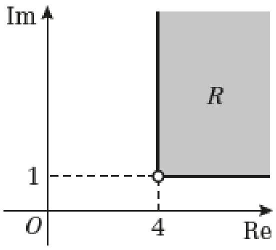
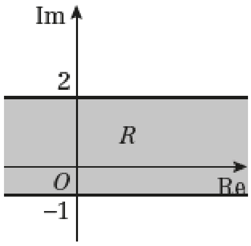
Exercise 4D

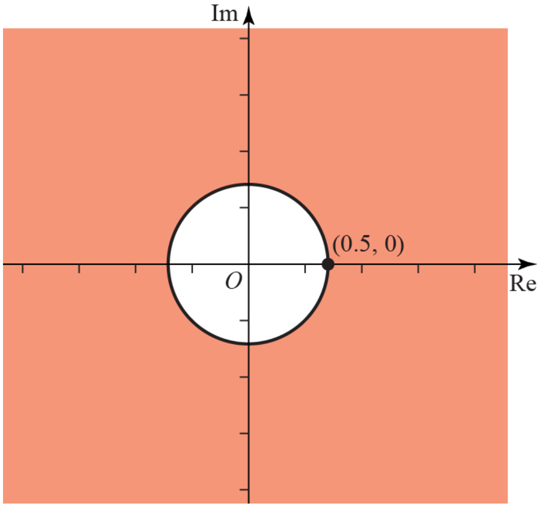
1 **a** The initial half-line goes through $z = 4 + \mathrm{i}$ and satisfies $\arg(z) = 0$ so the line is parallel to the real axis. The terminal half-line goes through z and satisfies $\arg(z) = \frac{\pi}{2}$, so it's perpendicular to the real axis. Because the inequalities are not strict, the half-lines are included in the region. Thus:



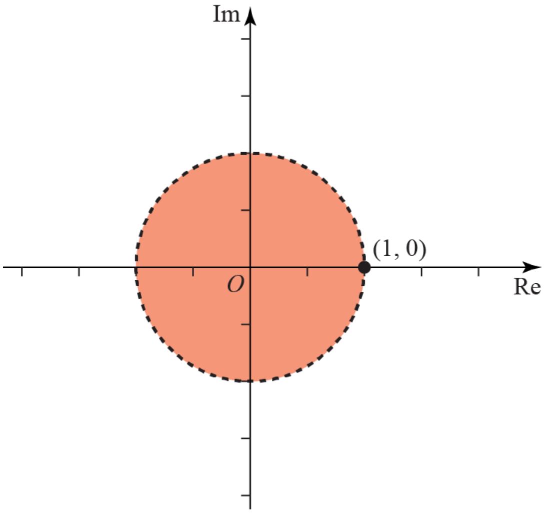
b $-1 \leqslant \mathrm{Im}(z) \leqslant 2$ describes two lines limiting the possible range of imaginary parts of z. The inequalities are not strict, so the half-lines are included in the region. Thus:



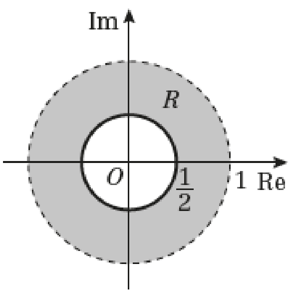
1 c $\frac{1}{2} \leqslant |z| < 1$. Each of these inequalities describes a circle centred at (0,0). $\frac{1}{2} \leqslant |z|$ gives the region outside of the circle centred at (0,0) with radius $r = \frac{1}{2}$, including the circle.



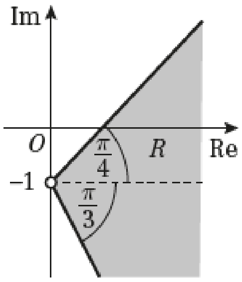
The second inequality, $|z| < 1$, describes the region inside the circle centred at (0,0) with radius $r = 1$ but excluding the circle itself, since the inequality is strict.



