

Chapter review 7

1 a $y = e^{1-2x}$

$$y' = -2e^{1-2x} = -2y$$

$$y'' = -2y' = (-2)^2 y$$

$$\frac{d^n y}{dx^n} = (-2)^n y$$

b $\frac{d^8 y}{dx^8} = (-2)^8 y$

When $x = \log 32$, $\frac{d^8 y}{dx^8} = 2^8 e^{1-2 \log 32}$

$$= 2^8 e(32)^{-2} = 2^{8-2 \times 5} e = \frac{e}{4}$$

2 a $f(x) = \ln(1 + e^x)$

so $f(0) = \ln 2$

$$f'(x) = \frac{e^x}{1 + e^x}$$

$$= 1 - \frac{1}{1 + e^x} = 1 - (1 + e^x)^{-1}$$

$$f'(0) = \frac{1}{2}$$

So $f''(x) = \frac{e^x}{(1 + e^x)^2}$ or use the quotient rule $f''(0) = \frac{1}{4}$

b $f'''(x) = \frac{(1 + e^x)^2 e^x - e^x 2(1 + e^x) e^x}{(1 + e^x)^4}$

Use the quotient rule and chain rule.

$$= \frac{(1 + e^x) e^x \{(1 + e^x) - 2e^x\}}{(1 + e^x)^4} = \frac{e^x(1 - e^x)}{(1 + e^x)^3} \quad f'''(0) = 0$$

c Using Maclaurin's expansion

$$\ln(1 + e^x) = \ln 2 + \frac{x}{2} + \frac{x^2}{8} + \dots$$

The expansion is valid for $-1 < e^x \leq 1 \Rightarrow 0, e^x \leq 1$ so for $x \leq 0$.

3 a $\cos 4x = 1 - \frac{(4x)^2}{2!} + \frac{(4x)^4}{4!} - \frac{(4x)^6}{6!} + \dots$

$$= 1 - 8x^2 + \frac{32}{3}x^4 - \frac{256}{45}x^6 + \dots$$

b $\cos 4x = 1 - 2 \sin^2 2x$,

so $2 \sin^2 2x = 1 - 2 \cos 4x = 8x^2 - \frac{32}{3}x^4 + \frac{256}{45}x^6 + \dots$

$$\sin^2 2x = 4x^2 - \frac{16}{3}x^4 + \frac{128}{45}x^6 + \dots$$

4 Using $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \dots$ and $\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \dots$

$$\begin{aligned} e^{\cos x} &= e^{\left(1 - \frac{x^2}{2} + \frac{x^4}{24}\right)} = e \times e^{-\frac{x^2}{2}} \times e^{\frac{x^4}{24}} \\ &= e \left(1 + \left(-\frac{x^2}{2}\right) + \frac{1}{2} \left(-\frac{x^2}{2}\right)^2 + \dots\right) \left(1 + \frac{x^4}{24} + \dots\right) \quad \text{no other terms required} \\ &= e \left(1 - \frac{x^2}{2} + \frac{x^4}{8} + \dots\right) \left(1 + \frac{x^4}{24} + \dots\right) \\ &= e \left(1 - \frac{x^2}{2} + \frac{x^4}{8} + \frac{x^4}{24} + \dots\right) = e \left(1 - \frac{x^2}{2} + \frac{x^4}{6} + \dots\right) \end{aligned}$$

5 Let $f(x) = \left(x - \frac{\pi}{4}\right) \cot x$ and $a = \frac{\pi}{4} \Rightarrow f(a) = 0$

$$f'(x) = \left(x - \frac{\pi}{4}\right) (-\operatorname{cosec}^2 x) + \cot x \Rightarrow f'(a) = 1$$

$$f''(x) = \left(x - \frac{\pi}{4}\right) 2 \cot x \operatorname{cosec}^2 x + (-2 \operatorname{cosec}^2 x) \Rightarrow f''(a) = -4$$

$$f'''(x) = \left(x - \frac{\pi}{4}\right) (-2 \operatorname{cosec}^4 x - 4 \cot^2 x \operatorname{cosec}^2 x) + 6 \cot x \operatorname{cosec}^2 x \Rightarrow f'''(a) = 12$$

Substituting into the Taylor series expansion gives

$$\begin{aligned} f(x) &= 0 + 1 \left(x - \frac{\pi}{4}\right) + \frac{-4}{2!} \left(x - \frac{\pi}{4}\right)^2 + \frac{12}{3!} \left(x - \frac{\pi}{4}\right)^3 + \dots \\ &= \left(x - \frac{\pi}{4}\right) - 2 \left(x - \frac{\pi}{4}\right)^2 + 2 \left(x - \frac{\pi}{4}\right)^3 + \dots \quad \text{as required} \end{aligned}$$

6 $\ln((1+x)^2(1-2x)) = 2\ln(1+x) + \ln(1-2x)$

$$\begin{aligned} &= 2 \left\{ x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots \right\} + \left\{ (-2x) - \frac{(-2x)^2}{2} - \frac{(-2x)^3}{3} - \frac{(-2x)^4}{4} + \dots \right\} \\ &= 2x - x^2 + \frac{2}{3}x^3 - \frac{1}{2}x^4 - 2x - 2x^2 - \frac{8}{3}x^3 - 4x^4 + \dots \\ &= -3x^2 - 2x^3 - \dots \end{aligned}$$

