Further Pure Maths 2   Solution Bank   Ⓟ Pearson

Exercise 7D

1 a $f(x) = \sqrt{x} = f(a) + f'(a)(x-a) + \dfrac{f''(a)}{2!}(x-a)^2 + \dfrac{f'''(a)}{3!}(x-a)^3 + \dfrac{f''''(a)}{4!}(x-a)^4 + ...,$

where $a = 1$

$f(x) = \sqrt{x}$ $\qquad\qquad\qquad\qquad$ $f(1) = 1$

$f'(x) = \dfrac{1}{2}x^{-\frac{1}{2}}$ $\qquad\qquad\qquad$ $f'(1) = \dfrac{1}{2}$

$f''(x) = -\dfrac{1}{4}x^{-\frac{3}{2}}$ $\qquad\qquad\qquad$ $f''(1) = -\dfrac{1}{4}$

$f'''(x) = \dfrac{3}{8}x^{-\frac{5}{2}}$ $\qquad\qquad\qquad$ $f'''(1) = \dfrac{3}{8}$

$f''''(x) = -\dfrac{15}{16}x^{-\frac{7}{2}}$ $\qquad\qquad\qquad$ $f''''(1) = -\dfrac{15}{16}$

So $\sqrt{x} = 1 + \dfrac{1}{2}(x-1) - \dfrac{1}{4 \times 2!}(x-1)^2 + \dfrac{3}{8 \times 3!}(x-1)^3 - \dfrac{15}{16 \times 4!}(x-1)^4 + ...$

$= 1 + \dfrac{1}{2}(x-1) - \dfrac{1}{8}(x-1)^2 + \dfrac{1}{16}(x-1)^3 - \dfrac{5}{128}(x-1)^4 + ...$

b $\sqrt{1.2} \approx 1 + \dfrac{1}{2}(0.2) - \dfrac{1}{8}(0.2)^2 + \dfrac{1}{16}(0.2)^3 - \dfrac{5}{128}(0.2)^4$

$\approx 1 + 0.1 - 0.005 + 0.0005 - 0.0000625$

$= 1.095 \, (3\,\text{d.p.})$

2 All solutions use the Taylor expansion in the form:

$$f(x) = f(a) + f'(a)(x-a) + \dfrac{f''(a)(x-a)^2}{2!} + \dfrac{f'''(a)(x-a)^3}{3!} + \dfrac{f''''(a)(x-a)^4}{4!} + ...$$

a Let $f(x) = \ln x$ $\qquad$ then $\qquad$ $f(a) = f(e) = \ln e = 1$

$f'(x) = \dfrac{1}{x}$ $\qquad\qquad\qquad$ $f'(a) = f'(e) = \dfrac{1}{e}$

$f''(x) = -\dfrac{1}{x^2}$ $\qquad\qquad\qquad$ $f''(a) = f''(e) = -\dfrac{1}{e^2}$

So $f(x) = \ln x = 1 + \dfrac{1}{e}(x-e) + \dfrac{\left(-\frac{1}{e^2}\right)}{2!}(x-e)^2 + ...$

$= 1 + \dfrac{(x-e)}{e} - \dfrac{(x-e)^2}{2e^2} + ...$

2 b Let $f(x) = \tan x$ $\qquad$ then $f(a) = f\left(\dfrac{\pi}{3}\right) = \sqrt{3}$

$f'(x) = \sec^2 x$ $\qquad$ $f'(a) = f'\left(\dfrac{\pi}{3}\right) = 4$

$f'(x) = 2\sec^2 x \tan x$ $\qquad$ $f''(a) = f''\left(\dfrac{\pi}{3}\right) = 2(4)(\sqrt{3}) = 8\sqrt{3}$

$f'''(x) = 2\sec^4 x + 2\tan x(2\sec^2 x \tan x)$ $\qquad$ $f'''(a) = f'''\left(\dfrac{\pi}{3}\right) = 2(16) + 4(4)(3) = 80$