Thus the region described by $\frac{1}{2} \leqslant |z| < 1$ is the following:



1 d $-\frac{\pi}{3} \leqslant \arg(z+i) \leqslant \frac{\pi}{4}$ describes the region between and including two half-lines. The initial one goes through $z=-i$ and satisfies $\arg(z+i) \leqslant -\frac{\pi}{3}$. The terminal half-line also goes through $z=-i$ and satisfies $\arg(z+i) \leqslant \frac{\pi}{4}$.



where $\sphericalangle BAC = \frac{\pi}{4}$ and $\sphericalangle CAD = \frac{\pi}{3}$

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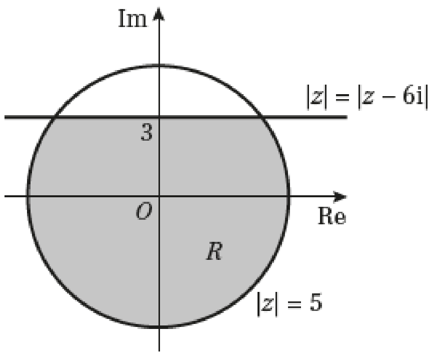
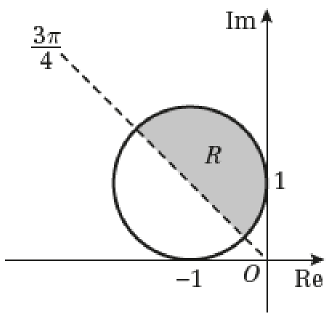


$|z| \leqslant 5$
$|z| \leqslant |z-6\text{i}|$

$|z| = 5$ represents a circle centre $(0, 0)$, radius 5

$|z| = |z-6\text{i}|$ represents a perpendicular bisector of the line joining $(0, 0)$, to $(0, 6)$ and has the equation $y = 3$.

3 $|z+1-\text{i}| \leqslant 1 \qquad |z-(-1+\text{i})| \leqslant 1$



Inside of a circle centre $(-1, 1)$ radius 1

$\arg z = \frac{3\pi}{4}$ is a half-line with equation $y=-x$, which goes through the centre of the circle, $(-1, 1)$.

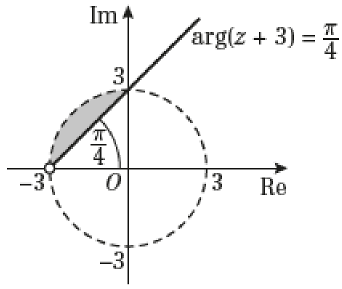
4 $|z| \leqslant 3$ and $\frac{\pi}{4} \leqslant \arg(z+3) \leqslant \pi$

$|z| = 3$ represents a circle centre (0, 0) radius 3.

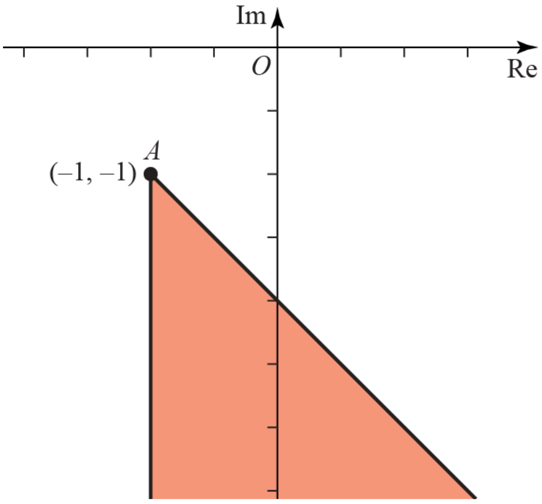
$\arg(z+3) = \frac{\pi}{4}$ is a half-line with equation $y - 0 = 1(x+3) \Rightarrow y = x+3, x > 0$.

Note it passes through the points (–3, 0) and (0, 3).

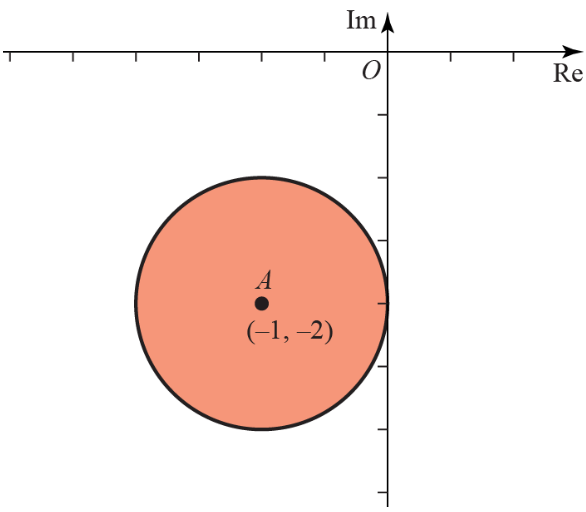
$\arg(z+3) = \pi$ is a half-line with equation $y = 0, x < -3$.