7 $f(x) = \ln(\sec x + \tan x)$ $f(0) = \ln 1 = 0$

$$f'(x) = \frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x} = \frac{\sec x (\tan x + \sec x)}{\sec x + \tan x} = \sec x \quad f'(0) = 1$$

$$f''(x) = \sec x \tan x \quad f''(0) = 0$$

$$f'''(x) = \sec x \sec^2 x + \sec x \tan x \tan x \quad f'''(0) = 1$$

Substituting into Maclaurin's expansion gives $y = x + \frac{x^3}{6} + \dots$

$$\begin{aligned}
 8 \text{ a } \frac{d}{dx}(e^x) &= \frac{d}{dx} \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots + \frac{x^r}{r!} + \frac{x^{r+1}}{(r+1)!} + \dots \right) \\
 &= 1 + \frac{2x}{2!} + \frac{3x^2}{3!} + \frac{4x^3}{4!} + \dots + \frac{(r+1)x^r}{(r+1)!} + \dots \\
 &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^r}{r!} + \dots \\
 &= e^x
 \end{aligned}$$

$$\begin{aligned}
 8 \text{ b } \frac{d}{dx}(\sin x) &= \frac{d}{dx} \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots + (-1)^r \frac{x^{2r+1}}{(2r+1)!} + \dots \right) \\
 &= 1 - \frac{3x^2}{3!} + \frac{5x^4}{5!} - \dots + (-1)^r \frac{(2r+1)x^{2r}}{(2r+1)!} + \dots \\
 &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots + (-1)^r \frac{x^{2r}}{(2r)!} + \dots = \cos x
 \end{aligned}$$

$$\begin{aligned}
 8 \text{ c } \frac{d}{dx}(\cos x) &= \frac{d}{dx} \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots + (-1)^r \frac{x^{2r}}{(2r)!} + (-1)^{r+1} \frac{x^{2r+2}}{(2r+2)!} + \dots \right) \\
 &= \left(-\frac{2x}{2!} + \frac{4x^3}{4!} - \frac{6x^5}{6!} + \dots + (-1)^r \frac{2rx^{2r-1}}{(2r)!} + (-1)^{r+1} \frac{(2r+2)x^{2r+1}}{(2r+2)!} + \dots \right) \\
 &= -x + \frac{x^3}{3!} - \frac{x^5}{5!} + \dots + (-1)^{r+1} \frac{x^{2r+1}}{(2r+1)!} + \dots \\
 &= - \left(x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \dots + \frac{(-1)^r}{(2r+1)!}x^{2r+1} + \dots \right) = -\sin x
 \end{aligned}$$

9 a You can write $\cos x = 1 - \left(\frac{x^2}{2} - \frac{x^4}{24} + \dots \right)$; it is not necessary to have higher powers

$$\sec x = \frac{1}{\cos x} = \frac{1}{1 - \left(\frac{x^2}{2} - \frac{x^4}{24} + \dots \right)} = \left\{ 1 - \left(\frac{x^2}{2} - \frac{x^4}{24} + \dots \right) \right\}^{-1}$$

Using the binomial expansion but only requiring powers up to x^4

$$\begin{aligned}
 \sec x &= 1 + (-1) \left\{ - \left(\frac{x^2}{2} - \frac{x^4}{24} \right) \right\} + \frac{(-1)(-2)}{2!} \left\{ - \left(\frac{x^2}{2} - \frac{x^4}{24} \right) \right\}^2 + \dots \\
 &= 1 + \left(\frac{x^2}{2} - \frac{x^4}{24} \right) + \frac{x^4}{4} + \text{higher powers of } x \\
 &= 1 + \frac{x^2}{2} + \frac{5}{24}x^4 + \dots
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{9\ b} \quad \tan x &= \frac{\sin x}{\cos x} = \sin x \times \sec x \\
 &= \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots \right) \left(1 + \frac{x^2}{2} + \frac{5}{24}x^4 + \dots \right) \\
 &= x + \frac{x^3}{2} + \frac{5}{24}x^5 - \frac{x^3}{3!} - \frac{1}{2(3!)}x^5 + \frac{x^5}{5!} + \dots \\
 &= x + \left(\frac{1}{2} - \frac{1}{6} \right) x^3 + \left(\frac{5}{24} - \frac{1}{12} + \frac{1}{120} \right) x^5 + \dots \\
 &= x + \frac{x^3}{3} + \frac{16}{120}x^5 + \dots \\
 &= x + \frac{x^3}{3} + \frac{2}{15}x^5 + \dots
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{10} \text{ Using } e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \text{ and } \cos 3x = 1 - \frac{(3x)^2}{2!} + \dots \\
 e^x \cos 3x &= \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \dots \right) \left(1 - \frac{9x^2}{2} + \dots \right) \\
 &= \left\{ 1 + x + \left(\frac{x^2}{2} - \frac{9x^2}{2} \right) + \left(\frac{x^3}{6} - \frac{9x^3}{2} \right) + \dots \right\} \\
 &= 1 + x - 4x^2 - \frac{13}{3}x^3 + \dots
 \end{aligned}$$