So $f(x) = \tan x = \sqrt{3} + 4\left(x - \dfrac{\pi}{3}\right) + \dfrac{8\sqrt{3}}{2!}\left(x - \dfrac{\pi}{3}\right)^2 + \dfrac{80}{3!}\left(x - \dfrac{\pi}{3}\right)^3 + \ldots$

$\qquad = \sqrt{3} + 4\left(x - \dfrac{\pi}{3}\right) + 4\sqrt{3}\left(x - \dfrac{\pi}{3}\right)^2 + \dfrac{40}{3}\left(x - \dfrac{\pi}{3}\right)^3 + \ldots$

c Let $f(x) = \cos x$ $\qquad$ then $\qquad$ $f(a) = f(1) = \cos 1$

$f'(x) = -\sin x$ $\qquad\qquad\qquad\qquad$ $f'(a) = f'(1) = -\sin 1$

$f''(x) = -\cos x$ $\qquad\qquad\qquad\qquad$ $f''(a) = f''(1) = -\cos 1$

$f'''(x) = \sin x$ $\qquad\qquad\qquad\qquad$ $f'''(a) = f'''(1) = \sin 1$

$f''''(x) = \cos x$ $\qquad\qquad\qquad\qquad$ $f''''(a) = f''''(1) = \cos 1$

So $f(x) = \cos x = \cos 1 - \sin 1(x-1) - \dfrac{(\cos 1)}{2}(x-1)^2 + \dfrac{(\sin 1)}{6}(x-1)^3 + \dfrac{(\cos 1)}{24}(x-1)^4 + \ldots$

3 a i Let $f(x) = \cos(x)$. Then $\cos\left(x + \dfrac{\pi}{4}\right) = f\left(x + \dfrac{\pi}{4}\right)$

$f'(x) = \cos(x) \Rightarrow f\left(\dfrac{\pi}{4}\right) = \dfrac{\sqrt{2}}{2}$

$f'(x) = -\sin(x) \Rightarrow f'\left(\dfrac{\pi}{4}\right) = -\dfrac{\sqrt{2}}{2}$

$f''(x) = -\cos(x) \Rightarrow f''\left(\dfrac{\pi}{4}\right) = -\dfrac{\sqrt{2}}{2}$

$f'''(x) = \sin(x) \Rightarrow f'''\left(\dfrac{\pi}{4}\right) = \dfrac{\sqrt{2}}{2}$

$f''''(x) = \cos(x) \Rightarrow f''''\left(\dfrac{\pi}{4}\right) = \dfrac{\sqrt{2}}{2}$

Using $f(x + a) = f(a) + f'(a)\,x + \dfrac{f''(a)x^2}{2!} + \dfrac{f'''(a)x^3}{3!} + \dfrac{f''''(a)x^4}{4!} + \ldots$

$\cos\left(x + \dfrac{\pi}{4}\right) = \dfrac{\sqrt{2}}{2}\left(1 - x - \dfrac{x^2}{2} + \dfrac{x^3}{6} + \dfrac{x^4}{24} - \ldots\right)$

3 a ii Let $f(x) = \ln(x)$. Then $\ln(x+5) = f(x+5)$

$f(x) = \ln(x) \Rightarrow f(5) = \ln 5$

$f'(x) = \dfrac{1}{x} \Rightarrow f'(5) = \dfrac{1}{5}$

$f''(x) = -\dfrac{1}{x^2} \Rightarrow f''(5) = -\dfrac{1}{25}$

$f'''(x) = \dfrac{2}{x^3} \Rightarrow f'''(5) = \dfrac{2}{125}$

$f''''(x) = -\dfrac{6}{x^4} \Rightarrow f''''(5) = -\dfrac{6}{625}$

Using $f(x+a) = f(a) + f'(a)x + \dfrac{f''(a)x^2}{2!} + \dfrac{f'''(a)x^3}{3!} + \dfrac{f''''(a)x^4}{4!} + \ldots$

