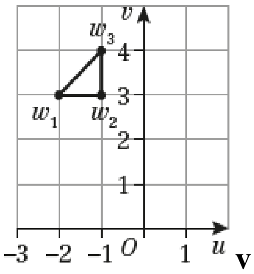
Exercise 4E

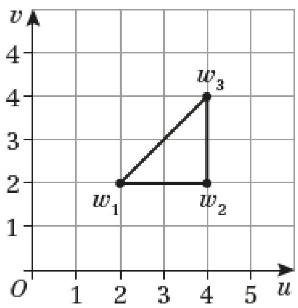
1 We have $z_1 = 1 + \mathrm{i}$, $z_2 = 2 + \mathrm{i}$ and $z_3 = 2 + 2\mathrm{i}$. The transformed triangle can be found by directly computing the transformed values w_1, w_2, w_3 of z_1, z_2, z_3:

a **i** $w = z - 3 + 2\mathrm{i}$



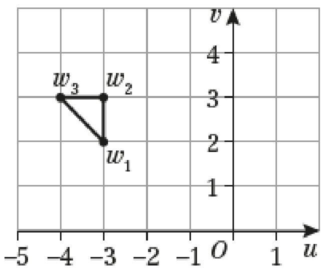
ii This transformation represents a translation by vector $\begin{pmatrix} -3 \\ 2 \end{pmatrix}$.

b **i** $w = 2z$



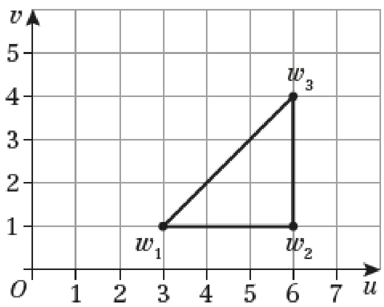
ii This transformation represents enlargement by a factor of 2 with centre (0,0).

c **i** $w = \mathrm{i}z - 2 + \mathrm{i}$



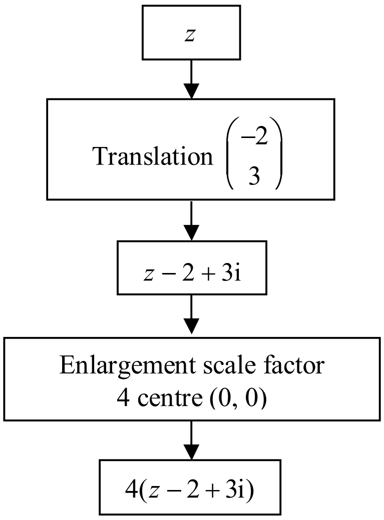
ii This transformation represents a rotation anticlockwise through $\frac{\pi}{2}$ and a translation by $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$

d **i** $w = 3z - 2\mathrm{i}$



1 d ii This transformation represents enlargement by a factor of 3 and translation by $\begin{pmatrix} 0 \\ -2 \end{pmatrix}$.

2



Hence $T: w = 4(z - 2 + 3\text{i})$
$= 4z - 8 + 12\text{i}$

The transformation T is $w = 4z - 8 + 12\text{i}$

Note: $a = 4, b = -8 + 12\text{i}$.

3 Rotation through $\frac{\pi}{2}$ around the origin is achieved by multiplying all values in the z-plane by i.

Enlargement by a scale factor of 4 is achieved by multiplying all values in the z-plane by 4.

Therefore this transformation can be written as $w = 4\text{i}z$.

4 z moves on a circle $|z-2|=4$

METHOD **(1)** $w=2z-5+3\text{i}$

$$\Rightarrow w+5-3\text{i}=2z$$

$$\Rightarrow \frac{w+5-3\text{i}}{2}=z$$

$$\Rightarrow \frac{w+5-3\text{i}}{2}-2=z-2$$

$$\Rightarrow \frac{w+5-3\text{i}-4}{2}=z-2$$

$$\Rightarrow \frac{w+1-3\text{i}}{2}=z-2$$

$$\Rightarrow \left|\frac{w+1-3\text{i}}{2}\right|=|z-2|$$

$$\Rightarrow \frac{|w+1-3\text{i}|}{|2|}=|z-2|$$

$$\Rightarrow |w+1-3\text{i}|=2|z-2|$$

$$\Rightarrow |w+1-3\text{i}|=2(4)$$

$$\Rightarrow |w+1-3\text{i}|=8$$

$$\Rightarrow |w-(-1+3\text{i})|=8$$

So the locus of w is a circle centre (–1, 3), radius 8 with equation $(u+1)^2+(v-3)^2=64$.

METHOD **(2)** $|z-2|=4$

z lies on a circle, centre (2, 0), radius 4

↓ enlargement scale factor 2, centre 0.

$2z$ lies on a circle, centre (4, 0), radius 8.

↓ translation by a translation vector $\begin{pmatrix}-5\\3\end{pmatrix}$.

$w=2z-5+3\text{i}$ lies on a circle centre (–1, 3), radius 8.

So the locus of w is a circle, centre (–1, 3), radius 8 with equation $(u+1)^2+(v-3)^2=64$.