$$\mathbf{11} \quad f(x) = (1+x)^2 \ln(1+x).$$

$$f'(x) = (1+x)^2 \frac{1}{1+x} + 2(1+x) \ln(1+x) = (1+x) \{1 + 2 \ln(1+x)\}$$

$$f''(x) = (1+x) \left(\frac{2}{1+x} \right) + \{1 + 2 \ln(1+x)\} = 3 + 2 \ln(1+x)$$

$$f'''(x) = \left(\frac{2}{1+x} \right)$$

$$f(0) = 0, f'(0) = 1, f''(0) = 3, f'''(0) = 2$$

Using Maclaurin's expansion

$$\begin{aligned}
 (1+x)^2 \ln(1+x) &= 0 + (1)x + \frac{3}{2!}x^2 + \frac{2}{3!}x^3 + \dots \\
 &= x + \frac{3}{2}x^2 + \frac{1}{3}x^3 + \dots
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{12\ a} \quad \ln(1 + \sin x) &= \ln \left\{ 1 + \left(x - \frac{x^3}{3!} + \dots \right) \right\} \\
 &= \left(x - \frac{x^3}{3!} + \dots \right) - \frac{1}{2} \left(x - \frac{x^3}{3!} + \dots \right)^2 + \frac{1}{3} \left(x - \frac{x^3}{3!} + \dots \right)^3 - \frac{1}{4} \left(x - \frac{x^3}{3!} + \dots \right)^4 + \dots \\
 &= \left(x - \frac{x^3}{6} + \dots \right) - \frac{1}{2} \left(x^2 - \frac{x^4}{3} + \dots \right) + \frac{1}{3} (x^3 + \dots) - \frac{1}{4} (x^4 + \dots) \quad \text{no other terms necessary} \\
 &= x - \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{12} + \dots
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad \int_0^{\frac{\pi}{6}} \ln(1 + \sin x) dx &\approx \int_0^{\frac{\pi}{6}} \left(x - \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{12} \right) dx \\
 &\approx \left[\frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{60} \right]_0^{\frac{\pi}{6}} = \frac{\pi^2}{72} - \frac{\pi^3}{1296} + \frac{\pi^4}{31104} - \frac{\pi^5}{466560} = 0.116 \text{ (3 d.p.)}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{13\ a} \quad f(x) = e^{\tan x} &= e^{x + \frac{x^3}{3} + \dots} = e^x \times e^{\frac{x^3}{3}} \quad (\text{As only terms up to } x^3 \text{ are required, only first two terms of } \tan x \text{ are needed.}) \\
 &= \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \right) \left(1 + \frac{x^3}{3} + \dots \right) \quad \text{no other terms required.} \\
 &= \left(1 + \frac{x^3}{3} + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \right) \\
 &= 1 + x + \frac{x^2}{2} + \frac{x^3}{2} + \dots
 \end{aligned}$$

$$\mathbf{b} \quad e^{-\tan x} = e^{\tan(-x)}, \text{ so replacing } x \text{ by } -x \text{ in } \mathbf{a} \text{ gives}$$

$$e^{-\tan x} = 1 - x + \frac{x^2}{2} - \frac{x^3}{2} + \dots$$

$$\begin{array}{ll}
 \mathbf{14\ a} \quad f(x) = \ln \cos x & f(0) = 0 \\
 f'(x) = \frac{-\sin x}{\cos x} = -\tan x & f'(0) = 0 \\
 f''(x) = -\sec^2 x & f''(0) = -1 \\
 f'''(x) = -2 \sec^2 x \tan x & f'''(0) = 0 \\
 f''''(x) = -2 \sec^4 x - 4 \sec^2 x \tan^2 x & f''''(0) = -2
 \end{array}$$

Substituting into Maclaurin:

$$\ln \cos x = (-1) \frac{x^2}{2!} + (-2) \frac{x^4}{4!} + \dots = -\frac{x^2}{2} - \frac{x^4}{12} - \dots$$

14 b Using $1 + \cos x = 2 \cos^2 \left(\frac{x}{2} \right)$, $\ln(1 + \cos x) = \ln 2 \cos^2 \left(\frac{x}{2} \right) = \ln 2 + 2 \ln \cos \left(\frac{x}{2} \right)$

$$\text{so } \ln(1 + \cos x) = \ln 2 + 2 \left\{ -\frac{1}{2} \left(\frac{x}{2} \right)^2 - \frac{1}{12} \left(\frac{x}{2} \right)^4 - \dots \right\} = \ln 2 - \frac{x^2}{4} - \frac{x^4}{96} - \dots$$

