Exercise 5B

1 a $x\dfrac{dy}{dx}+y=\cos x$

$\Rightarrow \dfrac{d}{dx}(xy)=\cos x$ as $\dfrac{d}{dx}(xy)=x\dfrac{dy}{dx}+y$ from the product rule

$\Rightarrow xy=\int \cos dx=\sin x+c$

So $y=\dfrac{1}{x}\sin x+\dfrac{c}{x}$

b $e^{-x}\dfrac{dy}{dx}-e^{-x}y=xe^{x}$

$\Rightarrow \dfrac{d}{dx}(e^{-x}y)=xe^{x}$ as $\dfrac{d}{dx}(e^{-x}y)=e^{-x}\dfrac{dy}{dx}-e^{-x}y$ from the product rule

$\Rightarrow e^{-x}y=\int xe^{x}dx=xe^{x}-\int e^{x}dx$ using integration by parts formula (with $u=x$, $v=e^{x}$)

$=xe^{x}-e^{x}+c$

So $y=xe^{2x}-e^{2x}+ce^{x}$ multiplying both sides by e^{x}

c $\sin x\dfrac{dy}{dx}+y\cos x=3$

$\Rightarrow \dfrac{d}{dx}(y\sin x)=3$ as $\dfrac{d}{dx}(y\sin x)=\sin x\dfrac{dy}{dx}+y\cos x$ from the product rule

$\Rightarrow y\sin x=\int 3\ dx$

$\Rightarrow y\sin x=3x+c$

So $y=\dfrac{3x}{\sin x}+\dfrac{c}{\sin x}=3x\operatorname{cosec}x+c\operatorname{cosec}x$

d $\dfrac{1}{x}\dfrac{dy}{dx}-\dfrac{1}{x^2}y=e^{x}$

$\Rightarrow \dfrac{d}{dx}\left(\dfrac{1}{x}y\right)=e^{x}$ as $\dfrac{d}{dx}\left(\dfrac{1}{x}y\right)=\dfrac{1}{x}\dfrac{dy}{dx}-\dfrac{1}{x^2}y$ from the product rule

$\Rightarrow \dfrac{1}{x}y=\int e^{x}\,dx=e^{x}+c$

So $y=xe^{x}+cx$

1 e Simplify the left-hand side by noting that from the product and chain rules

$$\frac{\mathrm{d}}{\mathrm{d}x}(x^2\mathrm{e}^y) = x^2\frac{\mathrm{d}(\mathrm{e}^y)}{\mathrm{d}x} + 2x\mathrm{e}^y \quad \text{the product rule}$$

$$= x^2\frac{\mathrm{d}(\mathrm{e}^y)}{\mathrm{d}y}\frac{\mathrm{d}y}{\mathrm{d}x} + 2x\mathrm{e}^y \quad \text{the chain rule}$$

$$= x^2\mathrm{e}^y\frac{\mathrm{d}y}{\mathrm{d}x} + 2x\mathrm{e}^y$$

So $x^2\mathrm{e}^y\frac{\mathrm{d}y}{\mathrm{d}x} + 2x\mathrm{e}^y = x \Rightarrow \frac{\mathrm{d}}{\mathrm{d}x}(x^2\mathrm{e}^y) = x$

$$\Rightarrow x^2\mathrm{e}^y = \int x\ \mathrm{d}x = \frac{x^2}{2} + c$$

$$\Rightarrow \mathrm{e}^y = \frac{1}{2} + \frac{c}{x^2}$$

So $y = \ln\left(\frac{1}{2} + \frac{c}{x^2}\right)$

f $4xy\frac{\mathrm{d}y}{\mathrm{d}x} + 2y^2 = x^2$

$$\Rightarrow \frac{\mathrm{d}}{\mathrm{d}x}(2xy^2) = x^2 \quad \text{us ing the product and chain rules}$$

$$\Rightarrow 2xy^2 = \int x^2\,\mathrm{d}x = \frac{1}{3}x^3 + c$$

$$\Rightarrow y^2 = \frac{1}{6}x^2 + \frac{c}{2x}$$

So $y = \pm\sqrt{\frac{1}{6}x^2 + \frac{c}{2x}}$

2 a The equation is in the form $\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = Q(x)$, so the integrating factor is $\mathrm{e}^{\int P(x)\mathrm{d}x} = \mathrm{e}^{\int 2x\mathrm{d}x} = \mathrm{e}^{x^2}$

Multiplying the equation by this factor gives:

$$\mathrm{e}^{x^2}\frac{\mathrm{d}y}{\mathrm{d}x} + \mathrm{e}^{x^2}2xy = 1$$

$$\Rightarrow \frac{\mathrm{d}}{\mathrm{d}x}(y\mathrm{e}^{x^2}) = 1$$

$$\Rightarrow y\mathrm{e}^{x^2} = \int 1\mathrm{d}x = x + c$$

So $y = \frac{x+c}{\mathrm{e}^{x^2}} = x\mathrm{e}^{-x^2} + c\mathrm{e}^{-x^2}$

b As $x \to \infty$, e^{x^2} becomes much larger than x; therefore, $y \to 0$.

