Exercise 6A

1 a $\frac{d^2y}{dx^2}+5\frac{dy}{dx}+6y=0$

$m^2+5m+6=0$

$(m+2)(m+3)=0$

$m=-2$ or $m=-3$

$y=Ae^{-2x}+Be^{-3x}$

b $\frac{d^2y}{dx^2}-8\frac{dy}{dx}+12y=0$

$m^2-8m+12=0$

$(m-2)(m-6)=0$

$m=2$ or $m=6$

$y=Ae^{2x}+Be^{6x}$

c $\frac{d^2y}{dx^2}+2\frac{dy}{dx}-15y=0$

$m^2+2m-15=0$

$(m-3)(m+5)=0$

$m=3$ or $m=-5$

$y=Ae^{3x}+Be^{-5x}$

d $\frac{d^2y}{dx^2}-3\frac{dy}{dx}-28y=0$

$m^2-3m-28=0$

$(m-7)(m+4)=0$

$m=7$ or $m=-4$

$y=Ae^{7x}+Be^{-4x}$

e $\frac{d^2y}{dx^2}+5\frac{dy}{dx}=0$

$m^2+5m=0$

$m(m+5)=0$

$m=0$ or $m=-5$

$y=A+Be^{-5x}$

f $3\frac{d^2y}{dx^2}+7\frac{dy}{dx}+2y=0$

$3m^2+7m+2=0$

$(3m+1)(m+2)=0$

$m=-\frac{1}{3}$ or $m=-2$

$y=Ae^{-\frac{1}{3}x}+Be^{-2x}$

1 g $4\frac{d^2y}{dx^2} - 7\frac{dy}{dx} - 2y = 0$

$4m^2 - 7m - 2 = 0$

$(4m+1)(m-2) = 0$

$m = -\frac{1}{4}$ or $m = 2$

$y = Ae^{-\frac{1}{4}x} + Be^{2x}$

1 h $15\frac{d^2y}{dx^2} - 7\frac{dy}{dx} - 2y = 0$

$15m^2 - 7m - 2 = 0$

$(3m-2)(5m+1) = 0$

$m = \frac{2}{3}$ or $m = -\frac{1}{5}$

$y = Ae^{\frac{2}{3}x} + Be^{-\frac{1}{5}x}$

2 a $\frac{d^2y}{dx^2} + 10\frac{dy}{dx} + 25y = 0$

$m^2 + 10m + 25 = 0$

$(m+5)(m+5) = 0$

$m = -5$

$y = (A + Bx)e^{-5x}$

b $\frac{d^2y}{dx^2} - 18\frac{dy}{dx} + 81y = 0$

$m^2 - 18m + 81 = 0$

$(m-9)(m-9) = 0$

$m = 9$

$y = (A + Bx)e^{9x}$

c $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 0$

$m^2 + 2m + 1 = 0$

$(m+1)(m+1) = 0$

$m = -1$

$y = (A + Bx)e^{-x}$

2 d $\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 16y = 0$

$m^2 - 8m + 16 = 0$

$(m-4)(m-4) = 0$

$m = 4$

$y = (A + Bx)e^{4x}$

e $16\frac{d^2y}{dx^2} + 8\frac{dy}{dx} + y = 0$

$16m^2 + 8m + 1 = 0$

$(4m+1)(4m+1) = 0$

$m = -\frac{1}{4}$

$y = (A + Bx)e^{-\frac{1}{4}x}$

f $4\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + y = 0$

$4m^2 - 4m + 1 = 0$

$(2m-1)(2m-1) = 0$

$m = \frac{1}{2}$

$y = (A + Bx)e^{\frac{1}{2}x}$

g $4\frac{d^2y}{dx^2} + 20\frac{dy}{dx} + 25y = 0$

$4m^2 + 20m + 25 = 0$

$(2m+5)(2m+5) = 0$

$m = -\frac{5}{2}$

$y = (A + Bx)e^{-\frac{5}{2}x}$

h $\frac{d^2y}{dx^2} + 2\sqrt{3}\frac{dy}{dx} + 3y = 0$

$m^2 + 2\sqrt{3}m + 3 = 0$

$(m+\sqrt{3})(m+\sqrt{3}) = 0$

$m = -\sqrt{3}$

$y = (A + Bx)e^{-\sqrt{3}x}$

3 **a** $\frac{d^2y}{dx^2}+25y=0$

$m^2+25=0$

$m=\pm 5i$

$y=A\sin 5x+B\cos 5x$

b $\frac{d^2y}{dx^2}+81y=0$

$m^2+81=0$

$m=\pm 9i$

$y=A\sin 9x+B\cos 9x$

c $\frac{d^2y}{dx^2}+y=0$

$m^2+1=0$

$m=\pm i$

$y=A\sin x+B\cos x$

d $9\frac{d^2y}{dx^2}+16y=0$

$9m^2+16=0$

$m^2=-\frac{16}{9}$

$m=\pm\frac{4}{3}i$

$y=A\sin\left(\frac{4}{3}x\right)+B\cos\left(\frac{4}{3}x\right)$