15 a

$$y = e^{3x} - e^{-3x}$$

$$y' = 3e^{3x} + 3e^{-3x}$$

$$y'' = 9e^{3x} - 9e^{-3x} = 9y$$

$$y''' = 9y', y'''' = 9y'' = 81y$$

b When $x = 0$,

$$y = 0$$

$$y' = 6$$

$$y'' = 9y = 0$$

$$y''' = 9y' = 54$$

$$y'''' = 81y = 0$$

$$y''''' = 81y' = 486$$

$$\text{so } y = 6x + \frac{54}{3!}x^3 + \frac{486}{5!}x^5 + \dots$$

$$= 6x + 9x^3 + \frac{81}{20}x^5 + \dots$$

c

$$e^{3x} = \sum_{n=0}^{\infty} \frac{(3x)^n}{n!}, e^{-3x} = \sum_{n=0}^{\infty} \frac{(-3x)^n}{n!}$$

$$y = e^{3x} - e^{-3x}$$

$$= \sum_{n=0}^{\infty} \frac{(3x)^n}{n!} - \frac{(-3x)^n}{n!} = \sum_{n=0}^{\infty} (1 - (-1)^n) \frac{(3x)^n}{n!}$$

$$n^{\text{th}} \text{ non-zero term is } (1 - (-1)^{2n-1}) \frac{3^{2n-1} x^{2n-1}}{(2n-1)!} = 2 \frac{3^{2n-1} x^{2n-1}}{(2n-1)!}$$

16 a $f(x) = \ln(1 + e^x)$

$$f'(x) = \frac{e^x}{1 + e^x}$$

$$= 1 - \frac{1}{1 + e^x} = 1 - (1 + e^x)^{-1}$$

so $f(0) = \ln 2$

$$f'(0) = \frac{1}{2}$$

So $f''(x) = \frac{e^x}{(1 + e^x)^2}$ or use the quotient rule

$$f''(0) = \frac{1}{4}$$

b $f'''(x) = \frac{(1 + e^x)^2 e^x - e^x 2(1 + e^x) e^x}{(1 + e^x)^4}$

$$= \frac{(1 + e^x) e^x \{(1 + e^x) - 2e^x\}}{(1 + e^x)^4} = \frac{e^x (1 - e^x)}{(1 + e^x)^3}$$

Use the quotient rule and chain rule.

$$f'''(0) = 0$$

16 c Using Maclaurin's expansion:

$$\ln(1 + e^x) = \ln 2 + \frac{x}{2} + \frac{x^2}{8} + \dots$$

The expansion is valid for $-1 < e^x \leq 1 \Rightarrow 0, e^x \leq 1$ so for $x \leq 0$

$$\begin{aligned} \mathbf{17 a} \quad \cos 4x &= 1 - \frac{(4x)^2}{2!} + \frac{(4x)^4}{4!} - \frac{(4x)^6}{6!} + \dots \\ &= 1 - 8x^2 + \frac{32}{3}x^4 - \frac{256}{45}x^6 + \dots \end{aligned}$$

$$\mathbf{b} \quad \cos 4x = 1 - 2\sin^2 2x,$$

$$\text{so } 2\sin^2 2x = 1 - \cos 4x = 8x^2 - \frac{32}{3}x^4 + \frac{256}{45}x^6 + \dots$$

$$\sin^2 2x = 4x^2 - \frac{16}{3}x^4 + \frac{128}{45}x^6 + \dots$$

18 Using $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \dots$ and $\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \dots$

$$\begin{aligned} e^{\cos x} &= e^{\left(1 - \frac{x^2}{2} + \frac{x^4}{24}\right)} = e \times e^{-\frac{x^2}{2}} \times e^{\frac{x^4}{24}} \\ &= e \left(1 + \left(-\frac{x^2}{2}\right) + \frac{1}{2} \left(-\frac{x^2}{2}\right)^2 + \dots \right) \left(1 + \frac{x^4}{24} + \dots \right) \quad \text{no other terms required} \\ &= e \left(1 - \frac{x^2}{2} + \frac{x^4}{8} + \dots \right) \left(1 + \frac{x^4}{24} + \dots \right) \\ &= e \left(1 - \frac{x^2}{2} + \frac{x^4}{8} + \frac{x^4}{24} + \dots \right) = e \left(1 - \frac{x^2}{2} + \frac{x^4}{6} + \dots \right) \end{aligned}$$

$$\mathbf{19 a} \quad \frac{dy}{dx} = 2 + x + \sin y \text{ and } x_0 = 0, y_0 = 0 \quad \mathbf{(1)} \quad \text{so } \left(\frac{dy}{dx}\right)_0 = 2$$

$$\text{Differentiating (1) gives } \frac{d^2y}{dx^2} = 1 + \cos y \frac{dy}{dx} \quad \mathbf{(2)}$$

$$\text{Substituting } x_0 = 0, y_0 = 0, \left(\frac{dy}{dx}\right)_0 = 2 \text{ into (2) gives } \left(\frac{d^2y}{dx^2}\right)_0 = 3$$

$$\text{Differentiating (2) gives } \frac{d^3y}{dx^3} = \cos y \frac{d^2y}{dx^2} - \sin y \left(\frac{dy}{dx}\right)^2 \quad \mathbf{(3)}$$

$$\text{Substituting } y_0 = 0, \left(\frac{dy}{dx}\right)_0 = 2, \left(\frac{d^2y}{dx^2}\right)_0 = 3 \text{ into (3) gives } \left(\frac{d^3y}{dx^3}\right)_0 = 3$$