3 a $x^2\frac{dy}{dx}+2xy=2x+1$

$\Rightarrow \frac{d}{dx}(x^2y)=2x+1$

$\Rightarrow x^2y=\int(2x+1)\,dx$

$\Rightarrow x^2y=x^2+x+c$

So $y=1+\frac{1}{x}+\frac{c}{x^2}$

b When $x=-\frac{1}{2}$, $y=0$, $1-2+4c=0\Rightarrow 4c=1\Rightarrow c=\frac{1}{4}$

So $y=1+\frac{1}{x}+\frac{1}{4x^2}$

When $x=-\frac{1}{2}$, $y=3$, $1-2+4c=3\Rightarrow 4c=4\Rightarrow c=1$

So $y=1+\frac{1}{x}+\frac{1}{x^2}$

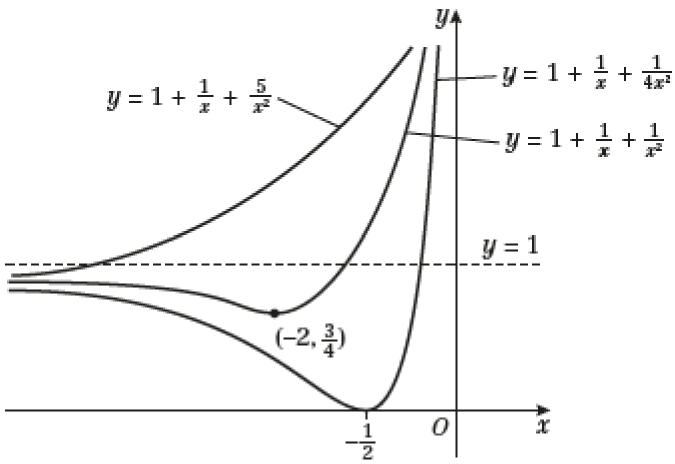
When $x=-\frac{1}{2}$, $y=19$, $1-2+4c=19\Rightarrow 4c=20\Rightarrow c=5$

So $y=1+\frac{1}{x}+\frac{5}{x^2}$

The curves have a horizontal asymptote at $y=1$ and a vertical asymptote at $x=0$

When $y=1$, $\frac{1}{x}+\frac{c}{x^2}=0\Rightarrow x=-c$. When $y=0$, $x^2+x+c=0$. There are no real roots for c $>\frac{1}{4}$.

So a sketch of the three curves for $x<0$ is



4 a $\ln x\dfrac{dy}{dx}+\dfrac{y}{x}=\dfrac{1}{(x+1)(x+2)}$

$\Rightarrow \dfrac{d}{dx}(y\ln x)=\dfrac{1}{(x+1)(x+2)}$ as $\dfrac{d}{dx}(y\ln x)=\ln x\dfrac{dy}{dx}+\dfrac{y}{x}$ using the product rule

$\Rightarrow y\ln x=\int\dfrac{1}{(x+1)(x+2)}dx$

$=\int\left(\dfrac{(x+2)-(x+1)}{(x+1)(x+2)}\right)dx=\int\left(\dfrac{1}{x+1}-\dfrac{1}{x+2}\right)dx$

$=\ln(x+1)-\ln(x+2)+\ln A$

So $y=\dfrac{\ln(x+1)-\ln(x+2)+\ln A}{\ln x}=\dfrac{\ln\dfrac{A(x+1)}{(x+2)}}{\ln x}$

b When $x=2,\ y=2,\ 2=\dfrac{\ln(\frac{3}{4}A)}{\ln 2}$

So $\ln(\frac{3}{4}A)=2\ln 2=\ln 4$

$\Rightarrow \frac{3}{4}A=4\Rightarrow A=\dfrac{16}{3}$

So the solution is $y=\dfrac{\ln\dfrac{16(x+1)}{3(x+2)}}{\ln x}$

5 a The integrating factor is $e^{\int 2dx}=e^{2x}$

Multiplying the equation by this factor gives:

$e^{2x}\dfrac{dy}{dx}+2e^{2x}y=e^{2x}e^{x}$

$\Rightarrow \dfrac{d}{dx}(e^{2x}y)=e^{3x}$

$\Rightarrow e^{2x}y=\int e^{3x}\,dx=\frac{1}{3}e^{3x}+c$

So $y=\frac{1}{3}e^{x}+ce^{-2x}$

b The integrating factor is $e^{\int\cot x\,dx}=e^{\ln\sin x}=\sin x$

Multiplying the equation by this factor gives:

$\sin x\dfrac{dy}{dx}+y\cos x=\sin x$

$\Rightarrow \dfrac{d}{dx}(y\sin x)=\sin x$

$\Rightarrow y\sin x=\int\sin x\,dx=-\cos x+c$

So $y=-\cot x+c\operatorname{cosec}x$

5 c The integrating factor is $e^{\int \sin x \, dx} = e^{-\cos x}$

Multiplying the equation by this factor gives:

$$e^{-\cos x}\frac{dy}{dx} + y\sin x e^{-\cos x} = e^{-\cos x}e^{\cos x}$$

$$\Rightarrow \frac{d}{dx}(ye^{-\cos x}) = 1$$

$$\Rightarrow ye^{-\cos x} = x + c$$

So $y = xe^{\cos x} + ce^{\cos x}$

d The integrating factor is $e^{\int -1 \, dx} = e^{-x}$

Note that $P(x) = -1$ and the minus sign is important.

Multiplying the equation by this factor gives:

$$e^{-x}\frac{dy}{dx} - ye^{-x} = e^{2x}e^{-x}$$

$$\Rightarrow \frac{d}{dx}(ye^{-x}) = e^{x}$$

$$\Rightarrow ye^{-x} = \int e^{x}\, dx = e^{x} + c$$

So $y = e^{2x} + ce^{x}$

e The integrating factor is $e^{\int \tan x \, dx} = e^{\ln \sec x} = \sec x$

Multiplying the equation by this factor gives:

$$\sec x\frac{dy}{dx} + y\sec x\tan x = x\cos x\sec x$$

$$\Rightarrow \frac{d}{dx}(y\sec x) = x$$

$$\Rightarrow y\sec x = \int x \, dx = \tfrac{1}{2}x^2 + c$$

So $y = \left(\tfrac{1}{2}x^2 + c\right)\cos x$

f The integrating factor is $e^{\int \frac{1}{x} dx} = e^{\ln x} = x$

Multiplying the equation by this factor gives:

$$x\frac{dy}{dx} + y = \frac{x}{x^2}$$

$$\Rightarrow \frac{d}{dx}(xy) = \frac{1}{x}$$

$$\Rightarrow xy = \int \frac{1}{x} dx = \ln|x| + c$$

So $y = \frac{1}{x}\ln|x| + \frac{c}{x}$

5 g Divide both sides by x^2 to get an equation in the form $\frac{dy}{dx}+P(x)y=Q(x)$. This gives:

$$\frac{dy}{dx}-\frac{1}{x}y=\frac{x}{x+2} \qquad \textbf{(1)}$$

The integrating factor is $e^{\int -\frac{1}{x}dx}=e^{-\ln x}=e^{\ln\frac{1}{x}}=\frac{1}{x}$

Multiplying equation (**1**) by this factor gives:

$$\frac{1}{x}\frac{dy}{dx}-\frac{1}{x^2}y=\frac{1}{x+2}$$

$$\Rightarrow \frac{d}{dx}\left(\frac{1}{x}y\right)=\frac{1}{x+2}$$

$$\Rightarrow \frac{1}{x}y=\int\frac{1}{x+2}dx=\ln(x+2)+c$$

So $y=x\ln(x+2)+cx$

h Divide both sides by x^2 to get an equation in the form $\frac{dy}{dx}+P(x)y=Q(x)$. This gives:

$$\frac{dy}{dx}+\frac{1}{3x}y=\frac{1}{3} \qquad \textbf{(1)}$$

The integrating factor is $e^{\int \frac{1}{3x}dx}=e^{\frac{1}{3}\ln x}=e^{\ln x^{\frac{1}{3}}}=x^{\frac{1}{3}}$

Multiplying equation (**1**) by $x^{\frac{1}{3}}$ gives:

$$x^{\frac{1}{3}}\frac{dy}{dx}+\frac{1}{3}x^{-\frac{2}{3}}y=\frac{1}{3}x^{\frac{1}{3}}$$

$$\Rightarrow \frac{d}{dx}\left(x^{\frac{1}{3}}y\right)=\frac{1}{3}x^{\frac{1}{3}}$$

$$\Rightarrow x^{\frac{1}{3}}y=\int\frac{1}{3}x^{\frac{1}{3}}\,dx=\frac{1}{4}x^{\frac{4}{3}}+c$$

So $y=\frac{1}{4}x+cx^{-\frac{1}{3}}$

i Dividing both sides by $(x + 2)$ gives:

$$\frac{dy}{dx}-\frac{1}{(x+2)}y=1 \qquad \textbf{(1)}$$

The integrating factor is $e^{\int\frac{-1}{(x+2)}dx}=e^{-\ln(x+2)}=e^{\ln\frac{1}{x+2}}=\frac{1}{x+2}$

Multiplying equation (**1**) by the integrating factor:

$$\frac{1}{(x+2)}\frac{dy}{dx}-\frac{1}{(x+2)^2}y=\frac{1}{(x+2)}$$

$$\Rightarrow \frac{d}{dx}\left[\frac{1}{(x+2)}y\right]=\frac{1}{x+2}$$

$$\Rightarrow \frac{1}{(x+2)}y=\int\frac{1}{x+2}\,dx=\ln(x+2)+c$$

So $y=(x+2)\ln(x+2)+c(x+2)$

5 **j** Dividing both sides by x gives:

$$\frac{dy}{dx}+\frac{4}{x}y=\frac{e^x}{x^3} \qquad (1)$$

The integrating factor is $e^{\int \frac{4}{x}dx}=e^{4\ln x}=e^{\ln x^4}=x^4$

Multiplying equation (**1**) by the integrating factor:

$$x^4\frac{dy}{dx}+4x^3y=xe^x$$

$$\Rightarrow \frac{d}{dx}(x^4y)=xe^x$$

$$\Rightarrow x^4y=\int xe^x dx=xe^x-\int e^x dx=xe^x-e^x+c$$

Integrating xe^x using integration by parts

So $y=\frac{1}{x^3}e^x-\frac{1}{x^4}e^x+\frac{c}{x^4}$

6 Dividing both sides by x gives:

$$\frac{dy}{dx}+\frac{2}{x}y=\frac{1}{x}e^x \qquad (1)$$

The integrating factor is $e^{\int \frac{2}{x}dx}=e^{2\ln x}=e^{\ln x^2}=x^2$

Multiplying equation (**1**) by x^2

$$x^2\frac{dy}{dx}+2xy=xe^x$$

$$\Rightarrow \frac{d}{dx}(x^2y)=xe^x$$

$$\Rightarrow x^2y=\int xe^x dx=xe^x-\int e^x dx=xe^x-e^x+c$$

So $y=\frac{1}{x}e^x-\frac{1}{x^2}e^x+\frac{c}{x^2}$

Given that $y=1$ when $x=1$, then $1=e-e+c\Rightarrow c=1$

So the required equation is $y=\frac{1}{x}e^x-\frac{1}{x^2}e^x+\frac{1}{x^2}$

7 Dividing both sides by x^3 gives:

$$\frac{dy}{dx} - \frac{1}{x}y = \frac{1}{x^3} \qquad \mathbf{(1)}$$

The integrating factor is $e^{-\int \frac{1}{x}dx} = e^{-\ln x} = e^{\ln \frac{1}{x}} = \frac{1}{x}$

Multiplying equation (**1**) by $\frac{1}{x}$

$$\frac{1}{x}\frac{dy}{dx} - \frac{1}{x^2}y = \frac{1}{x^4}$$

$$\Rightarrow \frac{d}{dx}\left(\frac{1}{x}y\right) = \frac{1}{x^4}$$

$$\frac{1}{x}y = \int \frac{1}{x^4}dx = \int x^{-4}\,dx = -\frac{1}{3}x^{-3} + c$$

So $y = -\frac{1}{3}x^{-2} + cx = -\frac{1}{3x^2} + cx$

But $y = 1$ when $x = 1$, so $1 = -\frac{1}{3} + c \Rightarrow c = \frac{4}{3}$

So the required equation is $y = -\frac{1}{3x^2} + \frac{4x}{3}$

8 a Dividing both sides by $\left(x + \frac{1}{x}\right)$ gives:

$$\frac{dy}{dx} + \frac{2}{\left(x+\frac{1}{x}\right)}y = \frac{2(x^2+1)^2}{\left(x+\frac{1}{x}\right)},$$ which simplifies to

$$\frac{dy}{dx} + \frac{2x}{x^2+1} \times y = 2x(x^2+1) \qquad \mathbf{(1)}$$

The integrating factor is $e^{\int \frac{2x}{x^2+1}dx} = e^{\ln(x^2+1)} = (x^2+1)$

