Exercise 6D

1 a $x^2\dfrac{d^2y}{dx^2}+6x\dfrac{dy}{dx}+4y=0$ *

First express $x\dfrac{dy}{dx}$ as $\dfrac{dy}{du}$ and $x\dfrac{d^2y}{dx^2}$ as $\dfrac{d^2y}{du^2}-\dfrac{dy}{du}$

As $x=e^u$, $\dfrac{dx}{du}=e^u=x$

Form the chain rule $\dfrac{dy}{du}=\dfrac{dy}{dx}\times\dfrac{dx}{du}$

$\therefore \quad \dfrac{dy}{du}=x\dfrac{dy}{dx}$ **(1)**

Also $\dfrac{d^2y}{du^2}=\dfrac{d}{du}\left(x\dfrac{dy}{dx}\right)$

$=\dfrac{dx}{du}\times\dfrac{dy}{dx}+x\dfrac{d^2y}{dx^2}\times\dfrac{dx}{du}$

$=\dfrac{dy}{du}+x^2\dfrac{d^2y}{dx^2}$

$\therefore \quad x^2\dfrac{d^2y}{dx^2}=\dfrac{d^2y}{du^2}-\dfrac{dy}{du}$ **(2)**

Use the results **(1)** and **(2)** to change the variable in *

$\therefore \dfrac{d^2y}{du^2}-\dfrac{dy}{du}+6\dfrac{dy}{du}+4y=0$

i.e. $\dfrac{d^2y}{du^2}+5\dfrac{dy}{du}+4y=0$

This has auxiliary equation

$m^2+5m+4=0$

$\therefore (m+4)(m+1)=0$

i.e. $m=-4$ or -1

$\therefore$ The solution of the differential equation † is

$y=Ae^{-4u}+Be^{-u}$

But $e^u=x$

$\therefore \quad e^{-u}=x^{-1}=\dfrac{1}{x}$

and $e^{-4u}=x^{-4}=\dfrac{1}{x^4}$

$\therefore \quad y=\dfrac{A}{x^4}+\dfrac{B}{x}$

1 b $x^2\dfrac{d^2y}{dx^2}+5x\dfrac{dy}{dx}+4y=0$ *

As $x=e^u$, $x\dfrac{dy}{dx}=\dfrac{dy}{du}$ and $x^2\dfrac{d^2y}{dx^2}=\dfrac{d^2y}{du^2}-\dfrac{dy}{du}$

(See solution to question 1 for proof this.)

Use these results to change the variable in *

$\therefore \quad \dfrac{d^2y}{du^2}-\dfrac{dy}{du}+5\dfrac{dy}{du}+4y=-0.$

$\therefore \quad \dfrac{d^2y}{du^2}+4\dfrac{dy}{du}+4y=0 \quad †$

This has auxiliary equation

$m^2+4m+4=0$

$\therefore \quad (m+2)^2=0$

$\therefore \quad m=-2$ only

The solution of the differential equation † is thus

$$y=(A+Bu)e^{-2u}$$

As $x=e^u$ $\therefore$ $e^{-2u}=x^{-2}=\dfrac{1}{x^2}$

and $u=\ln x$

$\therefore \quad y=(A+B\ln x)\times\dfrac{1}{x^2}$

Use $x\dfrac{dy}{dx}=\dfrac{dy}{du}$ and $x^2\dfrac{d^2y}{dx^2}=\dfrac{d^2y}{du^2}-\dfrac{dy}{du}$

Ensure that you can prove these two results.

1 c $x^2\frac{d^2y}{dx^2}+6x\frac{dy}{dx}+6y=0$ *

As $x=e^u$, $x\frac{dy}{dx}=\frac{dy}{du}$ and $x^2\frac{d^2y}{dx^2}=\frac{d^2y}{du^2}-\frac{dy}{du}$

(See solution to question 1 for proof of this.)

Use these results to change the variable in *

$\therefore \frac{d^2y}{du^2}-\frac{dy}{du}+6\frac{dy}{du}+6y=0$

$\therefore \quad \frac{d^2y}{du^2}+5\frac{dy}{du}+6y=0$ †

This has auxiliary equation

$m^2+5m+6=0$

$\therefore \quad (m+2)(m+3)=0$

$\therefore \quad m=-2 \text{ or } -3$

The solution of the differential equation † is thus

$y=Ae^{-2u}+Be^{-3u}$

As $\quad x=e^u,\ e^{-2u}=x^{-2}=\frac{1}{x^2}$

and $\quad e^{-3u}=x^{-3}=\frac{1}{x^3}$

$\therefore \quad y=\frac{A}{x^2}+\frac{B}{x^3}$

Use $x\frac{dy}{dx}=\frac{dy}{du}$ and $x^2\frac{d^2y}{dx^2}=\frac{d^2y}{du^2}-\frac{dy}{du}$

d $x^2\frac{d^2y}{dx^2}+4\frac{dy}{dx}-28y=0$ *

As $x=e^u$, $x\frac{dy}{dx}=\frac{dy}{du}$ and $x^2\frac{d^2y}{dx^2}=\frac{d^2y}{du^2}-\frac{dy}{du}$

Substitute these results into equation *

$\therefore \frac{d^2y}{du^2}-\frac{dy}{du}+4\frac{dy}{du}-28y=0$

$\therefore \quad \frac{d^2y}{du^2}+3\frac{dy}{du}-28y=0$ †

This has auxiliary equation:

$m^2+3m-28=0$

$\therefore \quad (m+7)(m-4)=0$

$\therefore \quad m=-7 \text{ or } 4$

$\therefore y=Ae^{-7u}+Be^{4u}$ is the solution to †

As $x=e^u$, $\therefore \ e^{-7u}=\frac{1}{x^7}$

and $\quad e^{4u}=x^4$

$\therefore \quad y=\frac{A}{x^7}+Bx^4$

Use $x\frac{dy}{dx}=\frac{dy}{du}$ and $x^2\frac{d^2y}{dx^2}=\frac{d^2y}{du^2}-\frac{dy}{du}$

1 e $x^2\frac{d^2y}{dx^2}-4x\frac{dy}{dx}-14y=0$ $*$

As $x=e^u$, $x\frac{dy}{dx}=\frac{dy}{du}$ and $x^2\frac{d^2y}{dx^2}=\frac{d^2y}{du^2}-\frac{dy}{du}$

Substituting these results into $*$ gives

$$\frac{d^2y}{du^2}-\frac{dy}{du}-4\frac{dy}{du}-14y=0$$

i.e. $\frac{d^2y}{du^2}-5\frac{dy}{du}-14y=0$ †

This has auxiliary equation:

$$m^2-5m-14=0$$

i.e. $(m-7)(m+2)=0$

$\therefore$ $m=7$ or -2

$\therefore$ The solution of the differential equation † is

$$y=Ae^{7u}+Be^{-2u}$$

But $x=e^u$, $\therefore$ $e^{7u}=x^7$

and $e^{-2u}=x^{-2}=\frac{1}{x^2}$

$\therefore$ $y=Ax^7+\frac{B}{x^2}$

Use $x\frac{dy}{dx}=\frac{dy}{du}$ and $x^2\frac{d^2y}{dx^2}=\frac{d^2y}{du^2}-\frac{dy}{du}$

f $x^2\frac{d^2y}{dx^2}+3x\frac{dy}{dx}+2y=0$ $*$

As $x=e^u$, $x\frac{dy}{dx}=\frac{dy}{du}$ and $x^2\frac{d^2y}{dx^2}=\frac{d^2y}{du^2}-\frac{dy}{du}$

Substitute these results into $*$ to give:

$$\frac{d^2y}{du^2}-\frac{dy}{du}+3\frac{dy}{du}+2y=0$$

i.e. $\frac{d^2y}{du^2}+2\frac{dy}{du}+2y=0$ †

This has auxiliary equation:

$$m^2+2m+2=0$$

$\therefore$ $m=\frac{-2\pm\sqrt{4-8}}{2}$

$=-1\pm i$

The solution of the differential equation † is thus

$$y=e^{-u}[A\cos u+B\sin u]$$

As $x=e^u$, $e^{-u}=x^{-1}=\frac{1}{x}$

and $u=\ln x$

$\therefore$ $y=\frac{1}{x}[A\cos\ln x+B\sin\ln x]$

Use $x\frac{dy}{dx}=\frac{dy}{du}$ and $x^2\frac{d^2y}{dx^2}=\frac{d^2y}{du^2}-\frac{dy}{du}$

