Further Pure Maths 2   Solution Bank   **P** **Pearson**

Chapter review 3

1 a We have $e^{i\theta} = \cos\theta + i\sin\theta$ hence

$\frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right) = \frac{1}{2}\left(\cos\theta + i\sin\theta + \cos(-\theta) + i\sin(-\theta)\right) = \frac{1}{2}\left(\cos\theta + i\sin\theta + \cos\theta - i\sin\theta\right) = \cos\theta$

Where we have used the fact that

$\cos\theta = \cos(-\theta)$

$\sin\theta = -\sin(-\theta)$

b We have

$\cos A\cos B = \frac{1}{2}\left(e^{iA} + e^{-iA}\right) \times \frac{1}{2}\left(e^{iB} + e^{-iB}\right) = \frac{1}{4}\left(e^{i(A+B)} + e^{i(A-B)} + e^{i(B-A)} + e^{-i(A+B)}\right)$

$= \frac{1}{4}\left(\cos(A+B) + i\sin(A+B) + \cos(A-B) + i\sin(A-B) + \cos(B-A) + i\sin(B-A) + \cos(-(A+B)) + i\sin(-(A+B))\right)$

$= \frac{1}{4}\left(2\cos(A+B) + 2\cos(A-B)\right) = \dfrac{\cos(A+B) + \cos(A-B)}{2}$

2 We want to prove by induction that if $z = r\left(\cos\theta + i\sin\theta\right)$ then for all $n \in$

$z^n = r^n\left(\cos n\theta + i\sin n\theta\right)$ clearly the base case $n = 1$ is true, now assume the statement is
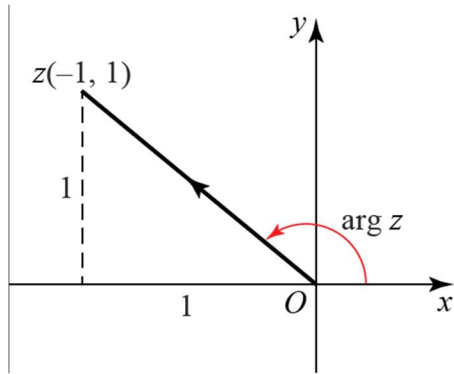
true for $n = k$ then we have

$z^{k+1} = z \times z^k = r\left(\cos\theta + i\sin\theta\right) \times r^k\left(\cos k\theta + i\sin k\theta\right) = r^{k+1}\left(\cos\theta + i\sin\theta\right)\left(\cos k\theta + i\sin k\theta\right)$

$= r^{k+1}\left(\cos\theta\cos k\theta - \sin k\theta\sin\theta + i\left(\cos\theta\sin k\theta + \cos k\theta\sin\theta\right)\right)$

$= r^{k+1}\left(\cos(k+1)\theta + i\sin(k+1)\theta\right)$

Hence the statement is true for $n = k+1$, this completes the induction.

3 $\dfrac{(\cos 3x + i\sin 3x)^2}{\cos x - i\sin x}$

$= \dfrac{(\cos 3x + i\sin 3x)^2}{\cos(-x) + i\sin(-x)}$

$= \dfrac{\cos 6x + i\sin 6x}{\cos(-x) + i\sin(-x)}$

$= \cos(6x - -x) + i\sin(6x - -x)$

$= \cos 7x + i\sin 7x$

4 a $(-1+i)^8$

If $z = -1+i,$ then

$r = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$

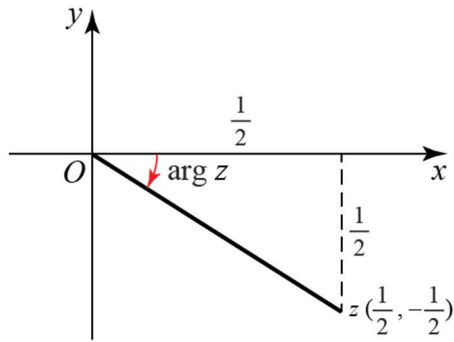$\theta = \arg z = \pi - \tan^{-1}\left(\dfrac{1}{1}\right) = \pi - \dfrac{\pi}{4} = \dfrac{3\pi}{4}$

So, $-1+i = \sqrt{2}\left(\cos\dfrac{3\pi}{4} + i\sin\dfrac{3\pi}{4}\right)$

$\therefore (-1+i)^8 = \left[\sqrt{2}\left(\cos\dfrac{3\pi}{4} + i\sin\dfrac{3\pi}{4}\right)\right]^8$

$= (\sqrt{2})^8 \left(\cos\dfrac{24\pi}{4} + i\sin\dfrac{24\pi}{4}\right)$

$= 16(\cos 6\pi + i\sin 6\pi)$

$= 16(1 + i(0))$

Therefore, $(-1+i)^8 = 16$

4 b $\dfrac{1}{\left(\frac{1}{2}-\frac{1}{2}i\right)^{16}}=\left(\dfrac{1}{2}-\dfrac{1}{2}i\right)^{-16}$

