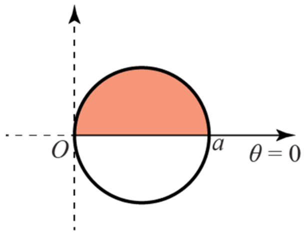
Exercise 8C

1 a



$r = a\cos\theta$

$$\text{Area } = \frac{1}{2}a^2\int_0^{\frac{\pi}{2}} \cos^2\theta \, d\theta$$

$$= \frac{a^2}{4}\int_0^{\frac{\pi}{2}} (\cos 2\theta + 1)$$

$$= \frac{a^2}{4}\left[\frac{1}{2}\sin 2\theta + \theta\right]_0^{\frac{\pi}{2}}$$

$$= \frac{a^2}{4}\left[\left(0 + \frac{\pi}{2}\right) - (0)\right]$$

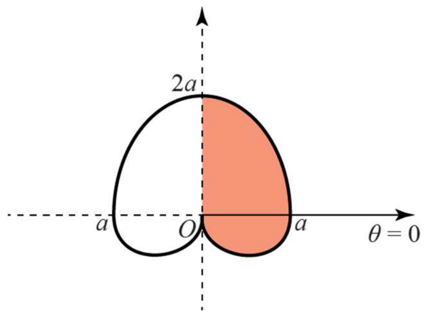
$$= \frac{\pi a^2}{8}$$

$\cos 2\theta = 2\cos^2\theta - 1$

$r = a\cos\theta$ is a circle centre $\left(\frac{a}{2}, 0\right)$ and radius $\frac{a}{2}$.

The area of the semicircle is $\therefore \frac{1}{2}\pi\frac{a^2}{4} = \frac{a^2\pi}{8}$.

1 b



$$r = a(1+\sin\theta)$$

$$\text{Area} = \frac{1}{2}a^2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(1+2\sin\theta+\sin^2\theta)\,d\theta$$

$$= \frac{1}{2}a^2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(1+2\sin\theta+\frac{1}{2}-\frac{1}{2}\cos 2\theta\right)d\theta$$

Use $\cos 2\theta = 1-2\sin^2\theta$.

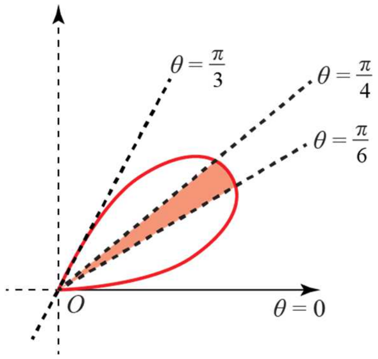
$$= \frac{1}{2}a^2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(\frac{3}{2}+2\sin\theta-\frac{1}{2}\cos 2\theta\right)d\theta$$

$$= \frac{1}{2}a^2\left[\frac{3}{2}\theta-2\cos\theta-\frac{1}{4}\sin 2\theta\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= \frac{1}{2}a^2\left[\left(\frac{3\pi}{4}-0-0\right)-\left(-\frac{3\pi}{4}-0-0\right)\right]$$

$$= \frac{3\pi a^2}{4}$$

1 c



$$r = a\sin 3\theta$$

$$\text{Area} = \frac{1}{2}a^2\int_{\frac{\pi}{6}}^{\frac{\pi}{4}}\sin^2 3\theta\,\mathrm{d}\theta$$

$$= \frac{a^2}{4}\int_{\frac{\pi}{6}}^{\frac{\pi}{4}}(1-\cos 6\theta)\,\mathrm{d}\theta$$

Use $\cos 6\theta = 1 - 2\sin^2 3\theta$.

$$= \frac{a^2}{4}\left[\theta - \frac{1}{6}\sin 6\theta\right]_{\frac{\pi}{6}}^{\frac{\pi}{4}}$$

