\theta$

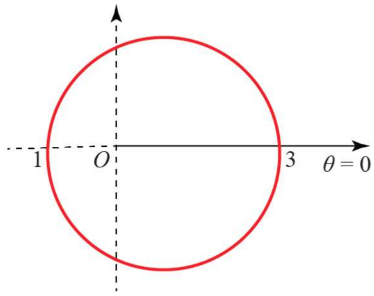
$x = r\cos\theta = 2\cos\theta + \cos^2\theta$

$\frac{dx}{d\theta} = 0 \Rightarrow 0 = -2\sin\theta - 2\cos\theta\sin\theta$

$0 = -2\sin\theta(1+\cos\theta)$

$\sin\theta = 0 \quad \Rightarrow \theta = 0, \pi$

$\cos\theta = -1 \Rightarrow \theta = \pi$



$\therefore$ tangents are perpendicular to the initial line at:

$(3, 0)$ and $(1, \pi)$

The equations are

$r\cos\theta = 3 \qquad r\cos\theta = -1$

$r = 3\sec\theta \qquad r = -\sec\theta$

6 $r = a(1+\tan\theta)$

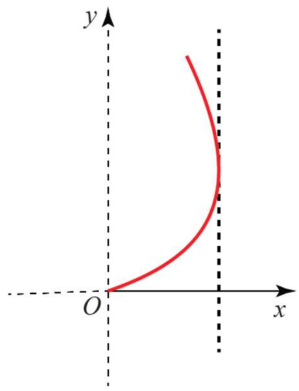
$x = r\cos\theta = a(\cos\theta + \sin\theta)$

$\frac{dx}{d\theta} = 0 \Rightarrow \quad 0 = a\left(-\sin\theta + \cos\theta\right)$

$\Rightarrow \tan\theta = 1$

$\Rightarrow \quad \theta = \frac{\pi}{4}$

$\therefore$ point is $\left(2a, \frac{\pi}{4}\right)$



7 In order to find the length of the line OA, first we must find where the point is.
We find an expression for

$y = r\sin\theta$

$= \sin\theta + 3\cos\theta\sin\theta.$

Now differentiating with respect to θ,

$\frac{dy}{d\theta} = \cos\theta + 3\cos^2\theta - 3\sin^2\theta.$

Rearrange in terms of $\cos\theta$, then set equal to 0 and solve

$6\cos^2\theta + \cos\theta - 3 = 0$

$\cos\theta = \frac{-1+\sqrt{73}}{12}$

We neglect the negative term since $0 \leqslant \theta \leqslant \frac{\pi}{2}$.

Substituting this back into the given expression for r, we find that

$r = 1 + 3\left(\frac{-1+\sqrt{73}}{12}\right)$

$= \frac{3+\sqrt{73}}{4}$

8 First we need to find the points for which the tangent to the curve is perpendicular to the initial line. We form an expression for x and differentiate with respect to θ.

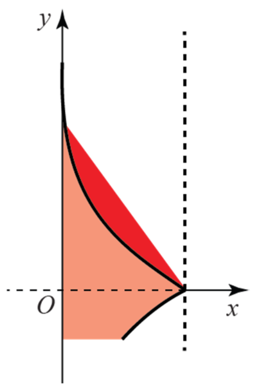
$$x = r\cos\theta$$
$$= 2\cos\theta + 2\cos^2\theta$$
$$\frac{dx}{d\theta} = -2\sin\theta - 4\cos\theta\sin\theta$$
$$= -2\sin\theta(1+2\cos\theta).$$

We now solve equal to 0 in order to find our required θ values. We choose to neglect the solutions coming from $\sin\theta = 0$ factor, although $\theta = k\pi$ are clearly tangent to the curve and perpendicular to the initial line, they are not the tangents we are looking for. This can be clearly seen by looking at the diagram.

So, $1+2\cos\theta = 0$ gives $\theta = \pm\frac{2\pi}{3}$.

Since we have symmetry about the horizontal axis, we may compute the top half of the region and then double it later on.

To find the area of the top region, we will need to find the area of the sector that lies between $\frac{2\pi}{3} \leqslant \theta \leqslant \pi$ as shown in the diagram (red region).



$$A_{sector} = \frac{1}{2}\int_{\frac{2\pi}{3}}^{\pi} (2(1+\cos\theta))^2 \, d\theta$$
$$= 2\int_{\frac{2\pi}{3}}^{\pi} (1+2\cos\theta+\cos^2\theta)d\theta$$
$$= 2\int_{\frac{2\pi}{3}}^{\pi} (1+2\cos\theta+\tfrac{1}{2}(1+\cos 2\theta))d\theta$$
$$= 2\left[\theta + 2\sin\theta + \tfrac{1}{2}(\theta + \tfrac{1}{2}\sin 2\theta)\right]_{\frac{2\pi}{3}}^{\pi}$$
$$= \pi - \frac{7\sqrt{3}}{4}$$

Now we find the area of the right-angle triangle bounded by the horizontal axis, the tangent and the line $\theta = \frac{2\pi}{3}$.

8 Continued

Using the formula

$$A_{tri} = \frac{1}{2} \times Base \times Height$$
$$= \frac{1}{2}|x||y|$$
$$= \frac{1}{2}r^2|\cos\theta||\sin\theta|$$
$$= 2(1+\cos\theta)^2|\cos\theta||\sin\theta|$$

and substituting in $\theta = \frac{2\pi}{3}$, we find that

$$A_{tri} = 2\left(1-\tfrac{1}{2}\right)^2\left(\tfrac{1}{2}\right)\left(\tfrac{\sqrt{3}}{2}\right)$$
$$= \frac{\sqrt{3}}{8}$$

So, our upper shaded region is

$$A_{top} = A_{tri} - A_{sector}$$
$$= \frac{\sqrt{3}}{8} - \left(\pi - \frac{7\sqrt{3}}{4}\right)$$
$$= \frac{15\sqrt{3}}{8} - \pi.$$

Which gives our total shaded region as

$$A = 2A_{top}$$
$$= \frac{15\sqrt{3}}{4} - 2\pi$$
$$\approx 0.212$$