$$\text{Substituting found values into } y = y_0 + x \left(\frac{dy}{dx}\right)_0 + \frac{x^2}{2!} \left(\frac{d^2y}{dx^2}\right)_0 + \frac{x^3}{3!} \left(\frac{d^3y}{dx^3}\right)_0 + \dots$$

$$y = 2x + \frac{3}{2}x^2 + \frac{1}{2}x^3 + \dots$$

$$19 \text{ b } \text{ At } x = 0.1, y \approx 2(0.1) + \frac{3}{2}(0.1)^2 + \frac{1}{2}(0.1)^3 = 0.2155$$

$$\begin{aligned}
 20 \ln((1+x)^2(1-2x)) &= 2\ln(1+x) + \ln(1-2x) \\
 &= 2\left\{x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots\right\} + \left\{(-2x) - \frac{(-2x)^2}{2} + \frac{(-2x)^3}{3} - \frac{(-2x)^4}{4} + \dots\right\} \\
 &= 2x - x^2 + \frac{2}{3}x^3 - \frac{1}{2}x^4 - 2x - 2x^2 - \frac{8}{3}x^3 - 4x^4 + \dots \\
 &= -3x^2 - 2x^3 - \dots
 \end{aligned}$$

$$21 \frac{d^2y}{dx^2} - (x+2)\frac{dy}{dx} + 3y = 0 \quad (1)$$

$$\text{Differentiating (1) gives } \frac{d^3y}{dx^3} - (x+2)\frac{d^2y}{dx^2} - \frac{dy}{dx} + 3\frac{dy}{dx} = 0$$

$$\text{So that } \frac{d^3y}{dx^3} - (x+2)\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 0 \quad (2)$$

$$\text{Substituting initial data in (1) gives } \left(\frac{d^2y}{dx^2}\right)_0 = 2$$

$$\text{Substituting known data in (2) gives } \left(\frac{d^3y}{dx^3}\right)_0 = -4$$

$$\begin{aligned}
 \text{So } y &= 2 + 4x + \frac{2x^2}{2!} - \frac{4x^3}{3!} + \dots \\
 &= 2 + 4x + x^2 - \frac{2}{3}x^3
 \end{aligned}$$

$$22 \text{ a } f(x) = \ln(\sec x + \tan x)$$

$$f(0) = \ln 1 = 0$$

$$f'(x) = \frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x} = \frac{\sec x(\tan x + \sec x)}{\sec x + \tan x} = \sec x$$

$$f'(0) = 1$$

$$f''(x) = \sec x \tan x$$

$$f''(0) = 0$$

$$f'''(x) = \sec x \sec^2 x + \sec x \tan x \tan x$$

$$f'''(0) = 1$$

$$\text{Substituting into Maclaurin's expansion gives } y = x + \frac{x^3}{6} + \dots$$

22 b We use the expansions:

$$\ln(\sec x + \tan x) = x + \frac{1}{6}x^3 + \dots$$

$$\sin x = x - \frac{1}{3!}x^3 + \dots$$

to see that:

$$\begin{aligned} & \frac{\sin x - \ln(\sec x + \tan x)}{x(\cos x - 1)} \\ &= \frac{\left(x - \frac{1}{3!}x^3 + \dots\right) - \left(x + \frac{1}{6}x^3 + \dots\right)}{x\left(1 - \frac{1}{2}x^2 + \dots - 1\right)} \\ &= \frac{-\frac{1}{3}x^3 + \dots}{-\frac{1}{2}x^3 + \dots} \\ \Rightarrow \lim_{x \rightarrow 0} \frac{\sin x - \ln(\sec x + \tan x)}{x(\cos x - 1)} \\ &= \lim_{x \rightarrow 0} \frac{-\frac{1}{3}x^3 + \dots}{-\frac{1}{2}x^3 + \dots} = \frac{2}{3} \end{aligned}$$

23 a We differentiate the respective Taylor series term by term and match that up with the derivative.

Firstly:

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \dots + \frac{1}{r!}x^r + \dots$$

$$\begin{aligned} \Rightarrow \frac{d}{dx} e^x &= 1 + \frac{2}{2!}x + \frac{3}{3!}x^2 + \frac{4}{4!}x^3 + \dots \\ &+ \frac{r}{r!}x^{r-1} + \frac{r+1}{(r+1)!}x^r + \dots \end{aligned}$$

$$\begin{aligned} \Rightarrow \frac{d}{dx} e^x &= 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \dots \\ &+ \frac{1}{(r-1)!}x^{r-1} + \frac{1}{r!}x^r + \dots = e^x \end{aligned}$$

$$\mathbf{b} \quad \frac{d}{dx} \sin x = 1 - \frac{3}{3!}x^2 + \dots + \frac{(-1)^r(2r+1)}{(2r+1)!}x^{2r} + \dots$$

$$\Rightarrow \frac{d}{dx} \sin x = 1 - \frac{1}{2!}x^2 + \dots + \frac{(-1)^r}{(2r)!}x^{2r} + \dots = \cos x$$

$$\mathbf{c} \quad \frac{d}{dx} \cos x = -\frac{2}{2!}x + \frac{4}{4!}x^3 + \dots + \frac{(-1)^r(2r)}{(2r)!}x^{2(r-1)+1} + \dots$$