Let $z=\dfrac{1}{2}-\dfrac{1}{2}i$, then

$r=\sqrt{\left(\dfrac{1}{2}\right)^{2}+\left(-\dfrac{1}{2}\right)^{2}}=\sqrt{\dfrac{1}{4}+\dfrac{1}{4}}=\sqrt{\dfrac{1}{2}}=\dfrac{1}{\sqrt{2}}$

$\theta=\tan^{-1}\left(\dfrac{\frac{1}{2}}{\frac{1}{2}}\right)=-\dfrac{\pi}{4}$

So $\dfrac{1}{2}-\dfrac{1}{2}i=\dfrac{1}{\sqrt{2}}\left[\cos\left(-\dfrac{\pi}{4}\right)+i\sin\left(-\dfrac{\pi}{4}\right)\right]$

$\left(\dfrac{1}{2}-\dfrac{1}{2}i\right)^{-16}=\left[\dfrac{1}{\sqrt{2}}\left(\cos\left(-\dfrac{\pi}{4}\right)+i\sin\left(-\dfrac{\pi}{4}\right)\right)\right]^{-16}$

$=\left(2^{-\frac{1}{2}}\right)^{-16}\left(\cos\left(\dfrac{16\pi}{4}\right)+i\sin\left(\dfrac{16\pi}{4}\right)\right)$

$=2^{8}(\cos 4\pi+i\sin 4\pi)$

$=256(1+i(0))$

$=256$

Therefore, $\dfrac{1}{\left(\frac{1}{2}-\frac{1}{2}i\right)^{16}}=256$

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5 a $z = \cos\theta + i\sin\theta$

$$z^n = (\cos\theta + i\sin\theta)^{-n}$$
$$= \cos n\theta + i\sin n\theta$$

> de Moivre's Theorem.

$$\frac{1}{z^n} = z^{-n} = (\cos\theta + i\sin\theta)^n$$
$$= \cos(-n\theta) + i\sin(-n\theta)$$
$$= \cos n\theta - i\sin n\theta$$

> de Moivre's Theorem.

> $\cos(-n\theta) = \cos n\theta$
> $\sin(-n\theta) = -\sin n\theta$

Therefore $z^n + \dfrac{1}{z^n} = \cos n\theta + i\sin n\theta + \cos n\theta - i\sin n\theta$

i.e. $z^n + \dfrac{1}{z^n} = 2\cos n\theta$ (as required)

b $\left(z^2 + \dfrac{1}{z^2}\right)^3 = (z^2)^3 + {}^3C_1(z^2)^2\left(\dfrac{1}{z^2}\right) + {}^3C_2(z^2)\left(\dfrac{1}{z^2}\right)^2 + \left(\dfrac{1}{z^2}\right)^3$

$$= z^6 + 3z^4\left(\dfrac{1}{z^2}\right) + 3z^2\left(\dfrac{1}{z^4}\right) + \left(\dfrac{1}{z^6}\right)$$

$$= z^6 + 3z^2 + \dfrac{3}{z^2} + \dfrac{1}{z^6}$$

$$= \left(z^6 + \dfrac{1}{z^6}\right) = 3\left(z^2 + \dfrac{1}{z^2}\right)$$

$$= 2\cos 6\theta + 3(2)\cos 2\theta$$

$$= 2\cos 6\theta + 6\cos 2\theta$$

Hence, $\left(z^2 + \dfrac{1}{z^2}\right)^3 = 2\cos 6\theta + 6\cos 2\theta$

c $\left(z^2 + \dfrac{1}{z^2}\right)^3 = (2\cos 2\theta)^3 = 8\cos^3 2\theta = 2\cos 6\theta + 6\cos 2\theta$

$\therefore \cos^3 2\theta = \frac{2}{8}\cos 6\theta + \frac{6}{8}\cos 2\theta$

Hence, $\cos^3 2\theta = \frac{1}{4}\cos 6\theta + \frac{3}{4}\cos 2\theta$

So $a = \frac{1}{4}, b = \frac{3}{4}$

5 d $\displaystyle\int_0^{\frac{\pi}{6}} \cos^3 2\theta \, d\theta = \int_0^{\frac{\pi}{6}} \frac{1}{4}\cos 6\theta + \frac{3}{4}\cos 2\theta \, d\theta$

$$= \left[\frac{1}{24}\sin 6\theta + \frac{3}{8}\sin 2\theta \right]_0^{\frac{\pi}{6}}$$

$$= \left(\frac{1}{24}\sin \pi + \frac{3}{8}\sin\left(\frac{\pi}{3}\right) \right) - \left(\frac{1}{24}\sin 0 + \frac{3}{8}\sin 0 \right)$$

$$= \left(\frac{1}{24}(0) + \frac{3}{8}\left(\frac{\sqrt{3}}{2}\right) \right) - (0)$$

$$= \frac{3}{16}\sqrt{3}$$

So, $\displaystyle\int_0^{\frac{\pi}{6}} \cos^3 2\theta \, d\theta = \frac{3}{16}\sqrt{3}$ and $k = \frac{3}{16}$

