Exercise 6B

1 a $\frac{d^2y}{dx^2}+6\frac{dy}{dx}+5y=10$ **(1)**

Let $y=\lambda$

$\frac{dy}{dx}=0$

$\frac{d^2y}{dx^2}=0$

Substituting into **(1)** gives:

$5\lambda=10$

$\lambda=2$

So the particular integral is 5

$m^2+6m+5=0$

$(m+1)(m+5)=0$

$m=-1$ or $m=-5$

So the complementary function is:

$y=Ae^{-x}+Be^{-5x}$

And the general solution is:

$y=Ae^{-x}+Be^{-5x}+2$

b $\frac{d^2y}{dx^2}-8\frac{dy}{dx}+12y=36x$ **(1)**

Let $y=\lambda x+\mu$

$\frac{dy}{dx}=\lambda$, $\frac{d^2y}{dx^2}=0$

Substituting into **(1)** gives:

$0-8\lambda+12(\lambda x+\mu)=36x$

$-8\lambda+12\lambda x+12\mu=36x$

Comparing coefficients:

For x:

$12\lambda=36$

$\lambda=3$

For the constant terms:

$-8\lambda+12\mu=0$

$\mu=\frac{2}{3}\lambda$

$\mu=2$

So the particular integral is $3x+2$

$m^2-8m+12=0$

$(m-2)(m-6)=0$

$m=2$ or $m=6$

So the complementary function is:

$y=Ae^{2x}+Be^{6x}$

And the general solution is:

$y=Ae^{2x}+Be^{6x}+3x+2$

1 c $\frac{d^2y}{dx^2}+\frac{dy}{dx}-12y=12e^{2x}$ **(1)**

Let $y=\lambda e^{2x}$

$\frac{dy}{dx}=2\lambda e^{2x}$

$\frac{d^2y}{dx^2}=4\lambda e^{2x}$

Substituting into **(1)** gives:

$4\lambda e^{2x}+2\lambda e^{2x}-12\lambda e^{2x}=12e^{2x}$

$-6\lambda e^{2x}=12e^{2x}$

$\lambda=-2$

So the particular integral is $-2e^{2x}$

$m^2+m-12=0$

$(m+4)(m-3)=0$

$m=-4$ or $m=3$

So the complementary function is:

$y=Ae^{-4x}+Be^{3x}$

And the general solution is:

$y=Ae^{-4x}+Be^{3x}-2e^{2x}$

d $\frac{d^2y}{dx^2}+2\frac{dy}{dx}-15y=5$ **(1)**

Let $y=\lambda$

$\frac{dy}{dx}=0$

$\frac{d^2y}{dx^2}=0$

Substituting into **(1)** gives:

$-15\lambda=5$

$\lambda=-\frac{1}{3}$

So the particular integral is $-\frac{1}{3}$

$m^2+2m-15=0$

$(m-3)(m+5)=0$

$m=3$ or $m=-5$

So the complementary function is:

$y=Ae^{3x}+Be^{-5x}$

And the general solution is:

$y=Ae^{3x}+Be^{-5x}-\frac{1}{3}$

1 e $\frac{d^2y}{dx^2}-8\frac{dy}{dx}+16y=8x+12$ **(1)**

Let $y=\lambda x+\mu$

$\frac{dy}{dx}=\lambda$

$\frac{d^2y}{dx^2}=0$

Substituting into **(1)** gives:

$0-8\lambda+16(\lambda x+\mu)=8x+12$

$-8\lambda+16\lambda x+16\mu=8x+12$

Comparing coefficients:

For x:

$16\lambda=8$

$\lambda=\frac{1}{2}$

For the constant terms:

$-8\lambda+16\mu=12$

$-4+16\mu=12$

$\mu=1$

So the particular integral is $\frac{1}{2}x+1$

$m^2-8m+16=0$

$(m-4)(m-4)=0$

$m=4$

So the complementary function is:

$y=(A+Bx)e^{4x}$

And the general solution is:

$y=(A+Bx)e^{4x}+\frac{1}{2}x+1$

1 f $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 25\cos 2x$ **(1)**

Let $y = \lambda \sin 2x + \mu \cos 2x$

$\frac{dy}{dx} = 2\lambda \cos 2x - 2\mu \sin 2x$

$\frac{d^2y}{dx^2} = -4\lambda \sin 2x - 4\mu \cos 2x$

Substituting into **(1)** gives:

$-4\lambda \sin 2x - 4\mu \cos 2x + 2(2\lambda \cos 2x - 2\mu \sin 2x) + \lambda \sin 2x + \mu \cos 2x = 25\cos 2x$

$-4\lambda \sin 2x - 4\mu \cos 2x + 4\lambda \cos 2x - 4\mu \sin 2x + \lambda \sin 2x + \mu \cos 2x = 25\cos 2x$