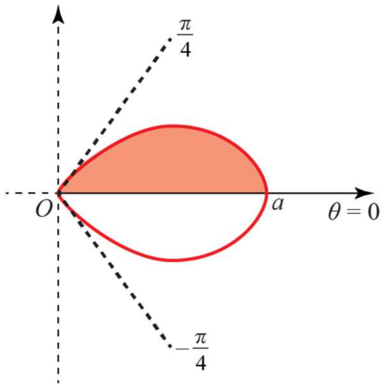
$$= \frac{a^2}{4}\left[\left(\frac{\pi}{4} - \frac{1}{6}\sin\frac{3\pi}{2}\right) - \left(\frac{\pi}{6} - \frac{1}{6}\sin\pi\right)\right]$$

$$= \frac{a^2}{4}\left(\frac{\pi}{4} + \frac{1}{6} - \frac{\pi}{6}\right)$$

$$= \frac{a^2}{4}\left(\frac{\pi}{12} + \frac{2}{12}\right)$$

$$= \frac{(\pi+2)a^2}{48}$$

1 d



$$r^2 = a^2 \cos 2\theta$$

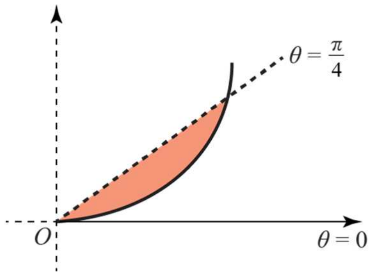
$$\text{Area} = \frac{1}{2}a^2\int_0^{\frac{\pi}{4}} \cos 2\theta \, d\theta$$

$$= \left[\frac{a^2}{4}\sin 2\theta\right]_0^{\frac{\pi}{4}}$$

$$= \left(\frac{a^2}{4}\sin\frac{\pi}{2}\right) - (0)$$

$$= \frac{a^2}{4}$$

e



$$r^2 = a^2 \tan\theta$$

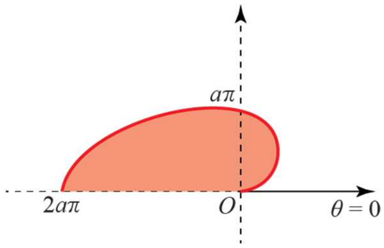
$$\text{Area} = \frac{1}{2}a^2\int_0^{\frac{\pi}{4}} \tan\theta \, d\theta$$

$$= \left[\frac{1}{2}a^2 \ln\sec\theta\right]_0^{\frac{\pi}{4}}$$

$$= \left(\frac{1}{2}a^2 \ln\sqrt{2}\right) - (0)$$

$$= \frac{a^2 \ln\sqrt{2}}{2} \quad \text{or} \quad \frac{a^2 \ln 2}{4}$$

1 f



$$r = 2a\theta$$

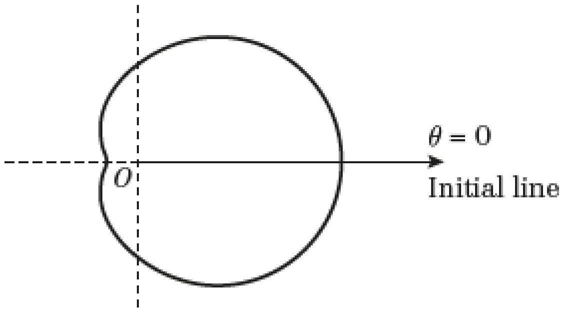
$$\text{Area} = \frac{1}{2}\int_0^{\pi} 4a^2\theta^2 \, d\theta$$

$$= 2a^2\left[\frac{\theta^3}{3}\right]_0^{\pi}$$

$$= 2a^2\left[\left(\frac{\pi^3}{3}\right) - (0)\right]$$

$$= \frac{2a^2\pi^3}{3}$$

g



$$r = a(3 + 2\cos\theta)$$

$$\text{Area} = \frac{a^2}{2}\int_0^{\frac{\pi}{2}} (9 + 12\cos\theta + 4\cos^2\theta) \, d\theta$$

$$= \frac{a^2}{2}\int_0^{\frac{\pi}{2}} (11 + 12\cos\theta + 2\cos 2\theta) \, d\theta$$