6 a We wish to show that $\cos^5\theta = \frac{1}{16}\left(\cos 5\theta + 5\cos 3\theta + 10\cos\theta\right)$

Starting with $\cos\theta = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)$ we have

$$\cos^5\theta = \frac{1}{32}\left(e^{i\theta} + e^{-i\theta}\right)^5 = \frac{1}{32}\left(e^{5i\theta} + 5e^{3i\theta} + 10e^{i\theta} + 10e^{-i\theta} + 5e^{-3i\theta} + e^{-5i\theta}\right)$$

$$= \frac{1}{32}\left(2\cos 5\theta + 10\cos 3\theta + 20\cos\theta\right) = \frac{1}{16}\left(\cos 5\theta + 5\cos 3\theta + 10\cos\theta\right)$$

b The area of the region is given by the integral

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^5 x \, dx = 2\int_0^{\frac{\pi}{2}} \cos^5 x \, dx$$

Using the first part of the question, this is equivalent to

$$\int_0^{\frac{\pi}{2}} \frac{1}{8}\left(\cos 5x + 5\cos 3x + 10\cos x\right) dx$$

$$= \left[\frac{1}{8}\left(\frac{1}{5}\sin 5x + \frac{5}{3}\sin 3x + 10\sin x\right) \right]_0^{\frac{\pi}{2}} = \frac{1}{8}\left(\frac{1}{5}\sin\frac{5\pi}{2} + \frac{5}{3}\sin\frac{3\pi}{2} + 10\sin\frac{\pi}{2}\right) = \frac{1}{8}\left(\frac{1}{5} - \frac{5}{3} + 10\right) = \frac{16}{15}$$

7 a We have $\sin\theta = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)$ so that

$$\sin^6\theta = -\frac{1}{64}\left(e^{i\theta} - e^{-i\theta}\right)^6$$

$$= -\frac{1}{64}\left(e^{6i\theta} - 6e^{4i\theta} + 15e^{2i\theta} - 20 + 15e^{2i\theta} - 6e^{4i\theta} + e^{6i\theta}\right)$$

$$= -\frac{1}{64}\left(2\cos 6\theta - 12\cos 4\theta + 30\cos 2\theta - 20\right) = -\frac{1}{32}\left(\cos 6\theta - 6\cos 4\theta + 15\cos 2\theta - 10\right)$$

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7 b Let $\alpha = \dfrac{\pi}{2} - \theta$ then recall that we have $\cos\alpha = \sin\theta$ hence we have

$$\cos^6\alpha = \sin^6\theta = -\frac{1}{32}\left(\cos 6\theta - 6\cos 4\theta + 15\cos 2\theta - 10\right)$$

$$= -\frac{1}{32}\left(\cos 6\left(\frac{\pi}{2} - \alpha\right) - 6\cos 4\left(\frac{\pi}{2} - \alpha\right) + 15\cos 2\left(\frac{\pi}{2} - \alpha\right) - 10\right)$$

$$= -\frac{1}{32}\left(\cos(3\pi - 6\alpha) - 6\cos(2\pi - 4\alpha) + 15\cos(\pi - \alpha) - 10\right)$$

$$= -\frac{1}{32}\left(-\cos 6\alpha - 6\cos 4\alpha - 15\cos\alpha - 10\right) = \frac{1}{32}\left(\cos 6\alpha + 6\cos 4\alpha + 15\cos\alpha + 10\right)$$

c We have $\displaystyle\int_0^a (\cos^6\theta + \sin^6\theta)\mathrm{d}\theta = \dfrac{5\pi}{32}$ using the previous two parts we can evaluate the integral as

$$\int_0^a (\cos^6\theta + \sin^6\theta)\mathrm{d}\theta = \int_0^a \frac{1}{32}(12\cos 4\theta + 20)\,\mathrm{d}\theta = \frac{1}{32}\left[3\sin 4\theta + 20\theta\right]_0^a$$

$$= \frac{1}{32}\left(3\sin 4a + 20a\right)$$

Hence we require $3\sin 4a + 20a = 5\pi$ and observe that $a = \frac{\pi}{4}$ solves this and this is the only
Solution since the function $3\sin 4a + 20a$ is increasing.