$\sin 2x(-3\lambda - 4\mu) + \cos 2x(4\lambda - 3\mu) = 25\cos 2x$

Comparing coefficients:

For cos $2x$:

$4\lambda - 3\mu = 25$

For sin $2x$:

$-3\lambda - 4\mu = 0$

$\mu = -\frac{3}{4}\lambda$

$4\lambda - 3\left(-\frac{3}{4}\lambda\right) = 25$

$\frac{25}{4}\lambda = 25$

$\lambda = 4$

$\mu = -3$

So the particular integral is $4\sin 2x - 3\cos 2x$

$m^2 + 2m + 1 = 0$

$(m+1)(m+1) = 0$

$m = -1$

So the complementary function is:

$y = (A + Bx)e^{-x}$

And the general solution is:

$y = (A + Bx)e^{-x} + 4\sin 2x - 3\cos 2x$

1 g $\frac{d^2y}{dx^2}+81y=15e^{3x}$ **(1)**

Let $y=\lambda e^{3x}$

$\frac{dy}{dx}=3\lambda e^{3x}$

$\frac{d^2y}{dx^2}=9\lambda e^{3x}$

Substituting into **(1)** gives:

$9\lambda e^{3x}+81\lambda e^{3x}=15e^{3x}$

$\lambda=\frac{1}{6}$

So the particular integral is $\frac{1}{6}e^{3x}$

$m^2+81=0$

$m^2=-81$

$m=\pm 9i$

So the complementary function is:

$y=A\cos 9x+B\sin 9x$

And the general solution is:

$y=A\cos 9x+B\sin 9x+\frac{1}{6}e^{3x}$

1 h $\frac{d^2y}{dx^2}+4y=\sin x$ **(1)**

Let $y=\lambda\sin x+\mu\cos x$

$\frac{dy}{dx}=\lambda\cos x-\mu\sin x$

$\frac{d^2y}{dx^2}=-\lambda\sin x-\mu\cos x$

Substituting into **(1)** gives:

$-\lambda\sin x-\mu\cos x+4(\lambda\sin x+\mu\cos x)=\sin x$

$-\lambda\sin x-\mu\cos x+4\lambda\sin x+4\mu\cos x=\sin x$

$3\lambda\sin x+3\mu\cos x=\sin x$

Comparing coefficients:

For $\cos x$:

$3\mu=0$

$\mu=0$

For $\sin x$:

$3\lambda=1$

$\lambda=\frac{1}{3}$

So the particular integral is $\frac{1}{3}\sin x$

$m^2+4=0$

$m^2=-4$

$m=\pm 2\text{i}$

So the complementary function is:

$y=A\cos 2x+B\sin 2x$

And the general solution is:

$y=A\cos 2x+B\sin 2x+\frac{1}{3}\sin x$

1 i $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 5y = 25x^2 - 7$ **(1)**

Let $y = \lambda x^2 + \mu x + \nu$

$\frac{dy}{dx} = 2\lambda x + \mu$

$\frac{d^2y}{dx^2} = 2\lambda$

Substituting into **(1)** gives:

$2\lambda - 4(2\lambda x + \mu) + 5(\lambda x^2 + \mu x + \nu) = 25x^2 - 7$

$2\lambda - 8\lambda x - 4\mu + 5\lambda x^2 + 5\mu x + 5\nu = 25x^2 - 7$

Comparing coefficients:

For x^2:

$5\lambda = 25$

$\lambda = 5$

For x:

$-8\lambda + 5\mu = 0$

$\mu = \frac{8}{5}\lambda$

$\mu = 8$

For the constant terms:

$2\lambda - 4\mu + 5\nu = -7$

$10 - 32 + 5\nu = -7$

$5\nu = 15$

$\nu = 3$

So the particular integral is $5x^2 + 8x + 3$

$m^2 - 4m + 5 = 0$

$m = \frac{4 \pm \sqrt{(-4)^2 - 4(1)(5)}}{2(1)}$

$= \frac{4 \pm \sqrt{-4}}{2}$

$= 2 \pm i$

So the complementary function is:

$y = e^{2x}(A\cos x + B\sin x)$

And the general solution is:

$y = e^{2x}(A\cos x + B\sin x) + 5x^2 + 8x + 3$

1 j $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 26y = e^x$ **(1)**

Let $y = \lambda e^x$

$\frac{dy}{dx} = \lambda e^x$

$\frac{d^2y}{dx^2} = \lambda e^x$

Substituting into **(1)** gives:

$\lambda e^x - 2\lambda e^x + 26\lambda e^x = e^x$

$25\lambda = 1$

$\lambda = \frac{1}{25}$

So the particular integral is $\frac{1}{25}e^x$

$m^2 - 2m + 26 = 0$

$m = \frac{2 \pm \sqrt{(-2)^2 - 4(1)(26)}}{2(1)}$

$= \frac{2 \pm \sqrt{-100}}{2}$

$= 1 \pm 5\text{i}$

So the complementary function is:

$y = e^x(A\cos 5x + B\sin 5x)$

And the general solution is:

$y = e^x(A\cos 5x + B\sin 5x) + \frac{1}{25}e^x$

2 a $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 4y = x^2 - 3x + 2$ **(1)**

Let $y = \lambda x^2 + \mu x + \nu$

$\frac{dy}{dx} = 2\lambda x + \mu$

$\frac{d^2y}{dx^2} = 2\lambda$

Substituting into **(1)** gives:

$2\lambda - 5(2\lambda x + \mu) + 4(\lambda x^2 + \mu x + \nu) = x^2 - 3x + 2$

$2\lambda - 10\lambda x - 5\mu + 4\lambda x^2 + 4\mu x + 4\nu = x^2 - 3x + 2$

Comparing coefficients:

For x^2:

$4\lambda = 1$

$\lambda = \frac{1}{4}$

For x:

$-10\lambda + 4\mu = -3$

$-\frac{10}{4} + 4\mu = -3$

$4\mu = -\frac{1}{2}$

$\mu = -\frac{1}{8}$

For the constant terms:

$2\lambda - 5\mu + 4\nu = 2$

$\frac{1}{2} + \frac{5}{8} + 4\nu = 2$

$4\nu = \frac{7}{8}$

$\nu = \frac{7}{32}$

So the particular integral is $\frac{1}{4}x^2 - \frac{1}{8}x + \frac{7}{32}$

b $m^2 - 5m + 4 = 0$

$(m-1)(m-4) = 0$

$m = 1$ or $m = 4$

So the complementary function is:

$y = Ae^x + Be^{4x}$

And the general solution is:

$y = Ae^x + Be^{4x} + \frac{1}{4}x^2 - \frac{1}{8}x + \frac{7}{32}$

3 a $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} = 2x^2 - x + 1$ **(1)**

$m^2 - 6m = 0$

$m(m-6) = 0$

$m = 0$ or $m = 6$

So the complementary function is:

$y = Ae^0 + Be^{6x}$

$= A + Be^{6x}$

b Let $y = \lambda x^3 + \mu x^2 + \nu x$

$\frac{dy}{dx} = 3\lambda x^2 + 2\mu x + \nu$

$\frac{d^2y}{dx^2} = 6\lambda x + 2\mu$

Substituting into **(1)** gives:

$6\lambda x + 2\mu - 6(3\lambda x^2 + 2\mu x + \nu) = 2x^2 - x + 1$

$6\lambda x + 2\mu - 18\lambda x^2 - 12\mu x - 6\nu = 2x^2 - x + 1$

Comparing coefficients:

For x^2:

$-18\lambda = 2$

$\lambda = -\frac{1}{9}$

For x:

$-\frac{2}{3} - 12\mu = -1$

$12\mu = \frac{1}{3}$

$\mu = \frac{1}{36}$

For the constant terms:

$2\mu - 6\nu = 1$

$\frac{1}{18} - 6\nu = 1$

$6\nu = -\frac{17}{18}$

$\nu = -\frac{17}{108}$

So the particular integral is $-\frac{1}{9}x^3 + \frac{1}{36}x^2 - \frac{17}{108}x$

And the general solution is:

$y = A + Be^{6x} - \frac{1}{9}x^3 + \frac{1}{36}x^2 - \frac{17}{108}x$

4 $\dfrac{d^2y}{dx^2}+4\dfrac{dy}{dx}=24x^2$ **(1)**

Let $y=\lambda x^3+\mu x^2+\nu x$

$\dfrac{dy}{dx}=3\lambda x^2+2\mu x+\nu$

$\dfrac{d^2y}{dx^2}=6\lambda x+2\mu$

Substituting into **(1)** gives:

$6\lambda x+2\mu+4\left(3\lambda x^2+2\mu x+\nu\right)=24x^2$

$6\lambda x+2\mu+12\lambda x^2+8\mu x+4\nu=24x^2$

Comparing coefficients:

For x^2:

$12\lambda=24$

$\lambda=2$

For x:

$6\lambda+8\mu=0$

$12+8\mu=0$

$\mu=-\dfrac{3}{2}$

For the constant terms:

$2\mu+4\nu=0$

$-3+4\nu=0$

$\nu=\dfrac{3}{4}$

So the particular integral is $2x^3-\dfrac{3}{2}x^2+\dfrac{3}{4}x$

$m^2+4m=0$

$m(m+4)=0$

$m=0$ or $m=-4$

So the complementary function is:

$y=Ae^0+Be^{-4x}$

$\quad=A+Be^{-4x}$

And the general solution is:

$y=A+Be^{-4x}+2x^3-\dfrac{3}{2}x^2+\dfrac{3}{4}x$

5 a $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = e^x$ **(1)**

Let $y = \lambda x e^x$

$\frac{dy}{dx} = \lambda x e^x + \lambda e^x$

$\frac{d^2y}{dx^2} = \lambda x e^x + \lambda e^x + \lambda e^x$

$= \lambda x e^x + 2\lambda e^x$

Substituting into **(1)** gives:

$\lambda x e^x + 2\lambda e^x - 2(\lambda x e^x + \lambda e^x) + \lambda x e^x = e^x$

$\lambda x e^x + 2\lambda e^x - 2\lambda x e^x - 2\lambda e^x + \lambda x e^x = e^x$

$0 = e^x$

This is not possible therefore $y = \lambda x e^x$ is not suitable

b $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = e^x$ **(1)**

Let $y = \lambda x^2 e^x$

$\frac{dy}{dx} = \lambda x^2 e^x + 2\lambda x e^x$

$\frac{d^2y}{dx^2} = \lambda x^2 e^x + 2\lambda x e^x + 2\lambda x e^x + 2\lambda e^x$

$= \lambda x^2 e^x + 4\lambda x e^x + 2\lambda e^x$

Substituting into **(1)** gives:

$\lambda x^2 e^x + 4\lambda x e^x + 2\lambda e^x - 2(\lambda x^2 e^x + 2\lambda x e^x) + \lambda x^2 e^x = e^x$

$\lambda x^2 e^x + 4\lambda x e^x + 2\lambda e^x - 2\lambda x^2 e^x - 4\lambda x e^x + \lambda x^2 e^x = e^x$

$2\lambda e^x = e^x$

$\lambda = \frac{1}{2}$

So the particular integral is $\frac{1}{2}x^2 e^x$

c $m^2 - 2m + 1 = 0$

$(m-1)(m-1) = 0$

$m = 1$

So the complementary function is:

$y = (A + Bx)e^x$

And the general solution is:

$y = (A + Bx)e^x + \frac{1}{2}x^2 e^x$

6 a $\frac{d^2y}{dt^2}+4\frac{dy}{dt}+3t=kt+5$ **(1)**

Let $y=\lambda t+\mu$

$\frac{dy}{dt}=\lambda$

$\frac{d^2y}{dt^2}=0$

Substituting into **(1)** gives:

$4\lambda+3(\lambda t+\mu)=kt+5$

$4\lambda+3\lambda t+3\mu=kt+5$

Comparing coefficients:

For t:

$3\lambda=k$

$\lambda=\frac{1}{3}k$

For the constant terms:

$4\lambda+3\mu=5$

$\frac{4k}{3}+3\mu=5$

$\mu=\frac{15-4k}{9}$

So the particular integral is $\frac{1}{3}kt+\frac{15-4k}{9}$

$m^2+4m+3=0$

$(m+1)(m+3)=0$

$m=-1$ or $m=-3$

So the complementary function is:

$y=Ae^{-3t}+Be^{-t}$

And the general solution is:

$y=Ae^{-3t}+Be^{-t}+\frac{1}{3}kt+\frac{15-4k}{9}$

b As $t\to\infty$, $Ae^{-3t}\to 0$ and $Be^{-t}\to 0$

$y=\frac{1}{3}kt+\frac{15-4k}{9}$

When $k=6$

$y=\frac{1}{3}(6)t+\frac{15-4(6)}{9}$

$=2t-1$

Challenge

$\frac{d^2y}{dx^2} + y = 5xe^{2x}$ **(1)**

Let $y = \lambda xe^{2x} + \mu e^{2x}$

$\frac{dy}{dx} = 2\lambda xe^{2x} + \lambda e^{2x} + 2\mu e^{2x}$

$\frac{d^2y}{dx^2} = 4\lambda xe^{2x} + 2\lambda e^{2x} + 2\lambda e^{2x} + 4\mu e^{2x}$

Substituting into **(1)** gives:

$4\lambda xe^{2x} + 4\lambda e^{2x} + 4\mu e^{2x} + \lambda xe^{2x} + \mu e^{2x} = 5xe^{2x}$

Comparing coefficients:

For x:

$4\lambda + \lambda = 5$

$\lambda = 1$

For the constant terms:

$4\lambda + 4\mu + \mu = 0$

$4 + 5\mu = 0$

$\mu = -\frac{4}{5}$

So the particular integral is $xe^{2x} - \frac{4}{5}e^{2x}$

$m^2 + 1 = 0$

$m = \pm i$

So the complementary function is:

$y = A\cos x + B\sin x$

And the general solution is:

$y = A\cos x + B\sin x + xe^{2x} - \frac{4}{5}e^{2x}$