5 $w = z - 1 + 2\text{i}$

a $|z - 1| = 3$ circle centre (1, 0) radius 3.

METHOD **(1)** $|z - 1| = 3$ is translated by a translation vector $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ to give a circle, centre (0, 2), radius 3, in the *w*-plane.

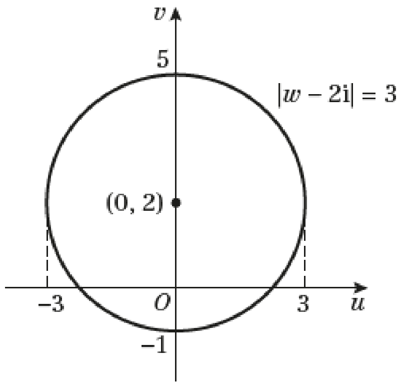
METHOD **(2)** $w = z - 1 + 2\text{i}$

$\Rightarrow w - 2\text{i} = z - 1$

$\Rightarrow |w - 2\text{i}| = |z - 1|$

$\Rightarrow |w - 2\text{i}| = 3$

The locus of *w* is a circle, centre (0, 2), radius 3.



5 b $\arg(z-1+\text{i})=\frac{\pi}{4}$ half-line from $(1, -1)$ at $\frac{\pi}{4}$ with the positive real axis.

METHOD **(1)** $\arg(z-1+\text{i})=\frac{\pi}{4}$ is translated by a translation vector $\begin{pmatrix}-1\\2\end{pmatrix}$ to give a half-line from $(0, 1)$ at $\frac{\pi}{4}$ with the positive real axis.

METHOD **(2)** $w=z-1+2\text{i}$

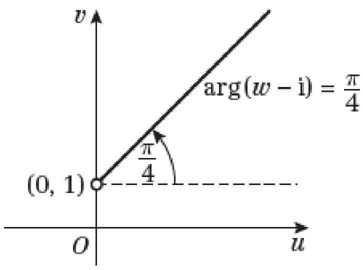
$\Rightarrow w+1-2\text{i}=z$

So $\arg(z-1+\text{i})=\frac{\pi}{4}$

becomes $\arg(w+1-2\text{i}-1+\text{i})=\frac{\pi}{4}$

$\Rightarrow \arg(w-\text{i})=\frac{\pi}{4}$

Therefore, the locus of w is a half-line from $(0, 1)$ at $\frac{\pi}{4}$ with the positive real axis.



c $y=2x$

$w=z-1+2\text{i}$

$\Rightarrow z=w+1-2\text{i}$

$\Rightarrow x+\text{i}y=u+\text{i}v+1-2\text{i}$

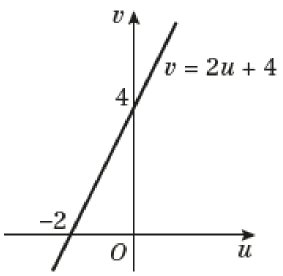
$\Rightarrow x+\text{i}y=u+1+\text{i}(v-2)$

So $y=2x \Rightarrow v-2=2(u+1)$

$\Rightarrow v-2=2u+2$

$\Rightarrow v=2u+4$

The locus of w is a line with equation $v=2u+4$.



6 $w = \frac{1}{z}, z \neq 0$

a z lies on a circle, $|z| = 2$

$$w = \frac{1}{z}$$

$$\Rightarrow |w| = \left|\frac{1}{z}\right|$$

$$\Rightarrow |w| = \frac{|1|}{|z|}$$

$$\Rightarrow |w| = \frac{1}{2}$$ ← apply $|z| = 2$

Therefore the locus of w is a circle, centre (0, 0), radius $\frac{1}{2}$, with equation $u^2 + v^2 = \frac{1}{4}$.

b z lies on the half-line, $\arg z = \frac{\pi}{4}$

$$w = \frac{1}{z} \Rightarrow wz = 1 \Rightarrow z = \frac{1}{w}$$

So $\arg z = \frac{\pi}{4}$, becomes $\arg\left(\frac{1}{w}\right) = \frac{\pi}{4}$

$$\Rightarrow \arg(1) - \arg(w) = \frac{\pi}{4}$$

$$\Rightarrow -\arg w = \frac{\pi}{4}$$ ← $\arg 1 = 0$

$$\Rightarrow \arg w = -\frac{\pi}{4}$$

Therefore the locus of w is a half-line from (0, 0) at an angle of $-\frac{\pi}{4}$ with the positive x-axis.

The locus of w has equation, $v = -u, u > 0, v < 0$.

