Exercise 7C

1 a $\frac{1}{e^x} = e^{-x} = 1 + (-x) + \frac{(-x)^2}{2!} + \frac{(-x)^3}{3!} +$

$= 1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \ldots$ valid for all values of x

b $\frac{e^{2x} \times e^{3x}}{e^x} = e^{4x} = 1 + (4x) + \frac{(4x)^2}{2!} + \frac{(4x)^3}{3!} +$

$= 1 + 4x + 8x^2 + \frac{32x^3}{3} + \ldots$ valid for all values of x

c $e^{1+x} = e \times e^x = e\left\{1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \ldots\right\}$ valid for all values of x

d $\ln(1-x) = (-x) - \frac{(-x)^2}{2} + \frac{(-x)^3}{3} + \frac{(-x)^4}{4} + \ldots$ $[-1 < -x \leqslant 1]$

$\Rightarrow 1 > x \geqslant -1$

$= -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \ldots$ $-1 \leqslant x < 1$

e $\sin\left(\frac{x}{2}\right) = \left(\frac{x}{2}\right) - \frac{\left(\frac{x}{2}\right)^3}{3!} + \frac{\left(\frac{x}{2}\right)^5}{5!} - \frac{\left(\frac{x}{2}\right)^7}{7!} + \ldots$

$= \frac{x}{2} - \frac{x^3}{48} + \frac{x^5}{3840} - \frac{x^7}{645\,120} + \ldots$ valid for all values of x

f $\ln(2+3x) = \ln\left\{2\left(1 + \frac{3x}{2}\right)\right\} = \ln 2 + \ln\left(1 + \frac{3x}{2}\right)$

$= \ln 2 + \frac{3x}{2} - \frac{\left(\frac{3x}{2}\right)^2}{2} + \frac{\left(\frac{3x}{2}\right)^3}{3} +$ $\left[-1 < \frac{3x}{2} \leqslant 1\right]$

$= \ln 2 + \frac{3x}{2} - \frac{9x^2}{8} + \frac{9x^3}{8} + \ldots$ $-\frac{2}{3} < x \leqslant \frac{2}{3}$

2 **a** $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \ldots, \qquad -1 < x \leqslant 1$

$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} - \ldots, \qquad -1 \leqslant x < 1$

$$\ln\left(\frac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x)$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \ldots - \left(-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} - \ldots\right)$$

$$= 2x + \frac{2x^2}{3} + \frac{2x^5}{5} + \ldots$$

$$= 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots\right)$$

As x must be in both the intervals $-1 < x \leqslant 1$ and $-1 \leqslant x < 1$ this expansion requires x to be in the interval $-1 < x < 1$.

b $\ln\sqrt{\frac{1+x}{1-x}} = \ln\left(\frac{1+x}{1-x}\right)^{\frac{1}{2}} = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$

so $\ln\sqrt{\frac{1+x}{1-x}} = \left(x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots\right), -1 < x < 1.$

c Solving $\left(\frac{1+x}{1-x}\right) = \frac{2}{3}$ gives $3 + 3x = 2 - 2x$

$5x = -1$

$x = -0.2$

This is a valid value of x.

So an approximation to $\ln\left(\frac{2}{3}\right)$ is $2\left(-0.2 - \frac{0.008}{3} - \frac{0.00032}{5}\right)$

$= 2(-0.2 - 0.0026666 - 0.000064)$

$= -0.4055$ (4 d.p.)

This is accurate to 4 d.p.

d $\ln\sqrt{\frac{1+x}{1-x}}$ with $x = \frac{3}{5}$ gives $\ln\sqrt{4} = \ln 2$

so $\ln 2 \approx 0.6 + \frac{(0.6)^3}{3} + \frac{(0.6)^5}{5}$

$\approx 0.687552\ldots = 0.69$ (2 d.p.)

Use the result in **b**.

3 $e^{2x} = 1 + 2x + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} + \ldots = 1 + 2x + 2x^2 + \frac{4x^3}{3} + \ldots$

$e^{-x} = 1 - x + \frac{(-x)^2}{2!} + \frac{(-x)^3}{3!} + \ldots = 1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \ldots$

So $e^{2x} - e^{-x} \approx 3x + \frac{3}{2}x^2$, if terms x^3 and above may be neglected.