8 Let $z = \mathrm{e}^{\mathrm{i}\theta}$ then we have

$$\sin 6\theta = \frac{1}{2\mathrm{i}}\left(z^6 - z^{-6}\right) = \frac{1}{2\mathrm{i}}\left((\cos\theta + \mathrm{i}\sin\theta)^6 - (\cos\theta - \mathrm{i}\sin\theta)^6\right)$$

When we expand the right hand side the only terms that survive are odd powers of sin hence we get

$$= \frac{1}{2\mathrm{i}}\left(12\mathrm{i}\cos^5\theta\sin\theta - 40\mathrm{i}\cos^3\theta\sin^3 + 12\mathrm{i}\cos\theta\sin^5\theta\right)$$

$$= 6\cos^5\theta\sin\theta - 20\cos^3\theta\sin^3 + 6\cos\theta\sin^5\theta$$

$$= 2\sin\theta\cos\theta\left(3\cos^4\theta - 10\cos^2\theta\sin^2\theta + 3\sin^4\theta\right)$$

$$= \sin 2\theta\left(3\cos^4\theta - 10\cos^2\theta\sin^2\theta + 3\sin^4\theta\right)$$

$$= \sin 2\theta\left(3\cos^4\theta - 10\cos^2\theta(1 - \cos^2\theta) + 3(1 - \cos^2\theta)^2\right)$$

$$= \sin 2\theta\left(3\cos^4\theta - 10\cos^2\theta(1 - \cos^2\theta) + 3(1 - 2\cos^2\theta + \cos^4\theta)\right)$$

$$= \sin 2\theta\left(16\cos^4\theta - 16\cos^2\theta + 3\right)$$

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9   **a**   Let $z = e^{i\theta}$ then we have

$$\cos 5\theta = \frac{1}{2i}\left(z^5 + z^{-5}\right) = \frac{1}{2i}\left(\left(\cos\theta + i\sin\theta\right)^5 + \left(\cos\theta - i\sin\theta\right)^5\right)$$

Now note the only terms that survive are even powers of sin so this becomes

$$\cos 5\theta = \frac{1}{2i}\left(z^5 + z^{-5}\right) = \frac{1}{2i}\left(\left(\cos\theta + i\sin\theta\right)^5 + \left(\cos\theta - i\sin\theta\right)^5\right)$$

$$= \frac{1}{2i}\left(2i\cos^5\theta - 20i\cos^3\theta\sin^2\theta + 10i\cos\theta\sin^4\theta\right)$$

$$= \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$$

$$= \cos\theta\left(\cos^4\theta - 10\cos^2\theta\sin^2\theta + 5\sin^4\theta\right)$$

$$= \cos\theta\left(\cos^4\theta - 10\cos^2\theta\left(1 - \cos^2\theta\right) + 5\left(1 - \cos^2\theta\right)^2\right)$$

$$\cos\theta\left(\cos^4\theta - 10\cos^2\theta + 10\cos^4\theta + 5\left(1 - 2\cos^2\theta + \cos^4\theta\right)\right)$$

$$\cos\theta\left(16\cos^4\theta - 20\cos^2\theta + 5\right)$$

  b   Now consider the equation

$16x^5 - 20x^3 + 5x + 1 = 0$

We make the substitution $x = \cos\theta$ so the equation becomes

$16\cos^5\theta - 20\cos^3\theta + 5\cos\theta = -1$

i.e. $\cos 5\theta = -1$ for which the general solution is $5\theta = \pi + 2k\pi$ for $k \in \mathbb{R}$ we take 5 different values of θ

$$\theta_1 = \frac{\pi}{5}$$

$$\theta_2 = \frac{3\pi}{5}$$

$$\theta_3 = \pi$$

$$\theta_4 = \frac{7\pi}{5}$$

$$\theta_5 = \frac{9\pi}{5}$$

Let $x_i = \cos\theta_i$ so

$$x_1 = \cos\frac{\pi}{5} = \frac{1 + \sqrt{5}}{4} = 0.809$$

$$x_2 = \cos\frac{3\pi}{5} = \frac{1 - \sqrt{5}}{4} = -0.309$$

$$x_3 = \cos\pi = -1$$

$$x_4 = \cos\frac{7\pi}{5} = \frac{1 - \sqrt{5}}{4} = -0.309$$

$$x_5 = \cos\frac{9\pi}{5} = \frac{1 + \sqrt{5}}{4} = 0.809$$

Now we only have 3 distinct solutions so we still have to check this is all solutions, factorising out $x + 1$ gives

$$16x^5 - 20x^3 + 5x + 1 = (x + 1)\left(4x^2 - 2x - 1\right)^2 = 0$$

9 b (Continued)

And observe that

$$\left(x-\left(\frac{1-\sqrt5}{4}\right)\right)\left(x-\left(\frac{1+\sqrt5}{4}\right)\right)=x^2-\frac12x-\frac14=\frac14\left(4x^2-2x-1\right)$$

In other words the 3 distinct roots that we have obtained completely factorise the equation and so these three solutions are the only solutions.