Multiplying equation (**1**) by (x^2+1)

$$(x^2+1)\frac{dy}{dx} + 2xy = 2x(x^2+1)^2$$

$$\Rightarrow \frac{d}{dx}\left((x^2+1)y\right) = 2x(x^2+1)^2$$

$$y(x^2+1) = \int 2x(x^2+1)^2\,dx = \frac{1}{3}(x^2+1)^3 + c$$

So $y = \frac{1}{3}(x^2+1)^2 + \frac{c}{(x^2+1)}$

b Given that $y = 1$ when $x = 1$, then $1 = \frac{1}{3} \times 4 + \frac{1}{2}c \Rightarrow c = -\frac{2}{3}$

So the required equation is $y = \frac{1}{3}(x^2+1)^2 - \frac{2}{3(x^2+1)}$

9 a Dividing both sides by $\cos x$ gives:

$$\frac{dy}{dx} + y\sec x = \sec x \qquad \textbf{(1)}$$

Using the standard result $\int \sec x \, dx = \ln(\sec x + \tan x)$, (you will not be expected to prove this result)

the integrating factor is $e^{\int \sec x \, dx} = e^{\ln(\sec x + \tan x)} = \sec x + \tan x$

Multiplying equation (**1**) by this factor gives:

$$(\sec x + \tan x)\frac{dy}{dx} + (\sec^2 x + \sec x \tan x)y = \sec^2 x + \sec x \tan x$$

$$\Rightarrow \frac{d}{dx}\big((\sec x + \tan x)y\big) = \sec^2 x + \sec x \tan x$$

$$\Rightarrow (\sec x + \tan x)y = \int \sec^2 x + \sec x \tan x \, dx = \tan x + \sec x + c$$

So $y = 1 + \dfrac{c}{\sec x + \tan x}$

b Given that $y = 2$ when $x = 0$, then $2 = 1 + \dfrac{c}{1+0} \Rightarrow c = 1$

So $y = 1 + \dfrac{1}{\sec x + \tan x}$

Dividing top and bottom by $\cos x$ gives $y = 1 + \dfrac{\cos x}{1 + \sin x}$

10 a Dividing both sides by cos x gives

$$\frac{dy}{dx} + y\tan x = \sec x \qquad \textbf{(1)}$$

The integrating factor is $e^{\int \tan x \, dx} = e^{\ln|\sec x|} = \sec x$

Multiply both sides of equation (**1**) by sec x

$$\sec x\frac{dy}{dx} + y\tan x = \sec^2 x$$

$$\frac{d}{dx}(y\sec x) = \sec^2 x$$

$$\Rightarrow y\sec x = \int \sec^2 x \, dx$$

$$\Rightarrow y\sec x = \tan x + c$$

$$\Rightarrow y = \sin x + c\cos x$$

b Given that $y = 3$ when $x = \pi$, then $3 = \sin\pi + c\cos\pi = 0 - c \Rightarrow c = -3$.

So the particular solution is $y = \sin x - 3\cos x$

10 c If $x = \dfrac{\pi}{2}$, then $y = \sin\dfrac{\pi}{2} + c\cos\dfrac{\pi}{2} = 1 + c \times 0 = 1$ for any value of c

Similarly if $x = \dfrac{3\pi}{2}$, then $y = \sin\dfrac{3\pi}{2} + c\cos\dfrac{3\pi}{2} = -1 + c \times 0 = -1$ for any value of c

So $\left(\dfrac{\pi}{2}, 1\right)$ and $\left(\dfrac{3\pi}{2}, -1\right)$ lie on all solution curves.

11 Dividing by a gives $\frac{dy}{dx} + \frac{b}{a}y = 0$

The integrating factor is $e^{\int \frac{b}{a}dx} = e^{\frac{bx}{a}}$

Multiplying by this factor gives

$e^{\frac{bx}{a}}\frac{dy}{dx} + \frac{b}{a}e^{\frac{bx}{a}}y = 0$

$\Rightarrow \frac{d}{dx}\left(e^{\frac{bx}{a}}y\right) = 0$

$\Rightarrow e^{\frac{bx}{a}}y = c$

So the general solution is $y = ce^{-\frac{bx}{a}}$