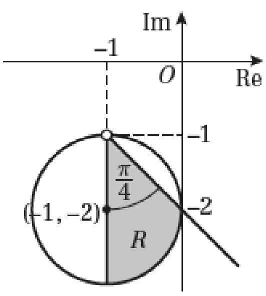
5 a The first region, $\left\{z \in \mathbb{R} : -\frac{\pi}{2} \leqslant \arg(z+1+\mathrm{i}) \leqslant -\frac{\pi}{4}\right\}$, describes all numbers lying between the two half-lines going through $z = -1 - \mathrm{i}$. The inequalities are not strict, so the half-lines are included in the region. The initial half-line satisfies $-\frac{\pi}{2} \leqslant \arg(z+1+\mathrm{i})$. The terminal half-line satisfies $\arg(z+1+\mathrm{i}) \leqslant -\frac{\pi}{4}$. Thus we have



The second region, $\{z \in \mathbb{R} : |z+1+2\mathrm{i}| \leqslant 1\}$, describes the inside of the circle centred at $(-1, -2)$ with radius $r = 1$ and includes the circle itself:



Thus the region inside both of the regions described above is as follows



5 b The first region describes a circle but we need to algebraically work out its radius and centre. To that end, represent z in real and imaginary parts and square both sides:

$z = x + y\text{i}$

$2|z-6| \leqslant |z-3|$

$2|x-6+y\text{i}| \leqslant |x-3+y\text{i}|$

$2\sqrt{(x-6)^2+y^2} \leqslant \sqrt{(x-3)^2+y^2}$

$4\left[(x-6)^2+y^2\right] \leqslant (x-3)^2+y^2$

$4\left[x^2-12x+36+y^2\right] \leqslant (x-3)^2+y^2$

$4x^2-48x+144+4y^2 \leqslant x^2-6x+9+y^2$

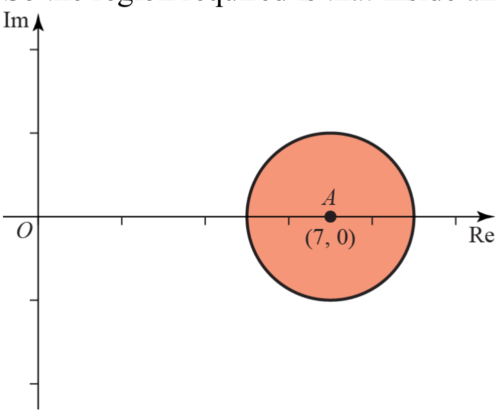
$3x^2-42x+3y^2+135 \leqslant 0$

$x^2-14x+y^2+45 \leqslant 0$

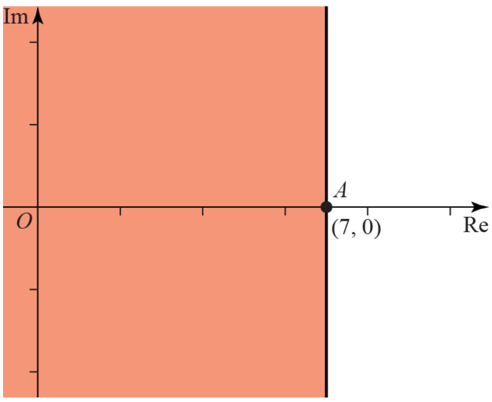
$(x-7)^2-49+y^2+45 \leqslant 0$

$(x-7)^2+y^2 \leqslant 4$

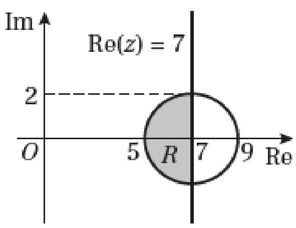
So the region required is that inside and including the circle centred at (7,0) with radius 2:



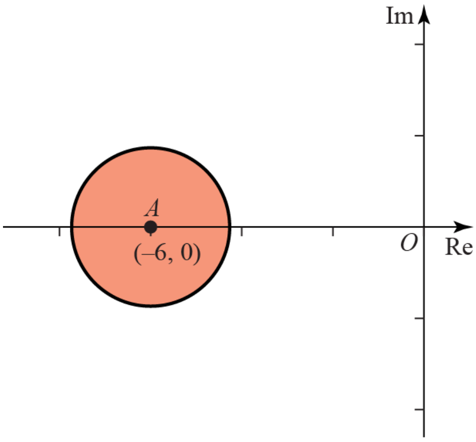
Now, the second regions describes all complex numbers whose real part is less than or equal to 7:



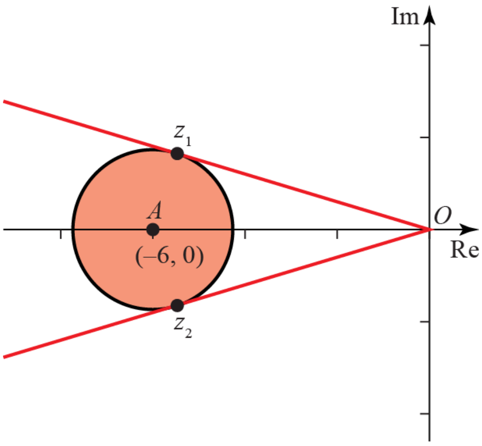
Numbers lying in both of these regions simultaneously are shown on the diagram below:



6 a The region $|z+6| \leqslant 3$ describes the inside of the circle centred at (−6, 0) with radius $r=3$ and includes the circle itself:



b Numbers z satisfying $|z+6| \leqslant 3$ lie in the region shaded above. The numbers with smallest and largest argument lie on the intersections of lines going through the origin and tangential to the circle:



Since $\angle AZ_1O = \frac{\pi}{2}$, we know that $\sin\theta = \frac{3}{6} = \frac{1}{2}$ where $\theta = Z_1\hat{O}A$.

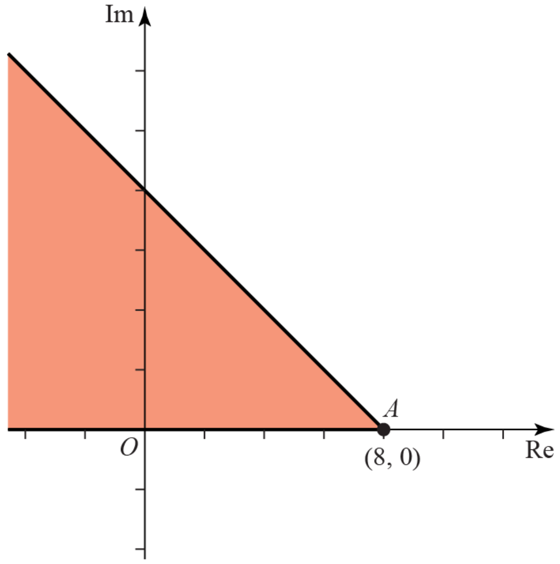
Hence $\theta = \frac{\pi}{6}$, and $\arg(z_1) = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$.

By symmetry, $\arg(z_2) = \arg(z_1) + 2\theta = \frac{7\pi}{6}$

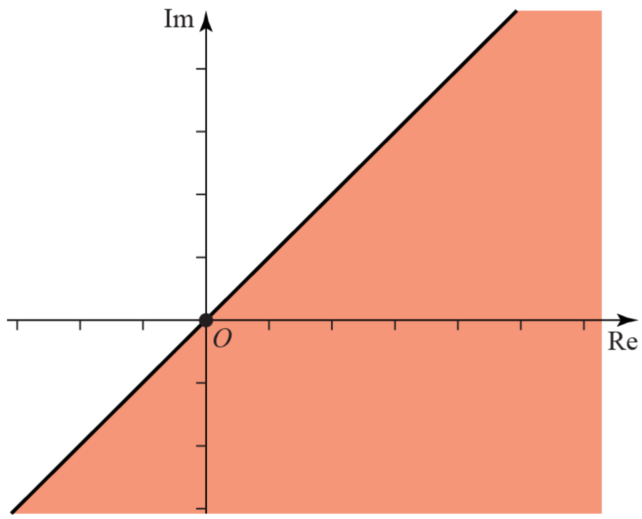
Thus for any z satisfying $|z+6| \leqslant 3$ we have $\frac{5\pi}{6} \leqslant \arg(z) \leqslant \frac{7\pi}{6}$

7 a $\frac{3\pi}{4} \leqslant \arg(z-8) \leqslant \pi$ describes the region between and including two half-lines going through (8,0). The initial half-line satisfies $\frac{3\pi}{4} = \arg(z-8)$ and the terminal one satisfies $\arg(z-8) = \pi$.

