

Exercise 7B

$$\begin{aligned}
 1 \text{ a } f(x) &= (1-x)^{-1} && \Rightarrow f(0) = 1 \\
 f'(x) &= -1(1-x)^{-2}(-1) = (1-x)^{-2} && \Rightarrow f'(0) = 1 \\
 f''(x) &= -2(1-x)^{-3}(-1) = 2(1-x)^{-3} && \Rightarrow f''(0) = 2 \\
 f'''(x) &= -3 \cdot 2(1-x)^{-4}(-1) = 3 \cdot 2(1-x)^{-4} && \Rightarrow f'''(0) = 3!
 \end{aligned}$$

General term: The pattern here is such that $f^{(r)}(x)$ can be written down

$$f^{(r)}(x) = r(r-1)\dots 2(1-x)^{-(r+1)} = r!(1-x)^{-(r+1)} \Rightarrow f^{(r)}(0) = r!$$

$$\text{Using } f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \dots + \frac{f^{(r)}(0)}{r!}x^r + \dots$$

$$(1-x)^{-1} = 1 + x + \frac{2}{2!}x^2 + \dots + \frac{r!}{r!}x^r + \dots = 1 + x + x^2 + \dots + x^r + \dots$$

b

$$\begin{aligned}
 f(x) &= \sqrt{1+x} = (1+x)^{\frac{1}{2}} && \Rightarrow f(0) = 1 \\
 f'(x) &= \frac{1}{2}(1+x)^{-\frac{1}{2}} && \Rightarrow f'(0) = \frac{1}{2} \\
 f''(x) &= \frac{1}{2}\left(-\frac{1}{2}\right)(1+x)^{-\frac{3}{2}} && \Rightarrow f''(0) = -\frac{1}{4} \\
 f'''(x) &= \frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)(1+x)^{-\frac{5}{2}} && \Rightarrow f'''(0) = \frac{3}{8}
 \end{aligned}$$

Using Maclaurin's expansion

$$\begin{aligned}
 \sqrt{1+x} &= 1 + \frac{1}{2}x + \frac{\left(-\frac{1}{4}\right)}{2!}x^2 + \frac{\left(\frac{3}{8}\right)}{3!}x^3 - \dots \\
 &= 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \dots
 \end{aligned}$$

$$\begin{aligned}
 2 \text{ } f(x) &= e^{\sin x} && \Rightarrow f(0) = 1 \\
 f'(x) &= \cos x e^{\sin x} && \Rightarrow f'(0) = 1 \\
 f''(x) &= \cos^2 x e^{\sin x} - \sin x e^{\sin x} && \Rightarrow f''(0) = 1
 \end{aligned}$$

Substituting into Maclaurin's expansion gives

$$\begin{aligned}
 e^{\sin x} &= 1 + \sin x + \frac{1}{2!}x^2 + \dots \\
 &= 1 + x + \frac{1}{2}x^2 + \dots
 \end{aligned}$$

$$\begin{aligned}
 3 \text{ a } f(x) &= \cos x & \Rightarrow f(0) &= 1 \\
 f'(x) &= -\sin x & \Rightarrow f'(0) &= 0 \\
 f''(x) &= -\cos x & \Rightarrow f''(0) &= -1 \\
 f'''(x) &= \sin x & \Rightarrow f'''(0) &= 0 \\
 f''''(x) &= \cos x & \Rightarrow f''''(0) &= 1
 \end{aligned}$$

The process repeats itself after every 4th derivative, like $\sin x$ does. Using Maclaurin's expansion, only even powers of x are produced.

$$\begin{aligned}
 \cos x &= 1 + \frac{(-1)}{2!}x^2 + \frac{1}{4!}x^4 + \dots + \frac{(-1)^r}{(2r)!}x^{2r} + \dots \\
 &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \dots + (-1)^r \frac{x^{2r}}{(2r)!} + \dots
 \end{aligned}$$

b Using $\cos x \approx 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$ with $x = \frac{\pi}{6}$ (must be in radians)

$$\cos \approx 1 - \frac{\pi^2}{72} + \frac{\pi^4}{31104} = 0.86605\dots \text{ which is correct to 3 d.p.}$$

4 a Substituting $x = 1$ into the Maclaurin expansion of e^x , gives

$$e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} + \frac{1}{7!} + \dots$$

The approximations, to 4 d.p. where necessary, using n terms of the series are

n	1	2	3	4	5	6	7	8	9	10
Approx.	1	2	2.5	2.6667	2.7083	2.7167	2.7181	2.7183	2.7183	2.7183

So $e = 2.718$ (3 d.p.)

b Substituting $x = 0.2$ into the Maclaurin expansion of $\ln(1+x)$, gives

$$\ln\left(\frac{6}{5}\right) = 0.2 - \frac{(0.2)^2}{2} + \frac{(0.2)^3}{3} - \frac{(0.2)^4}{4} + \frac{(0.2)^5}{5} - \frac{(0.2)^6}{6} + \frac{(0.2)^7}{7} - \dots$$

The approximations, to 4 d.p. where necessary, using n terms of the series are

n	1	2	3	4	5
Approximation	0.2	0.18	0.1827	0.1823	0.1823

So $\ln\left(\frac{6}{5}\right) = 0.182$ (3 d.p.)