$\ln(x+5) = \ln 5 + \dfrac{1}{5}x - \dfrac{1}{50}x^2 + \dfrac{1}{375}x^3 - \dfrac{1}{2500}x^4 + \ldots$

iii Let $f(x) = \sin(x)$. Then $\sin\left(x - \dfrac{\pi}{3}\right) = f\left(x - \dfrac{\pi}{3}\right)$

$f(x) = \sin(x) \Rightarrow f\left(-\dfrac{\pi}{3}\right) = -\dfrac{\sqrt{3}}{2}$

$f'(x) = \cos(x) \Rightarrow f'\left(-\dfrac{\pi}{3}\right) = \dfrac{1}{2}$

$f''(x) = -\sin(x) \Rightarrow f''\left(-\dfrac{\pi}{3}\right) = \dfrac{\sqrt{3}}{2}$

$f'''(x) = -\cos(x) \Rightarrow f'''\left(-\dfrac{\pi}{3}\right) = -\dfrac{1}{2}$

$f''''(x) = \sin(x) \Rightarrow f''''\left(-\dfrac{\pi}{3}\right) = -\dfrac{\sqrt{3}}{2}$

Using $f(x+a) = f(a) + f'(a)x + \dfrac{f''(a)x^2}{2!} + \dfrac{f'''(a)x^3}{3!} + \dfrac{f''''(a)x^4}{4!} + \ldots$

$\sin\left(x - \dfrac{\pi}{3}\right) = \dfrac{1}{2}\left(-\sqrt{3} + x + \dfrac{\sqrt{3}}{2!}x^2 - \dfrac{1}{3!}x^3 - \dfrac{\sqrt{3}}{4!}x^4 + \ldots\right)$

$= \dfrac{1}{2}\left(-\sqrt{3} + x + \dfrac{\sqrt{3}}{2}x^2 - \dfrac{1}{6}x^3 - \dfrac{\sqrt{3}}{24}x^4 + \ldots\right)$

b Taking $x = 0.2$,

$\ln 5.2 \approx \ln 5 + \dfrac{0.2}{5} - \dfrac{0.2^2}{50} + \dfrac{0.2^3}{375} - \dfrac{0.2^4}{2500}$

≈ 1.649 to 4 s.f.

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4 a $y = xe^x, \dfrac{dy}{dx} = xe^x + e^x = e^x(x+1)$

| Product rule. |

$$\frac{d^2y}{dx^2} = xe^x + e^x + e^x = e^x(x+2)$$

$$\frac{d^3y}{dx^3} = xe^x + 2e^x + e^x = e^x(x+3)$$

Each differentiation adds another e^x, so $\dfrac{d^n y}{dx^n} = (n+x)e^x$

So for $f(x) = xe^x, f^{(n)}(x) = (n+x)e^x$

b Using the Taylor series with $a = -1, f(-1) = -e^{-1}, \quad f'(-1) = 0, f''(-1) = e^{-1}$

$$f'''(-1) = 2e^{-1}, \; f''''(-1) = 3e^{-1}$$

So $xe^x = e^{-1}\left\{-1 + 0(x+1) + \dfrac{1}{2!}(x+1)^2 + \dfrac{2}{3!}(x+1)^3 + \dfrac{3}{4!}(x+1)^4 + ...\right\}$

$$= e^{-1}\left\{-1 + \frac{1}{2}(x+1)^2 + \frac{1}{3}(x+1)^3 + \frac{1}{8}(x+1)^4 + ...\right\}$$

5 a Let $f(x) = x^3 \ln x$ then as $a = 1$ $f(a) = f(1) = 0$

$f'(x) = 3x^2 \ln x + x^3 \times \dfrac{1}{x} = x^2(1 + 3\ln x)$ $f'(a) = f'(1) = 1$

$f''(x) = x^2 \times \dfrac{3}{x} + 2x(1 + 3\ln x) = x(5 + 6\ln x)$ $f''(a) = f''(1) = 5$

$f'''(x) = x \times \dfrac{6}{x} + (5 + 6\ln x) = (11 + 6\ln x)$ $f'''(a) = f'''(1) = 11$

