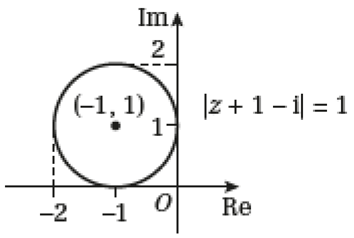
Chapter review 4

1 a $|z+1-i|=1$ is the circle with centre $(-1, 1)$ and radius 1

Therefore the Cartesian equation is:

$(x+1)^2+(y-1)^2=1$

b



c The distance OC where C is the centre of this circle is $\sqrt{2}$

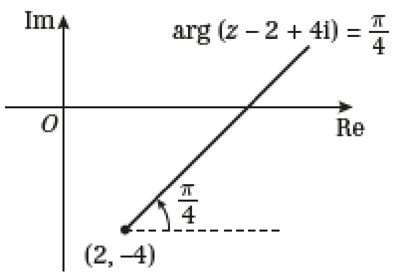
Therefore $|z|_{min} = \sqrt{2}-1$ and $|z|_{max} = \sqrt{2}+1$

d $|z-1|$ is the circle with centre $(-2, 1)$ and radius 1

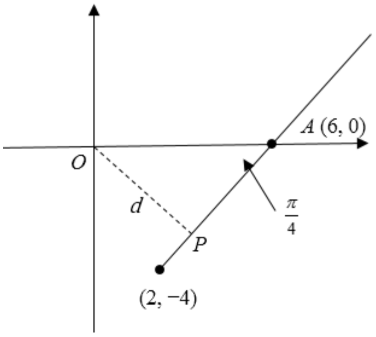
The distance OD where D is the centre of this circle is $\sqrt{5}$

Therefore $|z-1|_{min} = \sqrt{5}-1$ and $|z-1|_{max} = \sqrt{5}+1$

2 a $\arg(z-2+4i)=\frac{\pi}{4}$



2 b The half-line cuts the x-axis at (6, 0)



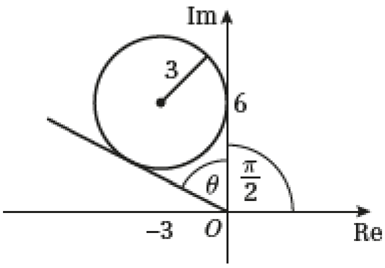
$$\sin\left(\frac{\pi}{4}\right) = \frac{d}{6}$$

$$d = 6\sin\left(\frac{\pi}{4}\right)$$

$$= 3\sqrt{2}$$

Therefore $|z|_{min} = 3\sqrt{2}$

3 $|z + 3 - 6i| = 3$ is the circle with centre (−3, 6) and radius 3



$$\sin\left(\frac{\theta}{2}\right) = \frac{3}{\sqrt{3^2 + 6^2}}$$

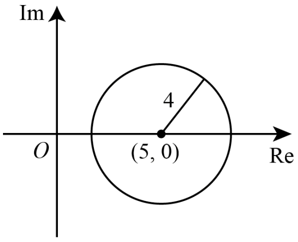
$$= \frac{1}{\sqrt{5}}$$

The maximum is at $\frac{\pi}{2} + \theta$

Therefore:

$$\frac{\pi}{2} + \theta = \frac{\pi}{2} + 2\sin^{-1}\left(\frac{1}{\sqrt{5}}\right) \text{ as required}$$

4 a $|z-5|=4$ is the circle with centre (5, 0) and radius 4



b $\arg(z+3\text{i})=\frac{\pi}{3}$ is the half-line originating at $(0, -3)$ at an angle of $\frac{\pi}{3}$ to the x-axis.

The half-line has a gradient of $\sqrt{3}$ and originates at (0, −3), therefore is part of the line

$y=\sqrt{3}x-3$

Substituting into $(x-5)^2+y^2=16$ gives:

$(x-5)^2+\left(\sqrt{3}x-3\right)^2=16$

$x^2-10x+25+3x^2-6\sqrt{3}x+9=16$

$4x^2-\left(10+6\sqrt{3}\right)x+18=0$

$$x=\frac{10+6\sqrt{3}\pm\sqrt{\left(10+6\sqrt{3}\right)^2-4(4)(18)}}{2(4)}$$

$$x=\frac{10+6\sqrt{3}\pm 11.306...}{8}$$

$x = 1.135\ldots$ or $x = 3.962\ldots$

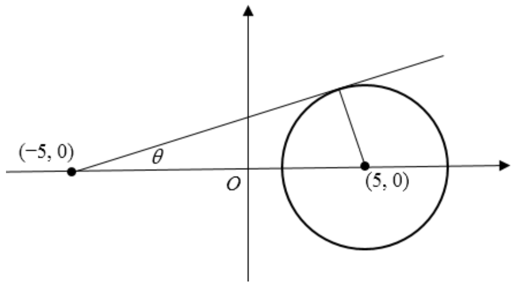
when $x = 1.135\ldots$, $y = -1.032\ldots$

when $x = 3.962\ldots$, $y = 3.863\ldots$

Therefore the required solutions are:

$z = 1.14 - 1.03\text{i}$ and $z = 3.96 + 3.86\text{i}$

4 c



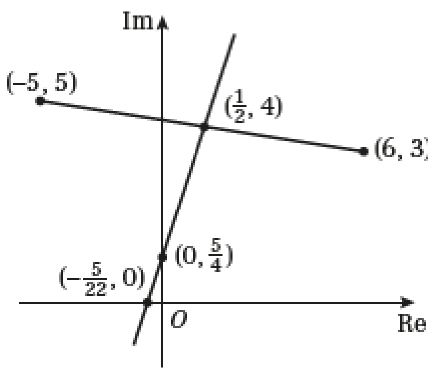
$\sin\theta = \frac{4}{10}$

$\theta = 0.4115...$

Therefore:

$0.412 < \theta < \pi$ and $-\pi < \theta < -0.412$

5 a



5 b $|z+5-5\text{i}| = |z-6-3\text{i}|$

$|x+y\text{i}+5-5\text{i}| = |x+y\text{i}-6-3\text{i}|$

$|(x+5)+\text{i}(y-5)| = |(x-6)+\text{i}(y-3)|$

$|(x+5)+\text{i}(y-5)|^2 = |(x-6)+\text{i}(y-3)|^2$

$(x+5)^2+(y-5)^2 = (x-6)^2+(y-3)^2$

$x^2+10x+25+y^2-10y+25 = x^2-12x+36+y^2-6y+9$

$22x+50 = 4y+45$

$y = \frac{11}{2}x+\frac{5}{4}$

c Since the locus of $|z+5-5\text{i}| = |z-6-3\text{i}|$ has gradient $\frac{11}{2}$,

a perpendicular to the locus will have a gradient of $-\frac{2}{11}$

Using $y-y_1 = m(x-x_1)$ at (0, 0) with gradient $-\frac{2}{11}$ gives:

$y = -\frac{2}{11}x$

To find the point of intersection, P, equate $y = -\frac{2}{11}x$ and $y = \frac{11}{2}x+\frac{5}{4}$