e $\frac{d^2y}{dx^2}+8\frac{dy}{dx}+17y=0$

$m^2+8m+17=0$

$m=\frac{-8\pm\sqrt{8^2-4(1)(17)}}{2(1)}$

$m=\frac{-8\pm\sqrt{-4}}{2}$

$=\frac{-8\pm 2i}{2}$

$m=-4\pm i$

$y=e^{-4x}(A\sin x+B\cos x)$

3 f $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 5y = 0$

$m = \frac{4 \pm \sqrt{(-4)^2 - 4(1)(5)}}{2(1)}$

$= \frac{4 \pm \sqrt{-4}}{2}$

$= 2 \pm i$

$y = e^{2x}(A\cos x + B\sin x)$

g $\frac{d^2y}{dx^2} + 20\frac{dy}{dx} + 109y = 0$

$m^2 + 20m + 109 = 0$

$m = \frac{-20 \pm \sqrt{20^2 - 4(1)(109)}}{2(1)}$

$m = \frac{-20 \pm \sqrt{-36}}{2}$

$= \frac{-20 \pm 6i}{2}$

$m = -10 \pm 3i$

$y = e^{-10x}(A\sin 3x + B\cos 3x)$

h $\frac{d^2y}{dx^2} + \sqrt{3}\frac{dy}{dx} + 3y = 0$

$m^2 + \sqrt{3}m + 3 = 0$

$m = \frac{-\sqrt{3} \pm \sqrt{(\sqrt{3})^2 - 4(1)(3)}}{2(1)}$

$m = \frac{-\sqrt{3} \pm \sqrt{-9}}{2}$

$= \frac{-\sqrt{3} \pm 3i}{2}$

$m = -\frac{\sqrt{3}}{2} \pm \frac{3}{2}i$

$y = e^{-\frac{\sqrt{3}}{2}x}\left(A\sin\left(\frac{3}{2}x\right) + B\cos\left(\frac{3}{2}x\right)\right)$

4 a $\frac{d^2y}{dx^2} + 14\frac{dy}{dx} + 49y = 0$

$m^2 + 14m + 49 = 0$

$(m+7)(m+7) = 0$

$m = -7$

$y = (A + Bx)e^{-7x}$

4 b $\frac{d^2y}{dx^2}+\frac{dy}{dx}-12y=0$

$m^2+m-12=0$

$(m+4)(m-3)=0$

$m=-4$ or $m=3$

$y=Ae^{3x}+Be^{-4x}$

c $\frac{d^2y}{dx^2}+4\frac{dy}{dx}+13y=0$

$m^2+4m+13=0$

$m=\frac{-4\pm\sqrt{4^2-4(1)(13)}}{2(1)}$

$m=\frac{-4\pm\sqrt{-36}}{2}$

$=\frac{-4\pm 6i}{2}$

$m=-2\pm 3i$

$y=e^{-2x}(A\sin 3x+B\cos 3x)$

d $16\frac{d^2y}{dx^2}-24\frac{dy}{dx}+9y=0$

$16m^2-24m+9=0$

$(4m-3)(4m-3)=0$

$m=\frac{3}{4}$

$y=(A+Bx)e^{\frac{3}{4}x}$

e $9\frac{d^2y}{dx^2}-6\frac{dy}{dx}+5y=0$

$9m^2-6m+5=0$

$m=\frac{6\pm\sqrt{(-6)^2-4(9)(5)}}{2(9)}$

$m=\frac{6\pm\sqrt{-144}}{18}$

$=\frac{6\pm 12i}{18}$

$m=\frac{1}{3}\pm\frac{2}{3}i$

$y=e^{\frac{1}{3}x}\left(A\sin\left(\frac{2}{3}x\right)+B\cos\left(\frac{2}{3}x\right)\right)$

4 f $6\frac{d^2y}{dx^2}-\frac{dy}{dx}-2y=0$

$6m^2-m-2=0$

$(3m-2)(2m+1)=0$

$m=\frac{2}{3}$ or $m=-\frac{1}{2}$

$y=Ae^{\frac{2}{3}x}+Be^{-\frac{1}{2}x}$

5 a i $\frac{d^2x}{dt^2}+2k\frac{dx}{dt}+9x=0$

$m^2+2km+9=0$

$m=\frac{-2k\pm\sqrt{(2k)^2-4(1)(9)}}{2(1)}$

$m=-k\pm\frac{1}{2}\sqrt{4(k^2-9)}$

$=-k\pm\sqrt{k^2-9}$

If $|k|>3$ then $k^2>9$

$m=-k+\sqrt{k^2-9}$ or $m=-k-\sqrt{k^2-9}$

$x=Ae^{\left(-k+\sqrt{k^2-9}\right)t}+Be^{\left(-k-\sqrt{k^2-9}\right)t}$

ii If $k<3$ then $k^2<9$

$m=-k+i\sqrt{|k^2-9|}$ or $m=-k-i\sqrt{|k^2-9|}$

$x=e^{-kt}\left[A\cos\left(\sqrt{|k^2-9|}\right)t+B\cos\left(\sqrt{|k^2-9|}\right)t\right]$