Hence the three solutions are given by $-1,\dfrac{1+\sqrt5}{4}=0.809,\dfrac{1-\sqrt5}{4}=-0.309$

10 a Let $z=\mathrm e^{\mathrm i\theta}=\cos\theta+\mathrm i\sin\theta$ then $\sin\theta=\dfrac{1}{2\mathrm i}\left(z-z^{-1}\right)$ so we have

$$\sin^5\theta=\frac{1}{32\mathrm i}\left(z-z^{-1}\right)^5=\frac{1}{32\mathrm i}\left(z^5-5z^3+10z-10z^{-1}+5z^{-3}-z^{-5}\right)$$

$$=\frac{1}{32\mathrm i}\left(2\mathrm i\sin5\theta-10\mathrm i\sin3\theta+20\mathrm i\sin\theta\right)=\frac{1}{16}\left(\sin5\theta-5\sin3\theta+10\sin\theta\right)$$

b We wish to solve
$\sin5\theta-5\sin3\theta+9\sin\theta=0$
Which we can rearrange to give
$\sin5\theta-5\sin3\theta+10\sin\theta=\sin\theta$
So $16\sin^5\theta=\sin\theta$
So we either have $\sin\theta=0$ in which case $\theta=0$ or $16\sin^4\theta=1$ hence $\sin\theta=\pm\frac12$ and

on $0\leqslant\theta<\pi$ the solutions correspond to $\theta=\dfrac{\pi}{6},\dfrac{5\pi}{6}$

11 a $(\cos\theta+\mathrm i\sin\theta)^5=\cos5\theta+\mathrm i\sin5\theta$ ┌──────────────────────┐ de Moivre's Theorem.

$$=\cos^5\theta+{}^5C_1\cos^4\theta(\mathrm i\sin\theta)+{}^5C_2\cos^3\theta(\mathrm i\sin\theta)^2$$

$$+{}^5C_3\cos^2\theta(\mathrm i\sin\theta)^3+{}^5C_4\cos^3\theta(\mathrm i\sin\theta)^4+\left(\mathrm i\sin\theta\right)^5$$

$$=\cos^5\theta+5\mathrm i\cos^4\theta\sin\theta+10\mathrm i^2\cos^3\theta\sin^2\theta$$ Binomial expansion.

$$+10\mathrm i^3\cos^2\theta\sin^3\theta+5\mathrm i^4\cos\theta\sin^4\theta+\mathrm i^5\sin^5\theta$$

Hence,
$$\cos5\theta+\mathrm i\sin5\theta=\cos^5\theta+5\mathrm i\cos^4\theta\sin\theta-10\cos^3\theta\sin\theta$$
$$-10\mathrm i\cos^2\theta\sin^3\theta+5\cos\theta\sin^4\theta+\mathrm i\sin^5\theta$$

Equating the real parts gives,
$$\cos5\theta=\cos^5\theta-10\cos^3\theta\sin^2\theta+5\cos\theta\sin^4\theta$$
$$=\cos\theta(\cos^4\theta-10\cos^2\theta\sin^2\theta+5\sin^4\theta)$$
$$=\cos\theta(\cos^4\theta-10\cos^2\theta(1-\cos^2\theta)+5(1-\cos^2\theta)^2) \quad\longleftarrow\quad \text{Applying}$$
$$=\cos\theta(\cos^4\theta-10\cos^2\theta+10\cos^4\theta+5(1-2\cos^2\theta+\cos^4\theta)) \quad \sin^2\theta=1-\cos^2\theta.$$
$$=\cos\theta(\cos^4\theta-10\cos^2\theta+10\cos^4\theta+5-10\cos^2\theta+5\cos^4\theta)$$
$$=\cos\theta(16\cos^4\theta-20\cos^2\theta+5)$$

Hence, $\cos5\theta=\cos\theta(16\cos^4\theta-20\cos^2\theta+5)$ (as required)

11 b $\cos 5\theta = 0$

$$\alpha = \frac{\pi}{2}$$
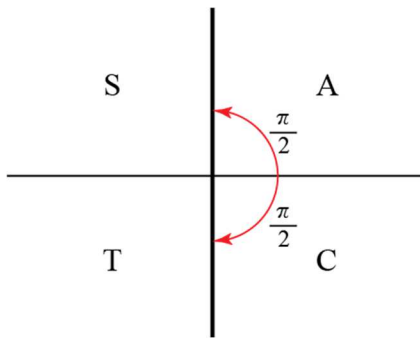
So $5\theta = \left\{ \dfrac{\pi}{2}, \dfrac{3\pi}{2}, \dfrac{5\pi}{2}, \dfrac{7\pi}{2}, \dfrac{9\pi}{2} \right\}$

$$\theta = \left\{ \frac{\pi}{10}, \frac{3\pi}{10}, \frac{5\pi}{10}, \frac{7\pi}{10}, \frac{9\pi}{10} \right\}$$

$$\theta = \left\{ \frac{\pi}{10}, \frac{3\pi}{10}, \frac{5\pi}{10}, \frac{7\pi}{10}, \frac{9\pi}{10} \right\} \quad \text{for } 0 < \theta \leqslant \pi$$

$\cos 5\theta = 0 \Rightarrow \cos\theta(16\cos^4\theta - 20\cos^2\theta + 5) = 0$