5 a $f(x) = e^{3x}, f^{(n)}(x) = 3^n e^{3x}$

So $f(0) = 1, f'(0) = 3, f''(0) = 3^2, f'''(0) = 3^3, f^{(4)}(0) = 3^4$

$$f(x) = e^{3x} = 1 + 3x + \frac{3^2}{2!}x^2 + \frac{3^3}{3!}x^3 + \frac{3^4}{4!}x^4 + \dots$$

$$= 1 + 3x + \frac{9x^2}{2} + \frac{9x^3}{2} + \frac{27}{8}x^4 + \dots \left[\text{Note: this is } 1 + 3x + \frac{(3x)^2}{2!} + \frac{(3x)^3}{3!} + \frac{(3x)^4}{4!} + \dots \right]$$

b As $f(x) = \ln(1 + 2x), \quad f(0) = \ln 1 = 0$

$$f'(x) = \frac{2}{1+2x} = 2(1+2x)^{-1}, \quad f'(0) = 2$$

$$f''(x) = -4(1+2x)^{-2}, \quad f''(0) = -4$$

$$f'''(x) = 16(1+2x)^{-3}, \quad f'''(0) = 16$$

$$f^{(4)}(x) = -96(1+2x)^{-4}, \quad f^{(4)}(0) = -96$$

So $\ln(1 + 2x) = 0 + 2x + \frac{(-4)}{2!}x^2 + \frac{(16)}{3!}x^3 + \frac{(-96)}{4!}x^4 + \dots$

$$= 2x - 2x^2 + \frac{8x^3}{3} - 4x^4 + \dots \left[\text{Note: this is } 2x - \frac{(2x)^2}{2} + \frac{(2x)^3}{3} - \frac{(2x)^4}{4} + \dots \right]$$

c $f(x) = \sin^2 x \quad f(0) = 0$

$$f'(x) = 2 \sin x \cos x = \sin 2x \quad f'(0) = 0$$

$$f''(x) = 2 \cos 2x \quad f''(0) = 2$$

$$f'''(x) = -4 \sin 2x \quad f'''(0) = 0$$

$$f^{(4)}(x) = -8 \cos 2x \quad f^{(4)}(0) = -8$$

So, $f(x) = \sin^2 x = 0 + 0x + \frac{2}{2!}x^2 + 0x^3 + \frac{(-8)}{4!}x^4 + \dots = x^2 - \frac{x^4}{3} + \dots$

6 $\cos\left(x - \frac{\pi}{4}\right) = \cos x \cos\left(\frac{\pi}{4}\right) + \sin x \sin\left(\frac{\pi}{4}\right)$

Use $\cos(A - B) = \cos A \cos B + \sin A \sin B$.

$$= \frac{1}{\sqrt{2}}(\cos x + \sin x)$$

$$= \frac{1}{\sqrt{2}} \left\{ \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots \right) + \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots \right) \right\}$$

$$= \frac{1}{\sqrt{2}} \left(1 + x - \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} + \dots \right)$$

7 a $f(x) = (1-x)^2 \ln(1-x)$

$$f'(x) = (1-x^2) \times \frac{(-1)}{1-x} + 2(1-x)(-1) \ln(1-x)$$

$$= x - 1 - 2(1-x) \ln(1-x)$$

$$f''(x) = 1 - 2 \left[(1-x) \times \frac{(-1)}{1-x} - \ln(1-x) \right] = 1 + 2 + 2 \ln(1-x) = 3 + 2 \ln(1-x)$$

Use the product rule.

$$7 \text{ b } f'''(x) = \frac{-2}{1-x}$$

Substituting $x = 0$ in all the results gives

$$f(0) = 0, f'(0) = -1, f''(0) = 3, f'''(0) = -2$$

$$\begin{aligned} c \quad f(x) &= (1-x)^2 \ln(1-x) = 0 + (-1)x + \frac{3}{2!}x^2 + \frac{(-2)}{3!}x^3 + \dots \\ &= -x + \frac{3x^2}{2} - \frac{1}{3}x^3 \end{aligned}$$

8 a Using the series expansions for $\sin x$ and $\cos x$ as far as the term in x^5 ,

