## Chapter review 5

**1** The integrating factor is $e^{\int \tan x \, dx} = e^{\ln \sec x} = \sec x$

Multiplying the equation by this factor gives:

$$\sec x \frac{dy}{dx} + y \sec x \tan x = 2\sec^2 x$$

$$\Rightarrow \frac{d}{dx}(y \sec x) = 2\sec^2 x$$

$$\Rightarrow y \sec x = \int 2\sec^2 x \, dx = 2\tan x + c$$

$$\Rightarrow y = 2\sin x + c \cos x$$

**2** Rewrite the equation as $\frac{dy}{dx} + \frac{x}{1-x^2} y = \frac{5x}{1-x^2}$

The integrating factor is $e^{\int \frac{x}{1-x^2} dx} = e^{-\frac{1}{2}\ln(1-x^2)} = e^{\ln(1-x^2)^{-\frac{1}{2}}} = (1-x^2)^{-\frac{1}{2}}$

Multiplying the equation by this factor gives:

$$\frac{1}{(1-x^2)^{\frac{1}{2}}} \frac{dy}{dx} + \frac{x}{(1-x^2)^{\frac{3}{2}}} y = \frac{5x}{(1-x^2)^{\frac{3}{2}}}$$

$$\Rightarrow \frac{dy}{dx} \frac{y}{(1-x^2)^{\frac{1}{2}}} = \frac{5x}{(1-x^2)^{\frac{3}{2}}}$$

$$\Rightarrow \frac{y}{(1-x^2)^{\frac{1}{2}}} = \int \frac{5x}{(1-x^2)^{\frac{3}{2}}} dx$$

$$\Rightarrow \frac{y}{(1-x^2)^{\frac{1}{2}}} = \frac{5}{(1-x^2)^{\frac{1}{2}}} + c$$

So $y = 5 + c(1-x^2)^{\frac{1}{2}}$

**3** Rewrite the equation as $x\frac{dy}{dx} + y = -x$

The left-hand side is the derivative of the product $xy$, so the equation is equivalent to

$$\frac{d}{dx}(xy) = -x$$

$$\Rightarrow xy = -\int x dx = -\frac{x^2}{2} + c$$

So $y = -\frac{x}{2} + \frac{c}{x}$

**4** The integrating factor is $e^{\int \frac{1}{x}dx} = e^{\ln x} = x$

Multiplying the equation by this factor gives:

$$x\frac{dy}{dx} + y = x\sqrt{x}$$

$$\Rightarrow \frac{d}{dx}(xy) = x\sqrt{x}$$

$$\Rightarrow xy = \int x\sqrt{x}\,dx = \frac{2}{5}x^{\frac{5}{2}} + c$$

So $y = \frac{2}{5}x^{\frac{3}{2}} + \frac{c}{x}$

**5** The integrating factor is $\mathbf{e}^{\int 2x\,dx} = \mathbf{e}^{x^2}$

Multiplying the equation by this factor gives:

$$e^{x^2}\frac{dy}{dx} + 2xe^{x^2}y = xe^{x^2}$$

$$\Rightarrow \frac{d}{dx}(e^{x^2}y) = xe^{x^2}$$

$$\Rightarrow e^{x^2}y = \int xe^{x^2}\,dx = \tfrac{1}{2}e^{x^2} + c$$

So $y = \frac{1}{2} + ce^{-x^2}$

**6** Rewrite the equation in the form $\frac{dy}{dx} + \frac{2x^2-1}{x(1-x^2)}y = \frac{2x^2}{(1-x^2)}$

The integrating factor is $\mathbf{e}^{\int \frac{2x^2-1}{x(1-x^2)}dx}$ and this simplifies as follows:

$$\int \frac{2x^2-1}{x(1-x^2)}dx = \int \frac{3x^2-1-x^2}{x-x^3}dx = \int \frac{3x^2-1}{x-x^3} - \frac{x}{1-x^2}dx$$

$$= -\ln(x-x^3) + \frac{1}{2}\ln(1-x^2) = \ln(1-x^2)^{\frac{1}{2}} - \ln(x-x^3)$$

$$= \ln\left(\frac{(1-x^2)^{\frac{1}{2}}}{(x-x^3)}\right) = \ln\left(\frac{1}{x(1-x^2)^{\frac{1}{2}}}\right)$$