6 c z lies on the line $y = 2x + 1$

$$w = \frac{1}{z} \Rightarrow wz = 1 \Rightarrow z = \frac{1}{w}.$$

$$\Rightarrow x + iy = \frac{1}{u + iv}$$

$$\Rightarrow x + iy = \frac{1}{(u + iv)}\frac{(u - iv)}{(u - iv)}$$

$$\Rightarrow x + iy = \frac{u - iv}{u^2 + v^2}$$

$$\Rightarrow x + iy = \frac{u}{u^2 + v^2} + i\left(\frac{-v}{u^2 + v^2}\right)$$

So $x = \frac{u}{u^2 + v^2}$ and $y = \frac{-v}{u^2 + v^2}$

Hence $y = 2x + 1$ becomes $\frac{-v}{u^2 + v^2} = \frac{2u}{u^2 + v^2} + 1 \qquad \times (u^2 + v^2)$

$$\Rightarrow -v = 2u + u^2 + v^2$$

$$\Rightarrow 0 = u^2 + 2u + v^2 + v$$

$$\Rightarrow (u + 1)^2 - 1 + \left(v + \frac{1}{2}\right)^2 - \frac{1}{4} = 0$$

$$\Rightarrow (u + 1)^2 + \left(v + \frac{1}{2}\right)^2 = \frac{5}{4}$$

$$\Rightarrow (u + 1)^2 + \left(v + \frac{1}{2}\right)^2 = \left(\frac{\sqrt{5}}{2}\right)^2$$

Therefore, the locus of w is a circle, centre $\left(-1, -\frac{1}{2}\right)$, radius $\frac{\sqrt{5}}{2}$, with equation

$$(u + 1)^2 + \left(v + \frac{1}{2}\right)^2 = \frac{5}{4}$$

7 $w = z^2$

a z moves once round a circle, centre (0, 0), radius 3.

The equation of the circle, $|z| = 3$ is also $r = 3$.

The equation of the circle can be written as $z = 3e^{i\theta}$

or $z = 3(\cos\theta + i\sin\theta)$

$$\Rightarrow w = z^2 = (3(\cos\theta + i\sin\theta))^2$$
$$= 3^2(\cos 2\theta + i\sin 2\theta)$$
$$= 9(\cos 2\theta + i\sin 2\theta)$$

de Moivre's Theorem.

So, $w = 9(\cos 2\theta + i\sin 2\theta)$ can be written as $|w| = 9$

Hence, as $|w| = 9$ and $\arg w = 2\theta$ then w moves twice round a circle, centre (0, 0), radius 9.

7 b z lies on the real-axis $\Rightarrow y = 0$

So $z = x + \text{i}y$ becomes $z = x$ (as $y = 0$)

$\Rightarrow w = z^2 = x^2$

$\Rightarrow u + \text{i}v = x^2 + \text{i}(0)$

$\Rightarrow u = x^2$ and $v = 0$

As $v = 0$ and $u = x^2 \geqslant 0$ then w lies on the positive real-axis including the origin, 0.

c z lies on the imaginary axis $\Rightarrow x = 0$

So $z = x + \text{i}y$ becomes $z = \text{i}y$ (as $x = 0$)

$\Rightarrow w = z^2 = (\text{i}y)^2 = -y^2$

$\Rightarrow u + \text{i}v = -y^2 + \text{i}(0)$

$\Rightarrow u = -y^2$ and $v = 0$

As $v = 0$ and $u = -y^2 \leqslant 0$ then w lies on the negative real-axis including the origin, 0.

8 We have transformation T given by $w=\frac{2}{\text{i}-2z}$, $z\neq\frac{\text{i}}{2}$

a i Rearrange the transformation to get an expression for z:

$$w=\frac{2}{\text{i}-2z}$$

$$w(\text{i}-2z)=2$$

$$\text{i}-2z=\frac{2}{w}$$

$$2z=\text{i}-\frac{2}{w}$$

$$z=\frac{\text{i}}{2}-\frac{1}{w}=\frac{\text{i}w-2}{2w}$$

Therefore we can write $|z|=\left|\frac{\text{i}w-2}{2w}\right|$.

Since $|z|=1$ we have that:

$$\left|\frac{\text{i}w-2}{2w}\right|=1$$

$$|\text{i}w-2|=|2w|$$

$$|\text{i}||w+2\text{i}|=2|w|$$

$$|w+2\text{i}|=2|w|$$

Write $w=u+\text{i}v$, substitute into the equation and square both sides:

$$|u+\text{i}v+2\text{i}|=2|u+\text{i}v|$$

$$|u+\text{i}v+2\text{i}|^2=4|u+\text{i}v|^2$$

$$u^2+(v+2)^2=4u^2+4v^2$$

$$3u^2+3v^2-4v-4=0$$

$$u^2+v^2-\frac{4}{3}v-\frac{4}{3}=0$$

Complete the square for v

$u^2+\left(v-\frac{2}{3}\right)^2=\frac{16}{9}$, which is the equation of a circle

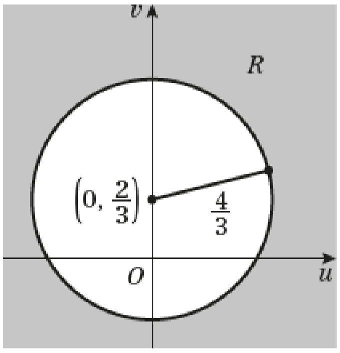
ii Since $u^2+\left(v-\frac{2}{3}\right)^2=\frac{16}{9}$, the circle is centred at $\left(0,\frac{2}{3}\right)$ and has radius $r=\frac{4}{3}$

8 b We showed that the region $|z| = 1$ is mapped to a circle centred at $\left(0, \frac{2}{3}\right)$ with radius $r = \frac{4}{3}$

Thus $|z| \leqslant 1$ will be either a circle and its interior, or a circle and its exterior.

The easiest way to check that is to pick a point inside the circle $|z| \leqslant 1$ and see where it maps to.

Pick $z_0 = 0$ (for example). Then $w_0 = \frac{2}{\text{i}} = -2\text{i}$. This lies outside of the circle centred at $\left(0, \frac{2}{3}\right)$ with radius $r = \frac{4}{3}$, so we see that the region $|z| \leqslant 1$ will be mapped to:



9 We want to show that the transformation T given by $w = \frac{1}{2-z}$, $z \neq 2$ transforms the circle centred at O, radius 2, $|z| = 2$ to a line. First, rearrange T to obtain an expression for z:

$$w(2-z) = 1$$

$$2 - z = \frac{1}{w}$$

$$z = 2 - \frac{1}{w}$$

$$z = \frac{2w-1}{w}$$

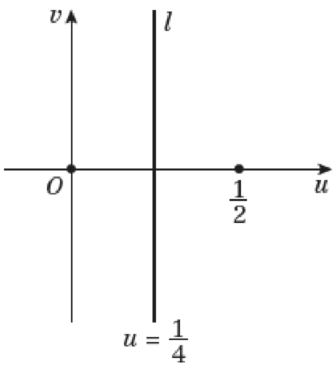
As $|z| = 2$, we can write:

$$2 = \frac{|2w-1|}{|w|}$$

$$2|w| = |2w-1|$$

$$|w| = \left|w - \tfrac{1}{2}\right|$$

This equation represents points on the perpendicular bisector of the line segment joining $(0,0)$ and $\left(\frac{1}{2}, 0\right)$. Therefore the line l has equation $u = \frac{1}{4}$:



10 We know that the transformation T is given by $w=\dfrac{z-\mathrm{i}}{z+\mathrm{i}}$, $z\neq -\mathrm{i}$

a We want to show that the circle $|z-\mathrm{i}|=1$ in z-plane is mapped to a circle in w-plane.

Begin by rearranging the transformation to obtain an expression for z:

$$w(z+\mathrm{i})=z-\mathrm{i}$$
$$wz+\mathrm{i}w=z-\mathrm{i}$$
$$z(1-w)=\mathrm{i}(w+1)$$
$$z=\frac{\mathrm{i}w+\mathrm{i}}{1-w}$$

We know that $|z-\mathrm{i}|=1$, so subtract i from both sides:

$$z-\mathrm{i}=\frac{\mathrm{i}w+\mathrm{i}}{1-w}-\mathrm{i}$$
$$z-\mathrm{i}=\frac{2\mathrm{i}w}{1-w}$$

Use $|z-\mathrm{i}|=1$:

$$1=\frac{|2\mathrm{i}w|}{|1-w|}$$
$$|1-w|=2|w|$$

Write $w=u+\mathrm{i}v$ and square both sides of the equation:

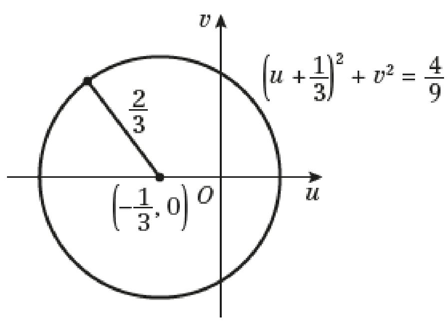
$$|1-u-\mathrm{i}v|^2=4|u+\mathrm{i}v|^2$$
$$(1-u)^2+v^2=4u^2+4v^2$$
$$1-2u+u^2=4u^2+3v^2$$
$$3u^2+2u+3v^2-1=0$$
$$u^2+\frac{2}{3}u+v^2-\frac{1}{3}=0$$

Complete the square

$$u^2+\frac{2}{3}u+v^2-\frac{1}{3}=0$$
$$\left(u+\frac{1}{3}\right)^2+v^2=\frac{4}{9}$$

Which represents a circle centred $\left(-\frac{1}{3},0\right)$, radius $r=\frac{2}{3}$ in the w-plane.

b



11 $T: w = \dfrac{3}{2-z}, z \neq 2$

$\Rightarrow w(2-z) = 3$

$\Rightarrow 2w - wz = 3$

$\Rightarrow 2w = 3 + wz$

$\Rightarrow 2w - 3 = wz$

$\Rightarrow \dfrac{2w-3}{w} = z$

$\Rightarrow z = \dfrac{2w-3}{w}$

$\Rightarrow z = \dfrac{2(u+\mathrm{i}v)-3}{u+\mathrm{i}v}$

$\Rightarrow z = \dfrac{(2u-3)+2\mathrm{i}v}{u+\mathrm{i}v}$

$\Rightarrow z = \dfrac{[(2u-3)+2\mathrm{i}v]}{u+\mathrm{i}v} \times \dfrac{[u-\mathrm{i}v]}{[u-\mathrm{i}v]}$

$\Rightarrow z = \dfrac{(2u-3)u - \mathrm{i}v(2u-3) + 2\mathrm{i}uv + 2v^2}{u^2+v^2}$

$\Rightarrow z = \dfrac{2u^2 - 3u - \cancel{2uv\mathrm{i}} + 3\mathrm{i}v + \cancel{2uv\mathrm{i}} + 2v^2}{u^2+v^2}$

$\Rightarrow z = \dfrac{2u^2-3u+2v^2}{u^2+v^2} + \mathrm{i}\left[\dfrac{3v}{u^2+v^2}\right]$

So, $x + \mathrm{i}y = \dfrac{2u^2-3u+2v^2}{u^2+v^2} + \mathrm{i}\left[\dfrac{3v}{u^2+v^2}\right]$

$\Rightarrow x = \dfrac{2u^2-3u+2v^2}{u^2+v^2}$

and $y = \dfrac{3v}{u^2+v^2}$

11 (continued)

As, $2y = x \Rightarrow 2\left(\frac{3v}{u^2+v^2}\right) = \frac{2u^2-3u+2v^2}{u^2+v^2}$

$$\Rightarrow \frac{6v}{u^2+v^2} = \frac{2u^2-3u+2v^2}{u^2+v^2}$$

$$\Rightarrow 6v = 2u^2 - 3u + 2v^2$$

$$\Rightarrow 0 = 2u^2 - 3u + 2v^2 - 6v$$

$$\Rightarrow 2u^2 - 3u + 2v^2 - 6v = 0 \quad (\div 2)$$

$$\Rightarrow u^2 - \frac{3}{2}u + v^2 - 3v = 0$$

$$\Rightarrow \left(u - \frac{3}{4}\right)^2 - \frac{9}{16} + \left(v - \frac{3}{2}\right)^2 - \frac{9}{4} = 0$$

$$\Rightarrow \left(u - \frac{3}{4}\right)^2 + \left(v - \frac{3}{2}\right)^2 = \frac{9}{16} + \frac{9}{4}$$

$$\Rightarrow \left(u - \frac{3}{4}\right)^2 + \left(v - \frac{3}{2}\right)^2 = \frac{45}{16}$$

$$\Rightarrow \left(u - \frac{3}{4}\right)^2 + \left(v - \frac{3}{2}\right)^2 = \left(\frac{3\sqrt{5}}{4}\right)^2$$

The image under T of $2y = x$ is a circle centre $\left(\frac{3}{4}, \frac{3}{2}\right)$, radius $\frac{3\sqrt{5}}{4}$, as required.

12 $T: w = \dfrac{-\mathrm{i}z + \mathrm{i}}{z+1}, z \neq -1$

a Circle with equation $x^2 + y^2 = 1 \Rightarrow |z| = 1$

$$w = \frac{-\mathrm{i}z + \mathrm{i}}{z+1}$$
$$\Rightarrow w(z+1) = -\mathrm{i}z + \mathrm{i}$$
$$\Rightarrow wz + w = -\mathrm{i}z + \mathrm{i}$$
$$\Rightarrow wz + \mathrm{i}z = \mathrm{i} - w$$
$$\Rightarrow z(w + \mathrm{i}) = \mathrm{i} - w$$
$$\Rightarrow z = \frac{\mathrm{i} - w}{w + \mathrm{i}}$$
$$\Rightarrow |z| = \left|\frac{\mathrm{i} - w}{w + \mathrm{i}}\right|$$
$$\Rightarrow |z| = \frac{|\mathrm{i} - w|}{|w + \mathrm{i}|}$$

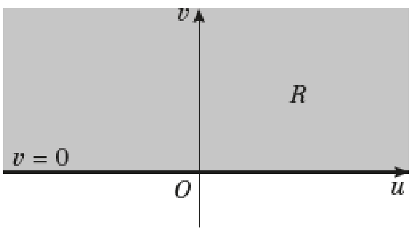
Applying $|z| = 1 \Rightarrow 1 = \dfrac{|\mathrm{i} - w|}{|w + \mathrm{i}|}$

$$\Rightarrow |w + \mathrm{i}| = |\mathrm{i} - w|$$
$$\Rightarrow |w + \mathrm{i}| = |(-1)(w - \mathrm{i})|$$
$$\Rightarrow |w + \mathrm{i}| = |(-1)||(w - \mathrm{i})|$$
$$\Rightarrow |w + \mathrm{i}| = |w - \mathrm{i}|$$

The image under T of $x^2 + y^2 = 1$ is the perpendicular bisector of the line segment joining (0, –1) to (0, 1). Therefore the line l has equation $v = 0$. (i.e. the u-axis.)

b $|z| \leqslant 1 \Rightarrow 1 \geqslant \dfrac{|\mathrm{i} - w|}{|w + \mathrm{i}|}$

$$\Rightarrow |w + \mathrm{i}| \geqslant |\mathrm{i} - w|$$
$$\Rightarrow |w + \mathrm{i}| \geqslant |w - \mathrm{i}|$$