$-\frac{2}{11}x = \frac{11}{2}x+\frac{5}{4}$

$-\frac{125}{22}x = \frac{5}{4}$

$x = -\frac{11}{50}$ and $y = \frac{1}{25}$

$|OP| = \sqrt{\left(-\frac{11}{50}\right)^2+\left(\frac{1}{25}\right)^2}$

$= \frac{\sqrt{5}}{10}$

6 a $|z-4|=|z-8\text{i}|$

$|x+y\text{i}-4|=|x+y\text{i}-8\text{i}|$

$|(x-4)+y\text{i}|=|x+\text{i}(y-8)|$

$|(x-4)+y\text{i}|^2=|x+\text{i}(y-8)|^2$

$(x-4)^2+y^2=x^2+(y-8)^2$

$x^2-8x+16+y^2=x^2+y^2-16y+64$

$-8x+16=-16y+64$

$16y=8x+48$

$y=\frac{1}{2}x+3$

b $\arg z=\frac{\pi}{4}$ is the half-line originating at the point (0, 0) at an angle of $\frac{\pi}{4}$ to positive x-axis, therefore the half-line is part of the line $y=x$

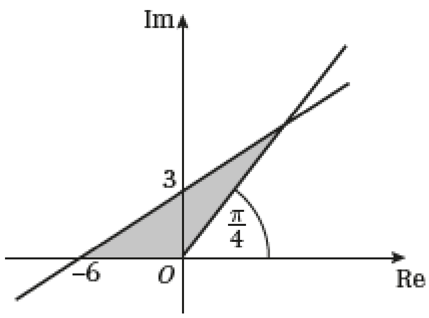
Equating $y=\frac{1}{2}x+3$ and $y=x$ gives:

$x=\frac{1}{2}x+3$

$x=6$ and $y=6$

Therefore $z=6+6\text{i}$

c $\{z\in\mathbb{C}:|z-4|\le|z-8\text{i}|\}\cap\left\{z\in\mathbb{C}:\frac{\pi}{4}\le\arg(z)\le\pi\right\}$



7 a i $|z-3+\text{i}|=|z-1-\text{i}|$

$|x+y\text{i}-3+\text{i}|=|x+y\text{i}-1-\text{i}|$

$|(x-3)+\text{i}(y+1)|=|(x-1)+\text{i}(y-1)|$

$|(x-3)+\text{i}(y+1)|^2=|(x-1)+\text{i}(y-1)|^2$

$(x-3)^2+(y+1)^2=(x-1)^2+(y-1)^2$

$x^2-6x+9+y^2+2y+1=x^2-2x+1+y^2-2y+1$

$-6x+2y+10=-2x-2y+2$

$4y=4x-8$

$y=x-2$

ii $|x+y\text{i}-2|=2\sqrt{2}$

$|(x-2)+y\text{i}|=2\sqrt{2}$

$|(x-2)+y\text{i}|^2=8$

$(x-2)^2+y^2=8$

b Substituting $y=x-2$ into $(x-2)^2+y^2=8$ gives:

$(x-2)^2=4$

$x-2=\pm 2$

$x=2\pm 2$

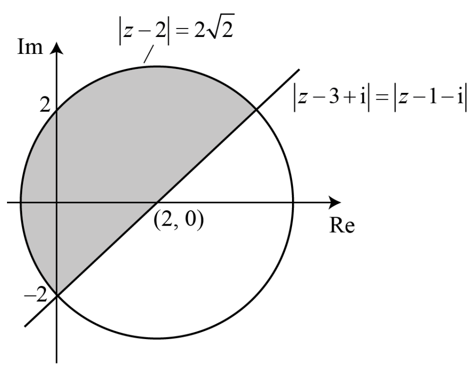
$x=0$ or $x=4$

When $x=0$, $y=-2$

When $x=4$, $y=2$

Therefore $z=-2\text{i}$ and $z=4+2\text{i}$

c



8 a i $|z| = |z-4|$

$|x+yi| = |x+yi-4|$

$|x+yi| = |(x-4)+yi|$

$|x+yi|^2 = |(x-4)+yi|^2$

$x^2+y^2 = (x-4)^2+y^2$

$x^2+y^2 = x^2-8x+16+y^2$

$0 = -8x+16$

$x = 2$

ii The perpendicular bisector of (0, 0) and (4, 0)

b i $|z| = 2|z-4|$

$|x+yi| = 2|(x-4)+yi|$

$|x+yi|^2 = 4|(x-4)+yi|^2$

$x^2+y^2 = 4(x-4)^2+4y^2$

$x^2+y^2 = 4x^2-32x+64+4y^2$

$3x^2-32x+3y^2 = -64$

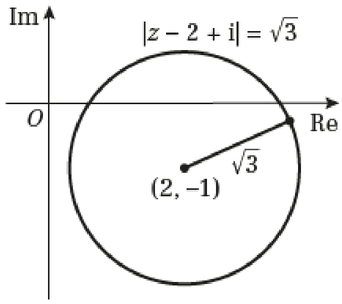
$x^2-\frac{32}{3}x+y^2 = -\frac{64}{3}$

$\left(x-\frac{32}{6}\right)^2-\left(\frac{32}{6}\right)^2+y^2 = -\frac{64}{3}$

$\left(x-\frac{16}{3}\right)^2+y^2 = \frac{64}{9}$

ii Circle with centre $\left(\frac{16}{3},0\right)$ and radius $\frac{8}{3}$

9 **a** The equation $|z-2+i|=\sqrt{3}$ describes a circle centred at $(2,-1)$, radius $r=\sqrt{3}$:



b The half-line L $y=mx-1,\ x\geqslant 0,\ m>0$ is tangent to the circle from part **a**. This means that it has to lie on the circle and thus satisfy the equation $(x-2)^2+(y+1)^2=3$. Substituting the expression for y into this equation gives:

$(x-2)^2+(mx)^2=3$

$x^2-4x+4+m^2x^2-3=0$

$x^2(1+m^2)-4x+1=0$

$x^2-\frac{4}{1+m^2}+\frac{1}{1+m^2}=0$

This equation must have exactly one solution, as the line and the circle only touch at one point. Therefore, it needs to be of the form:

$\left(x-\frac{2}{1+m^2}\right)^2=0$ which means that $\left(\frac{2}{1+m^2}\right)^2=\frac{1}{1+m^2}$

Solving this gives:

$\frac{4}{(1+m^2)^2}=\frac{1}{1+m^2}$

$4=1+m^2$

$m^2=3$

$m=\sqrt{3}$

since we know that $m>0$.

So L is given by $y=x\sqrt{3}-1$.

c For $x=0$, $y=-1$ so the line goes through $(0,-1)$. The gradient is equal to $\sqrt{3}$, so $\tan\theta=\sqrt{3}$.

Therefore $\theta=\frac{\pi}{3}$. So the equation for L can be written as $\arg(z+i)=\frac{\pi}{3}$.

d Using $\left(x-\frac{2}{1+m^2}\right)^2=0$ from part **b** and substituting $m=\sqrt{3}$ we obtain $\left(x-\frac{1}{2}\right)^2=0$, so $x=\frac{1}{2}$.

Now substituting this value into $y=x\sqrt{3}-1$ we see $y=\frac{\sqrt{3}-2}{2}$. So $a=\frac{1}{2}+\left(\frac{\sqrt{3}-2}{2}\right)\mathrm{i}$.