So the integrating factor simplifies to $\frac{1}{\mathbf{x(1-x^2)^{\frac{1}{2}}}}$

Multiplying the equation by this factor gives:

$$\frac{1}{x(1-x^2)^{\frac{1}{2}}}\frac{dy}{dx} + \frac{2x^2-1}{x^2(1-x^2)^{\frac{3}{2}}}y = \frac{2x}{(1-x^2)^{\frac{3}{2}}}$$

$$\Rightarrow \frac{d}{dx}\left(\frac{y}{x(1-x^2)^{\frac{1}{2}}}\right) = \frac{2x}{(1-x^2)^{\frac{3}{2}}}$$

$$\Rightarrow \frac{y}{x(1-x^2)^{\frac{1}{2}}} = \int \frac{2x}{(1-x^2)^{\frac{3}{2}}}dx = \frac{2}{(1-x^2)^{\frac{1}{2}}} + c$$

So $y = 2x + cx(1-x^2)^{\frac{1}{2}} = 2x + cx\sqrt{1-x^2}$

**7 a** Given that $\frac{dy}{dx} - ay = ke^{\lambda x}$, the integrating factor is $e^{\int -a\,dx} = e^{-ax}$

Multiplying the equation by this factor gives:

$$e^{-ax}\frac{dy}{dx} - ae^{-ax}y = ke^{\lambda x}e^{-ax}$$

$$\Rightarrow \frac{d}{dx}(ye^{-ax}) = ke^{\lambda x}e^{-ax}$$

$$\Rightarrow ye^{-ax} = \int ke^{(\lambda - a)x}\,dx = \frac{ke^{(\lambda - a)x}}{\lambda - a}e^{(\lambda - a)x} + c$$

So $y = \frac{ke^{(\lambda - a)x}e^{ax}}{\lambda - a} + ce^{ax} = \frac{ke^{\lambda x}}{\lambda - a} + ce^{ax}$ for $\lambda \neq a$

**b** When $Q(x) = kx^n e^{ax}$

$$\frac{d}{dx}(ye^{-ax}) = ke^{\lambda x}e^{ax}e^{-ax} = ke^{\lambda x}$$

$$\Rightarrow ye^{-ax} = \int kx^n dx = \frac{kx^{n+1}}{n+1} + c$$

So $y = \frac{kx^{n+1}}{n+1}e^{ax} + ce^{ax}$

**8** Rewrite the equation in the form $\frac{dy}{dx} + \frac{y}{\tan x} = 2\cos x$

The integrating factor is $e^{\int \frac{1}{\tan x}dx} = e^{\int \cot x\,dx} = e^{\ln \sin x} = \sin x$

Multiplying both sides by this factor gives:

$\sin x\frac{dy}{dx} + y\cos x = 2\cos x \sin x = \sin 2x$ using a trigonometric identity

$\Rightarrow \frac{d}{dx}(y\sin x) = \sin 2x$

$\Rightarrow y\sin x = \int \sin 2x dx = -\frac{1}{2}\cos 2x + c = -\frac{1}{2} + \sin^2 x + c$ using $\cos 2x = 1 - \sin^2 x$

So $y = \sin x + A\operatorname{cosec} x$ defining $A = c - \frac{1}{2}$

Note that without making the simplification using $\cos 2x = 1 - \sin^2 x$, an alternative but correct answer is obtained.