$$\begin{aligned} \Rightarrow \frac{d}{dx} \cos x &= -x + \frac{1}{3!}x^3 + \dots + \frac{(-1)^r}{(2r-1)!}x^{2(r-1)+1} + \dots \\ &= -\left(x - \frac{1}{3!}x^3 + \dots + \frac{(-1)^{r-1}}{(2r-1)!}x^{2(r-1)+1} + \dots\right) = -\sin x \end{aligned}$$

$$24 \quad \frac{d^2 y}{dx^2} + y \frac{dy}{dx} = x \quad (1)$$

$$\text{Differentiating } \frac{d^2 y}{dx^2} + y \frac{dy}{dx} = x, \text{ gives } \frac{d^3 y}{dx^3} + y \frac{d^2 y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 1 \quad (2)$$

$$\text{Substituting initial values into (1) gives } \left(\frac{d^2 y}{dx^2}\right)_1 = 1$$

$$\text{Substituting } \left(\frac{dy}{dx}\right)_1 = 2 \text{ and } \left(\frac{d^2 y}{dx^2}\right)_1 = 1 \text{ into (2) gives } \left(\frac{d^3 y}{dx^3}\right)_1 = -3.$$

Using Taylor's expansion in the form with $x_0 = 1$

$$\begin{aligned} y &= 0 + 2(x-1) + \frac{(1)}{2!}(x-1)^2 + \frac{(-3)}{3!}(x-1)^3 + \dots \\ &= 2(x-1) + \frac{1}{2}(x-1)^2 - \frac{1}{2}(x-1)^3 + \dots \end{aligned}$$

25 a You can write $\cos x = 1 - \left(\frac{x^2}{2} - \frac{x^4}{24} + \dots\right)$; it is not necessary to have higher powers

$$\sec x = \frac{1}{\cos x} = \frac{1}{1 - \left(\frac{x^2}{2} - \frac{x^4}{24} + \dots\right)} = \left\{1 - \left(\frac{x^2}{2} - \frac{x^4}{24} + \dots\right)\right\}^{-1}$$

Using the binomial expansion but only requiring powers up to x^4

$$\begin{aligned} \sec x &= 1 + (-1) \left\{-\left(\frac{x^2}{2} - \frac{x^4}{24}\right)\right\} + \frac{(-1)(-2)}{2!} \left\{-\left(\frac{x^2}{2} - \frac{x^4}{24}\right)\right\}^2 + \dots \\ &= 1 + \left(\frac{x^2}{2} - \frac{x^4}{24}\right) + \frac{x^4}{4} + \text{higher powers of } x \\ &= 1 + \frac{x^2}{2} + \frac{5}{24}x^4 + \dots \end{aligned}$$

$$b \quad \tan x = \frac{\sin x}{\cos x} = \sin x \times \sec x$$

$$\begin{aligned} &= \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots\right) \left(1 + \frac{x^2}{2} + \frac{5}{24}x^4 + \dots\right) \\ &= x + \frac{x^3}{2} + \frac{5}{24}x^5 - \frac{x^3}{3!} - \frac{1}{2(3!)}x^5 + \frac{x^5}{5!} + \dots \\ &= x + \left(\frac{1}{2} - \frac{1}{6}\right)x^3 + \left(\frac{5}{24} - \frac{1}{12} + \frac{1}{120}\right)x^5 + \dots \\ &= x + \frac{x^3}{3} + \frac{16}{120}x^5 + \dots \\ &= x + \frac{x^3}{3} + \frac{2}{15}x^5 + \dots \end{aligned}$$

26 Using $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$ and $\cos 3x = 1 - \frac{(3x)^2}{2!} + \dots$

$$\begin{aligned} e^x \cos 3x &= \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \dots\right) \left(1 - \frac{9x^2}{2} + \dots\right) \\ &= \left\{1 + x + \left(\frac{x^2}{2} - \frac{9x^2}{2}\right) + \left(\frac{x^3}{6} - \frac{9x^3}{2}\right) + \dots\right\} \\ &= 1 + x - 4x^2 - \frac{13}{3}x^3 + \dots \end{aligned}$$

27 a Differentiating $\frac{d^2y}{dx^2} + x^2 \frac{dy}{dx} + y = 0$ (1) with respect to x , gives:

$$\frac{d^3y}{dx^3} + 2x \frac{dy}{dx} + x^2 \frac{d^2y}{dx^2} + \frac{dy}{dx} = 0 \quad (2)$$

Substituting given data $x_0 = 0$, $y_0 = 2$ and $\left(\frac{dy}{dx}\right)_0 = 1$ into (1) gives $\left(\frac{d^2y}{dx^2}\right)_0 = -2$

Substituting $x_0 = 0$, $\left(\frac{dy}{dx}\right)_0 = 1$ and $\left(\frac{d^2y}{dx^2}\right)_0 = -2$ into (2) gives $\left(\frac{d^3y}{dx^3}\right)_0 = -1$

