Exercise 6C

1 a $\frac{d^2y}{dx^2}+5\frac{dy}{dx}+6y=12e^x$ **(1)**

Let $y = Ae^x$

$\frac{dy}{dx} = Ae^x$

$\frac{d^2y}{dx^2} = Ae^x$

Substituting into **(1)** gives:

$Ae^x+5Ae^x+6Ae^x=12e^x$

$12A=12$

$A = 1$

Hence the particular integral is e^x

$m^2+5m+6=0$

$(m+2)(m+3)=0$

$m = -2$ or $m = -3$

So the complementary function is:

$y=Be^{-3x}+Ce^{-2x}$

And the general solution is:

$y=Be^{-3x}+Ce^{-2x}+e^x$

b $y=Be^{-3x}+Ce^{-2x}+e^x$ **(1)**

$\frac{dy}{dx}=-3Be^{-3x}-2Ce^{-2x}+e^x$ **(2)**

When $x = 0$, $y = 1$ and $\frac{dy}{dx}=0$

Substituting into **(1)** gives:

$1=B+C+1$

$B=-C$

Substituting into **(2)** gives:

$0=-3B-2C+1$

$3B+2C=1$ **(3)**

Substituting $B=-C$ into **(3)** gives:

$-3C+2C=1$

$C=-1$ and $B = 1$

Therefore:

$y=e^{-3x}-e^{-2x}+e^x$

2 a $\frac{d^2y}{dx^2}+2\frac{dy}{dx}=12e^{2x}$ **(1)**

Let $y = Ae^{2x}$

$\frac{dy}{dx}=2Ae^{2x}$

$\frac{d^2y}{dx^2}=4Ae^{2x}$

Substituting into **(1)** gives:

$4Ae^{2x}+4Ae^{2x}=12e^{2x}$

$8A=12$

$A=\frac{3}{2}$

Hence the particular integral is $\frac{3}{2}e^{2x}$

$m^2+2m=0$

$m(m+2)=0$

$m = 0$ or $m = -2$

So the complementary function is:

$y=B+Ce^{-2x}$

And the general solution is:

$y=B+Ce^{-2x}+\frac{3}{2}e^{2x}$

b $y=B+Ce^{-2x}+\frac{3}{2}e^{2x}$ **(1)**

$\frac{dy}{dx}=-2Ce^{-2x}+3e^{2x}$ **(2)**

When $x = 0$, $y = 2$ and $\frac{dy}{dx}=6$

Substituting into **(1)** gives:

$2=B+C+\frac{3}{2}$

$B+C=\frac{1}{2}$ **(3)**

Substituting into **(2)** gives:

$6=-2C+3$

$C=-\frac{3}{2}$

Substituting $C=-\frac{3}{2}$ into **(3)** gives:

$B-\frac{3}{2}=\frac{1}{2}$

B = 2 and $C=-\frac{3}{2}$

Therefore:

$y=2-\frac{3}{2}e^{-2x}+\frac{3}{2}e^{2x}$

3 $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 42y = 14$ **(1)**

Let $y = A$

$\frac{dy}{dx} = 0$

$\frac{d^2y}{dx^2} = 0$

Substituting into **(1)** gives:

$-42A = 14$

$A = -\frac{1}{3}$

Hence the particular integral is $-\frac{1}{3}$

$m^2 - m - 42 = 0$

$(m-7)(m+6) = 0$

$m = 7$ or $m = -6$

So the complementary function is:

$y = Be^{-6x} + Ce^{7x}$

And the general solution is:

$y = Be^{-6x} + Ce^{7x} - \frac{1}{3}$

$\frac{dy}{dx} = -6Be^{-6x} + 7Ce^{7x}$

When $x = 0$, $y = 0$ and $\frac{dy}{dx} = \frac{1}{6}$

$0 = B + C - \frac{1}{3}$

$B = \frac{1}{3} - C$ **(2)**

$\frac{1}{6} = -6B + 7C$ **(3)**

Substituting **(2)** into **(3)** gives:

$\frac{1}{6} = -6\left(\frac{1}{3} - C\right) + 7C$

$\frac{1}{6} = -2 + 6C + 7C$

$13C = \frac{13}{6}$

$C = \frac{1}{6}$ and $B = \frac{1}{6}$

Therefore:

$y = \frac{1}{6}e^{-6x} + \frac{1}{6}e^{7x} - \frac{1}{3}$

4 a $\frac{d^2y}{dx^2}+9y=16\sin x$ **(1)**

Let $y=A\sin x+B\cos x$

$\frac{dy}{dx}=A\cos x-B\sin x$

$\frac{d^2y}{dx^2}=-A\sin x-B\cos x$

Substituting into **(1)** gives:

$-A\sin x-B\cos x+9(A\sin x+B\cos x)=16\sin x$

$-A\sin x-B\cos x+9A\sin x+9B\cos x=16\sin x$

$A\sin x+B\cos x=2\sin x$

Comparing coefficients:

For $\sin x$:

$A=2$

For $\cos x$:

$B=0$

Hence the particular integral is $2\sin x$

$m^2+9=0$

$m=\pm 3\text{i}$

So the complementary function is:

$y=C\sin 3x+D\cos 3x$

And the general solution is:

$y=C\sin 3x+D\cos 3x+2\sin x$

b $y=C\sin 3x+D\cos 3x+2\sin x$

$\frac{dy}{dx}=3C\cos 3x-3D\sin 3x+2\cos x$

When $x=0$, $y=1$ and $\frac{dy}{dx}=8$

$D=1$

$8=3C+2$

$C=2$

Therefore:

$y=2\sin 3x+\cos 3x+2\sin x$

5 **a** $4\frac{d^2y}{dx^2}+4\frac{dy}{dx}+5y=\sin x+4\cos x$ **(1)**

Let $y=A\sin x+B\cos x$

$\frac{dy}{dx}=A\cos x-B\sin x$

$\frac{d^2y}{dx^2}=-A\sin x-B\cos x$

Substituting into **(1)** gives:

$4(-A\sin x-B\cos x)+4(A\cos x-B\sin x)+5(A\sin x+B\cos x)=\sin x+4\cos x$

$-4A\sin x-4B\cos x+4A\cos x-4B\sin x+5A\sin x+5B\cos x=\sin x+4\cos x$

$A\sin x-4B\sin x+4A\cos x+B\cos x=\sin x+4\cos x$

Comparing coefficients:

$\sin x(A-4B)+\cos x(4A+B)=\sin x+4\cos x$

Comparing coefficients:

For $\sin x$:

$A-4B=1$ **(1)**

For $\cos x$:

$4A+B=4$ **(2)**

Adding 4 × **(2)** and **(1)** gives:

$17A=17$

$A=1$

$B=0$

Hence the particular integral is $\sin x$

$4m^2+4m+5=0$

$m=\frac{-4\pm\sqrt{(4)^2-4(4)(5)}}{2(4)}$

$=\frac{-4\pm\sqrt{-64}}{8}$

$=\frac{-4\pm 8\text{i}}{8}$

$=-\frac{1}{2}\pm\text{i}$

So the complementary function is:

$y=\text{e}^{-\frac{1}{2}x}(A\cos x+B\sin x)$

And the general solution is:

$y=\text{e}^{-\frac{1}{2}x}(A\cos x+B\sin x)+\sin x$

5 b $y = e^{-\frac{1}{2}x}(A\cos x + B\sin x) + \sin x$

$\frac{dy}{dx} = e^{-\frac{1}{2}x}(-A\sin x + B\cos x) + \cos x - \frac{1}{2}e^{-\frac{1}{2}x}(A\cos x + B\sin x)$

When $x = 0$, $y = 0$ and $\frac{dy}{dx} = 0$

$A = 0$

$B + 1 - \frac{1}{2}A = 0$

$B = -1$

Therefore:

$y = -e^{-\frac{1}{2}x}\sin x + \sin x$

$= \sin x\left(1 - e^{-\frac{1}{2}x}\right)$

6 a $\frac{d^2x}{dt^2} - 3\frac{dx}{dt} + 2x = 2t - 3$ **(1)**

Let $x = At + B$

$\frac{dx}{dt} = A$

$\frac{d^2x}{dt^2} = 0$

Substituting into **(1)** gives:

$-3A + 2(At + B) = 2t - 3$

$-3A + 2At + 2B = 2t - 3$

For t:

$2A = 2$

$A = 1$

For constant terms:

$-3A + 2B = -3$

$-3 + 2B = -3$

$B = 0$

Hence the particular integral is t

$m^2 - 3m + 2 = 0$

$(m-1)(m-2) = 0$

$m = 1$ or $m = 2$

So the complementary function is:

$x = Ce^{2t} + De^{t}$

And the general solution is:

$x = Ce^{2t} + De^{t} + t$

6 b $x = Ce^{2t} + De^{t} + t$

when $t = 0$, $x = 1$ and when $t = 1$, $x = 2$

$C + D = 1$

$C = 1 - D$ **(1)**

$Ce^2 + De + 1 = 2$

$Ce^2 + De = 1$ **(2)**

Substituting **(1)** into **(2)** gives:

$(1 - D)e^2 + De = 1$

$e^2 - De^2 + De = 1$

$De(1 - e) = 1 - e^2$

$D = \frac{1 - e^2}{e(1 - e)}$

$= \frac{1 + e}{e}$

$C = 1 - \frac{1 + e}{e}$

$= \frac{e - 1 - e}{e}$

$= -\frac{1}{e}$

Therefore:

$x = -\frac{1}{e}e^{2t} + \left(\frac{1 + e}{e}\right)e^{t} + t$

7 $\frac{d^2x}{dt^2} - 9x = 10\sin t$ **(1)**

Let $x = A\cos t + B\sin t$

$\frac{dx}{dt} = -A\sin t + B\cos t$

$\frac{d^2x}{dt^2} = -A\cos t - B\sin t$

Substituting into **(1)** gives:

$-A\cos t - B\sin t - 9(A\cos t + B\sin t) = 10\sin t$

$-A\cos t - B\sin t - 9A\cos t - 9B\sin t = 10\sin t$

$-10A\cos t - 10B\sin t = 10\sin t$

For sin t:

$-10B = 10$

$B = -1$

For cos t:

$-10A = 0$

$A = 0$

Hence the particular integral is $-\sin t$

$m^2 - 9 = 0$

$m = \pm 3$

So the complementary function is:

$x = Ce^{3t} + De^{-3t}$

And the general solution is:

$x = Ce^{3t} + De^{-3t} - \sin t$

$\frac{dx}{dt} = 3Ce^{3t} - 3De^{-3t} - \cos t$

When $t = 0$, $x = 2$ and $\frac{dx}{dt} = -1$

$C + D = 2$

$3C - 3D - 1 = -1$

$3C - 3D = 0$

$C = D$

$C = 1$ and $D = 1$

Therefore:

$x = e^{3t} + e^{-3t} - \sin t$

8 a i $\frac{d^2x}{dt^2} - 4\frac{dx}{dt} + 4x = 3te^{2t}$ **(1)**

$x = \lambda t^3 e^{2t}$

$\frac{dx}{dt} = 3\lambda t^2 e^{2t} + 2\lambda t^3 e^{2t}$

$\frac{d^2x}{dt^2} = 6\lambda te^{2t} + 6\lambda t^2 e^{2t} + 6\lambda t^2 e^{2t} + 4\lambda t^3 e^{2t}$

$= 6\lambda te^{2t} + 12\lambda t^2 e^{2t} + 4\lambda t^3 e^{2t}$

Substituting into **(1)** gives:

$6\lambda te^{2t} + 12\lambda t^2 e^{2t} + 4\lambda t^3 e^{2t} - 4\left(3\lambda t^2 e^{2t} + 2\lambda t^3 e^{2t}\right) + 4\lambda t^3 e^{2t} = 3te^{2t}$

$6\lambda te^{2t} + 12\lambda t^2 e^{2t} + 4\lambda t^3 e^{2t} - 12\lambda t^2 e^{2t} - 8\lambda t^3 e^{2t} + 4\lambda t^3 e^{2t} = 3te^{2t}$

$6\lambda t + 12\lambda t^2 + 4\lambda t^3 - 12\lambda t^2 - 8\lambda t^3 + 4\lambda t^3 = 3t$

$6\lambda t = 3t$

$\lambda = \frac{1}{2}$

Hence the particular integral is $\frac{1}{2}t^3 e^{2t}$

ii $m^2 - 4m + 4 = 0$

$(m-2)(m-2) = 0$

$m = 2$

So the complementary function is:

$x = (A + Bt)e^{2t}$

And the general solution is:

$x = (A + Bt)te^{2t} + \frac{1}{2}t^3 e^{2t}$

b $x = (A + Bt)e^{2t} + \frac{1}{2}t^3 e^{2t}$

$\frac{dx}{dt} = 2(A + Bt)e^{2t} + Be^{2t} + t^3 e^{2t} + \frac{3}{2}t^2 e^{2t}$

When $t = 0$, $x = 0$ and $\frac{dx}{dt} = 1$

$A = 0$

$2A + B = 1$

$B = 1$

Therefore:

$x = te^{2t} + \frac{1}{2}t^3 e^{2t}$

$= te^{2t}\left(1 + \frac{1}{2}t^2\right)$

9 $25\frac{d^2x}{dt^2}+36x=18$ **(1)**

Let $x=A$

$\frac{dx}{dt}=0$

$\frac{d^2x}{dt^2}=0$

Substituting into **(1)** gives:

$36A=18$

$A=\frac{1}{2}$

Hence the particular integral is $\frac{1}{2}$

$25m^2+36=0$

$m^2=-\frac{36}{25}$

$m=\pm\frac{6}{5}\text{i}$

So the complementary function is:

$x=B\cos\left(\frac{6}{5}t\right)+C\sin\left(\frac{6}{5}t\right)$

And the general solution is:

$x=B\cos\left(\frac{6}{5}t\right)+C\sin\left(\frac{6}{5}t\right)+\frac{1}{2}$

$\frac{dx}{dt}=-\frac{6}{5}B\sin\left(\frac{6}{5}t\right)+\frac{6}{5}C\cos\left(\frac{6}{5}t\right)$

when $t=0$, $x=1$ and $\frac{dx}{dt}=0.6$

$B+\frac{1}{2}=1$

$B=\frac{1}{2}$

$\frac{6}{5}C=\frac{6}{10}$

$C=\frac{1}{2}$

Therefore:

$x=\frac{1}{2}\cos\left(\frac{6}{5}t\right)+\frac{1}{2}\sin\left(\frac{6}{5}t\right)+\frac{1}{2}$

$=\frac{1}{2}\left(\cos\left(\frac{6}{5}t\right)+\sin\left(\frac{6}{5}t\right)+1\right)$

10 a $\frac{d^2x}{dt^2} - 2\frac{dx}{dt} + 2x = 2t^2$ **(1)**

Let $x = At^2 + Bt + C$

$\frac{dx}{dt} = 2At + B$

$\frac{d^2x}{dt^2} = 2A$

Substituting into **(1)** gives:

$2A - 2(2At + B) + 2(At^2 + Bt + C) = 2t^2$

$2A - 4At - 2B + 2At^2 + 2Bt + 2C = 2t^2$

For t^2:

$2A = 2$

$A = 1$

For t:

$-4A + 2B = 0$

$-4 + 2B = 0$

$B = 2$

For constant terms:

$2A - 2B + 2C = 0$

$2 - 4 + 2C = 0$

$C = 1$

Hence the particular integral is $t^2 + 2t + 1$

$m^2 - 2m + 2 = 0$

$m = \frac{2 \pm \sqrt{(-2)^2 - 4(1)(2)}}{2(1)}$

$= \frac{2 \pm \sqrt{-4}}{2}$

$m = 1 \pm i$

So the complementary function is:

$x = (D\cos t + E\sin t)e^t$

And the general solution is:

$x = (D\cos t + E\sin t)e^t + t^2 + 2t + 1$

10 b $x = (D\cos t + E\sin t)e^t + t^2 + 2t + 1$

$\frac{dx}{dt} = (-D\sin t + E\cos t)e^t + (D\cos t + E\sin t)e^t + 2t + 2$

when $t = 0$, $x = 1$ and $\frac{dx}{dt} = 3$

$D + 1 = 1$
$D = 0$
$E + D + 2 = 3$
$E = 1$
Therefore:
$x = e^t \sin t + t^2 + 2t + 1$
$= e^t \sin t + (t + 1)^2$

11 a $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2 = 3e^{2x}$ **(1)**
Let $y = \lambda xe^{2x}$
$\frac{dy}{dx} = 2\lambda xe^{2x} + \lambda e^{2x}$
$\frac{d^2y}{dx^2} = 4\lambda xe^{2x} + 2\lambda e^{2x} + 2\lambda e^{2x}$
$= 4\lambda xe^{2x} + 4\lambda e^{2x}$
Substituting into **(1)** gives:
$4\lambda xe^{2x} + 4\lambda e^{2x} - 3(2\lambda xe^{2x} + \lambda e^{2x}) + 2\lambda xe^{2x} = 3e^{2x}$
$4\lambda xe^{2x} + 4\lambda e^{2x} - 6\lambda xe^{2x} - 3\lambda e^{2x} + 2\lambda xe^{2x} = 3e^{2x}$
$\lambda = 3$
Hence the particular integral is $3xe^{2x}$
$m^2 - 3m + 2 = 0$
$(m - 1)(m - 2) = 0$
$m = 1$ or $m = 2$
So the complementary function is:
$y = Ae^x + Be^{2x}$
And the general solution is:
$y = Ae^x + Be^{2x} + 3xe^{2x}$

11 b $y = Ae^x + Be^{2x} + 3xe^{2x}$

$\frac{dy}{dx} = Ae^x + 2Be^{2x} + 3e^{2x} + 6xe^{2x}$

When $x = 0$, $y = 0$ and $\frac{dy}{dx} = 0$

Substituting into **(1)** gives:

$A + B = 0$

$A = -B$

$A + 2B + 3 = 0$

$A + 2B = -3$

$B = -3$

$A = 3$

Therefore:

$y = 3e^x - 3e^{2x} + 3xe^{2x}$

12 $\frac{d^2y}{dx^2}+9y=\sin 3x$ **(1)**

Let $y=Ax\cos 3x+Bx\sin 3x$

$\frac{dy}{dx}=-3Ax\sin 3x+3Bx\cos 3x+A\cos 3x+B\sin 3x$

$\frac{d^2y}{dx^2}=-9Ax\cos 3x-9Bx\sin 3x-6A\sin 3x+6B\cos 3x$

Substituting into **(1)** gives:

$-9Ax\cos 3x-9Bx\sin 3x-6A\sin 3x+6B\cos 3x+9\left(Ax\cos 3x+Bx\sin 3x\right)=\sin 3x$

$-9Ax\cos 3x-9Bx\sin 3x-6A\sin 3x+6B\cos 3x+9Ax\cos 3x+9Bx\sin 3x=\sin 3x$

$-6A\sin 3x+6B\cos 3x=\sin 3x$

Comparing coefficients:

For sin $3x$:

$-6A=1$

$A=-\frac{1}{6}$

For cos $3x$:

$6B=0$

$B=0$

Hence the particular integral is $-\frac{1}{6}x\cos 3x$

$m^2+9=0$

$m=\pm 3\text{i}$

So the complementary function is:

$y=C\sin 3x+D\cos 3x$

And the general solution is:

$y=C\sin 3x+D\cos 3x-\frac{1}{6}x\cos 3x$

$\frac{dy}{dx}=3C\cos 3x-3D\sin 3x-\frac{1}{6}\cos 3x+\frac{1}{2}x\sin 3x$

When $x=0$, $y=0$ and $\frac{dy}{dx}=0$

$D=0$

$3C-\frac{1}{6}=0$

$C=\frac{1}{18}$

Therefore:

$y=\frac{1}{18}\sin 3x-\frac{1}{6}x\cos 3x$

13 a $\frac{d^2x}{dt^2}+5\frac{dx}{dt}+6x=2e^{-t}$ **(1)**

$x=\lambda e^{-t}$

$\frac{dx}{dt}=-\lambda e^{-t}$

$\frac{d^2x}{dt^2}=\lambda e^{-t}$

Substituting into **(1)** gives:

$\lambda e^{-t}-5\lambda e^{-t}+6\lambda e^{-t}=2e^{-t}$

$2\lambda=2$

$\lambda=1$

Hence the particular integral is e^{-t}

$m^2+5m+6=0$

$(m+2)(m+3)=0$

$m=-2$ or $m=-3$

So the complementary function is:

$x=Ae^{-2t}+Be^{-3t}$

And the general solution is:

$x=Ae^{-2t}+Be^{-3t}+e^{-t}$

$\frac{dx}{dt}=-2Ae^{-2t}-3Be^{-3t}-e^{-t}$

When $t=0$, $x=0$ and $\frac{dx}{dt}=2$

$A+B+1=0$

$A+B=-1$ (1)

$-2A-3B-1=2$

$-2A-3B=3$ (2)

Adding 2 × **(1)** and **(2)** gives:

$B=-1$

$A=0$

Therefore:

$x=-e^{-3t}+e^{-t}$

13 b $x=-e^{-3t}+e^{-t}$

$\frac{dx}{dt}=3e^{-3t}-e^{-t}$

At a maximum $\frac{dx}{dt}=0$

Therefore:

$3e^{-3t}-e^{-t}=0$

$3e^{-3t}=e^{-t}$

$\frac{e^{-3t}}{e^{-t}}=\frac{1}{3}$

$\ln\left(e^{-2t}\right)=\ln\left(\frac{1}{3}\right)$

$-2t=\ln\left(\frac{1}{3}\right)$

$t=-\frac{1}{2}\ln\left(\frac{1}{3}\right)$

$=\frac{1}{2}\ln 3$

$\frac{d^2x}{dt^2}=-9e^{-3t}+e^{-t}$

Since $t=\frac{1}{2}\ln 3$

$\frac{d^2x}{dt^2}=-9e^{-\frac{3}{2}\ln 3}+e^{-\frac{1}{2}\ln 3}$

$=-9e^{\ln 3^{-\frac{3}{2}}}+e^{\ln 3^{-\frac{1}{2}}}$

$=-9\times 3^{-\frac{3}{2}}+3^{-\frac{1}{2}}$

$=-1.154..$

Since this is negative $t=\frac{1}{2}\ln 3$ is a maximum

$x=-e^{-\frac{3}{2}\ln 3}+e^{-\frac{1}{2}\ln 3}$

$=-3^{-\frac{3}{2}}+3^{-\frac{1}{2}}$

$=\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{27}}$

$=\frac{1}{\sqrt{3}}-\frac{1}{3\sqrt{3}}$

$=\frac{1}{\sqrt{3}}\left(1-\frac{1}{3}\right)$

$=\frac{2}{3\sqrt{3}}$

$=\frac{2\sqrt{3}}{9}$