Exercise 7A

1

	$f'(x)$	$f''(x)$	$f'''(x)$	$f^{(n)}(x)$
a	$2e^{2x}$	$2^2e^{2x}=4e^{2x}$	$2^3e^{2x}=8e^{2x}$	2^ne^{2x}
b	$n(1+x)^{n-1}$	$n(n-1)(1+x)^{n-2}$	$n(n-1)(n-2)(1+x)^{n-3}$	$n!$
c	e^x+xe^x	$e^x+(e^x+xe^x)=2e^x+xe^x$	$2e^x+(e^x+xe^x)=3e^x+xe^x$	ne^x+xe^x
d	$(1+x)^{-1}$	$-(1+x)^{-2}$	$(-1)(-2)(1+x)^{-3}=2(1+x)^{-3}$	$(-1)^{n-1}(n-1)!(1+x)^{-n}$

2 a $y=e^{2+3x}$, so $\frac{dy}{dx}=3e^{2+3x}$, $\frac{d^2y}{dx^2}=3^2e^{2+3x}$, $\frac{d^3y}{dx^3}=3^3e^{2+3x}$, and so on.

It follows that $\frac{d^ny}{dx^n}=3^ne^{2+3x}=3^ny$ as $y=e^{2+3x}$.

b

$y=e^{2+3x}$

$y'=3e^{2+3x}=3y$

$y''=3y'=3^2y$

$\frac{d^ny}{dx^n}=3^ny$

When $x=\log\frac{1}{9}$, $\frac{d^6y}{dx^6}=3^6e^{2+3\log\frac{1}{9}}$

$=3^6e^2\left(\frac{1}{9}\right)^3=e^2$

3 a $y=\sin^2 3x=(\sin 3x)^2$, so $\frac{dy}{dx}=2(\sin 3x)(3\cos 3x)$

$=3(2\sin 3x\cos 3x)$

$=3\sin 6x$

Use $\frac{du^n}{dx}=mu^{n-1}\frac{du}{dx}$.

Use $\sin 2A=2\sin A\cos A$.

b $\frac{d^2y}{dx^2}=18\cos 6x$, $\frac{d^3y}{dx^3}=-108\sin 6x$, $\frac{d^4y}{dx^4}=-648\cos 6x$

c $\frac{d^4y}{dx^4}=-648\cos 6x$

When $x=\frac{\pi}{6}$, $\frac{d^4y}{dx^4}=-648\cos\pi=648$

4 a $f'(x)=2xe^{-x}-x^2e^{-x}$

$f''(x)=(2e^{-x}-2xe^{-x})-(2xe^{-x}-x^2e^{-x})=e^{-x}(2-4x+x^2)$

$f'''(x)=e^{-x}(-4+2x)-e^{-x}(2-4x+x^2)=e^{-x}(-6+6x-x^2)$

b $f''''(x)=e^{-x}(6-2x)-e^{-x}(-6+6x-x^2)=e^{-x}(12-8x+x^2)$

so $f''''(2)=e^{-2}(12-16+4)=0$

5 **a** Given that $y = \sec x$, so $\frac{dy}{dx} = \sec x \tan x$

$$\frac{d^2y}{dx^2} = \sec x(\sec^2 x) + (\sec x \tan x)\tan x$$

Use the product rule.

$$= \sec x(\sec^2 x + \tan^2 x)$$
$$= \sec x(\sec^2 x + \sec^2 x - 1)$$

Use $1 + \tan^2 A = \sec^2 A$.

$$= 2\sec^3 x - \sec x$$

b $y'' = 2y^3 - y$

$y''' = 6y^2 y' - y'$

$= (6\sec^2 x - 1)\tan x \sec x$

When $x = \frac{\pi}{4}$, $y''' = \left(6\sec^2\frac{\pi}{4} - 1\right)\tan\frac{\pi}{4}\sec\frac{\pi}{4}$

$$= \left(6\left(\sqrt{2}\right)^2 - 1\right) \times 1 \times \sqrt{2} = 11\sqrt{2}$$

6 **a** $\frac{d}{dx}\left(y^2\right) = \frac{d}{dx}\left(y^2\right)\frac{dy}{dx} = 2y\frac{dy}{dx}$

Use the chain rule.

$$\frac{d^2}{dx^2}\left(y^2\right) = \frac{d}{dx}\left(2y\frac{dy}{dx}\right) = 2y\frac{d^2y}{dx^2} + 2\frac{dy}{dx}\frac{dy}{dx} = 2y\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2$$

Use the product rule.

b $\frac{d^3}{dx^3}\left(y^2\right) = \frac{d}{dx}\left(2y\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2\right)$

$$= 2\left\{y\frac{d^3y}{dx^3} + \frac{dy}{dx} \times \frac{d^2y}{dx^2} + 2\frac{dy}{dx} \times \frac{d^2y}{dx^2}\right\}$$
$$= 2\left\{y\frac{d^3y}{dx^3} + 3\frac{dy}{dx} \times \frac{d^2y}{dx^2}\right\}$$

7 $f(x) = \ln\left(x + \sqrt{1+x^2}\right)$

a $f'(x) = \dfrac{1}{x + \sqrt{1+x^2}} \times \left(1 + \dfrac{x}{\sqrt{1+x^2}}\right),$

$$= \frac{1}{\cancel{x + \sqrt{1+x^2}}} \times \left(\frac{\cancel{\sqrt{1+x^2} + x}}{\sqrt{1+x^2}}\right) = \frac{1}{\sqrt{1+x^2}}$$

Use $\dfrac{d}{dx}(\ln u) = \dfrac{1}{u}\dfrac{du}{dx}$.

So $\sqrt{1+x^2}\ f'(x) = 1$

b Differentiating this equation w.r.t.x, using the product rule

$$\sqrt{1+x^2}\ f''(x) + \frac{x}{\sqrt{1+x^2}} f'(x) = 0$$

So $\left(1+x^2\right) f''(x) + xf'(x) = 0$

Multiply through by $\sqrt{1+x^2}$.

c Differentiating this results w.r.t. x

$$\left(\left(1+x^2\right) f'''(x) + 2xf''(x)\right) + \left(f'(x) + xf''(x)\right) = 0$$

giving

$$\left(1+x^2\right) f'''(x) + 3xf''(x) + f'(x) = 0$$

d $f'(0) = \dfrac{1}{\sqrt{1+0}} = 1$

Using $\left(1+x^2\right) f''(x) + xf'(x) = 0$ with $x = 0$ and $f'(0) = 1$

$$f''(0) + (0)(1) = 0 \Rightarrow f''(0) = 0$$

Using $\left(1+x^2\right) f'''(x) + 3xf''(x) + f'(x) = 0$ with $x = 0,\ f'(0) = 1$ and $f''(0) = 0$

$$f'''(0) + (0)(0) + 1 = 0 \Rightarrow f'''(0) = -1$$