Five solutions must come from: $\cos\theta(16\cos^4\theta - 20\cos^2\theta + 5) = 0$

Solution **(1)** $\cos\theta = 0$

$$\alpha = \frac{\pi}{2}$$

For $0 < \theta \leqslant \pi$, $\theta = \dfrac{\pi}{2}$ (as found earlier)

The final 4 solutions come from: $16\cos^4\theta - 20\cos^2\theta + 5 = 0$

$$\cos^2\theta = \frac{20 \pm \sqrt{400 - 4(16)(5)}}{32}$$
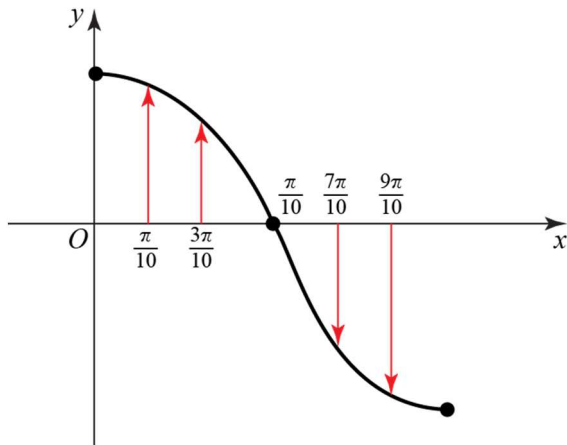
$$= \frac{20 \pm \sqrt{400 - 320}}{32}$$

$$= \frac{20 \pm \sqrt{80}}{32}$$

$$= \frac{20 \pm \sqrt{16}\sqrt{5}}{32}$$

$$= \frac{20 \pm 4\sqrt{5}}{32}$$

$$\therefore \cos^2\theta = \frac{5 + \sqrt{5}}{8}$$

11 b (Continued)

Due to symmetry and as $\cos\left(\dfrac{\pi}{10}\right) > \cos\left(\dfrac{3\pi}{10}\right)$

$$\cos^2\left(\frac{\pi}{10}\right) = \cos^2\left(\frac{9\pi}{10}\right) > \cos^2\left(\frac{3\pi}{10}\right) = \cos^2\left(\frac{7\pi}{10}\right)$$

$$\therefore \cos^2\left(\frac{\pi}{10}\right) = \frac{5+\sqrt{5}}{8}$$

c $\cos^2\left(\dfrac{3\pi}{10}\right) = \dfrac{5-\sqrt{5}}{8}$

$$\cos^2\left(\frac{7\pi}{10}\right) = \cos^2\left(\frac{3\pi}{10}\right) = \frac{5-\sqrt{5}}{8}$$

$$\cos^2\left(\frac{9\pi}{10}\right) = \cos^2\left(\frac{\pi}{10}\right) = \frac{5+\sqrt{5}}{8}$$

Therefore, $\cos^2\left(\dfrac{3\pi}{10}\right) = \dfrac{5-\sqrt{5}}{8}$, $\cos^2\left(\dfrac{7\pi}{10}\right) = \dfrac{5-\sqrt{5}}{8}$, $\cos^2\left(\dfrac{9\pi}{10}\right) = \dfrac{5+\sqrt{5}}{8}$

12 a   Let $z = \mathrm{e}^{\mathrm{i}\theta} = \cos\theta + \mathrm{i}\sin\theta$ then we have

$$\tan 3\theta = \frac{\sin 3\theta}{\cos 3\theta} = \frac{\dfrac{1}{2\mathrm{i}}\left(z^3 - z^{-3}\right)}{\dfrac{1}{2}\left(z^3 + z^{-3}\right)} = \frac{z^3 - z^{-3}}{\mathrm{i}\left(z^3 + z^{-3}\right)} = \frac{\left(\cos\theta + \mathrm{i}\sin\theta\right)^3 - \left(\cos\theta - \mathrm{i}\sin\theta\right)^3}{\mathrm{i}\left(\left(\cos\theta + \mathrm{i}\sin\theta\right)^3 + \left(\cos\theta - \mathrm{i}\sin\theta\right)^3\right)}$$

$$= \frac{6\mathrm{i}\cos^2\theta\sin\theta - 2\mathrm{i}\sin^3\theta}{\mathrm{i}\left(2\cos^3\theta - 6\cos\theta\sin^2\theta\right)} = \frac{6\cos^2\theta\sin\theta - 2\sin^3\theta}{2\cos^3\theta - 6\cos\theta\sin^2\theta}$$

Now divide top and bottom by $\cos^3\theta$ to give

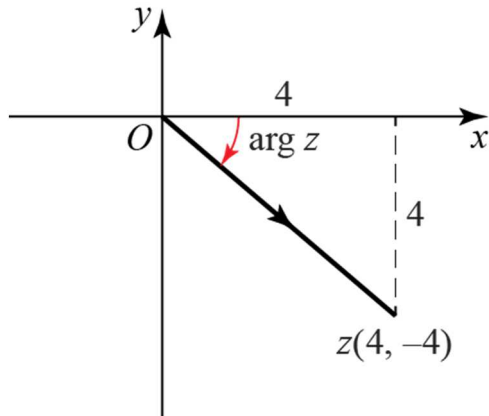$$\tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$$