So using Taylor series $y = y_0 + x\left(\frac{dy}{dx}\right)_0 + \frac{x^2}{2!}\left(\frac{d^2y}{dx^2}\right)_0 + \frac{x^3}{3!}\left(\frac{d^3y}{dx^3}\right)_0 + \dots$

$$y = 2 + x - x^2 - \frac{x^3}{6} + \dots$$

b Differentiating (2) with respect to x gives:

$$\frac{d^4y}{dx^4} + 2x \frac{d^2y}{dx^2} + 2 \frac{dy}{dx} + x^2 \frac{d^3y}{dx^3} + 2x \frac{d^2y}{dx^2} + \frac{d^2y}{dx^2} = 0 \quad (3)$$

Substituting $x = 0$, $\left(\frac{dy}{dx}\right)_0 = 1$, $\left(\frac{d^2y}{dx^2}\right)_0 = -2$ and $\left(\frac{d^3y}{dx^3}\right)_0 = -1$ into (3) gives,

$$\text{at } x = 0, \frac{d^4y}{dx^4} + 2(1) + (-2) = 0, \text{ so } \frac{d^4y}{dx^4} = 0$$

28 a $f(x) = (1+x)^2 \ln(1+x)$

$$f'(x) = (1+x)^2 \frac{1}{1+x} + 2(1+x) \ln(1+x) = (1+x)(1 + 2 \ln(1+x))$$

$$f''(x) = (1+x) \left(\frac{2}{1+x}\right) + (1 + 2 \ln(1+x)) = 3 + 2 \ln(1+x)$$

$$f'''(x) = \left(\frac{2}{1+x}\right)$$

$$f(0) = 0, f'(0) = 1, f''(0) = 3, f'''(0) = 2$$

b Using Maclaurin's expansion

$$(1+x)^2 \ln(1+x) = 0 + (1)x + \frac{3}{2!}x^2 + \frac{2}{3!}x^3 + \dots$$

$$= x + \frac{3}{2}x^2 + \frac{1}{3}x^3 + \dots$$

$$\begin{aligned}
 \mathbf{29\ a} \quad \ln(1 + \sin x) &= \ln \left\{ 1 + \left(x - \frac{x^3}{3!} + \dots \right) \right\} \\
 &= \left(x - \frac{x^3}{3!} + \dots \right) - \frac{1}{2} \left(x - \frac{x^3}{3!} + \dots \right)^2 + \frac{1}{3} \left(x - \frac{x^3}{3!} + \dots \right)^3 - \frac{1}{4} \left(x - \frac{x^3}{3!} + \dots \right)^4 + \dots \\
 &= \left(x - \frac{x^3}{6} + \dots \right) - \frac{1}{2} \left(x^2 - \frac{x^4}{3} + \dots \right) + \frac{1}{3} (x^3 + \dots) - \frac{1}{4} (x^4 + \dots) \text{ no other terms necessary} \\
 &= x - \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{12} + \dots
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad \int_0^{\frac{\pi}{6}} \ln(1 + \sin x) dx &\approx \int_0^{\frac{\pi}{6}} \left(x - \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{12} \right) dx \\
 &\approx \left[\frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{60} \right]_0^{\frac{\pi}{6}} = \frac{\pi^2}{72} - \frac{\pi^3}{1296} + \frac{\pi^4}{31104} - \frac{\pi^5}{466560} = 0.116 \text{ (3 d.p.)}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{30\ a} \quad f(x) = e^{\tan x} &= e^{x + \frac{x^3}{3} + \dots} = e^x \times e^{\frac{x^3}{3}} \quad (\text{As only terms up to } x^3 \text{ are required, only first two terms of } \tan x \text{ are needed.)} \\
 &= \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \right) \left(1 + \frac{x^3}{3} + \dots \right) \text{ no other terms required.} \\
 &= \left(1 + \frac{x^3}{3} + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \right) \\
 &= 1 + x + \frac{x^2}{2} + \frac{x^3}{2} + \dots
 \end{aligned}$$

$$\mathbf{b} \quad e^{-\tan x} = e^{\tan(-x)}, \text{ so replacing } x \text{ by } -x \text{ in a gives}$$

$$e^{-\tan x} = 1 - x + \frac{x^2}{2} - \frac{x^3}{2} + \dots$$

31 a Differentiating the given differential equation with respect to x gives:

$$y \frac{d^3 y}{dx^3} + \frac{dy}{dx} \frac{d^2 y}{dx^2} + 2 \frac{dy}{dx} \frac{d^2 y}{dx^2} + \frac{dy}{dx} = 0$$

$$\text{So } \frac{d^3 y}{dx^3} = -\frac{1}{y} \left\{ \frac{dy}{dx} \left(3 \frac{d^2 y}{dx^2} + 1 \right) \right\}$$

31 b Given that $y_0 = 1$, $\left(\frac{dy}{dx}\right)_0 = 1$ at $x = 0$,

$$\left(\frac{d^2y}{dx^2}\right)_0 + (1)^2 + (1) = 0, \text{ so } \left(\frac{d^2y}{dx^2}\right)_0 = -2,$$

$$\text{And } \left(\frac{d^3y}{dx^3}\right)_0 = -\frac{1}{(1)}(1)(3(-2)+1), \text{ so } \left(\frac{d^3y}{dx^3}\right)_0 = 5$$

$$\text{So } y = 1 + (1)x + \frac{(-2)}{2!}x^2 + \frac{5}{3!}x^3 + \dots = 1 + x - x^2 + \frac{5x^3}{6} + \dots$$

c The approximation is best for small values of x (closed to 0): $x = 0.2$, therefore, would be acceptable, but not $x = 50$