Use $\cos 2\theta = 2\cos^2\theta - 1$.

$$= \frac{a^2}{2}\left[11\theta + 12\sin\theta + \sin 2\theta\right]_0^{\frac{\pi}{2}}$$

$$= \frac{a^2}{2}\left[\left(\frac{11\pi}{2} + 12 + 0\right) - (0)\right]$$

$$= \frac{a^2}{4}(11\pi + 24)$$

2 Area $= \frac{1}{2}a^2\int_0^{2\pi}(p^2 + 2pq\cos\theta + q^2\cos^2\theta)\,d\theta$

$= \frac{1}{2}a^2\int_0^{2\pi}\left(p^2 + 2pq\cos\theta + \frac{q^2}{2}\cos 2\theta + \frac{q^2}{2}\right)d\theta$

Use $\cos 2\theta = 2\cos^2\theta - 1$.

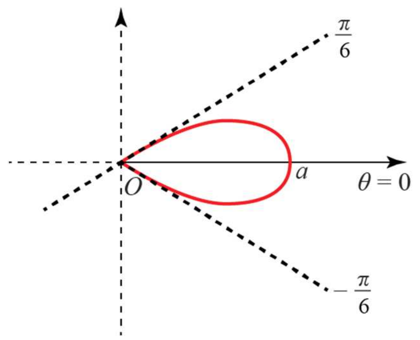
$= \frac{1}{2}a^2\int_0^{2\pi}\left(\left[\frac{2p^2 + q^2}{2}\right] + 2pq\cos\theta + \frac{q^2}{2}\cos 2\theta\right)d\theta$

$= \frac{1}{2}a^2\left[\left[\frac{2p^2 + q^2}{2}\right]\theta + 2pq\sin\theta + \frac{q^2}{4}\sin 2\theta\right]_0^{2\pi}$

$= \frac{1}{2}a^2\left[\left(\left[\frac{2p^2 + q^2}{\not 2}\right]\pi \times \not 2 + 0 + 0\right) - (0)\right]$

$= \frac{a^2(2p^2 + q^2)\pi}{2}$

3



Area $= \frac{1}{2}a^2\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}}\cos^2 3\theta\,d\theta = 2 \times \frac{1}{2}a^2\int_0^{\frac{\pi}{6}}\cos^2 3\theta\,d\theta$

$= \frac{a^2}{2}\int_0^{\frac{\pi}{6}}(1 + \cos 6\theta)\,d\theta$

Use $\cos 6\theta = 2\cos^2 3\theta - 1$.

$= \frac{a^2}{2}\left[\theta + \frac{1}{6}\sin 6\theta\right]_0^{\frac{\pi}{6}}$

$= \frac{a^2}{2}\left[\left(\frac{\pi}{6} + 0\right) - (0)\right]$

$= \frac{\pi a^2}{12}$

4 In order to find the area enclosed by a single loop, we calculate

$$\frac{1}{2}\int_0^{2\pi} r^2 \, d\theta = \frac{1}{2}\int_0^{2\pi} (a+5\sin\theta)^2 \, d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} \left(a^2 + 10a\sin\theta + 25\sin^2\theta\right) d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} \left(a^2 + 10a\sin\theta + \frac{25}{2}(1-\cos 2\theta)\right) d\theta$$

$$= \frac{1}{2}\left[a^2\theta - 10a\cos\theta + \frac{25}{2}\theta - \frac{25}{4}\sin 2\theta\right]_0^{2\pi}$$

$$= \frac{1}{2}\left(2\pi a^2 + 25\pi\right).$$

In order to have $\frac{1}{2}\left(2\pi a^2 + 25\pi\right) = \frac{187\pi}{2}$ and $a > 0$, we solve to obtain $a = 9$

5 First we find the intersection points of these two curves.

$a\sin 4\theta = a\sin 2\theta$

$\sin 4\theta = \sin 2\theta$

$4\theta = 2\theta + 2k\pi$ or $4\theta = \pi - 2\theta + 2k\pi$

$2\theta = 2k\pi$ or $6\theta = \pi(1+2k)$

$\theta = k\pi$ or $\theta = \frac{\pi}{6}(1+2k)$

($k \in \mathbb{R}$).

Since we are working in the range $0 \leqslant \theta \leqslant \frac{\pi}{2}$, we only care about the intersections occurring at $\theta = 0$, $\theta = \frac{\pi}{6}$ and $\theta = \frac{\pi}{2}$. In fact we only have a positive value for $a\sin 4\theta$ between $0 \leqslant \theta \leqslant \frac{\pi}{4}$ and so the loop is only defined in this range. Note that $a\sin 4\theta = 0$ when $\theta = \frac{\pi}{4}$, which means that it intersects the origin.

$$A = \frac{1}{2}\int_0^{\frac{\pi}{6}} (a\sin 2\theta)^2 \, d\theta + \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} (a\sin 4\theta)^2 \, d\theta$$

$$= \frac{a^2}{4}\left(\int_0^{\frac{\pi}{6}} (1-\cos 4\theta)\, d\theta + \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} (1-\cos 8\theta)\, d\theta\right)$$

Now we calculate the area to be

$$= \frac{a^2}{4}\left(\left[\theta - \frac{\sin 4\theta}{4}\right]_0^{\frac{\pi}{6}} + \left[\theta - \frac{\sin 8\theta}{8}\right]_{\frac{\pi}{6}}^{\frac{\pi}{4}}\right)$$

$$= \frac{a^2}{4}\left(\left(\frac{\pi}{6} - \frac{\sqrt{3}}{8} - 0\right) + \left(\frac{\pi}{4} - \left(\frac{\pi}{6} + \frac{\sqrt{3}}{16}\right)\right)\right)$$

$$= \frac{a^2}{4}\left(\frac{\pi}{4} - \frac{3\sqrt{3}}{16}\right).$$

6 The points of intersection are given by $1+\sin\theta = 3\sin\theta$

So $2\sin\theta = 1$ which means $\theta = \arcsin\left(\frac{1}{2}\right) = \frac{\pi}{6}$ and $\frac{5\pi}{6}$.

So the polar coordinates of intersection are $\left(\frac{3}{2}, \frac{\pi}{6}\right)$ and $\left(\frac{3}{2}, \frac{5\pi}{6}\right)$.

Since there is symmetry about the vertical axis, we may compute the area as

$$\text{Area} = 2\times\left(\frac{1}{2}\int_0^{\frac{\pi}{6}}(3\sin\theta)^2\,d\theta + \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}(1+\sin\theta)^2\,d\theta\right)$$

$$= \int_0^{\frac{\pi}{6}}\left(9\sin^2\theta\right)d\theta + \int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\left(1+2\sin\theta+\sin^2\theta\right)d\theta$$