4 a $3x\sin 2x = 3x\left((2x) - \frac{(2x)^3}{3!} + \ldots\right) = 6x^2 - 4x^4 + \ldots$

$\cos 3x = \left(1 - \frac{(3x)^2}{2!} + \frac{(3x)^4}{4!} - \ldots\right) = 1 - \frac{9}{2}x^2 + \frac{27}{8}x^4 - \ldots$

So $3x\sin 2x - \cos 3x = 6x^2 - 4x^4 + \ldots - \left(1 - \frac{9}{2}x^2 + \frac{27}{8}x^4 - \ldots\right)$

$= -1 + \frac{21}{2}x^2 - \frac{59}{8}x^4 + \ldots$

b $\frac{3x\sin 2x - \cos 3x + 1}{x^2} = \frac{21}{2} - \frac{59}{8}x^2 + \text{terms in higher powers of } x$

As $x \to 0$, so $\frac{3x\sin 2x - \cos 3x + 1}{x^2}$ tends to $\frac{21}{2}$.

5 a $\ln\left(1 + x - 2x^2\right) = \ln(1-x)(1+2x) = \ln(1-x) + \ln(1+2x)$

$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \ldots, \qquad -1 \leqslant x < 1$

$\ln(1+2x) = (2x) - \frac{(2x)^2}{2} + \frac{(2x)^3}{3} - \frac{(2x)^4}{4} + \ldots, \qquad -\frac{1}{2} < x \leqslant \frac{1}{2}$

$= 2x - 2x^2 + \frac{8x^3}{3} - 4x^4$

So $\ln\left(1 + x - 2x^2\right) = \ln(1-x) + \ln(1+2x)$

$= x - \frac{5x^2}{2} + \frac{7x^3}{3} - \frac{17x^4}{4} + \ldots, \qquad -\frac{1}{2} < x \leqslant \frac{1}{2}$ (smaller interval)

b $\ln\left(9 + 6x + x^2\right) = \ln(3+x)^2 = 2\ln(3+x) = 2\ln 3\left(1 + \frac{x}{3}\right) = 2\left\{\ln 3 + \ln\left(1 + \frac{x}{3}\right)\right\}$

The expansion of $\ln\left(1 + \frac{x}{3}\right)$ is $= \left(\frac{x}{3}\right) - \frac{\left(\frac{x}{3}\right)^2}{2} + \frac{\left(\frac{x}{3}\right)^3}{3} - \frac{\left(\frac{x}{3}\right)^4}{4} + \ldots, \qquad \left[-1 < \frac{x}{3} \leqslant 1\right]$

$= \frac{x}{3} - \frac{x^2}{18} + \frac{x^3}{81} - \frac{x^4}{324} + \ldots, \qquad -3 < x \leqslant 3$

So $\ln\left(9 + 6x + x^2\right) = 2\left\{\ln 3 + \ln\left(1 + \frac{x}{3}\right)\right\}$

$= 2\ln 3 + \frac{2x}{3} - \frac{x^2}{9} + \frac{2x^3}{81} - \frac{x^4}{162} + \ldots, \qquad -3 < x \leqslant 3$

6 **a** $\cos 2x = \left(1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} + \frac{(2x)^8}{8!} - \ldots\right)$

$= 1 - 2x^2 + \frac{2x^4}{3} - \frac{4x^6}{45} + \frac{2x^8}{315} - \ldots$

b Using $\cos 2x = 1 - 2\sin^2 x$,

$2\sin^2 x = 1 - \cos 2x = 2x^2 - \frac{2x^4}{3} + \frac{4x^6}{45} - \frac{2x^8}{315} + \ldots$

So $\sin^2 x = x^2 - \frac{x^4}{3} + \frac{2x^6}{45} - \frac{x^8}{315} + \ldots$

[Alternative: write out expansion of sin x as far as term in x^7, square it, and collect together appropriate terms!]

7 $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$

$(x-1)(e^x - 1) = (x-1)\left(x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots\right)$

$= x^2 + \frac{x^3}{2} + \frac{x^4}{6} \ldots - \left(x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots\right)$

$= -x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{8} + \ldots$

So $\ln(1+x) + (x-1)(e^x - 1) = \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\right) + \left(-x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{8} + \ldots\right)$

$= \frac{2x^3}{3} - \frac{x^4}{8} + \ldots \qquad \Rightarrow p = \frac{2}{3},\ q = -\frac{1}{8}$

8 **a** Only terms up to and including x^4 in the product are required, so using

$\sin x = x - \frac{x^3}{3!} + \ldots$ (next term is kx^5)

and the binomial expansion of $(1-x)^{-2}$, with terms up to and including x^3.
(It is not necessary to use the term in x^4, because it will be multiplied by expansion of sin x.)