$y\sin x = \int \sin 2x dx = -\frac{1}{2}\cos 2x + c$

$\Rightarrow y = -\frac{1}{2}\cos 2x \operatorname{cosec} x + c\operatorname{cosec} x$

**9** **a** $x\frac{dy}{dx}+y=y^2\ln x$

Use the transformation $z=y^{-1}$:

$\frac{dz}{dx}=-y^{-2}\frac{dy}{dx}$

$\frac{dy}{dx}=-y^2\frac{dz}{dx}$

$\frac{dy}{dx}=-z^{-2}\frac{dz}{dx}$

$z=y^{-1}\Rightarrow y=z^{-1}$

$z^2=y^{-2}\Rightarrow y^2=z^{-2}$

Transform the equation:

$-xz^{-2}\frac{dz}{dx}+z^{-1}=-z^{-2}\ln x$

Divide by $-xz^{-2}$:

$\frac{dz}{dx}-\frac{z}{x}=-\frac{\ln x}{x}$ as required

**b** $e^{-\int\frac{1}{x}dx}=e^{-\ln x}=\frac{1}{x}$

Thus:

$\frac{1}{x}\frac{dz}{dx}-\frac{z}{x^2}=-\frac{\ln x}{x}$

$\frac{z}{x}=\int-\frac{\ln x}{x}dx$

$\int -x^{-2}\ln x dx=x^{-1}\ln x-\int x^{-2}dx$

$=x^{-1}\ln x+x^{-1}$

$\frac{1}{x}z=\frac{1}{x}+\frac{\ln x}{x}+c$

$z=1+\ln x+cx$

Since $z=\frac{1}{y}$

$y=\frac{1}{1+\ln x+cx}$

**10 a** $z = y^2 \Rightarrow z^{-1} = y^{-2}$

$z^{\frac{1}{2}} = y \Rightarrow z^{-\frac{1}{2}} = y^{-1}$

$\frac{dz}{dx} = 2y\frac{dy}{dx}$

$\frac{dy}{dx} = \frac{1}{2y}\frac{dz}{dx}$

$= \frac{1}{2z^{\frac{1}{2}}}\frac{dz}{dx}$

The original equation is:

$2\cos x\frac{dy}{dx} - y\sin x + y^{-1} = 0$

Transform the equation:

$2\cos x \times \frac{1}{2z^{\frac{1}{2}}}\frac{dz}{dx} - z^{\frac{1}{2}}\sin x + z^{-\frac{1}{2}} = 0$

$\cos x\frac{dz}{dx} - z\sin x + 1 = 0$

$\cos x\frac{dz}{dx} - z\sin x = -1$ as required

**b** $\cos x\frac{dz}{dx} - z\sin x = -1$

$z\cos x = -x + c$

$z = c\sec x - x\sec x$

**c** Since $z = y^2$

$y^2 = c\sec x - x\sec x$

**11 a** $z=\frac{y}{x}\Rightarrow y=xz$

$\frac{\mathrm{d}y}{\mathrm{d}x}=x\frac{\mathrm{d}z}{\mathrm{d}x}+z$

The original equation is:

$\left(x^2-y^2\right)\frac{\mathrm{d}y}{\mathrm{d}x}-xy=0$

Transform the equation:

$\left(x^2-x^2z^2\right)\left[x\frac{\mathrm{d}z}{\mathrm{d}x}+z\right]-x^2z=0$

$\left(1-z^2\right)\left[x\frac{\mathrm{d}z}{\mathrm{d}x}+z\right]-z=0$

$\left(1-z^2\right)\left[x\frac{\mathrm{d}z}{\mathrm{d}x}+z\right]=z$

$x\frac{\mathrm{d}z}{\mathrm{d}x}+z=\frac{z}{1-z^2}$

$x\frac{\mathrm{d}z}{\mathrm{d}x}=\frac{z}{1-z^2}-z$

$=\frac{z-z\left(1-z^2\right)}{1-z^2}$

$=\frac{z^3}{1-z^2}$ as required

**b** $\int\frac{1-z^2}{z^3}\,\mathrm{d}z=\int\frac{1}{x}\,\mathrm{d}x$

$\int\frac{1}{z^3}\,\mathrm{d}z-\int\frac{1}{z}\,\mathrm{d}z=\int\frac{1}{x}\,\mathrm{d}x$