12 b We have

$$\cot 3\theta = \frac{1}{\tan 3\theta} = \frac{1 - 3\tan^2 \theta}{3\tan \theta - \tan^3 \theta}$$

Now divide top and bottom by $\tan^3 \theta$ to give

$$\cot 3\theta = \frac{\cot^3 \theta - 3\cot \theta}{3\cot^2 \theta - 1}$$

13 a $4 - 4i$

modulus $r = \sqrt{4^2 + (-4)^2} = \sqrt{16 + 16} = \sqrt{32} = \sqrt{16}\sqrt{2} = 4\sqrt{2}$

argument $= \theta = -\tan^{-1}\left(\frac{4}{4}\right) = -\frac{\pi}{4}$

$\therefore 4 - 4i = 4\sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right)$

13 b $z^5 = 4 - 4i$

for $4 - 4i$, $r = 4\sqrt{2}$, $\theta = -\frac{\pi}{4}$

So, $z^5 = 4\sqrt{2}\, e^{i\left(-\frac{\pi}{4}\right)}$

$z^5 = 4\sqrt{2}\, e^{i\left(-\frac{\pi}{4} + 2k\pi\right)}$, $k \in \mathbb{R}$

Hence, $z = \left[4\sqrt{2}\, e^{i\left(-\frac{\pi}{4} + 2k\pi\right)}\right]^{\frac{1}{5}}$

$= \left(4\sqrt{2}\right)^{\frac{1}{5}} e^{i\left(\frac{-\frac{\pi}{4} + 2k\pi}{5}\right)}$

$= \sqrt{2}\, e^{i\left(-\frac{\pi}{20} + \frac{2k\pi}{5}\right)}$

> de Moivre's Theorem.

> $4\sqrt{2} = 2^{\frac{5}{2}}$
> So, $(4\sqrt{2})^{\frac{1}{5}} = (2^{\frac{5}{2}})^{\frac{1}{5}}$
> $= 2^{\frac{1}{2}} = \sqrt{2}$

13 b (continued)

$$k = 0, \ z_1 = \sqrt{2} \ e^{i\left(-\frac{\pi}{20}\right)}$$

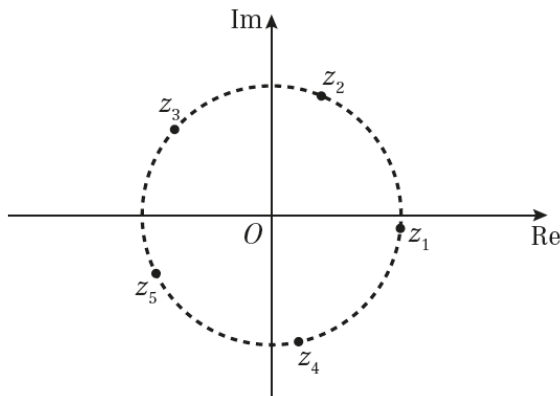$$k = 1, \ z_2 = \sqrt{2} \ e^{i\left(\frac{7\pi}{20}\right)}$$

$$k = 2, \ z_3 = \sqrt{2} \ e^{i\left(\frac{3\pi}{4}\right)}$$

$$k = -1, \ z_4 = \sqrt{2} \ e^{i\left(-\frac{9\pi}{20}\right)}$$

$$k = -2, \ z_5 = \sqrt{2} \ e^{i\left(-\frac{17\pi}{20}\right)}$$

Therefore, $z = \sqrt{2} \ e^{-\frac{\pi i}{20}}, \ \sqrt{2} \ e^{\frac{7\pi i}{20}}, \ \sqrt{2} \ e^{\frac{3\pi i}{4}}, \ \sqrt{2} \ e^{-\frac{9\pi i}{20}}, \ \sqrt{2} \ e^{-\frac{17\pi i}{20}}$
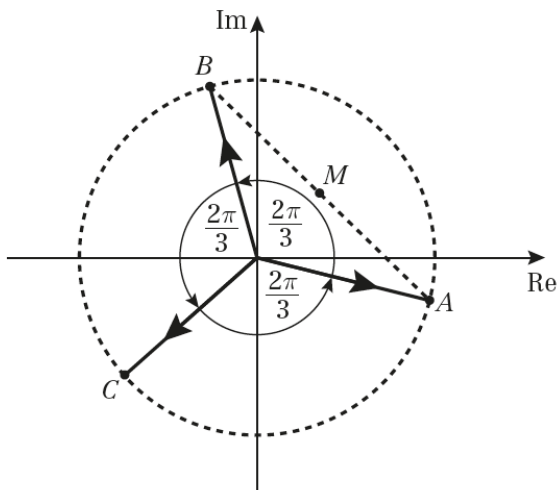
c