$(1-x)^{-2} = 1 + (-2)(-x) + (-2)(-3)\frac{(-x)^2}{2!} + (-2)(-3)(-4)\frac{(-x)^3}{3!} + \ldots$

$= 1 + 2x + 3x^2 + 4x^3 + \ldots$

So $\frac{\sin x}{(1-x)^2} = \left(x - \frac{x^3}{6} + \ldots\right)(1 + 2x + 3x^2 + 4x^3 + \ldots)$

$= x + 2x^2 + 3x^3 + 4x^4 + \ldots - \left(\frac{x^3}{6} + \frac{x^4}{3} + \ldots\right)$

$= x + 2x^2 + \frac{17x^3}{6} + \frac{11x^4}{3} + \ldots$

8 b $y = \dfrac{\sin x}{(1-x)^2} = x + 2x^2 + \dfrac{17x^3}{6} + \dfrac{11x^4}{3} + \ldots$

So $\dfrac{dy}{dx} = 1 + 4x +$ higher powers of $x \Rightarrow$ at the origin the gradient of tangent $= 1$.

9 a $(1-3x)\ln(1+2x) = (1-3x)\left(2 - 2x^2 + \dfrac{8x^3}{3} - 4x^4 + \ldots\right)$

$$= \left(2x - 2x^2 + \frac{8x^3}{3} - 4x^4 + \ldots\right) - (6x^2 - 6x^3 + 8x^4 - \ldots)$$

$$= 2x - 8x^2 + \tfrac{26}{3}x^3 - 12x^4 + \ldots$$

b $e^{2x}\sin x = \left\{1 + (2x) + \dfrac{(2x)^2}{2!} + \dfrac{(2x)^3}{3!} + \dfrac{(2x)^4}{4!} + \ldots\right\}\left\{x - \dfrac{x^3}{3!} + \ldots\right\}$ [only terms up to x^4]

$$= \left(1 + 2x + 2x^2 + \frac{4x^3}{3} + \frac{2x^4}{3} + \ldots\right)\left(x - \frac{x^3}{6} + \ldots\right)$$

$$= \left(x + 2x^2 + 2x^3 + \frac{4x^4}{3}\right) + \left(-\frac{x^3}{6} - \frac{x^4}{3}\right) + \ldots$$

$$= x + 2x^2 + \tfrac{11}{6}x^3 + x^4 + \ldots$$

c $\sqrt{1+x^2}\,e^{-x} = (1+x^2)^{\frac{1}{2}}e^{-x}$

$$= \left\{1 + \frac{1}{2}x^2 + \left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\frac{(x^2)^2}{2!} + \ldots\right\}\left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots\right)$$

$$= \left(1 + \frac{x^2}{2} - \frac{x^4}{8} + \ldots\right)\left(1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} + \ldots\right)$$

$$= \left\{1 - x + \left(\frac{1}{2} + \frac{1}{2}\right)x^2 + \left(-\frac{1}{2} - \frac{1}{6}\right)x^3 + \left(\frac{1}{24} + \frac{1}{4} - \frac{1}{8}\right)x^4 + \ldots\right\}$$

$$= 1 - x + x^2 - \tfrac{2}{3}x^3 + \tfrac{1}{6}x^4 + \ldots$$

10 a $e^{-\frac{x^2}{2}} = 1 + \left(-\dfrac{x^2}{2}\right) + \dfrac{\left(-\frac{x^2}{2}\right)^2}{2!} + \dfrac{\left(-\frac{x^2}{2}\right)^3}{3!} + \dfrac{\left(-\frac{x^2}{2}\right)^4}{4!} + \ldots$

$$= 1 - \frac{x^2}{2} + \frac{x^4}{8} - \frac{x^6}{48} + \frac{x^8}{384} - \ldots$$

10 b Area under the curve $= \int_{-1}^{1} e^{-\frac{x^2}{2}}\,dx = 2\int_{0}^{1} e^{-\frac{x^2}{2}}\,dx$

$$= 2\left[x - \frac{x^3}{6} + \frac{x^5}{40} - \frac{x^7}{336} + \frac{x^9}{3456} - \dots\right]_0^1$$

Integrate the result from **a.**

$$\approx 2\left[1 - \frac{1}{6} + \frac{1}{40} - \frac{1}{336} + \frac{1}{3456}\right]$$

$$\approx 1.711 \text{ (3 d.p.)}$$

11 a $e^{px}\sin 3x = \left\{1 + (px) + \frac{(px)^2}{2!} + \frac{(px)^3}{3!} + \dots\right\}\left\{(3x) - \frac{9x^3}{2} + \dots\right\}$

$$= \left(1 + px + \frac{p^2x^2}{2} + \frac{p^3x^3}{6} + \dots\right)\left(3x - \frac{9x^3}{2} + \dots\right)$$

$$= \left(3x + 3px^2 + \frac{3p^2x^3}{2} + \dots\right) + \left(-\frac{9x^3}{2} + \dots\right)$$

$$= 3x + 3px^2 + \frac{3(p^2-3)x^3}{2} + \dots$$

b $\ln(1+qx) = \left\{(qx) - \frac{(qx)^2}{2} + \frac{(qx)^3}{3} - \dots\right\}$