$f''''(x) = \dfrac{6}{x}$ $f''''(a) = f''''(1) = 6$

Using Taylor, form **ii**

$$f(x) = x^3 \ln x = 0 + 1(x-1) + \frac{5}{2!}(x-1)^2 + \frac{11}{3!}(x-1)^3 + \frac{6}{4!}(x-1)^4 + ...$$

$$= (x-1) + \frac{5}{2}(x-1)^2 + \frac{11}{6}(x-1)^3 + \frac{1}{4}(x-1)^4 + ...$$

b Substituting $x = 1.5$ in series in **a**, gives

$$\frac{27}{8}\ln 1.5 \approx 0.5 + \frac{5}{2}(0.5)^2 + \frac{11}{6}(0.5)^s + \frac{1}{4}(0.5)^4 + ...$$

$$\approx 0.5 + 0.625 + 0.22916... + 0.015625 (= 1.369791...)$$

So this gives an approximation for $\ln 1.5$ of $\dfrac{8}{27}(1.369791...) = 0.4059$ (4 d.p.)

6 We look to find the Taylor series for the -function $\tan(x-\alpha)$ about $x=0$ where, $\alpha = \arctan\left(\dfrac{3}{4}\right)$

Let $f(x) = \tan(x-\alpha)$ then;

$f(0) = \tan(-\alpha) = -\tan(\alpha) = -\dfrac{3}{4}$

$f'(x) = \sec^2(x-\alpha) \Rightarrow f'(0) = \sec^2(-\alpha) = \sec^2\alpha = \dfrac{25}{16}$

Now we use the fact that;

$f''(x) = 2\tan(x-\alpha)\sec^2(x-\alpha) \Rightarrow f''(0) = -2\tan\alpha\sec^2\alpha$

$\Rightarrow f''(0) = -2 \cdot \dfrac{3}{4} \cdot \dfrac{25}{16} = -\dfrac{75}{32}$

Finally, using $f(x) = f(0) + f'(0)x + \dfrac{1}{2!}f''(0)x^2 + \dots(x^3)$, we find that

$f(x) = \tan(x-\alpha) = -\dfrac{3}{4} + \dfrac{25}{16} \cdot x + \dfrac{1}{2!} \cdot \dfrac{-75}{32} \cdot x^2 + \dots(x^3)$

$\Rightarrow \tan(x-\alpha) = -\dfrac{3}{4} + \dfrac{25}{16}x - \dfrac{75}{64}x^2 + \dots(x^3)$

7 Let $f(x) = \sin 2x$, then we calculate the required derivatives;

$f\left(\dfrac{\pi}{6}\right) = \sin\dfrac{\pi}{3} = \dfrac{\sqrt{3}}{2}$

$f'(x) = 2\cos 2x \Rightarrow f'\left(\dfrac{\pi}{6}\right) = 2\cos\dfrac{\pi}{3} = 1$

$f''(x) = -4\sin 2x \Rightarrow f''\left(\dfrac{\pi}{6}\right) = -4\sin\dfrac{\pi}{3} = -\dfrac{4\sqrt{3}}{2} = -2\sqrt{3}$

$f'''(x) = -8\cos 2x \Rightarrow f'''\left(\dfrac{\pi}{6}\right) = -8\cos\dfrac{\pi}{3} = -4$

$f^{(4)}(x) = 16\sin 2x \Rightarrow f^{(4)}\left(\dfrac{\pi}{6}\right) = 8\sqrt{3}$

Then, using the Taylor expansion:

$$f(x) = f\left(\dfrac{\pi}{6}\right) + f'\left(\dfrac{\pi}{6}\right)\left(x-\dfrac{\pi}{6}\right) + \dfrac{1}{2!}f''\left(\dfrac{\pi}{6}\right)\left(x-\dfrac{\pi}{6}\right)^2$$

$$+ \tfrac{1}{3!}f'''\left(\dfrac{\pi}{6}\right)\left(x-\dfrac{\pi}{6}\right)^3 + \tfrac{1}{4!}f^{(4)}\left(\dfrac{\pi}{6}\right)\left(x-\dfrac{\pi}{6}\right)^4 + \dots\left(\left(x-\dfrac{\pi}{6}\right)^5\right)$$

We find that

$$\sin 2x = \dfrac{\sqrt{3}}{2} + 1 \cdot \left(x-\dfrac{\pi}{6}\right) + \dfrac{-2\sqrt{3}}{2!} \cdot \left(x-\dfrac{\pi}{6}\right)^2$$

$$+ \dfrac{-4}{3!} \cdot \left(x-\dfrac{\pi}{6}\right)^3 + \dfrac{8\sqrt{3}}{4!} \cdot \left(x-\dfrac{\pi}{6}\right)^4 + \dots\left(\left(x-\dfrac{\pi}{6}\right)^5\right)$$

$$\Rightarrow \sin 2x = \dfrac{\sqrt{3}}{2} + \left(x-\dfrac{\pi}{6}\right) - \sqrt{3}\left(x-\dfrac{\pi}{6}\right)^2$$

$$- \dfrac{2}{3}\left(x-\dfrac{\pi}{6}\right)^3 + \dfrac{\sqrt{3}}{3}\left(x-\dfrac{\pi}{6}\right)^4 + \dots\left(\left(x-\dfrac{\pi}{6}\right)^5\right)$$

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8 a Given $y = \dfrac{1}{\sqrt{(1+x)}} = (1+x)^{-\frac{1}{2}}$

$y_3 (= \text{value of } y \text{ when } x = 3) = \dfrac{1}{2}$

$\dfrac{dy}{dx} = -\dfrac{1}{2}(1+x)^{-\frac{3}{2}}$

$\left(\dfrac{dy}{dx}\right)_3 = -\dfrac{1}{2} \times \dfrac{1}{8} = -\dfrac{1}{16}$

$\dfrac{d^2y}{dx^2} = \dfrac{3}{4}(1+x)^{-\frac{5}{2}}$

$\left(\dfrac{d^2y}{dx^2}\right)_3 = \dfrac{3}{4} \times \dfrac{1}{32} = \dfrac{3}{128}$

b So using

$$f(x) = f(3) + f'(3)(x-3) + \dfrac{f''(3)}{2!}(x-3)^2 + \dots \qquad \text{with } f^{(n)}(3) \equiv \left(\dfrac{d^n y}{dx^n}\right)_3$$

$$y = \dfrac{1}{\sqrt{(1+x)}} = \dfrac{1}{2} - \dfrac{1}{16}(x-3) + \dfrac{3}{256}(x-3)^2 + \dots$$

Challenge

a Let $f(x) = \ln(\cos 2x)$, then we will use:

$$f(x) = f(\pi) + f'(\pi)(x-\pi) + \dfrac{1}{2!}f''(\pi)(x-\pi)^2$$
$$+ \dfrac{1}{3!}f'''(\pi)(x-\pi)^3 + \dots \left((x-\pi)^4\right)$$

Differentiating:
$f(\pi) = \ln(\cos 2\pi) = 0$

$f'(x) = -\dfrac{2\sin 2x}{\cos 2x} = -2\tan 2x \Rightarrow f'(\pi) = 0$

$f''(x) = -4\sec^2 2x \Rightarrow f''(\pi) = -\dfrac{4}{1^2} = -4$

$f'''(x) = -16\tan 2x\sec^2 2x \Rightarrow f'''(\pi) = 0$

So, $\ln(\cos 2x) = -2(x-\pi)^2 - \dfrac{4}{3}(x-\pi)^4 - \dots$

b Note that $\cos\left(2 \cdot \dfrac{13\pi}{12}\right) = \dfrac{\sqrt{3}}{2}$, so we take the leading order term in the expansion above setting

$x = \dfrac{13\pi}{12}$

$\ln\left(\cos \dfrac{13\pi}{6}\right) = \ln \dfrac{\sqrt{3}}{2} \approx -2\left(\dfrac{13\pi}{12} - \pi\right)^2 = -0.1433 \ (4\,\text{d.p.})$