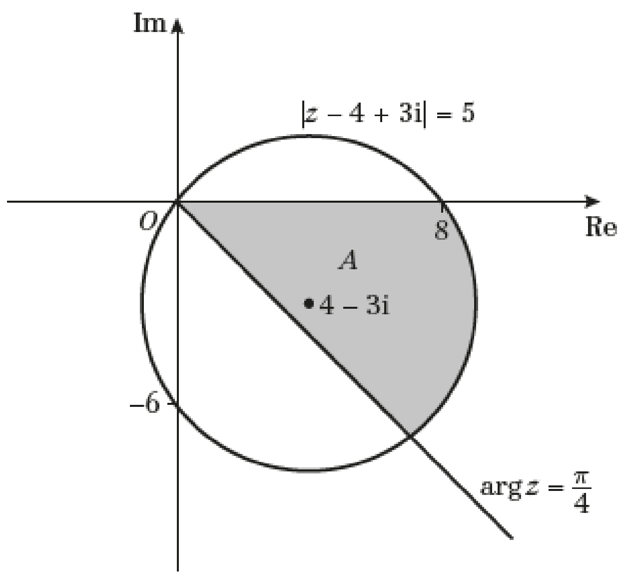
$$= \int_0^{\frac{\pi}{6}}\frac{9}{2}(1-\cos 2\theta)\,d\theta + \int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\left(1+2\sin\theta+\frac{1}{2}(1-\cos 2\theta)\right)d\theta$$

$$= \left[\frac{9}{2}\theta - \frac{9\sin 2\theta}{4}\right]_0^{\frac{\pi}{6}} + \left[\frac{3}{2}\theta - 2\cos\theta - \frac{\sin 2\theta}{4}\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$$

$$= \frac{3\pi}{4} - \frac{9\sqrt{3}}{8} + \frac{\pi}{2} + \frac{9\sqrt{3}}{8}$$

$$= \frac{5\pi}{4}$$

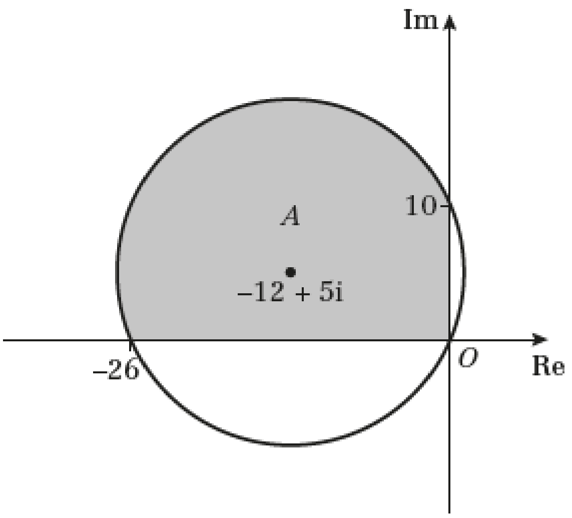
7 **a** The set of points $A = \left\{z : -\frac{\pi}{4} \leqslant \arg z \leqslant 0\right\} \cap \{z : |z-4+3i| \leqslant 5\}$ define a segment of a circle, centred at $(4,-3)$ with radius 5.



7 b The equation for a circle centred at $(4,-3)$ with radius 5, in Cartesian coordinates is $(x-4)^2+(y+3)^2=25$. To find the area of the region bounded by A, $\theta=-\frac{\pi}{4}$ and $\theta=0$, we convert into polar coordinates, obtaining $(r\cos\theta-4)^2+(r\sin\theta+3)^2=25$. This simplifies to $r=8\cos\theta-6\sin\theta$ when $r\neq 0$. Now we calculate

$$A=\frac{1}{2}\int_{-\frac{\pi}{4}}^{0}(8\cos\theta-6\sin\theta)^2\,\mathrm{d}\theta$$
$$=\frac{1}{2}\int_{-\frac{\pi}{4}}^{0}(64\cos^2\theta-96\cos\theta\sin\theta+36\sin^2\theta)\mathrm{d}\theta$$
$$=\frac{1}{2}\int_{-\frac{\pi}{4}}^{0}(32(1+\cos2\theta)-48\sin2\theta+18(1-\cos2\theta))\mathrm{d}\theta$$
$$=\frac{1}{2}\left[(32(\theta+\tfrac{1}{2}\sin2\theta)+24\cos2\theta+18(\theta-\tfrac{1}{2}\sin2\theta))\right]_{-\frac{\pi}{4}}^{0}$$
$$=\frac{1}{2}\left(24+7+\frac{25\pi}{2}\right)$$
$$\approx 35.1$$

8 a The set of points $A=\left\{z:\frac{\pi}{2}\leqslant\arg z\leqslant\pi\right\}\cap\{z:|z+12-5i|\leqslant 13\}$ define a region of a circle, centred at $(-12,5)$ with radius 13.



8 b To find the area of the region bounded by A, $\theta = \frac{\pi}{2}$ and $\theta = \pi$, we use the polar form $r = -24\cos\theta + 10\sin\theta$ to calculate

$$A = \frac{1}{2}\int_{\frac{\pi}{2}}^{\pi}(-24\cos\theta + 10\sin\theta)^2\,\mathrm{d}\theta$$

$$= \frac{1}{2}\int_{\frac{\pi}{2}}^{\pi}(576\cos^2\theta - 480\cos\theta\sin\theta + 100\sin^2\theta)\,\mathrm{d}\theta$$

$$= \frac{1}{2}\int_{\frac{\pi}{2}}^{\pi}(288(1+\cos 2\theta) - 240\sin 2\theta + 50(1-\cos 2\theta))\,\mathrm{d}\theta$$

$$= \frac{1}{2}\left[(288(\theta + \tfrac{1}{2}\sin 2\theta) + 120\cos 2\theta + 50(\theta - \tfrac{1}{2}\sin 2\theta))\right]_{\frac{\pi}{2}}^{\pi}$$

$$= \frac{1}{2}\left(338\pi + 120 - 169\pi + 120\right)$$

$$\approx 385.$$

9 In order to find the area of the shaded region, we must find the area of the sector bounded by the curve and the line OA, then subtract the area of the triangle OAB. The value of θ at the point A can be found by solving $r = 1 + \cos 3\theta = \frac{2+\sqrt{2}}{2}$ in order to get $\theta = \frac{\pi}{12}$.

We now find the area of the sector bounded by the curve and the line OA.

$$\begin{aligned}A_{sector} &= \frac{1}{2}\int_0^{\frac{\pi}{12}} (1+\cos 3\theta)^2 \, d\theta \\ &= \frac{1}{2}\int_0^{\frac{\pi}{12}} (1 + 2\cos 3\theta + \cos^2 3\theta) d\theta \\ &= \frac{1}{2}\int_0^{\frac{\pi}{12}} (1 + 2\cos 3\theta + \tfrac{1}{2}(\cos 6\theta + 1)) d\theta \\ &= \frac{1}{2}\left[(\theta + \tfrac{2}{3}\sin 3\theta + \tfrac{1}{2}(\tfrac{1}{6}\sin 6\theta + \theta))\right]_0^{\frac{\pi}{12}} \\ &= \frac{1}{2}\left(\frac{\pi}{8} + \frac{\sqrt{2}}{3} + \frac{1}{12}\right) \\ &\approx 0.47372.\end{aligned}$$

Now we find the area of the triangle OAB by using the formula

$Area_{OAB} = \frac{1}{2}ab\sin C$, where

a is the length of OA,
b is the length of OB and
C is the angle between OA and OB.

$$\begin{aligned}Area_{OAB} &= \frac{1}{2}ab\sin C \\ &= \frac{1}{2} \times \frac{2+\sqrt{2}}{2} \times \frac{2+\sqrt{2}}{2} \times \sin\left(\frac{\pi}{12}\right) \\ &\approx 0.37713.\end{aligned}$$

Thus, the area of the shaded region is found to be

$$\begin{aligned}A &= Area_{sector} - Area_{OAB} \\ &\approx 0.47372 - 0.37713 \\ &= 0.0966 \text{ (3sf)}\end{aligned}$$

10 Note that there is symmetry about the vertical axis and so we may compute one side and multiply by 2. In order to find the area of the shaded region, we first must find the area of the sector of $r = 1 + \sin\theta$ between $\theta = -\pi$ and the right hand side intersection point of the two curves.