32 a $f(x) = \ln \cos x$ $f(0) = 0$

$$f'(x) = \frac{-\sin x}{\cos x} = -\tan x$$
 $f'(0) = 0$

$$f''(x) = -\sec^2 x$$
 $f''(0) = -1$

$$f'''(x) = -2\sec^2 x \tan x$$
 $f'''(0) = 0$

$$f''''(x) = -2\sec^4 x - 4\sec^2 x \tan^2 x$$
 $f''''(0) = -2$

Substituting into Maclaurin:

$$\ln \cos x = (-1)\frac{x^2}{2!} + (-2)\frac{x^4}{4!} + \dots = -\frac{x^2}{2} - \frac{x^4}{12} - \dots$$

b Using $1 + \cos x = 2 \cos^2\left(\frac{x}{2}\right)$, $\ln(1 + \cos x) = \ln 2 \cos^2\left(\frac{x}{2}\right) = \ln 2 + 2 \ln \cos\left(\frac{x}{2}\right)$

$$\text{so } \ln(1 + \cos x) = \ln 2 + 2 \left\{ -\frac{1}{2}\left(\frac{x}{2}\right)^2 - \frac{1}{12}\left(\frac{x}{2}\right)^4 - \dots \right\} = \ln 2 - \frac{x^2}{4} - \frac{x^4}{96} - \dots$$

33 a Let $y = 3^x$, then $\ln y = \ln 3^x = x \ln 3 \Rightarrow y = e^{x \ln 3}$ so $3^x = e^{x \ln 3}$

$$\begin{aligned} 3^x &= e^{x \ln 3} = 1 + (x \ln 3) + \frac{(x \ln 3)^2}{2!} + \frac{(x \ln 3)^3}{3!} + \dots \\ &= 1 + x \ln 3 + \frac{x^2 (\ln 3)^2}{2} + \frac{x^3 (\ln 3)^3}{6} + \dots \end{aligned}$$

b Put $x = \frac{1}{2}$: $\sqrt{3} \approx 1 + \frac{\ln 3}{2} + \frac{(\ln 3)^2}{8} + \frac{(\ln 3)^3}{48} = 1.73$ (3 s.f.)

34 a We take $f(x) = \ln\left(1 + 2\cos\left(\frac{\pi x}{2}\right)\right)$, then differentiating:

$$f'(x) = \frac{1}{1 + 2\cos\left(\frac{\pi x}{2}\right)} \cdot \left(-2 \cdot \frac{\pi}{2} \sin\left(\frac{\pi x}{2}\right)\right) = -\frac{\pi \sin\left(\frac{\pi x}{2}\right)}{1 + 2\cos\left(\frac{\pi x}{2}\right)}$$

$$\begin{aligned} f''(x) &= -\pi \left(\frac{\pi \cos\left(\frac{\pi x}{2}\right)}{2(1 + 2\cos\left(\frac{\pi x}{2}\right))} + \frac{\pi \sin^2\left(\frac{\pi x}{2}\right)}{(1 + 2\cos\left(\frac{\pi x}{2}\right))^2} \right) \\ &= -\frac{\pi^2(2 + \cos\left(\frac{\pi x}{2}\right))}{2(1 + 2\cos\left(\frac{\pi x}{2}\right))^2} \end{aligned}$$

b Evaluating the above at $x = 1$, we find:

$$f(1) = \ln 1 = 0, f'(1) = -\pi, f''(1) = -\pi^2$$

Hence the Taylor expansion about $x = 1$ is:

$$f(x) = -\pi(x-1) + \frac{1}{2!}(-\pi^2)(x-1)^2 + \dots$$

$$\Rightarrow \ln\left(1 + 2\cos\left(\frac{\pi x}{2}\right)\right) = -\pi(x-1) - \frac{\pi^2}{2}(x-1)^2 + \dots$$

Challenge

a We have $\frac{d}{dx} \ln x = \frac{1}{x} = (-1)^{1+1} \frac{(1-1)!}{x^1}$, so holds for $n = 1$

Assume true for $n = k$ where $k \geq 1$

Then:

$$\begin{aligned} \frac{d^{k+1}}{dx^{k+1}} \ln x &= \frac{d}{dx} (-1)^{k+1} \cdot (k-1)! \cdot x^{-k} \\ &= -k(-1)^{k+1} \cdot (k-1)! \cdot x^{-k-1} = (-1)^{(k+1)+1} \frac{((k+1)-1)!}{x^{k+1}} \end{aligned}$$

So true for $n = k + 1$

The result then follows by induction.

b Hence the Taylor series about $x = a$, $a > 0$ is:

$$\begin{aligned} \ln x &= \ln a + \sum_{n=1}^{\infty} \frac{(-1)^{n+1} (n-1)!}{n! a^n} (x-a)^n \\ &= \ln a + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n a^n} (x-a)^n \end{aligned}$$