iii If $k=|3|$ then $k^2-9=0$ and $m=-k$

$x=(A+Bt)e^{-kt}$

b i $m=-2\pm i\sqrt{|k^2-9|}$

$=-2\pm\sqrt{5}i$

$x=e^{-2t}\left(A\cos\sqrt{5}t+B\cos\sqrt{5}t\right)$

ii As $t\to\infty$, $x(t)\to 0$

6 $y = (A + Bx)e^{\alpha x}$

$$\frac{dy}{dx} = \alpha(A + Bx)e^{\alpha x} + Be^{\alpha x}$$

$$\frac{d^2y}{dx^2} = \alpha^2(A + Bx)e^{\alpha x} + B\alpha e^{\alpha x} + B\alpha e^{\alpha x}$$

$$= \alpha^2(A + Bx)e^{\alpha x} + 2B\alpha e^{\alpha x}$$

$am^2 + bm + c = 0$, $m = \alpha$

Therefore:

$y = (A + Bx)e^{\alpha x}$

Since the auxiliary equation has repeated roots $b^2 - 4ac = 0$ and $\alpha = -\frac{b}{2a}$

$$b^2 - 4ac = 0 \Rightarrow c = \frac{b^2}{4a} \qquad \textbf{(1)}$$

$$\alpha = -\frac{b}{2a} \Rightarrow a = -\frac{b}{2\alpha}$$

Substituting $a = -\frac{b}{2\alpha}$ into **(1)** gives:

$$c = \frac{b^2}{4\left(-\frac{b}{2\alpha}\right)}$$

$$= -\frac{2\alpha b^2}{4b}$$

$$= -\frac{1}{2}\alpha b$$

Thus

$ay'' + by' + cy$

$$= -\frac{b}{2\alpha}y'' + by' - \frac{b\alpha}{2}y$$

$$= by'' - 2\alpha by' + b\alpha^2 y$$

$$= y'' - 2\alpha y' + \alpha^2 y$$

$$= \alpha^2(A + Bx)e^{\alpha x} + 2B\alpha e^{\alpha x} - 2\alpha\left(\alpha(A + Bx)e^{\alpha x} + Be^{\alpha x}\right) + \alpha^2(A + Bx)e^{\alpha x}$$

$$= \alpha^2(A + Bx)e^{\alpha x} + 2B\alpha e^{\alpha x} - 2\alpha^2(A + Bx)e^{\alpha x} - 2\alpha Be^{\alpha x} + \alpha^2(A + Bx)e^{\alpha x}$$

$$= 0$$

Thus

$y = (A + Bx)e^{\alpha x}$

is a solution.

7 $ay''+by'+cy=0$ **(1)**

Has solutions f(x) and g(x)

Consider

$A\text{f}(x)+B\text{g}(x)$

$$a\left(A\text{f}''(x)+B\text{g}''(x)\right)+b\left(A\text{f}'(x)+B\text{g}'(x)\right)+c\left(A\text{f}(x)+B\text{g}(x)\right)$$
$$=A\left(a\text{f}''(x)+b\text{f}'(x)+c\text{f}(x)\right)+B\left(a\text{g}''(x)+b\text{g}'(x)+c\text{g}(x)\right)$$

But f(x) and g(x) are solutions to (1) therefore:

$$A\left(a\text{f}''(x)+b\text{f}'(x)+c\text{f}(x)\right)+B\left(a\text{g}''(x)+b\text{g}'(x)+c\text{g}(x)\right)=0$$

and

$y=A\text{f}(x)+B\text{g}(x)$

is a solution to **(1)**

Challenge

$$\begin{aligned}Ae^{\alpha x}+Be^{\beta x}&=Ae^{(p+qi)x}+Be^{(p-qi)x}\\&=Ae^{px}e^{qix}+Be^{px}e^{-qix}\\&=e^{px}\left(Ae^{qix}+Be^{-qix}\right)\\&=e^{px}\left(A\cos qx+iA\sin qx+B\cos qx-iB\sin qx\right)\\&=e^{px}\left((A+B)\cos qx+i(A-B)\sin qx\right)\end{aligned}$$

$A, B \in C$ so let $A=r+is$ and let $B=r-is$

$A+B=2r$ and $A-B=2is$

So

$$\begin{aligned}Ae^{\alpha x}+Be^{\beta x}&=e^{px}\left(2r\cos qx+i(2is)\sin qx\right)\\&=e^{px}\left(2r\cos qx-2s\sin qx\right)\end{aligned}$$

let $C=2r$ and let $D=-2s$

then:

$Ae^{\alpha x}+Be^{\beta x}=e^{px}\left(C\cos qx-D\sin qx\right)$ as required.