A proof of these results is given in the book in Section 5.6

2 **a** $y = \frac{z}{x}$ implies $xy = z$

$\therefore \quad x\frac{dy}{dx} + y = \frac{dz}{dx}$

Also $x\frac{d^2y}{dx^2} + \frac{dy}{dx} + \frac{dy}{dx} = \frac{d^2z}{dx^2}$

$\therefore$ The equation $x\frac{d^2y}{dx^2} + (2-4x)\frac{dy}{dx} - 4y = 0$

becomes $\frac{d^2z}{dx^2} - 4\left(\frac{dz}{dx} - y\right) - 4y = 0$

i.e. $\frac{d^2z}{dx^2} - 4\frac{dz}{dx} = 0 \quad *$

as required.

Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in terms of $\frac{dz}{dx}$ and $\frac{d^2z}{dx^2}$

b $m^2 - 4m = 0$

$\Rightarrow m(m-4) = 0$

$\Rightarrow m = 0$ or 4

So general solution is

$z = A + Be^{4x}$

c $yx = A + Be^{4x}$

$\Rightarrow y = \frac{A}{x} + \frac{B}{x}e^{4x}$

3 **a** $y = \frac{z}{x^2}$ implies $z = yx^2$ or $x^2y = z$

$\therefore \quad x^2\frac{dy}{dx} + 2xy = \frac{dz}{dx} \quad$ **(1)**

Also $x^2\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} + 2x\frac{dy}{dx} + 2y = \frac{d^2z}{dx^2} \quad$ **(2)**

Express $\frac{dz}{dx}$ and $\frac{d^2z}{dx^2}$ in terms of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ respectively.

The differential equation:

$x^2\frac{d^2y}{dx^2} + 2x(x+2)\frac{dy}{dx} + 2(x+1)^2 y = e^{-x}$ can be written

$\left(x^2\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + 2y\right) + \left(2x^2\frac{dy}{dx} + 4xy\right) + 2x^2y = e^{-x}$

Using the results **(1)** and **(2)**

$\frac{d^2z}{dx^2} + 2\frac{dz}{dx} + 2z = e^{-x} \quad \dagger$

as required.

3 b This has auxiliary equation as

$m^2 + 2m + 2 = 0$

$\therefore \quad m = \dfrac{-2 \pm \sqrt{4-8}}{2}$

$m = -1 \pm \mathrm{i}$

$\therefore \; z = \mathrm{e}^{-x}(A\cos x + B\sin x)$ is the complementary function

A particular integral of † is $z = \lambda \mathrm{e}^{-x}$

$\therefore \; \dfrac{\mathrm{d}z}{\mathrm{d}x} = -\lambda \mathrm{e}^{-x}$ and $\dfrac{\mathrm{d}^2 z}{\mathrm{d}x^2} = \lambda \mathrm{e}^{-x}$

Substituting into †

$(\lambda - 2\lambda + 2\lambda)\mathrm{e}^{-x} = \mathrm{e}^{-x}$

$\therefore \quad \lambda = 1$

So $z = \mathrm{e}^{-x}$ is a particular integral.

$\therefore$ The general solution of † is

$z = \mathrm{e}^{-x}(A\cos x + B\sin x + 1)$

c Now $z = x^2 y$

$\therefore \quad y = \dfrac{\mathrm{e}^{-x}}{x^2}(A\cos x + B\sin x + 1)$ is the general solution of the given differential equation.

4 a $z = \sin x$ implies $\dfrac{\mathrm{d}z}{\mathrm{d}x} = \cos x$

$\therefore \; \dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{\mathrm{d}y}{\mathrm{d}z} \times \cos x$

and $\dfrac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \dfrac{\mathrm{d}^2 y}{\mathrm{d}z^2}\cos^2 x - \dfrac{\mathrm{d}y}{\mathrm{d}z}\sin x$

> Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ in terms of $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ and find $\dfrac{\mathrm{d}^2 y}{\mathrm{d}x^2}$ in terms of $\dfrac{\mathrm{d}^2 y}{\mathrm{d}z^2}$ and $\dfrac{\mathrm{d}y}{\mathrm{d}z}$