12 c Circle with equation $x^2 + y^2 = 4 \Rightarrow |z| = 2$

from part **a** $w = \dfrac{-\mathrm{i}z + \mathrm{i}}{z+1}$

$$\Rightarrow z = \frac{\mathrm{i} - w}{w + \mathrm{i}}$$

$$\Rightarrow |z| = \frac{|\mathrm{i} - w|}{|w + \mathrm{i}|}$$

Applying $|z| = 2 \Rightarrow 2 = \dfrac{|\mathrm{i} - w|}{|w + \mathrm{i}|}$

$$\Rightarrow 2|w + \mathrm{i}| = |\mathrm{i} - w|$$

$$\Rightarrow 2|w + \mathrm{i}| = |(-1)(w - \mathrm{i})|$$

$$\Rightarrow 2|w + \mathrm{i}| = |(-1)||(w - \mathrm{i})|$$

$$\Rightarrow 2|w + \mathrm{i}| = |w - \mathrm{i}|$$

$$\Rightarrow 2|u + \mathrm{i}v + \mathrm{i}| = |u + \mathrm{i}v - \mathrm{i}|$$

$$\Rightarrow 2|u + \mathrm{i}(v+1)| = |u + \mathrm{i}(v-1)|$$

$$\Rightarrow 2^2|u + \mathrm{i}(v+1)|^2 = |u + \mathrm{i}(v-1)|^2$$

$$\Rightarrow 4[u^2 + (v+1)^2] = u^2 + (v-1)^2$$

$$\Rightarrow 4[u^2 + v^2 + 2v + 1] = u^2 + v^2 - 2v + 1$$

$$\Rightarrow 4u^2 + 4v^2 + 8v + 4 = u^2 + v^2 - 2v + 1$$

$$\Rightarrow 3u^2 + 3v^2 + 10v + 3 = 0$$

$$\Rightarrow u^2 + v^2 + \frac{10}{3}v + 1 = 0$$

$$\Rightarrow u^2 + \left(v + \frac{5}{3}\right)^2 - \frac{25}{9} + 1 = 0$$

$$\Rightarrow u^2 + \left(v + \frac{5}{3}\right)^2 = \frac{25}{9} - 1$$

$$\Rightarrow u^2 + \left(v + \frac{5}{3}\right)^2 = \frac{16}{9}$$

$$\Rightarrow u^2 + \left(v + \frac{5}{3}\right)^2 = \left(\frac{4}{3}\right)^2$$

The image under T of $x^2 + y^2 = 4$ is a circle C with centre $\left(0, -\dfrac{5}{3}\right)$, radius $\dfrac{4}{3}$.

Therefore, the equation of C is $u^2 + \left(v + \dfrac{5}{3}\right)^2 = \dfrac{16}{9}$.

13 $T: w = \dfrac{4z-3\text{i}}{z-1}, z \neq 1$

Circle with equation $|z| = 3$

$$w = \frac{4z-3\text{i}}{z-1},$$
$$\Rightarrow w(z-1) = 4z-3\text{i}$$
$$\Rightarrow wz - w = 4z - 3\text{i}$$
$$\Rightarrow wz - 4z = w - 3\text{i}$$
$$\Rightarrow z(w-4) = w - 3\text{i}$$
$$\Rightarrow z = \frac{w-3\text{i}}{w-4}$$
$$\Rightarrow |z| = \left|\frac{w-3\text{i}}{w-4}\right|$$

Applying $|z| = 3 \Rightarrow 3 = \dfrac{|w-3\text{i}|}{|w-4|}$

$$\Rightarrow 3|w-4| = |w-3\text{i}|$$
$$\Rightarrow 3|u+\text{i}v-4| = |u+\text{i}v-3\text{i}|$$
$$\Rightarrow 3|(u-4)+\text{i}v| = |u+\text{i}(v-3)|$$
$$\Rightarrow 3^2|(u-4)+\text{i}v|^2 = |u+\text{i}(v-3)|^2$$
$$\Rightarrow 9[(u-4)^2+v^2] = u^2+(v-3)^2$$
$$\Rightarrow 9[u^2-8u+16+v^2] = u^2+v^2-6v+9$$
$$\Rightarrow 9u^2-72u+144+9v^2 = u^2+v^2-6v+9$$
$$\Rightarrow 8u^2-72u+8v^2+6v+144-9 = 0$$
$$\Rightarrow 8u^2-72u+8v^2+6v+135 = 0 \quad (\div 8)$$
$$\Rightarrow u^2-9u+v^2+\frac{3}{4}v+\frac{135}{8} = 0$$
$$\Rightarrow \left(u-\frac{9}{2}\right)^2-\frac{81}{4}+\left(v+\frac{3}{8}\right)^2-\frac{9}{64}+\frac{135}{8} = 0$$
$$\Rightarrow \left(u-\frac{9}{2}\right)^2+\left(v+\frac{3}{8}\right)^2 = \frac{81}{4}+\frac{9}{64}-\frac{135}{8}$$
$$\Rightarrow \left(u-\frac{9}{2}\right)^2+\left(v+\frac{3}{8}\right)^2 = \frac{225}{64}$$
$$\Rightarrow \left(u-\frac{9}{2}\right)^2+\left(v+\frac{3}{8}\right)^2 = \left(\frac{15}{8}\right)^2$$

Therefore, the circle with equation $|z| = 1$ is mapped onto a circle C with centre $\left(\frac{9}{2}, -\frac{3}{8}\right)$, radius $\frac{15}{8}$.

14 $T: w = \dfrac{1}{z+\mathrm{i}}, z \neq -\mathrm{i}$

a Real axis in the z-plane $\Rightarrow y = 0$

$$w = \frac{1}{z+\mathrm{i}}$$
$$\Rightarrow w(z+\mathrm{i}) = 1$$
$$\Rightarrow wz + \mathrm{i}w = 1$$
$$\Rightarrow wz = 1 - \mathrm{i}w$$
$$\Rightarrow z = \frac{1-\mathrm{i}w}{w}$$
$$\Rightarrow z = \frac{1-\mathrm{i}(u+\mathrm{i}v)}{u+\mathrm{i}v}$$
$$\Rightarrow z = \frac{1-\mathrm{i}u+v}{u+\mathrm{i}v}$$
$$\Rightarrow z = \frac{((1+v)-\mathrm{i}u)}{(u+\mathrm{i}v)} \times \frac{(u-iv)}{(u-iv)}$$
$$\Rightarrow z = \frac{(1+v)u - \mathrm{i}v(1+v) - \mathrm{i}u^2 - uv}{u^2+v^2}$$
$$\Rightarrow z = \frac{(1+v)u - uv}{u^2+v^2} + \frac{\mathrm{i}(-v(1+v)-u^2)}{u^2+v^2}$$
$$\Rightarrow z = \frac{u+uv-uv}{u^2+v^2} + \frac{\mathrm{i}(-v-v^2-u^2)}{u^2+v^2}$$
$$\Rightarrow z = \frac{u}{u^2+v^2} + \frac{\mathrm{i}(-v-v^2-u^2)}{u^2+v^2}$$
$$\text{So } x+\mathrm{i}y = \frac{u}{u^2+v^2} + \frac{\mathrm{i}(-v-v^2-u^2)}{u^2+v^2}$$
$$\Rightarrow x = \frac{u}{u^2+v^2} \text{ and } y = \frac{-v-v^2-u^2}{u^2+v^2}$$
$$\text{As } y = 0, \frac{-v-v^2-u^2}{u^2+v^2} = 0$$
$$\Rightarrow -v-v^2-u^2 = 0$$
$$\Rightarrow u^2+v^2+v = 0$$
$$\Rightarrow u^2 + \left(v+\tfrac{1}{2}\right)^2 - \tfrac{1}{4} = 0$$
$$\Rightarrow u^2 + \left(v+\tfrac{1}{2}\right)^2 = \tfrac{1}{4}$$
$$\Rightarrow u^2 + \left(v+\tfrac{1}{2}\right)^2 = \left(\tfrac{1}{2}\right)^2$$

Therefore, the image under T of the real axis in the z-plane is a circle C_1 with centre $\left(0, -\frac{1}{2}\right)$, radius $\frac{1}{2}$. The equation of C_1 is $u^2 + \left(v+\frac{1}{2}\right)^2 = \frac{1}{4}$.

14 b

As $x = 4, \dfrac{u}{u^2 + v^2} = 4$

$$\Rightarrow u = 4(u^2 + v^2)$$

$$\Rightarrow u = 4u^2 + 4v^2$$

$$\Rightarrow 0 = 4u^2 - u + 4v^2 \quad (\div 4)$$

$$\Rightarrow 0 = u^2 - \tfrac{1}{4}u + v^2$$

$$\Rightarrow 0 = \left(u - \tfrac{1}{8}\right)^2 - \tfrac{1}{64} + v^2$$

$$\Rightarrow \left(u - \tfrac{1}{8}\right)^2 + v^2 = \tfrac{1}{64}$$

$$\Rightarrow \left(u - \tfrac{1}{8}\right)^2 + v^2 = \left(\tfrac{1}{8}\right)^2$$

Therefore, the image under T of the line $x = 4$ is a circle C_2 with centre $\left(\tfrac{1}{8}, 0\right)$, radius $\tfrac{1}{8}$.

The equation of C_2 is $\left(u - \tfrac{1}{8}\right)^2 + v^2 = \tfrac{1}{64}$.

15 $T: w = z + \frac{4}{z}, z \neq 0$

Circle with equation $|z| = 2 \Rightarrow x^2 + y^2 = 4$

$$w = z + \frac{4}{z}$$

$$\Rightarrow w = \frac{z^2 + 4}{z}$$

$$\Rightarrow w = \frac{(x + \mathrm{i}y)^2 + 4}{x + \mathrm{i}y}$$

$$\Rightarrow w = \frac{x^2 + 2xy\mathrm{i} - y^2 + 4}{x + \mathrm{i}y}$$

$$\Rightarrow w = \frac{[(x^2 - y^2 + 4) + \mathrm{i}(2xy)]}{x + \mathrm{i}y}$$

$$\Rightarrow w = \frac{[(x^2 - y^2 + 4) + \mathrm{i}(2xy)]}{(x + \mathrm{i}y)} \times \frac{(x - \mathrm{i}y)}{(x - \mathrm{i}y)}$$

$$\Rightarrow w = \frac{x^3 - xy^2 + 4x + 2xy^2 + \mathrm{i}(2x^2y - x^2y + y^3 - 4y)}{x^2 + y^2}$$

$$\Rightarrow w = \left(\frac{x^3 + xy^2 + 4x}{x^2 + y^2}\right) + \mathrm{i}\left(\frac{y^3 + x^2y - 4y}{x^2 + y^2}\right)$$

$$\Rightarrow w = \frac{x(x^2 + y^2 + 4)}{x^2 + y^2} + \frac{\mathrm{i}y(x^2 + y^2 - 4)}{x^2 + y^2}$$

Apply $x^2 + y^2 + 4 \Rightarrow w = \frac{x(4 + 4)}{4} + \frac{\mathrm{i}y(4 - 4)}{4}$

$$\Rightarrow w = \frac{8x}{4} + \frac{\mathrm{i}y(0)}{4}$$

$$\Rightarrow w = 2x + 0\mathrm{i}$$

$$\Rightarrow u + \mathrm{i}v = 2x + 0\mathrm{i}$$

$$\Rightarrow u = 2x, v = 0$$

As $|z| = 2 \Rightarrow -2 \leqslant x \leqslant 2$

So $-4 \leqslant 2x \leqslant 4$

and $-4 \leqslant u \leqslant 4$

Therefore the transformation T maps the points on a circle $|z| = 2$ in the z-plane to points in the interval [–4, 4] on the real axis in the w-plane. Hence $k = 4$.

16 $T: w=\dfrac{1}{z+3}, z\neq -3$

Line with equation $2x-2y+7=0$ in the z-plane

$$w=\frac{1}{z+3}$$
$$\Rightarrow w(z+3)=1$$
$$\Rightarrow wz+3w=1$$
$$\Rightarrow wz=1-3w$$
$$\Rightarrow z=\frac{1-3w}{w}$$
$$\Rightarrow z=\frac{1-3(u+\mathrm{i}v)}{u+\mathrm{i}v}$$
$$\Rightarrow z=\frac{1-3u-3\mathrm{i}v}{u+\mathrm{i}v}$$
$$\Rightarrow z=\frac{[(1-3u)-(3v)\mathrm{i}]}{(u+\mathrm{i}v)}\times\frac{(u-\mathrm{i}v)}{(u-\mathrm{i}v)}$$
$$\Rightarrow z=\frac{(1-3u)u-3v^2-\mathrm{i}v(1-3u)-\mathrm{i}(3uv)}{u^2+v^2}$$
$$\Rightarrow z=\frac{u-3u^2-3v^2}{u^2+v^2}+\frac{\mathrm{i}(-v+3uv-3uv)}{u^2+v^2}$$
$$\Rightarrow z=\frac{u-3u^2-3v^2}{u^2+v^2}+\frac{\mathrm{i}(-v)}{u^2+v^2}$$
$$\text{So, } x+\mathrm{i}y=\frac{u-3u^2-3v^2}{u^2+v^2}+\frac{\mathrm{i}(-v)}{u^2+v^2}$$
$$\Rightarrow x=\frac{u-3u^2-3v^2}{u^2+v^2}$$
$$\text{and } y=\frac{-v}{u^2+v^2}$$

As $2x-2y+7=0$, then

$$2\left(\frac{u-3u^2-3v^2}{u^2+v^2}\right)-2\left(\frac{-v}{u^2+v^2}\right)+7=0$$
$$\Rightarrow \frac{2u-6u^2-6v^2}{u^2+v^2}+\frac{2v}{u^2+v^2}+7=0 \quad (\times(u^2+v^2))$$
$$\Rightarrow 2u-6u^2-6v^2+2v+7(u^2+v^2)=0$$
$$\Rightarrow 2u-6u^2-6v^2+2v+7u^2+7v^2=0$$
$$\Rightarrow u^2+2u+v^2+2v=0$$
$$\Rightarrow (u+1)^2-1+(v+1)^2-1=0$$
$$\Rightarrow (u+1)^2+(v+1)^2=2$$
$$\Rightarrow (u+1)^2+(v+1)^2=\left(\sqrt{2}\right)^2$$

16 (continued)

Therefore the transformation T maps the line $2x-2y+7=0$ in the z-plane to a circle C with centre (–1, –1), radius $\sqrt{2}$ in the w-plane.

Challenge

We know that the transformation T given by $w=az+b$ maps points $z_1=0$, $z_2=1$ and $z_3=1+\mathrm{i}$ to $w_1=2\mathrm{i}$, $w_2=3\mathrm{i}$ and $w_3=-1+3\mathrm{i}$ respectively.

Substitute z_1, w_1 into T to get $2\mathrm{i}=b$.

Next, substitute z_2, w_2 and b into T:

$3\mathrm{i}=a+2\mathrm{i}$

$a=\mathrm{i}$

Using z_3, w_3 we can check the result (although it is not necessary):

$az_3+b=\mathrm{i}(1+\mathrm{i})+2\mathrm{i}=\mathrm{i}-1+2\mathrm{i}=-1+3\mathrm{i}=w_3$

Thus T can be written as $w=\mathrm{i}z+2\mathrm{i}$