$-\frac{1}{2z^2}-\ln z=\ln x+c$

$-\frac{x^2}{2y^2}-\ln\left(\frac{y}{x}\right)=\ln x+c$

$-\frac{x^2}{2y^2}-\left(\ln y-\ln x\right)=\ln x+c$

$-\frac{x^2}{2y^2}+\ln x-\ln y=\ln x+c$

$-\frac{x^2}{2y^2}-\ln y=c$

$-x^2-2y^2\ln y-2y^2c=0$

$x^2+2y^2\ln y+2y^2c=0$

$x^2+2y^2\left(\ln y+c\right)=0$

**12 a** $z = \frac{y}{x} \Rightarrow y = xz$

$\frac{dy}{dx} = x\frac{dz}{dx} + z$

The original equation is:

$\frac{dy}{dx} = \frac{y(x+y)}{x(y-x)}$

Transform the equation:

$x\frac{dz}{dx} + z = \frac{xz(x+xz)}{x(xz-x)}$

$= \frac{z(z+1)}{z-1}$

$x\frac{dz}{dx} = \frac{z(z+1)}{z-1} - z$

$= \frac{z(z+1)-z(z-1)}{z-1}$

$= \frac{2z}{z-1}$ as required

**b** $\int \frac{z-1}{2z}\,dz = \int \frac{1}{x}\,dx$

$\frac{1}{2}\int dz - \frac{1}{2}\int \frac{1}{z}\,dz = \int \frac{1}{x}\,dx$

$\frac{1}{2}z - \frac{1}{2}\ln z = \ln x + c$

$\frac{1}{2}\left(\frac{y}{x}\right) - \frac{1}{2}\ln\left(\frac{y}{x}\right) = \ln x + c$

$\frac{y}{x} - \ln\left(\frac{y}{x}\right) = 2\ln x + 2c$

$\frac{y}{x} - (\ln y - \ln x) = 2\ln x + 2c$

$\frac{y}{x} + \ln x - \ln y = 2\ln x + 2c$

$\frac{y}{2x} - \frac{1}{2}\ln y = \frac{1}{2}\ln x + c$

**13 a** $z=\frac{y}{x}\Rightarrow y=xz$

$\frac{dy}{dx}=x\frac{dz}{dx}+z$

The original equation is:

$\frac{dy}{dx}=\frac{-3xy}{y^2-3x^2}$

Transform the equation:

$z+x\frac{dz}{dx}=\frac{-3x^2z}{x^2z^2-3x^2}$

$=-\frac{3z}{z^2-3}$

$x\frac{dz}{dx}=-\frac{3z}{z^2-3}-z$

$=\frac{-3z-z\left(z^2-3\right)}{z^2-3}$

$=\frac{-z^3}{z^2-3}$ as required

**b** $\frac{z^2-3}{-z^3}\frac{dz}{dx}=\frac{1}{x}$

$-\int\frac{z^2-3}{z^3}dz=\int\frac{1}{x}dx$

$-\int\frac{1}{z}dz+3\int\frac{1}{z^3}dz=\int\frac{1}{x}dx$

$-\ln z-\frac{3}{2z^2}=\ln x+c$

$-\ln\left(\frac{y}{x}\right)-\frac{3}{2}\left(\frac{y}{x}\right)^{-2}=\ln x+c$

$-\ln y+\ln x-\frac{3x^2}{2y^2}=\ln x+c$

$-\ln y-\frac{3x^2}{2y^2}=c$

$\ln y+\frac{3x^2}{2y^2}=c$

**14 a** $u = x + y \Rightarrow y = u - x$

$\frac{dy}{dx} = \frac{du}{dx} - 1$

The original equation is:

$\frac{dy}{dx} = (x + y + 1)(x + y - 1)$

Transform the equation:

$$\frac{du}{dx} - 1 = (x + u - x + 1)(x + u - x - 1)$$
$$= (u + 1)(u - 1)$$
$$= u^2 - 1$$