14 a   We have $z = 2 - 2i = 2\sqrt{2} e^{-\frac{i\pi}{4}}$, now suppose $\omega^3 = z$ then let $\omega = re^{i\theta}$ then we have $r^3 = \sqrt{8}$ so

$r = \left(\sqrt{8}\right)^{\frac{1}{3}} = \left(2^{\frac{3}{2}}\right)^{\frac{1}{3}} = \sqrt{2}$ and $3\theta = -\frac{\pi}{4} + 2k\pi$ hence three distinct roots are given by

$$\omega_1 = \sqrt{2} e^{-\frac{i\pi}{12}}$$

$$\omega_2 = \sqrt{2} e^{\frac{7i\pi}{12}}$$

$$\omega_1 = \sqrt{2} e^{-\frac{9i\pi}{12}}$$

14 b

c   We have $w = \dfrac{1}{2}\left(\sqrt{2}e^{\frac{7i\pi}{12}} + \sqrt{2}e^{-\frac{i\pi}{12}} \right)$ but by geometrical considerations the argument of w is just

$\dfrac{1}{2}\left(\dfrac{-\pi}{12} + \dfrac{7\pi}{12} \right) = \dfrac{3\pi}{12} = \dfrac{\pi}{4}$ and by considering the triangle with vertices at the origin, A, B we have

that the modulus is given by $\sqrt{2}\sin\dfrac{\pi}{6} = \dfrac{\sqrt{2}}{2}$ hence we have $w = \dfrac{\sqrt{2}}{2}e^{\frac{i\pi}{4}}$

d   We have

$$w^6 = \left(\dfrac{\sqrt{2}}{2}e^{\frac{i\pi}{4}} \right)^6 = \dfrac{8}{64}e^{\frac{6i\pi}{4}} = \dfrac{1}{8}e^{\frac{3i\pi}{2}} = \dfrac{1}{8}e^{\frac{-i\pi}{2}} = -\dfrac{i}{8}$$

Challenge

We consider the solutions of $\left(\dfrac{z+1}{z} \right)^6 = 1$, let $\omega = \dfrac{z+1}{z}$ then we have $\omega^6 = 1$ so $\omega = e^{\frac{2ki\pi}{6}}$ for

$k \in \mathbb{R}$ hence we have

$\dfrac{z+1}{z} = e^{\frac{2ki\pi}{6}}$

So

$z+1 = ze^{\frac{2ki\pi}{6}}$

So

$z\left(e^{\frac{2ki\pi}{6}} - 1 \right) = 1$

Therefore

$$z = \dfrac{1}{e^{\frac{ki\pi}{3}} - 1} = \dfrac{e^{-\frac{ki\pi}{3}} - 1}{\left(e^{\frac{ki\pi}{3}} - 1 \right)\left(e^{-\frac{ki\pi}{3}} - 1 \right)} = \dfrac{\cos\dfrac{k\pi}{3} - 1 - i\sin\dfrac{k\pi}{3}}{\left(\cos\dfrac{k\pi}{3} - 1 \right)^2 + \sin\dfrac{k\pi}{3}} = \dfrac{\cos\dfrac{k\pi}{3} - 1 - i\sin\dfrac{k\pi}{3}}{2 - 2\cos\dfrac{k\pi}{3}}$$

Challenge (continued)

Assuming $e^{\frac{2ki\pi}{6}} - 1 \neq 0$ i.e. $1 \leqslant k \leqslant 5$, when $k = 0$ the equation becomes $z = z + 1$ which has no solutions, however, for $1 \leqslant k \leqslant 5$ we get a different solution. So our six values of z are

$$z_0 = -\frac{1}{2}$$

$$z_1 = \frac{\cos\frac{\pi}{3} - 1 - i\sin\frac{\pi}{3}}{2 - 2\cos\frac{\pi}{3}} = -\frac{1}{2} - \frac{i\sqrt{3}}{2}$$

$$z_2 = \frac{\cos\frac{2\pi}{3} - 1 - i\sin\frac{2\pi}{3}}{2 - 2\cos\frac{2\pi}{3}} = -\frac{1}{2} - \frac{i\sqrt{3}}{6}$$

$$z_3 = \frac{\cos\pi - 1 - i\sin\pi}{2 - 2\cos\pi} = -\frac{1}{2}$$

$$z_4 = \frac{\cos\frac{4\pi}{3} - 1 - i\sin\frac{4\pi}{3}}{1 - 2\cos\frac{4\pi}{3}} = -\frac{1}{2} + \frac{i\sqrt{3}}{6}$$

$$z_5 = \frac{\cos\frac{5\pi}{3} - 1 - i\sin\frac{5\pi}{3}}{1 - 2\cos\frac{5\pi}{3}} = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$$

which all lie on the straight line $\text{Re}(z) = -\frac{1}{2}$