So $e^{px}\sin 3x + \ln(1+qx) - x = 3x + 3px^2 + \frac{3(p^2-3)x^3}{2} + qx - \frac{q^2x^2}{2} + \frac{q^3x^3}{3} - x + \dots$

$$= (2+q)x + \left(3p - \frac{q^2}{2}\right)x^2 + \left(\frac{3p^2}{2} + \frac{q^3}{3} - \frac{9}{2}\right)x^3 + \dots$$

Coefficient of x is zero, so $q = -2$.

Coefficient of x^2 is zero, so $3p - 2 = 0 \Rightarrow p = \frac{2}{3}$

Coefficient of $x^3 = \frac{2}{3} - \frac{8}{3} - \frac{9}{2} = -\frac{13}{2}$, so $k = -\frac{13}{2}$

12 a $e^{x-\ln x} = e^x \times e^{-\ln x} = e^x \times e^{-\ln x^{-1}}$ Using $e^{a+b} = e^a \times e^b$

$= e^x \times x^{-1}$ using $e^{\ln k} = k$

$= \dfrac{e^x}{x}$

$e^{x-\ln x}\sin x = \dfrac{e^x \sin x}{x}$, and so, using the expansions of e^x and $\sin x$,

$$f(x) = e^{x-\ln x}\sin x = \frac{\left(1+x+\frac{x^2}{2}+\frac{x^3}{6}+\ldots\right)\left(x-\frac{x^3}{6}+\ldots\right)}{x}, x>0$$

$$= \left(1+x+\frac{x^2}{2}+\frac{x^3}{6}+\ldots\right)\left(1-\frac{x^2}{6}+\ldots\right)$$

$$= \left(1+x+\frac{x^2}{2}+\frac{x^3}{6}\right)-\left(\frac{x^2}{6}+\frac{x^3}{6}\right)$$ ignoring terms in x^4 and above.

$$= 1+x+\frac{x^2}{3}$$ There is no term in x^3.

b $f(0.1) = \dfrac{e^{0.1}\sin 0.1}{0.1} = 1.103329\ldots$

The result in **a** gives an approximation for $f(0.1)$ of $1+0.1+0.00333333 = 1.103333\ldots$ which is correct to 6 s.f.

13 a

$y = \sin 2x - \cos 2x$

$y' = 2\cos 2x + 2\sin 2x$

$y'' = -4\sin 2x + 4\cos 2x = -4y$

$y''' = -4y'$

$y'''' = -4y'' = 16y$

b

$$\sin 2x = (2x) - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} - \ldots$$

$$\cos 2x = 1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \ldots$$

$$y = -1 + 2x + \frac{4x^2}{2!} - \frac{8x^3}{3!} - \frac{16x^4}{4!} + \ldots$$

$$= -1 + 2x + 2x^2 - \frac{4x^3}{3} - \frac{2x^4}{3} + \ldots$$

Challenge

a

$y = (1-\beta^2)^{-\frac{1}{2}}$

$f(x) = (1+x)^a$

$f'(x) = a(1+x)^{a-1}, f''(x) = a(a-1)(1+x)^{a-2}$

$f(x) = 1 + ax + \frac{a(a-1)}{2!}x^2 + \dots$

So $y = 1 + \left(-\frac{1}{2}\right)(-\beta^2) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)(-\beta^2)^2}{2!} + \dots$

$= 1 + \frac{1}{2}\beta^2 + \frac{3}{8}\beta^4 + \dots$

b $\beta = \frac{v}{c} = \frac{\frac{4.2c}{20}}{c} = 0.21$

$\lambda \approx 1 + \frac{1}{2}0.21^2 + \frac{3}{8}0.21^4 = 1.02278$ (5 d.p.)

Observed journey time is $\frac{20}{\lambda} = 19.55$ years

c $\gamma = \frac{1}{\sqrt{1-0.21^2}} = 1.02281$ (5 d.p.)

$\left(1 + \frac{1}{2}0.21^2 + \frac{3}{8}0.21^4\right) / \frac{1}{\sqrt{1-0.21^2}} = 0.999973$ (6 d.p.)

Percentage error is

$= \frac{1.02281 - 0.999973}{1.02281} \times 100$

$= 2\%$

d The velocity will be larger and so $\beta = \frac{v}{c}$ will be larger making the error in γ larger. Hence the approximation would be less accurate. if ship $3\times$ faster