This intersection point occurs when $1 + \sin\theta = 3\sin\theta$ i.e. when $\theta = \frac{\pi}{6}$.

So now we find the area of the sector of $r = 1 + \sin\theta$ for $-\pi \leqslant \theta \leqslant \frac{\pi}{6}$. We will denote this area A_1.

$$A_1 = \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{6}} (1+\sin\theta)^2 \, d\theta$$

$$= \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{6}} (1+2\sin\theta+\sin^2\theta) d\theta$$

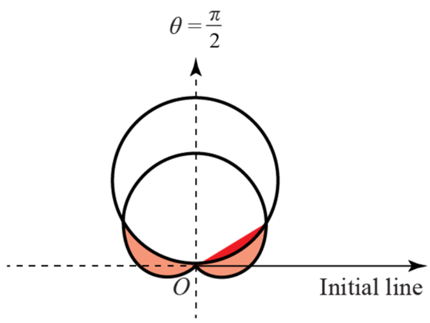
$$= \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{6}} (1+2\sin\theta+\tfrac{1}{2}(1-\cos 2\theta)) d\theta$$

$$= \frac{1}{2}\left[(\theta - 2\cos\theta + \tfrac{1}{2}(\theta - \tfrac{1}{2}\sin 2\theta))\right]_{-\frac{\pi}{2}}^{\frac{\pi}{6}}$$

$$= \frac{1}{2}\left(\pi - \frac{9\sqrt{3}}{8}\right)$$

$$\approx 0.59652.$$

This integral has included a small extra region we do not want (the red region in the image)



We find the value of the unwanted regions area (which we will denote A_2) by

$$A_2 = \frac{1}{2}\int_0^{\frac{\pi}{6}} (3\sin\theta)^2 d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{6}} (9\sin^2\theta) d\theta$$

$$= \frac{9}{4}\int_0^{\frac{\pi}{6}} (1-\cos 2\theta) d\theta$$

$$= \frac{9}{4}\left[(\theta - \tfrac{1}{2}\sin 2\theta)\right]_0^{\frac{\pi}{6}}$$

$$= \frac{9}{4}\left(\frac{\pi}{6} - \frac{\sqrt{3}}{4}\right)$$

$$\approx 0.20382.$$

So the right hand side of the shaded region has area

$$A_{right} \approx 0.59652 - 0.20382$$

$$= 0.3927.$$

Remember that this is only half of the total region we want to find, so that means we just need to double A_{right} in order to find $A_{total} = 0.79$ (2 d.p.)

Challenge

a The far left hand side of the shell is a point which occurs when $\theta = 3\pi$ and so we can set $r_{left} = 3k\pi$.

Similarly for the far right hand side of the shell, which occurs at $\theta = 4\pi$, giving $r_{right} = 4k\pi$.

Thus the horizontal diameter is

$d = 7k\pi = 3$ cm, so we conclude that $k = \dfrac{3}{7\pi}$

b In order to find the total area of the cross section, we must ensure that all angles are covered exactly once. So we choose to integrate between $2\pi \leqslant \theta \leqslant 4\pi$.

$$\begin{aligned}
A &= \frac{1}{2}\int_{2\pi}^{4\pi}\left(\frac{3\theta}{7\pi}\right)^2 \mathrm{d}\theta \\
&= \frac{1}{2}\int_{2\pi}^{4\pi}\left(\frac{9\theta^2}{49\pi^2}\right)\mathrm{d}\theta \\
&= \frac{1}{2}\left[\frac{3\theta^3}{49\pi^2}\right]_{2\pi}^{4\pi} \\
&= \frac{3}{98}(64\pi - 8\pi) \\
&= \frac{12\pi}{7}\text{ cm}^2
\end{aligned}$$