$\frac{du}{dx} = u^2$ as required

**b** $\int \frac{1}{u^2}\,du = \int dx$

$-\frac{1}{u} = x + c$

$\frac{1}{u} = -x - c$

$u = -\frac{1}{x + c}$

$x + y = -\frac{1}{x + c}$

$y = -\frac{1}{x + c} - x$

**15 a** $u = y - x - 2$

$y = u + x + 2$

$\frac{dy}{dx} = \frac{du}{dx} + 1$

The original equation is:

$\frac{dy}{dx} = (y - x - 2)^2$

Transform the equation:

$$\frac{du}{dx} + 1 = ((u + x + 2) - x - 2)^2$$
$$= u^2$$

$\frac{du}{dx} = u^2 - 1$ as required

**15 b** $\int \frac{1}{u^2-1}\,du = \int dx$

$\frac{1}{u^2-1} = \frac{A}{u-1} + \frac{B}{u+1}$

$1 = A(u+1) + B(u-1)$

When $u = -1$

$-2B = 1 \Rightarrow B = -\frac{1}{2}$

When $u = 1$

$2A = 1 \Rightarrow A = \frac{1}{2}$

$\frac{1}{u^2-1} = \frac{1}{2(u-1)} - \frac{1}{2(u+1)}$

$\frac{1}{2}\int \frac{1}{(u-1)}\,du - \frac{1}{2}\int \frac{1}{(u+1)}\,du = \int dx$

$\frac{1}{2}\ln(u-1) - \frac{1}{2}\ln(u+1) = x + c$

$\frac{1}{2}\left(\ln(u-1) - \ln(u+1)\right) = x + c$

$\frac{1}{2}\ln\left(\frac{u-1}{u+1}\right) = x + c$

$\ln\sqrt{\frac{u-1}{u+1}} = x + c$

$\sqrt{\frac{u-1}{u+1}} = Be^x$ where $B = e^c$

$\frac{u-1}{u+1} = B^2e^{2x}$

$= Ae^{2x}$ where $A = B^2$

$u - 1 = (u+1)Ae^{2x}$

$= uAe^{2x} + Ae^{2x}$

$u - uAe^{2x} = 1 + Ae^{2x}$

$u\left(1 - Ae^{2x}\right) = 1 + Ae^{2x}$

$u = \frac{1 + Ae^{2x}}{1 - Ae^{2x}}$

Since $u = y - x - 2$

$y - x - 2 = \frac{1 + Ae^{2x}}{1 - Ae^{2x}}$

$y = x + 2 + \frac{1 + Ae^{2x}}{1 - Ae^{2x}}$

**16 a** $u = v^{-2}$

$u^{-\frac{1}{2}} = v$

$\frac{dv}{dt} = -\frac{1}{2}u^{-\frac{3}{2}}\frac{du}{dt}$

The original equation is:

$t\frac{dv}{dt} + v = 2t^3v^3$

Transform the equation:

$t\left[-\frac{1}{2}u^{-\frac{3}{2}}\frac{du}{dt}\right] + u^{-\frac{1}{2}} = 2t^3u^{-\frac{3}{2}}$

$-\frac{1}{2}t\frac{du}{dt} + u = 2t^3$

$\frac{du}{dt} - \frac{2u}{t} = -4t^2$ as required

**16 b** $e^{-2\int \frac{1}{t}\mathrm{d}t} = e^{-2\ln t} = \frac{1}{t^2}$

$\frac{u}{t^2} = -4t + c$

$u = t^2(c - 4t)$

Since $u = v^{-2}$

$v^{-2} = t^2(c - 4t)$

$v^2 = \frac{1}{t^2(c-4t)}$

$v = \sqrt{\frac{1}{t^2(c-4t)}}$

When $t = 1$, $v = \frac{1}{2}$

$\frac{1}{2} = \sqrt{\frac{1}{(c-4)}}$

$\sqrt{c-4} = 2$

$c - 4 = 4$

$c = 8$

So:

$v = \sqrt{\frac{1}{t^2(8-4t)}}$

$= \sqrt{\frac{1}{4t^2(2-t)}}$

$= \frac{1}{2t}\sqrt{\frac{1}{(2-t)}}$

$= \frac{1}{2t\sqrt{(2-t)}}$