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots = x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \dots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \dots$$

$$\begin{aligned} \text{so } 3\sin x - 4x\cos x + x &= 3\left(x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \dots\right) - 4x\left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \dots\right) + x \\ &= 3x - \frac{1}{2}x^3 + \frac{1}{40}x^5 - 4x + 2x^3 - \frac{1}{6}x^5 + x + \dots \end{aligned}$$

$$3\sin x - 4x\cos x + x = \frac{3}{2}x^3 - \frac{17}{120}x^5 + \dots$$

$$b \quad \frac{3\sin x - 4x\cos x + x}{x^3} = \frac{3}{2} - \frac{17}{120}x^2 + \text{higher powers in } x \text{ using a}$$

Hence, the limit, as $x \rightarrow 0$, is $\frac{3}{2}$.

$$9 \text{ a } f(x) = \ln \cos x \quad \Rightarrow f(0) = 0$$

$$\begin{aligned} f'(x) &= \frac{1}{\cos x} \times (-\sin x) \quad \Rightarrow f'(0) = 0 \\ &= -\tan x \end{aligned}$$

$$b \quad f''(x) = -\sec^2 x \quad \Rightarrow f''(0) = -1$$

$$f'''(x) = -2\sec x(\sec x \tan x) = -2\sec^2 x \tan x \quad \Rightarrow f'''(0) = 0$$

$$f''''(x) = -2\{\sec^2 x(\sec^2 x) + \tan x(2\sec^2 x \tan x)\} \quad \Rightarrow f''''(0) = -2$$

c Substituting into Maclaurin's expansion

$$\begin{aligned} \ln \cos x &= 0 + 0x + \frac{(-1)}{2!}x^2 + 0x^3 + \frac{(-2)}{4!}x^4 + \dots \\ &= -\frac{x^2}{2} - \frac{x^4}{12} + \dots \end{aligned}$$

9 d Substituting $x = \frac{\pi}{4}$ gives $\ln\left(\frac{1}{\sqrt{2}}\right) = -\frac{1}{2}\left(\frac{\pi^2}{16}\right) - \frac{1}{12}\left(\frac{\pi^4}{256}\right)$

but $\ln\left(\frac{1}{\sqrt{2}}\right) = \ln 2^{-\frac{1}{2}} = -\frac{1}{2}\ln 2$,

so $-\frac{1}{2}\ln 2 = -\frac{\pi^2}{2 \cdot 16} - \frac{\pi^4}{12 \cdot 256} + \dots$

$\Rightarrow \ln 2 = \frac{\pi^2}{16} + \frac{\pi^4}{6.256}$, using only first two terms.
 $= \frac{\pi^2}{16} \left(1 + \frac{\pi^2}{96}\right)$

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$$f(x) = \tan x$$

$$\Rightarrow f(0) = 0$$

$$f'(x) = \sec^2 x$$

$$\Rightarrow f'(0) = 1$$

$$f''(x) = 2 \sec x (\sec x \tan x) = 2 \sec^2 x \tan x$$

$$\Rightarrow f''(0) = 0$$

$$f'''(x) = 2 \sec^4 x + 4 \sec^2 x \tan^2 x$$

$$\Rightarrow f'''(0) = 2$$

$$f''''(x) = 16 \sec^4 x \tan x + 8 \sec^2 x \tan^3 x$$

$$\Rightarrow f''''(0) = 0 \text{ as } \tan(0) = 0$$

$$f''''''(x) = 16 \sec^2 x \tan^4 x + 88 \sec^4 x \tan^2 x + 16 \sec^6 x$$

$$\Rightarrow f''''''(0) = 16 \text{ as } \tan(0) = 0 \text{ and } \sec(0) = 1$$

Substituting into Maclaurin's expansion gives

$$\tan x = 0 + 1x + \frac{0}{2!}x^2 + \frac{2}{3!}x^3 + \frac{0}{4!}x^4 + \frac{16}{5!}x^5 + \dots$$

$$= x + \frac{x^3}{3} + \frac{2}{15}x^5 + \dots$$

Challenge

a $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

$$\lim_{n \rightarrow \infty} \left| \frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}} \right| = \lim_{n \rightarrow \infty} \left| \frac{x}{n+1} \right| = 0, \forall x \in \mathbb{R}$$

so the Maclaurin expansion of e^x converges for all $x \in \mathbb{R}$

b $\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}$

$$\lim_{n \rightarrow \infty} \left| \frac{\frac{(-1)^n x^{n+1}}{n+1}}{\frac{(-1)^{n-1} x^n}{n}} \right| = \lim_{n \rightarrow \infty} \left| -x \frac{n}{n+1} \right| = |x| \lim_{n \rightarrow \infty} \left| \frac{n}{n+1} \right| = |x|$$

so the Maclaurin expansion of

$\ln(1+x)$ converges for $|x| < 1$

and diverges for $|x| > 1$