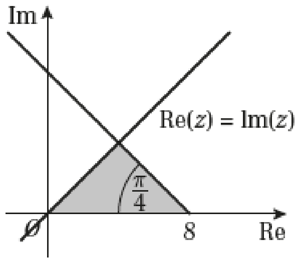
Thus:



$\text{Im}(z) \leqslant \text{Re}(z)$ describes numbers whose imaginary part is less than or equal to their real part:



Numbers that belong to both these regions are shown on the diagram below:



b The two lines above intersect at (4,4) creating a triangle with height $h = 4$ and base $a = 8$. Thus the area of that region is $\text{Area} = \frac{1}{2} \times 8 \times 4 = 16$.

8 a $\{z : |z-3+2\mathrm{i}| \geqslant \sqrt{2}|z-1|\}$ describes a circle and we need to algebraically find its centre and radius. Write $z = x + \mathrm{i}y$:

$|x-3+2\mathrm{i}+\mathrm{i}y| \geqslant \sqrt{2}|x-1+\mathrm{i}y|$

$\sqrt{(x-3)^2+(2+y)^2} \geqslant \sqrt{2}\cdot\sqrt{(x-1)^2+y^2}$

$x^2-6x+9+4+4y+y^2 \geqslant 2x^2-4x+2+2y^2$

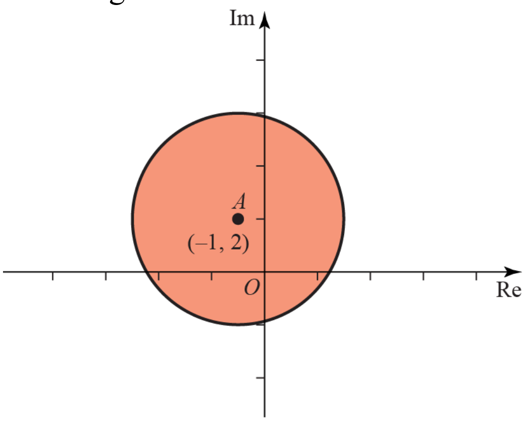
$x^2+2x+y^2-4y-11 \leqslant 0$

Complete the squares:

$(x+1)^2-1+(y-2)^2-4-11 \leqslant 0$

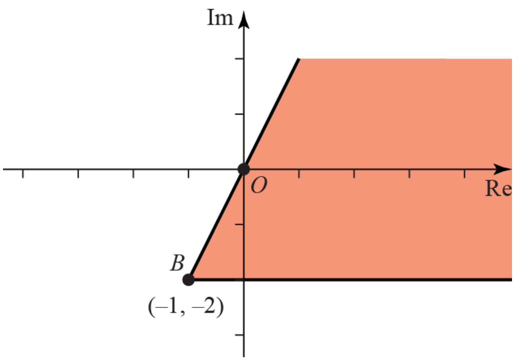
$(x+1)^2+(y-2)^2 \leqslant 16$

So the region described by this equation is the inside of the circle centred at (−1, 2) with radius $r = 4$ together with the circle.

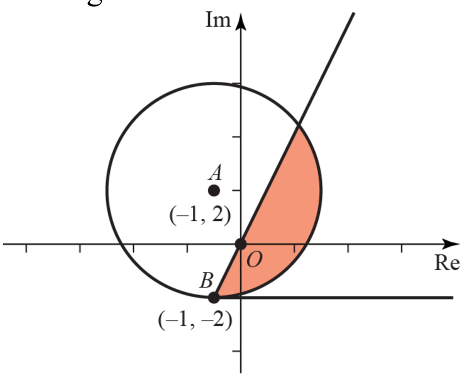


$\left\{z : 0 \leqslant \arg(z+1+2\mathrm{i}) \leqslant \frac{\pi}{3}\right\}$ describes the region between and including the two half-lines going through $z = -1-2\mathrm{i}$. The initial line satisfies $\arg(z+1+2\mathrm{i}) = 0$, so it is parallel to the real axis.