$\therefore$ The equation $\cos x \dfrac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \sin x \dfrac{\mathrm{d}y}{\mathrm{d}x} - 2y\cos^3 x = 2\cos^5 x$

becomes $\cos^3 x \dfrac{\mathrm{d}^2 y}{\mathrm{d}z^2} - \cos x \sin x \dfrac{\mathrm{d}y}{\mathrm{d}z} + \cos x \sin x \dfrac{\mathrm{d}y}{\mathrm{d}z} - 2y\cos^3 x = 2\cos^5 x$

$\therefore$ Divide by $\cos^3 x$ gives:

$\dfrac{\mathrm{d}^2 y}{\mathrm{d}z^2} - 2y = 2\cos^2 x$

$= 2(1 - z^2)$ † $\quad$ [as $\cos^2 x = 1 - \sin^2 x = 1 - z^2$]

4 b First solve $\frac{d^2y}{dz^2} - 2y = 0$

This has auxiliary equation

$m^2 - 2 = 0$

$\therefore \quad m = \pm\sqrt{2}$

$\therefore$ The complementary function is $y = Ae^{\sqrt{2}z} + Be^{-\sqrt{2}z}$

Let $y = \lambda z^2 + \mu z + v$ be a particular integral of the differential equation †

Then $\frac{dy}{dz} = 2\lambda z + \mu$ and $\frac{d^2y}{dz^2} = 2\lambda$

Substitute into †

Then $2\lambda - 2(\lambda z^2 + \mu z + v) = 2(1 - z^2)$

Compare coefficients of z^2: $-2\lambda = -2 \quad \therefore \quad \lambda = 1$

Compare coefficients of z: $-2\mu = 0 \quad \therefore \quad \mu = 0$

Compare constants: $2\lambda - 2v = 2 \quad \therefore \quad v = 0$

$\therefore z^2$ is the particular integral.

$\therefore$ The general solution of † is

$y = Ae^{\sqrt{2}z} + Be^{-\sqrt{2}z} + z^2$.

But $z = \sin x$

$\therefore \quad y = Ae^{\sqrt{2}\sin x} + Be^{-\sqrt{2}\sin x} + \sin^2 x$

5 a $x = ut, \frac{dx}{dt} = u + t\frac{du}{dt}, \frac{d^2x}{dt^2} = 2\frac{du}{dt} + t\frac{d^2u}{dt^2}$

So the differential equation becomes

$$t^2\left(2\frac{du}{dt} + t\frac{d^2u}{dt^2}\right) - 2t\left(u + t\frac{du}{dt}\right) = -2\left(1 - 2t^2\right)ut$$

which rearranges to give

$$t^3\left(\frac{d^2u}{dt^2} - 4u\right) = 0$$

$$\Rightarrow \frac{d^2u}{dt^2} - 4u = 0$$

b $m^2 - 4 = 0$

$\Rightarrow m = \pm 2$

So general solution is

$u = Ae^{2t} + Be^{-2t}$

$\Rightarrow x = t\left(Ae^{2t} + Be^{-2t}\right)$

5 c $\frac{dx}{dt} = Ae^{2t} + Be^{-2t} + t\left(2Ae^{2t} + -2Be^{-2t}\right)$

$x = 2$ at $t = 1 \Rightarrow Ae^{2} + Be^{-2} = 2$

$\frac{dx}{dt} = 1$ at $t = 1 \Rightarrow Ae^{2} + Be^{-2} + 2Ae^{2} - 2Be^{-2} = 1$

$\Rightarrow 3Ae^{2} - Be^{-2} = 1$

Adding the equations we obtain

$4Ae^{2} = 3$

$\Rightarrow A = \frac{3}{4e^{2}}$

and then $B = \frac{5}{4e^{-2}} = \frac{5e^{2}}{4}$

so the particular solution is

$x = t\left(\frac{3}{4e^{2}}e^{2t} + \frac{5e^{2}}{4}e^{-2t}\right)$

Challenge

$u = \frac{dy}{dx} \Rightarrow \frac{du}{dx} = \frac{d^{2}y}{dx^{2}}$

So the equation becomes

$x\frac{du}{dx} + u = 12x$

$\Rightarrow \frac{du}{dx} + \frac{1}{x}u = 12$

So the integrating factor is $e^{\int \frac{1}{x}dx} = e^{\ln x} = x$

This gives $\frac{d}{dx}(xu) = 12x$

$\Rightarrow xu = 6x^{2} + A$

$\Rightarrow u = 6x + \frac{A}{x}$

$\Rightarrow y = 3x^{2} + A\ln x + B$