10 a $|z+2|=|2z-1|$

$\Rightarrow |x+\mathrm{i}y+2| = |2(x+\mathrm{i}y)-1|$

$\Rightarrow |x+\mathrm{i}y+2| = |2x+2\mathrm{i}y-1|$

$\Rightarrow |(x+2)+\mathrm{i}y| = |(2x-1)+\mathrm{i}(2y)|$

$\Rightarrow |(x+2)+\mathrm{i}y|^2 = |(2x-1)+\mathrm{i}(2y)|^2$

$\Rightarrow (x+2)^2+y^2=(2x-1)^2+(2y)^2$

$\Rightarrow x^2+4x+4+y^2=4x^2-4x+1+4y^2$

$\Rightarrow 0=3x^2-8x+3y^2+1-4$

$\Rightarrow 3x^2-8x+3y^2-3=0$

$\Rightarrow x^2-\frac{8}{3}x+y^2-1=0$

$\Rightarrow \left(x-\frac{4}{3}\right)^2-\frac{16}{9}+y^2-1=0$

$\Rightarrow \left(x-\frac{4}{3}\right)^2+y^2=\frac{16}{9}+1$

$\Rightarrow \left(x-\frac{4}{3}\right)^2+y^2=\frac{25}{9}$

$\Rightarrow \left(x-\frac{4}{3}\right)^2+y^2=\left(\frac{5}{3}\right)^2$

This is a circle, centre $\left(\frac{4}{3},0\right)$, radius $\frac{5}{3}$.

The Cartesian equation of the locus of points representing $|z+2|=|2z-1|$ is $\left(x-\frac{4}{3}\right)^2+y^2=\frac{25}{9}$.

10 b $|z+2|=|2z-1| \Rightarrow \left(x-\frac{4}{3}\right)^2+y^2=\frac{25}{9}$ **(1)**

$\arg z=\frac{\pi}{4} \Rightarrow \arg(x+\mathrm{i}y)=\frac{\pi}{4}$

$\Rightarrow \frac{y}{x}=\tan\frac{\pi}{4}$

$\Rightarrow \frac{y}{x}=1$

$\Rightarrow y=x \quad \text{where } x>0, y>0$ **(2)**

Solving simultaneously:

$$\left(x-\frac{4}{3}\right)^2+x^2=\frac{25}{9}$$

$$\Rightarrow x^2-\frac{4}{3}x-\frac{4}{3}x+\frac{16}{9}+x^2=\frac{25}{9}$$

$$\Rightarrow 2x^2-\frac{8}{3}x=\frac{25}{9}-\frac{16}{9}$$

$$\Rightarrow 2x^2-\frac{8}{3}x=\frac{9}{9}$$

$$\Rightarrow 2x^2-\frac{8}{3}x=1 \qquad (\times 3)$$

$$\Rightarrow 6x^2-8x=3$$

$$\Rightarrow 6x^2-8x-3=0$$

$$\Rightarrow x=\frac{8\pm\sqrt{64-4(6)(-3)}}{2(6)}$$

$$\Rightarrow x=\frac{8\pm\sqrt{136}}{12}$$

$$\Rightarrow x=\frac{8\pm 2\sqrt{34}}{12}$$

$$\Rightarrow x=\frac{4\pm\sqrt{34}}{6}$$

As $x>0$ then we reject $x=\frac{4-\sqrt{34}}{6}$

and accept $x=\frac{4+\sqrt{34}}{6}$

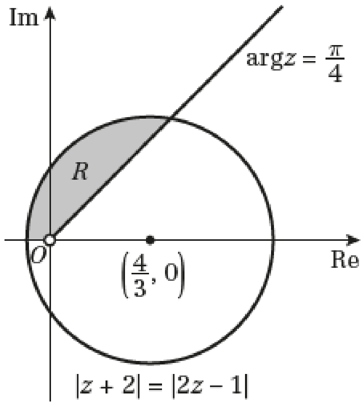
as $y=x$, then $y=\frac{4+\sqrt{34}}{6}$

So $z=\left(\frac{4+\sqrt{34}}{6}\right)+\left(\frac{4+\sqrt{34}}{6}\right)\mathrm{i}$

The value of z satisfying $|z+2|=|2z-1|$ and $\arg z=\frac{\pi}{4}$

is $z=\left(\frac{4+\sqrt{34}}{6}\right)+\left(\frac{4+\sqrt{34}}{6}\right)\mathrm{i}$ OR $z=1.64+1.64\mathrm{i}$ (2 d.p.)

10 c



The region R (shaded) satisfies both $|z+2| \geqslant |2z-1|$ and $\frac{\pi}{4} \leqslant \arg z \leqslant \pi$.

Note that $|z+2| \geqslant |2z-1|$

$\Rightarrow (x+2)^2 + y^2 \geqslant (2x-1)^2 + (2y)^2$

$\Rightarrow 0 \geqslant 3x^2 - 8x + 3y^2 - 3$

$\Rightarrow 0 \geqslant \left(x - \frac{4}{3}\right)^2 - \frac{16}{9} + y^2 - 1$

$\Rightarrow \frac{25}{9} \geqslant \left(x - \frac{4}{3}\right)^2 + y^2$

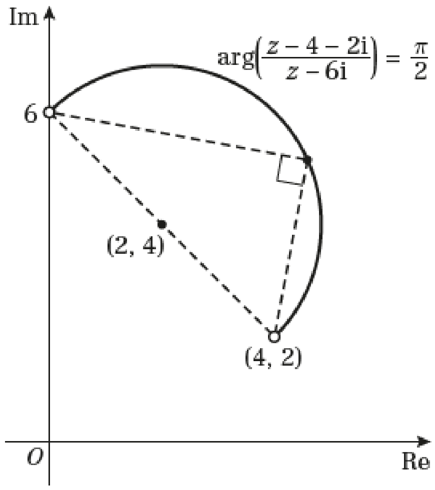
$\Rightarrow \left(x - \frac{4}{3}\right)^2 + y^2 \leqslant \frac{25}{9}$

represents region inside and bounded by the circle, centre $\left(\frac{4}{3}, 0\right)$, radius $\frac{5}{3}$.

11 a $\arg\left(\frac{z-4-2i}{z-6i}\right)=\frac{\pi}{2}$

$\Rightarrow \arg(z-4-2i)-\arg(z-6i)=\frac{\pi}{2}$

$\Rightarrow \theta-\phi=\frac{\pi}{2}$, where $\arg(z-4-2i)=\theta$ and $\arg(z-6i)=\phi$.



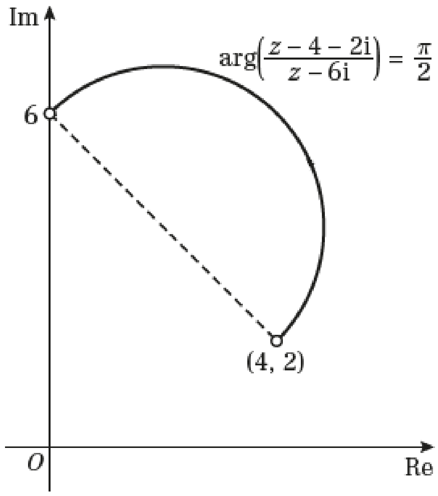
Using geometry,

$\Rightarrow A\hat{P}B=-\phi+\theta$

$\Rightarrow A\hat{P}B=\theta-\phi$

$\Rightarrow A\hat{P}B=\frac{\pi}{2}$

The locus of z is the arc of a circle (in this case, a semi-circle) cut off at (4, 2) and (0, 6) as shown below.



11 b $|z-2-4\mathrm{i}|$ is the distance from the point (2, 4) to the locus of points P.

Note, as the locus is a semi-circle, its centre is $\left(\frac{4+0}{2}, \frac{2+6}{2}\right) = (2, 4)$.

Therefore $|z-2-4\mathrm{i}|$ is the distance from the centre of the semi-circle to points on the locus of points P.

Hence $|z-2-4\mathrm{i}| = \text{radius of semi-circle}$

$$= \sqrt{(0-2)^2 + (6-4)^2}$$
$$= \sqrt{4+4}$$
$$= \sqrt{8}$$
$$= 2\sqrt{2}$$

The exact value of $|z-2-4\mathrm{i}|$ is $2\sqrt{2}$

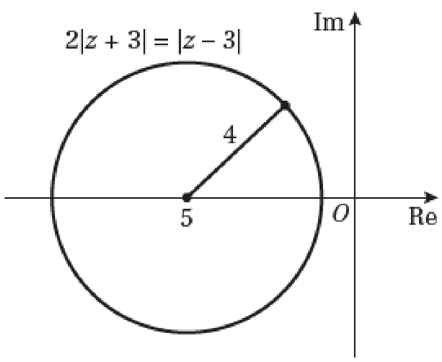
12 We have $2|z+3| = |z-3|$

a To show that this describes a circle, write $z = x + \mathrm{i}y$ and square both sides:

$$2|x+3+iy| = |x-3+iy|$$
$$4|x+3+\mathrm{i}y|^2 = |x-3+\mathrm{i}y|^2$$
$$4(x+3)^2 + 4y^2 = (x-3)^2 + y^2$$
$$4x^2 + 24x + 36 + 3y^2 = x^2 - 6x + 9$$
$$3x^2 + 30x + 27 + 3y^2 = 0$$
$$x^2 + y^2 + 10x + 9 = 0$$

as required.

b



12 c L is given by $bz^* + b^*z = 0$ where $b, z \in \mathbb{R}$.
We know that L is tangent to the circle and that $\arg b = \theta$.
We want to find possible values of $\tan\theta$.
Write $z = x + \mathrm{i}y$, $b = u + \mathrm{i}v$.
Then the equation for L becomes:

$$b(x - \mathrm{i}y) + b^*(x + \mathrm{i}y) = 0$$

$$(u + \mathrm{i}v)(x - \mathrm{i}y) + (u - \mathrm{i}v)(x + \mathrm{i}y) = 0$$

$$ux + vy = 0$$

$$y = -\frac{ux}{v},\ v \neq 0$$

If $v = 0$ then $ux = 0$, so either $u = 0$ i.e. $b = 0$, which means that the line does not exist, or $x = 0$ but this is not tangent to the circle.
So we can assume $v \neq 0$.

Now, since $b = u + \mathrm{i}v$, $\tan\theta = \frac{v}{u}$.

So L can be written as $y = -\frac{x}{\tan\theta}$

We want to find $\tan\theta$ such that the line is tangent to the circle.
Therefore, it has to satisfy the equation for the circle $x^2 + y^2 + 10x + 9 = 0$:

$$x^2 + \frac{x^2}{\tan^2\theta} + 10x + 9 = 0$$

$$x^2\tan^2\theta + x^2 + 10x\tan^2\theta + 9\tan^2\theta = 0$$

$$x^2(\tan^2\theta + 1) + 10x\tan^2\theta + 9\tan^2\theta = 0$$

Let $a = \tan^2\theta$. Then $x^2(a + 1) + 10xa + 9a = 0$.

Since the line is tangent to the circle, this equation can only have one solution.
Therefore we need $\Delta = 0$.
Therefore

$$100a^2 - 36a(a + 1) = 0$$

$$100a^2 - 36a^2 - 36a = 0$$

$$a(64a - 36) = 0$$

$$a = 0 \quad \text{or} \quad a = \tfrac{36}{64} = \tfrac{9}{16}$$

Recall that $a = \tan^2\theta = \frac{v^2}{u^2}$

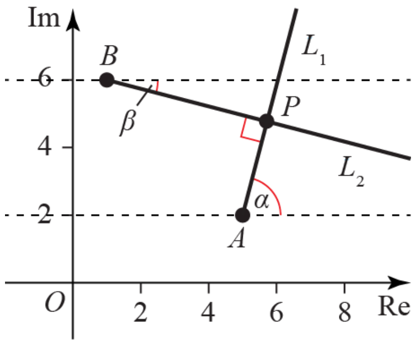
Since $v \neq 0$, we have $a \neq 0$
So $a = \frac{9}{16}$ and we can solve for $\tan\theta$:

$$\tan^2\theta = \tfrac{9}{16}$$

$$\tan\theta = \pm\tfrac{3}{4}$$

13 a Let $\arg(z-5-2\text{i})=\alpha$ and $\arg(z-1-6\text{i})=\beta$.

Then we have $\arg\left(\frac{z-5-2\text{i}}{z-1-6\text{i}}\right)=\arg(z-5-2\text{i})-\arg(z-1-6\text{i})=\alpha-\beta=\frac{\pi}{2}$



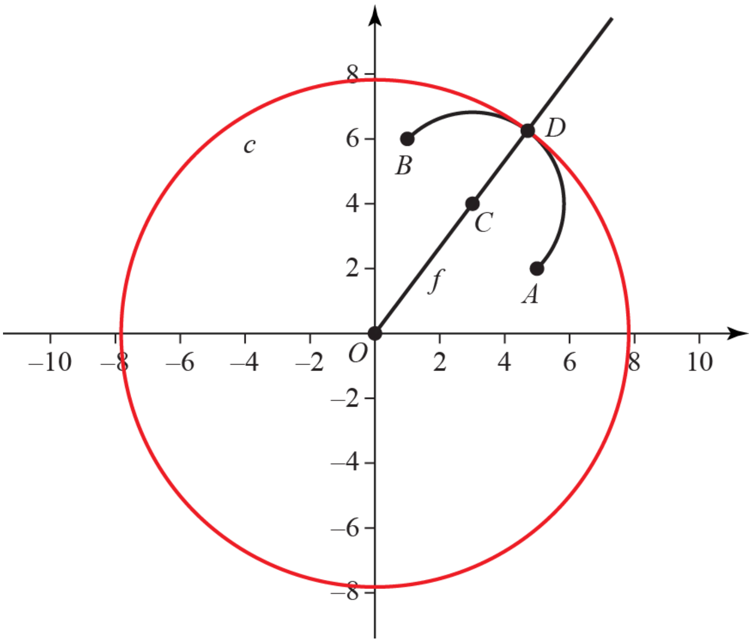
As α,β vary, P i.e. the intersection of L_1 and L_2 creates an arc.

Since $A\hat{P}B=\frac{\pi}{2}$, this arc will be a semicircle and the line segment AB is its diameter.

The centre of the circle lies in the middle of the line segment AB. $A=(5,2)$ and $B=(1,6)$ so the midpoint is $C=(3,4)$.

The radius is the distance CA. $CA=\sqrt{(5-3)^2+(4-2)^2}=2\sqrt{2}$, so $r=2\sqrt{2}$.

13 b The maximum value will lie on the line connecting the centre of this circle with the origin. This is represented by point D on the diagram below:



Thus D satisfies both the equation of the semicircle described in part **a**, $(x-3)^2+(y-4)^2=8$, and the equation of the line going through the origin and the centre of that semicircle, $y=\frac{4x}{3}$.

Substituting this into the equation for the circle we obtain:

$$(x-3)^2+\left(\frac{4x}{3}-4\right)^2=8$$

$$x^2-6x+9+\frac{16x^2}{9}-\frac{32x}{3}+16-8=0$$

$$\frac{25x^2}{9}-\frac{50x}{3}+17=0$$

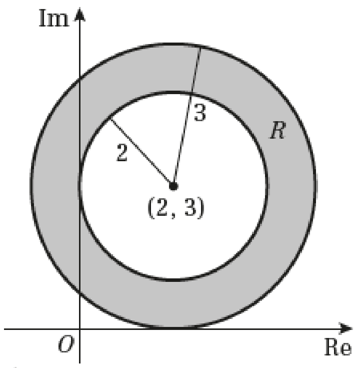
$$x^2-6x+\frac{153}{25}=0$$

$$x=3\pm\frac{6\sqrt{2}}{5}$$

We're looking for the larger value, so $x=3+\frac{6\sqrt{2}}{5}$, $y=4+\frac{8\sqrt{2}}{5}$ and so:

$$|z|=\sqrt{x^2+y^2}=\sqrt{\left(5+2\sqrt{2}\right)^2}=5+2\sqrt{2}.$$

14 a First note that both $2 = |z - 2 - 3\text{i}|$ and $3 = |z - 2 - 3\text{i}|$ represent circles centred at (2,3) with radius $r = 2$ and $r = 3$ respectively.
Thus $2 \leqslant |z - 2 - 3\text{i}| \leqslant 3$ represents the region between these two circles, including the circles since the inequalities are not strict.



b The area of this region can be found by subtracting the area of the smaller circle from the area of the larger circle $P_{\text{region}} = P_{\text{large}} - P_{\text{small}} = 9\pi - 4\pi = 5\pi$.

c We want to determine whether $z = 4 + \text{i}$ lies within the region.
We have $|4 + \text{i} - 2 - 3\text{i}| = |2 - 2\text{i}| = 2|1 - \text{i}| = 2\sqrt{2}$.
Since $2 \leqslant 2\sqrt{2} \leqslant 3,$ the point lies in the region.

15 $T : w = \dfrac{1}{z}$

a line $x = \frac{1}{2}$ in the z-plane

$w = \dfrac{1}{z}$

$\Rightarrow wz = 1$

$\Rightarrow z = \dfrac{1}{w}$

$\Rightarrow z = \dfrac{1}{u + \mathrm{i}v}$

$\Rightarrow z = \dfrac{1}{(u + \mathrm{i}v)} \times \dfrac{(u - \mathrm{i}v)}{(u - \mathrm{i}v)}$

$\Rightarrow z = \dfrac{u - \mathrm{i}v}{u^2 + v^2}$

$\Rightarrow z = \dfrac{u}{u^2 + v^2} + \mathrm{i}\left(\dfrac{-v}{u^2 + v^2}\right)$

So, $x + \mathrm{i}y = \dfrac{u}{u^2 + v^2} + \mathrm{i}\left(\dfrac{-v}{u^2 + v^2}\right)$

$\Rightarrow x = \dfrac{u}{u^2 + v^2}$ and $y \dfrac{-v}{u^2 + v^2}$

As $x = \dfrac{1}{2}$, then $\dfrac{1}{2} = \dfrac{u}{u^2 + v^2}$

$\Rightarrow u^2 + v^2 = 2u$

$\Rightarrow u^2 - 2u + v^2 = 0$

$\Rightarrow (u - 1)^2 - 1 + v^2 = 0$

$\Rightarrow (u - 1)^2 + v^2 = 1$

Therefore the transformation T maps the line $x = \frac{1}{2}$ in the z-plane to a circle C, with centre (1, 0), radius 1. The equation of C is $(u - 1)^2 + v^2 = 1$.

15 b $x \geqslant \frac{1}{2}$

$$\frac{u}{u^2+v^2} \geqslant \frac{1}{2}$$

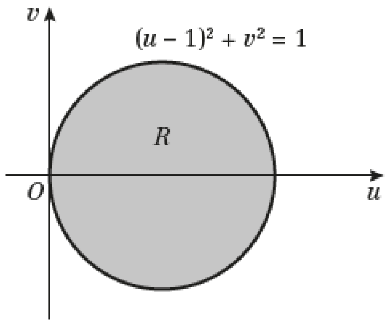
$$\Rightarrow 2u \geqslant u^2+v^2$$

$$\Rightarrow 0 \geqslant u^2+v^2-2u$$

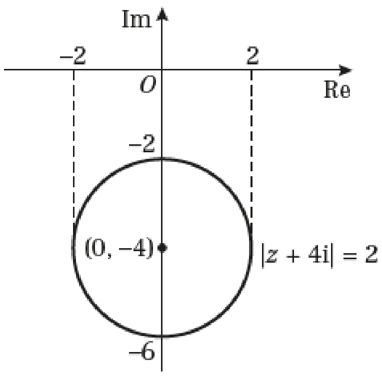
$$\Rightarrow 0 \geqslant (u-1)^2+v^2-1$$

$$\Rightarrow 1 \geqslant (u-1)^2+v^2$$

$$\Rightarrow (u-1)^2+v^2 \leqslant 1$$



16 a $|z+4\text{i}|=2$ is represented by a circle centre (0, –4), radius 2.



b $|z|$ represents the distance from (0, 0) to points on the locus of *P*.

Hence $|z|_{\max}$ occurs when $z=-6\text{i}$

Therefore $|z|_{\max}=|-6\text{i}|=6$.

16 c i $T_1 : w = 2z$

METHOD **(1)** z lies on circle with equation $|z + 4\text{i}| = 2$

$$\Rightarrow w = 2z$$

$$\Rightarrow \frac{w}{2} = z$$

$$\Rightarrow \frac{w}{2} + 4\text{i} = z + 4\text{i}$$

$$\Rightarrow \frac{w + 8\text{i}}{2} = z + 4\text{i}$$

$$\Rightarrow \left|\frac{w + 8\text{i}}{2}\right| = |z + 4\text{i}|$$

$$\Rightarrow \frac{|w + 8\text{i}|}{|2|} = |z + 4\text{i}|$$

$$\Rightarrow \frac{|w + 8\text{i}|}{2} = 2$$

$$\Rightarrow |w + 8\text{i}| = 4$$

So the locus of the image of P under T_1 is a circle centre (0, –8), radius 4, with equation $u^2 + (v + 8)^2 = 16$.

METHOD **(2)** z lies on circle centre (0, –4), radius 2

↓ enlargement scale factor 2, centre 0.

$w = 2z$ lies on circle centre (0,–8), radius 4.

So the locus of the image of P under T_1 is a circle centre (0, –8), radius 4, with equation $u^2 + (v + 8)^2 = 16$.

16 c ii $T_2 : w = \mathrm{i}z$

z lies on a circle with equation $|z + 4\mathrm{i}| = 2$

$$w = \mathrm{i}z$$
$$\Rightarrow \frac{w}{\mathrm{i}} = z$$
$$\Rightarrow \frac{w}{\mathrm{i}}\left(\frac{\mathrm{i}}{\mathrm{i}}\right) = z$$
$$\Rightarrow \frac{w\mathrm{i}}{(-1)} = z$$
$$\Rightarrow -w\mathrm{i} = z$$
$$\Rightarrow z = -w\mathrm{i}$$

Hence $|z + 4\mathrm{i}| = 2 \Rightarrow |-w\mathrm{i} + 4\mathrm{i}| = 2$
$$\Rightarrow |(-\mathrm{i})(w-4)| = 2$$
$$\Rightarrow |(-\mathrm{i})||w-4| = 2$$
$$\Rightarrow |w-4| = 2$$

So the locus of the image of P under T_2 is a circle centre (4, 0), radius 2, with equation $(u-4)^2 + v^2 = 4$.

iii $T_3 : w = -\mathrm{i}z$

z lies on a circle with equation $|z + 4\mathrm{i}| = 2$

$$w = -\mathrm{i}z$$
$$\Rightarrow \mathrm{i}w = \mathrm{i}(-\mathrm{i}z)$$
$$\Rightarrow \mathrm{i}w = z$$
$$\Rightarrow z = \mathrm{i}w$$

Hence $|z + 4\mathrm{i}| = 2 \Rightarrow |\mathrm{i}w + 4\mathrm{i}| = 2$
$$\Rightarrow |\mathrm{i}(w+4)| = 2$$
$$\Rightarrow |\mathrm{i}||w+4| = 2$$
$$\Rightarrow |w+4| = 2 \leftarrow |\mathrm{i}| = 1$$

So the locus of the image of P under T_3 is a circle centre (–4, 0), radius 2, with equation $(u+4)^2 + v^2 = 4$.

16 c iv $T_4 : w = z^*$

z lies on a circle with equation $|z + 4\text{i}| = 2$

$w = z^* \Rightarrow u + \text{i}v = x - \text{i}y$ ← $z = x + \text{i}y \Rightarrow z^* = x - \text{i}y$

So $u = x,\ v = -y$ and $x = u$ and $y = -v$

$$|z + 4\text{i}| = 2 \Rightarrow |x + \text{i}y + 4\text{i}| = 2$$
$$\Rightarrow |x + \text{i}(y + 4)| = 2$$
$$\Rightarrow |u + \text{i}(-v + 4)| = 2$$
$$\Rightarrow |u + \text{i}(4 - v)| = 2$$
$$\Rightarrow |u + \text{i}(4 - v)|^2 = 2^2$$
$$\Rightarrow u^2 + (4 - v)^2 = 4$$
$$\Rightarrow u^2 + (v - 4)^2 = 4$$

So the locus of the image of P under T_4 is a circle centre (0, 4), radius 2, with equation $u^2 + (v - 4)^2 = 4$.

17 $T : w = \dfrac{z+2}{z+\mathrm{i}}, z \neq -\mathrm{i}$

a the imaginary axis in z-plane $\Rightarrow x = 0$

$$w = \frac{z+2}{z+\mathrm{i}}$$

$$\Rightarrow w(z+\mathrm{i}) = z+2$$

$$\Rightarrow wz + \mathrm{i}w = z + 2$$

$$\Rightarrow wz - z = 2 - \mathrm{i}w$$

$$\Rightarrow z(w-1) = 2 - \mathrm{i}w$$

$$\Rightarrow z = \frac{2-\mathrm{i}w}{w-1}$$

$$\Rightarrow z = \frac{2-\mathrm{i}(u+\mathrm{i}v)}{u+\mathrm{i}v-1}$$

$$\Rightarrow z = \frac{2-\mathrm{i}u+v}{(u-1)+\mathrm{i}v}$$

$$\Rightarrow z = \left[\frac{(2+v)-\mathrm{i}u}{(u-1)+\mathrm{i}v}\right] \times \left[\frac{(u-1)-\mathrm{i}v}{(u-1)-\mathrm{i}v}\right]$$

$$\Rightarrow z = \frac{(2+v)(u-1)-uv-\mathrm{i}v(2+v)-\mathrm{i}u(u-1)}{(u-1)^2+v^2}$$

$$\Rightarrow z = \frac{(2+v)(u-1)-uv}{(u-1)^2+v^2} - \mathrm{i}\left(\frac{v(2+v)+u(u-1)}{(u-1)^2+v^2}\right)$$

So $x + \mathrm{i}y = \dfrac{(2+v)(u-1)-uv}{(u-1)^2+v^2} - \mathrm{i}\left(\dfrac{v(2+v)+u(u-1)}{(u-1)^2+v^2}\right)$

$\Rightarrow x = \dfrac{(2+v)(u-1)-uv}{(u-1)^2+v^2}$ and $y = \dfrac{-v(2+v)-u(u-1)}{(u-1)^2+v^2}$

As $x = 0$, then

$$\frac{(2+v)(u-1)-uv}{(u-1)^2+v^2} = 0$$

$$\Rightarrow (2+v)(u-1)-uv = 0$$

$$\Rightarrow 2u - 2 + vu - v - uv = 0$$

$$\Rightarrow 2u - 2 - v = 0$$

$$\Rightarrow v = 2u - 2$$

The transformation T maps the imaginary axis in the z-plane to the line l with equation $v = 2u - 2$ in the w-plane.

17 b As $y = x$, then

$$\frac{-v(2+v)-u(u-1)}{(u-1)^2+v^2} = \frac{(2+v)(u-1)-uv}{(u-1)^2+v^2}$$

$$\Rightarrow -v(2+v)-u(u-1) = (2+v)(u-1)-uv$$

$$\Rightarrow -2v-v^2-u^2+u = 2u-2+vu-v-uv$$

$$\Rightarrow -2v-v^2-u^2+u = 2u-2-v$$

$$\Rightarrow 0 = u^2+v^2+u+v-2$$

$$\Rightarrow \left(u+\frac{1}{2}\right)^2-\frac{1}{4}+\left(v+\frac{1}{2}\right)^2-\frac{1}{4}-2=0$$

$$\Rightarrow \left(u+\frac{1}{2}\right)^2+\left(v+\frac{1}{2}\right)^2=\frac{5}{2}$$

$$\Rightarrow \left(u+\frac{1}{2}\right)^2+\left(v+\frac{1}{2}\right)^2=\left(\frac{\sqrt{10}}{2}\right)^2$$

$$\sqrt{\frac{5}{2}} = \frac{\sqrt{5}}{\sqrt{2}} = \frac{\sqrt{5}}{\sqrt{2}}\frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{10}}{2} = \frac{1}{2}\sqrt{10}$$

The transformation T maps the line $y = x$ in the z-plane to the circle C with centre $\left(-\frac{1}{2}, -\frac{1}{2}\right)$, radius $\frac{\sqrt{10}}{2}$ in the w-plane.

18 $T: \; w = \dfrac{4-z}{z+\mathrm{i}} \quad z \neq -\mathrm{i}$

circle with equation $|z| = 1$ in the z-plane.

$$w = \frac{4-z}{z+\mathrm{i}}$$
$$\Rightarrow w(z+\mathrm{i}) = 4-z$$
$$\Rightarrow wz + \mathrm{i}w = 4-z$$
$$\Rightarrow wz + z = 4-\mathrm{i}w$$
$$\Rightarrow z(w+1) = 4-\mathrm{i}w$$
$$\Rightarrow z = \frac{4-\mathrm{i}w}{w+1}$$
$$\Rightarrow |z| = \left|\frac{4-\mathrm{i}w}{w+1}\right|$$
$$\Rightarrow |z| = \frac{|4-\mathrm{i}w|}{|w+1|}$$

Applying $|z| = 1$ gives $1 = \dfrac{|4-\mathrm{i}w|}{|w+1|}$

$$\Rightarrow |w+1| = |4-\mathrm{i}w|$$
$$\Rightarrow |w+1| = |-\mathrm{i}(w+4\mathrm{i})|$$
$$\Rightarrow |w+1| = |-\mathrm{i}||w+4\mathrm{i}|$$
$$\Rightarrow |w+1| = |w+4\mathrm{i}|$$
$$\Rightarrow |u+\mathrm{i}v+1| = |u+\mathrm{i}v+4\mathrm{i}|$$
$$\Rightarrow |(u+1)+\mathrm{i}v| = |u+\mathrm{i}(v+4)|$$
$$\Rightarrow |(u+1)+\mathrm{i}v|^2 = |u+\mathrm{i}(v+4)|^2$$
$$\Rightarrow (u+1)^2 + v^2 = u^2 + (v+4)^2$$
$$\Rightarrow u^2 + 2u + 1 + v^2 = u^2 + v^2 + 8v + 16$$
$$\Rightarrow 2u + 1 = 8v + 16$$
$$\Rightarrow 2u - 8v - 15 = 0$$

The circle $|z| = 1$ is mapped by T onto the line l: $2u - 8v - 15 = 0$ (i.e. $a = 2$, $b = -8$, $c = -15$).

19 $T: \; w = \dfrac{3\text{i}z+6}{1-z}; \quad z \neq 1$

circle with equation $|z| = 2$

$$w = \frac{3\text{i}z+6}{1-z}$$
$$\Rightarrow w(1-z) = 3\text{i}z+6$$
$$\Rightarrow w - wz = 3\text{i}z+6$$
$$\Rightarrow w - 6 = 3\text{i}z + wz$$
$$\Rightarrow w - 6 = z(3\text{i}+w)$$
$$\Rightarrow \frac{w-6}{w+3\text{i}} = z$$
$$\Rightarrow \left|\frac{w-6}{w+3\text{i}}\right| = |z|$$
$$\Rightarrow \frac{|w-6|}{|w+3\text{i}|} = |z|$$

Applying $|z| = 2 \Rightarrow \dfrac{|w-6|}{|w+3\text{i}|} = 2$

$$\Rightarrow |w-6| = 2|w+3\text{i}|$$
$$\Rightarrow |u+\text{i}v-6| = 2|u+\text{i}v+3\text{i}|$$
$$\Rightarrow |(u-6)+\text{i}v| = 2|u+\text{i}(v+3)|$$
$$\Rightarrow |(u-6)+\text{i}v|^2 = 2^2|u+\text{i}(v+3)|^2$$
$$\Rightarrow (u-6)^2 + v^2 = 4[u^2+(v+3)^2]$$
$$\Rightarrow u^2 - 12u + 36 + v^2 = 4[u^2 + v^2 + 6v + 9]$$
$$\Rightarrow u^2 - 12u + 36 + v^2 = 4u^2 + 4v^2 + 24v + 36$$
$$\Rightarrow 0 = 3u^2 + 12u + 3v^2 + 24v$$
$$\Rightarrow 0 = u^2 + 4u + v^2 + 8v$$
$$\Rightarrow 0 = (u+2)^2 - 4 + (v+4)^2 - 16$$
$$\Rightarrow 20 = (u+2)^2 + (v+4)^2$$
$$\Rightarrow (u+2)^2 + (v+4)^2 = (2\sqrt{5})^2$$

$\sqrt{20} = \sqrt{4}\sqrt{5} = 2\sqrt{5}$

Therefore the circle with equation $|z| = 2$ is mapped onto a circle C, centre (–2, –4), radius $2\sqrt{5}$. So $k = 2$.

20 a We know that the transformation T given by $w=\dfrac{az+b}{z+c}$ maps the origin onto itself, so,

substituting $w=z=0$ into T we get $0=\dfrac{b}{c}$.

We have to assume $c\neq 0$ or else it is not possible to map the origin onto itself.
Therefore $b=0$.
We also know that this mapping reflects $z_1=1+2\mathrm{i}$ in the real axis, i.e. $w_1=1-2\mathrm{i}$. Substituting these values into T we obtain:

$$1-2\mathrm{i}=\frac{a+2a\mathrm{i}}{1+2\mathrm{i}+c}$$

$$(1-2\mathrm{i})(1+2\mathrm{i}+c)=a+2a\mathrm{i}$$

$$1+2\mathrm{i}+c-2\mathrm{i}+4-2c\mathrm{i}=a+2ai$$

$$1+4+c-2ci=a+2ai$$

$$5+c-2ci=a+2ai$$

We now equate the real and complex parts:

$$5+c=a$$

$$-c=a$$

Solving simultaneously gives:

$$2c=-5$$

$$c=-\tfrac{5}{2}$$

$$a=\tfrac{5}{2}$$

So $a=\tfrac{5}{2}$, $b=0$ and $c=-\tfrac{5}{2}$.

b We know that another complex number, ω, is mapped onto itself. i.e. we have:

$$\omega=\frac{\tfrac{5}{2}\omega}{\omega-\tfrac{5}{2}}$$

$$\omega^2-\tfrac{5}{2}\omega=\tfrac{5}{2}\omega$$

$$\omega^2=5\omega$$

$$\omega^2-5\omega=0$$

$$\omega(\omega-5)=0$$

$\omega=0$ or $\omega=5$

$\omega=0$ is the origin, so the other number mapped onto itself is $\omega=5$.

21 a $w = \dfrac{az+b}{z+c} \quad a, b, c \in \mathbb{R}.$

$w = 1$ when $z = 0$ **(1)**

$w = 3 - 2\text{i}$ when $z = 2 + 3\text{i}$ **(2)**

(1) $\Rightarrow 1 = \dfrac{a(0)+b}{0+c} \Rightarrow 1 = \dfrac{b}{c} \Rightarrow c = b$ **(3)**

(3) $\Rightarrow w = \dfrac{az+b}{z+b}$

(2) $\Rightarrow 3 - 2\text{i} = \dfrac{a(2+3\text{i})+b}{2+3\text{i}+b}$

$3 - 2\text{i} = \dfrac{(2a+b)+3a\text{i}}{(2+b)+3\text{i}}$

$(3-2\text{i})[(2+b)+3\text{i}] = 2a + b + 3a\text{i}$

$6 + 3b + 9\text{i} - 4\text{i} - 2b\text{i} + 6 = 2a + b + 3a\text{i}$

$(12+3b) + (5-2b)\text{i} = (2a+b) + 3a\text{i}$

Equate real parts: $12 + 3b = 2a + b$

$\Rightarrow \quad 12 = 2a - 2b$ **(4)**

Equate imaginary parts: $5 - 2b = 3a$

$\Rightarrow \quad 5 = 3a + 2b$ **(5)**

(4) + **(5)**: $17 = 5a$

$\Rightarrow \frac{17}{5} = a$

(5) $\Rightarrow 5 = \frac{51}{5} + 2b$

$\Rightarrow -\frac{26}{5} = 2b$

$\Rightarrow -\frac{13}{5} = b$

As $b = c$ then $c = -\frac{13}{5}$

The values are $a = \frac{17}{5},\ b = -\frac{13}{5},\ c = -\frac{13}{5}$

21 b $w = \dfrac{\frac{17}{5}z - \frac{13}{5}}{z - \frac{13}{5}}$

$w = \dfrac{17z - 13}{5z - 13}$

invariant points $\Rightarrow z = \dfrac{17z - 13}{5z - 13}$

$z(5z - 13) = 17z - 13$

$5z^2 - 13z = 17z - 13$

$5z^2 - 30z + 13 = 0$

$z = \dfrac{30 \pm \sqrt{900 - 4(5)(13)}}{10}$

$z = \dfrac{30 \pm \sqrt{900 - 260}}{10}$

$z = \dfrac{30 \pm \sqrt{640}}{10}$

$z = \dfrac{30 \pm \sqrt{64}\sqrt{10}}{10}$

$z = \dfrac{30 \pm 8\sqrt{10}}{10} = 3 \pm \dfrac{4\sqrt{10}}{5}$

The exact values of the two points which remain invariant are

$z = 3 + \dfrac{4\sqrt{10}}{5}$ and $z = 3 - \dfrac{4\sqrt{10}}{5}$

22 $T: \quad w = \dfrac{z+\mathrm{i}}{z}, \quad z \neq 0.$

a the line $y = x$ in the z-plane other than (0, 0)

$$w = \frac{z+\mathrm{i}}{z}$$

$$\Rightarrow wz = z + \mathrm{i}$$

$$\Rightarrow wz - z = \mathrm{i}$$

$$\Rightarrow z(w-1) = \mathrm{i}$$

$$\Rightarrow z = \frac{i}{w-1}$$

$$\Rightarrow z = \frac{\mathrm{i}}{(u+\mathrm{i}v)-1} = \frac{\mathrm{i}}{(u-1)+\mathrm{i}v}$$

$$\Rightarrow z = \left[\frac{\mathrm{i}}{(u-1)+\mathrm{i}v}\right]\left[\frac{(u-1)-\mathrm{i}v}{(u-1)-\mathrm{i}v}\right]$$

$$\Rightarrow z = \frac{\mathrm{i}(u-1)+v}{(u-1)^2+v^2}$$

$$\Rightarrow z = \frac{v}{(u-1)^2+v^2} + \mathrm{i}\frac{(u-1)}{(u-1)^2+v^2}$$

So $x + \mathrm{i}y = \dfrac{v}{(u-1)^2+v^2} + \mathrm{i}\dfrac{(u-1)}{(u-1)^2+v^2}$

$\Rightarrow x = \dfrac{v}{(u-1)^2+v^2}$ and $y = \dfrac{u-1}{(u-1)^2+v^2}$

Applying $y = x$, gives $\dfrac{u-1}{(u-1)^2+v^2} = \dfrac{v}{(u-1)^2+v^2}$

$$\Rightarrow u - 1 = v$$

$$\Rightarrow v = u - 1$$

Therefore the line l has equation $v = u - 1$.

22 b The line with equation $x+y+1=0$ in the z-plane

$$x+y+1=0 \Rightarrow \frac{v}{(u-1)^2+v^2}+\frac{u-1}{(u-1)^2+v^2}+1=0 \; [\times(u-1)^2+v^2]$$

$$\Rightarrow v+(u-1)+(u-1)^2+v^2=0$$

$$\Rightarrow v+u-1+u^2-2u+1+v^2=0$$

$$\Rightarrow u^2+v^2-u+v=0$$

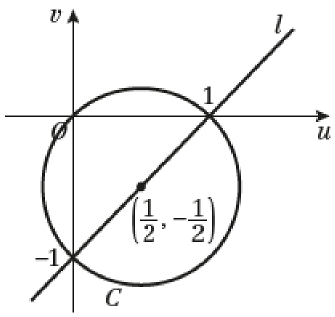
$$\Rightarrow \left(u-\frac{1}{2}\right)^2-\frac{1}{4}+\left(v+\frac{1}{2}\right)^2-\frac{1}{4}=0$$

$$\Rightarrow \left(u-\frac{1}{2}\right)^2+\left(v+\frac{1}{2}\right)^2=\frac{1}{2}$$

$$\Rightarrow \left(u-\frac{1}{2}\right)^2+\left(v+\frac{1}{2}\right)^2=\left(\frac{\sqrt{2}}{2}\right)^2$$

The image of $x+y+1=0$ under T is a circle C, centre $\left(\frac{1}{2},-\frac{1}{2}\right)$, radius $\frac{\sqrt{2}}{2}$ with equation $u^2+v^2-u+v=0$, as required.

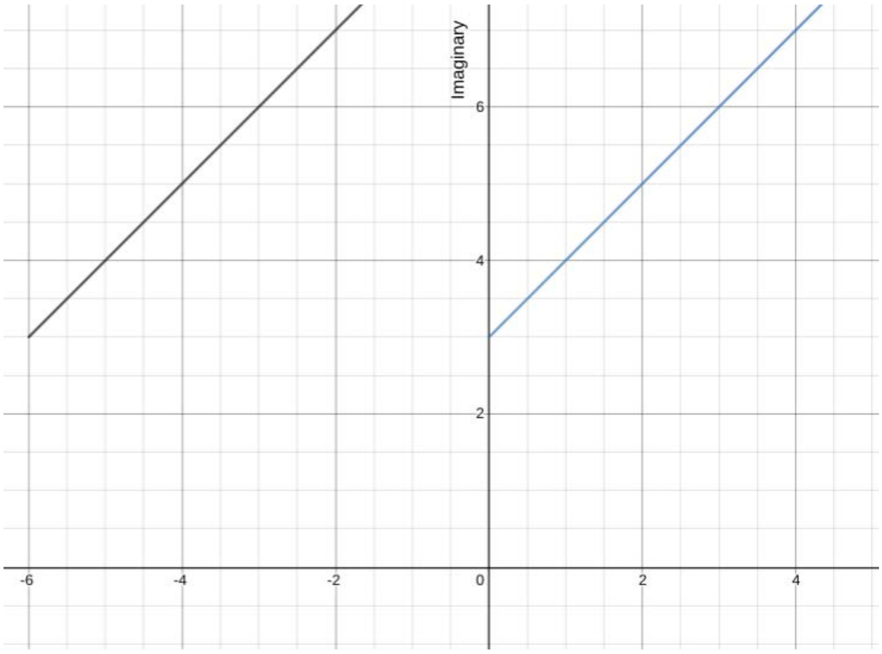
c



Challenge

1 a As $|w - z| = 3$, we can substitute in $z = 3 + w$ and $z = w - 3$ to provide two locuses.

(1) $\arg(w + 3i) = \pi/4$
(2) $\arg(w - 6 + 3i) = \pi/4$



b At its minimum, $|w| = 3$

2 We want to find a transformation of the form $\mathrm{f}(z) = az^* + b$, which reflects the z-plane in the line $x + y = 1$.

First note that points lying on the line will be mapped onto themselves.

This means that for any $z = x + \mathrm{i}y$ such that $x + y = 1$ we have $\mathrm{f}(z) = z$.

Choose $z_1 = 1,\ z_2 = \mathrm{i}$.

Both these points satisfy $x + y = 1$, so $\mathrm{f}(z_1) = z_1$ and $\mathrm{f}(z_2) = z_2$.

Therefore we have:

$\mathrm{f}(1) = a + b = 1$

and $\mathrm{f}(\mathrm{i}) = -a\mathrm{i} + b = \mathrm{i}$.

Solving simultaneously:

$-(1-b)\mathrm{i} + b = \mathrm{i}$

$-\mathrm{i} + b\mathrm{i} + b = \mathrm{i}$

$b(1+\mathrm{i}) = 2\mathrm{i}$

$$b = \frac{2\mathrm{i}}{1+\mathrm{i}} = \frac{2\mathrm{i}}{1+\mathrm{i}}\left(\frac{1-\mathrm{i}}{1-\mathrm{i}}\right) = \frac{2\mathrm{i}(1-\mathrm{i})}{2} = \mathrm{i}(1-\mathrm{i})$$

$b = 1 + \mathrm{i}$

So $a = 1 - b = -\mathrm{i}$ and $\mathrm{f}(z) = -\mathrm{i}z^* + 1 + \mathrm{i}$