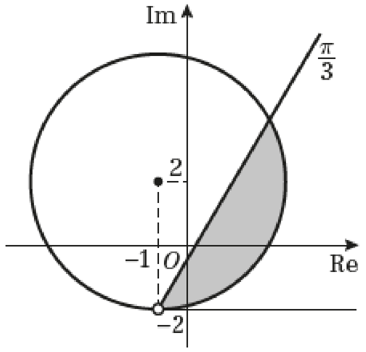
The terminal line satisfies $\arg(z+1+2\mathrm{i}) = \frac{\pi}{3}$:



Finding the intersection of the two regions gives:



8 b To find the area of the shaded region we first need to find the angle $C\hat{A}B = \theta$.



Since $D\hat{B}O = \frac{\pi}{3}$ and $D\hat{B}A = \frac{\pi}{2}$, we have that $O\hat{B}A = \frac{\pi}{6}$. Since the triangle ABC is isosceles, $B\hat{C}A = \frac{\pi}{6}$ as well and therefore $C\hat{A}B = \frac{2\pi}{3}$. Thus the area can be calculated as

$$\text{Area} = \frac{r^2}{2}(\theta - \sin\theta) = \frac{16}{2}\left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right) = \frac{16\pi}{3} - 4\sqrt{3}.$$

c The point with the largest imaginary value lies where the line BO intersects the circle, i.e. point C. Line BO satisfies $y = 2x$. Substituting this into the equation of the circle gives:

$$x^2 + 2x + 4x^2 - 8x - 11 = 0$$

$$5x^2 - 6x - 11 = 0$$

$$5(x+1)\left(x - \tfrac{11}{5}\right) = 0$$

The point with $x = -1$ is represented by B, so we are interested in the point with $x = \frac{11}{5}$ and $y = \frac{22}{5}$.

Thus the maximum value of $\text{Im}(z) = \frac{22}{5}$

Challenge

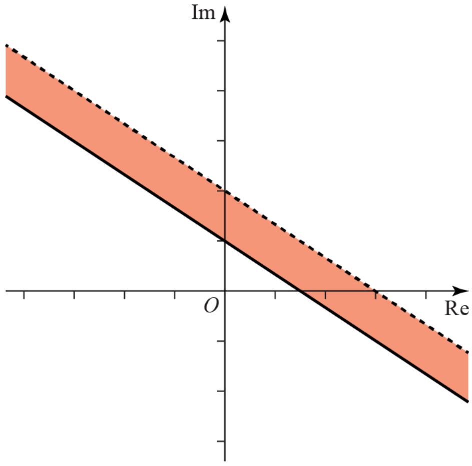
We want to find the region defined by $\{z \in \mathbb{R} : 6 \leqslant \text{Re}((2-3\text{i})z) < 12\}$. Write $z = x + \text{i}y$. Then:

$6 \leqslant \text{Re}((2-3\text{i})(x+\text{i}y)) < 12$

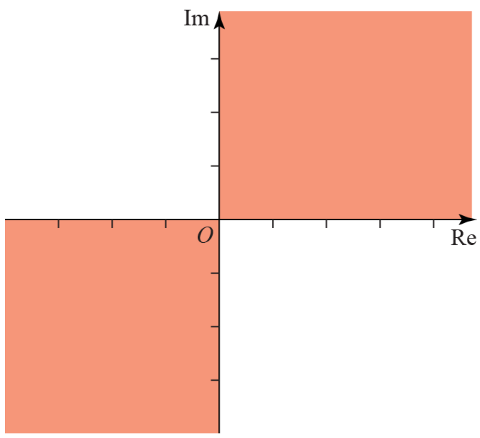
$6 \leqslant \text{Re}(2x+3y+\text{i}(2y-3x)) < 12$

$6 \leqslant 2x+3y < 12$

So the initial line is described by $6 \leqslant 2\,\text{Re}(z) + 3\,\text{Im}(z)$. Rearranging we get $\text{Im}(z) \geqslant 2 - \frac{2}{3}\text{Re}(z)$. Note that the inequality is not strict, so the line will be included in the region. Similarly, for the other inequality we get $\text{Im}(z) < 4 - \frac{2}{3}\text{Re}(z)$. Here the inequality is strict, so the line will not be included in the region:



$\{z \in \mathbb{R} : (\text{Re}\,z)(\text{Im}\,z) \geqslant 0\}$. For a product of two numbers to be positive, they both need to be positive, or both need to be negative. Hence this region looks as follows:



Since the inequality is not strict, both axes are included in the region.
The intersection of the two regions is as follows: