

Exercise 3C

$$1 \text{ a } (\cos \theta + i \sin \theta)^6 = \cos 6\theta + i \sin 6\theta$$

$$\text{b } (\cos 3\theta + i \sin 3\theta)^4 = \cos 12\theta + i \sin 12\theta$$

c

$$\begin{aligned} \left(\cos \frac{\pi}{6} + i \sin \frac{\pi}{6} \right)^5 &= \cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6} \\ &= -\frac{\sqrt{3}}{2} + \frac{i}{2} \end{aligned}$$

d

$$\begin{aligned} \left(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3} \right)^8 &= \cos \frac{8\pi}{3} + i \sin \frac{8\pi}{3} = \cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3} \\ &= -\frac{1}{2} + \frac{i\sqrt{3}}{2} \end{aligned}$$

e

$$\begin{aligned} \left(\cos \frac{2\pi}{5} + i \sin \frac{2\pi}{5} \right)^5 &= \cos \frac{10\pi}{5} + i \sin \frac{10\pi}{5} = \cos 2\pi + i \sin 2\pi \\ &= 1 \end{aligned}$$

f

$$\begin{aligned} \left(\cos \frac{\pi}{10} - i \sin \frac{\pi}{10} \right)^{15} &= \left(\cos \frac{-\pi}{10} + i \sin \frac{-\pi}{10} \right)^{15} = \cos \left(-\frac{3\pi}{2} \right) + i \sin \left(-\frac{3\pi}{2} \right) \\ &= i \end{aligned}$$

$$2 \text{ a } \frac{\cos 5\theta + i \sin 5\theta}{(\cos 2\theta + i \sin 2\theta)^2} = \frac{(\cos \theta + i \sin \theta)^5}{(\cos \theta + i \sin \theta)^4} = \cos \theta + i \sin \theta = e^{i\theta}$$

$$\text{b } \frac{(\cos 2\theta + i \sin 2\theta)^7}{(\cos 4\theta + i \sin 4\theta)^3} = \frac{(\cos \theta + i \sin \theta)^{14}}{(\cos \theta + i \sin \theta)^{12}} = (\cos \theta + i \sin \theta)^2 = e^{2i\theta}$$

$$\text{c } \frac{1}{(\cos 2\theta + i \sin 2\theta)^3} = \frac{1}{(\cos \theta + i \sin \theta)^6} = \frac{1}{e^{6i\theta}} = e^{-6i\theta}$$

$$\text{d } \frac{(\cos 2\theta + i \sin 2\theta)^4}{(\cos 3\theta + i \sin 3\theta)^3} = \frac{(\cos \theta + i \sin \theta)^8}{(\cos \theta + i \sin \theta)^9} = \frac{1}{\cos \theta + i \sin \theta} = \frac{1}{e^{i\theta}} = e^{-i\theta}$$

$$\text{e } \frac{\cos 5\theta + i \sin 5\theta}{(\cos 3\theta - i \sin 3\theta)^2} = \frac{(\cos \theta + i \sin \theta)^5}{(\cos(-3\theta) + i \sin(-3\theta))^2} = \frac{(\cos \theta + i \sin \theta)^5}{(\cos(-\theta) + i \sin(-\theta))^6} = \frac{e^{5i\theta}}{e^{-6i\theta}} = e^{11i\theta}$$

$$2 \text{ f } \frac{\cos \theta - i \sin \theta}{(\cos 2\theta - i \sin 2\theta)^3} = \frac{\cos(-\theta) + i \sin(-\theta)}{(\cos(-2\theta) + i \sin(-2\theta))^3} = \frac{e^{-i\theta}}{(e^{-i\theta})^3} = \frac{e^{-i\theta}}{e^{-3i\theta}} = e^{2i\theta}$$

3 a

$$\frac{\left(\cos \frac{7\pi}{13} + i \sin \frac{7\pi}{13}\right)^4}{\left(\cos \frac{4\pi}{13} - i \sin \frac{4\pi}{13}\right)^6} = \frac{\cos \frac{28\pi}{13} + i \sin \frac{28\pi}{13}}{\cos \frac{-24\pi}{13} + i \sin \frac{-24\pi}{13}} = \frac{e^{\frac{28\pi i}{13}}}{e^{\frac{-24\pi i}{13}}} = e^{\frac{52\pi i}{13}} = e^{4\pi i} = 1$$

b Recall the identity $\sin(\pi + x) = -\sin(\pi - x)$ then

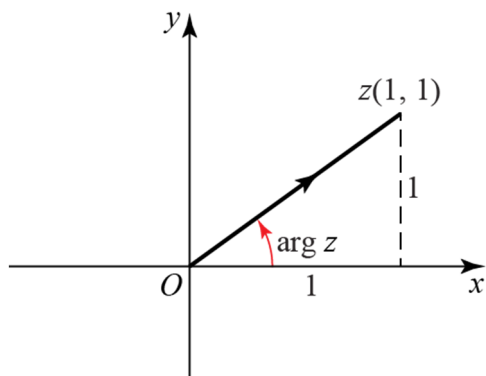
$$\frac{\left(\cos \frac{3\pi}{7} - i \sin \frac{11\pi}{7}\right)^3}{\left(\cos \frac{15\pi}{7} + i \sin \frac{\pi}{7}\right)^2} = \frac{\left(\cos \frac{3\pi}{7} + i \sin \frac{3\pi}{7}\right)^3}{\left(\cos \frac{\pi}{7} + i \sin \frac{\pi}{7}\right)^2} = \frac{\cos \frac{9\pi}{7} + i \sin \frac{9\pi}{7}}{\cos \frac{2\pi}{7} + i \sin \frac{2\pi}{7}} = \frac{e^{\frac{9\pi i}{7}}}{e^{\frac{2\pi i}{7}}} = e^{\frac{7\pi i}{7}} = e^{\pi i} = -1$$

c

$$\frac{\left(\cos \frac{4\pi}{3} - i \sin \frac{2\pi}{3}\right)^7}{\left(\cos \frac{10\pi}{3} + i \sin \frac{4\pi}{3}\right)^4} = \frac{\left(\cos \frac{4\pi}{3} + i \sin \frac{4\pi}{3}\right)^7}{\left(\cos \frac{4\pi}{3} + i \sin \frac{4\pi}{3}\right)^4} = \frac{\cos \frac{28\pi}{3} + i \sin \frac{28\pi}{3}}{\cos \frac{16\pi}{3} + i \sin \frac{16\pi}{3}} = \frac{e^{\frac{28\pi i}{3}}}{e^{\frac{16\pi i}{3}}} = e^{\frac{12\pi i}{3}} = e^{4\pi i} = 1$$

4 a $(1+i)^5$

If $z = 1+i$, then



$$r = \sqrt{1^2 + 1^2} = \sqrt{2}$$

$$\theta = \arg z = \tan^{-1}\left(\frac{1}{1}\right) = \frac{\pi}{4}$$

$$\text{So, } 1+i = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

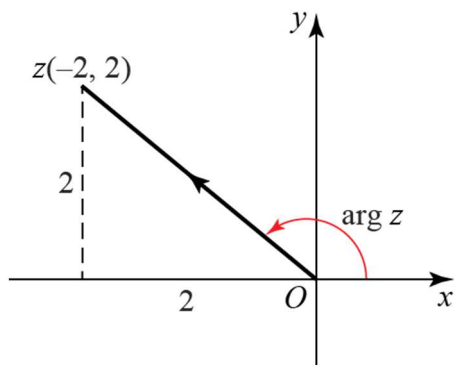
$$\begin{aligned} \therefore (1+i)^5 &= \left[\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\right]^5 \\ &= (\sqrt{2})^5\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right) \\ &= 4\sqrt{2}\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right) \\ &= 4\sqrt{2}\left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right) \\ &= -4 - 4i \end{aligned}$$

$$\begin{aligned} (\sqrt{2})^5 &= \sqrt{2}\sqrt{2}\sqrt{2}\sqrt{2}\sqrt{2} \\ &= 4\sqrt{2} \end{aligned}$$

Therefore, $(1+i)^5 = -4 - 4i$

4 b $(-2 + 2i)^8$

If $z = -2 + i$, then



$$r = \sqrt{(-2)^2 + 2^2} = \sqrt{4 + 4} = \sqrt{8} = \sqrt{4}\sqrt{2} = 2\sqrt{2}$$

$$\theta = \arg z = \pi - \tan^{-1}\left(\frac{2}{2}\right) = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$$

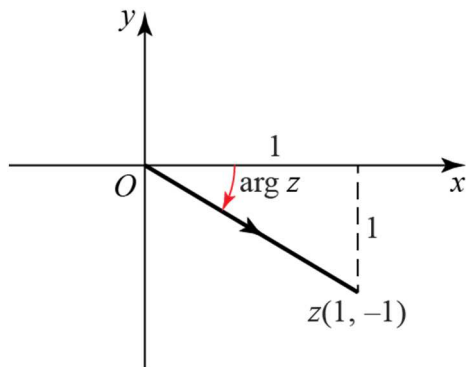
$$\text{So, } -2 + 2i = 2\sqrt{2} \left(\cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4} \right)$$

$$\begin{aligned} \therefore (-2 + 2i)^8 &= \left[2\sqrt{2} \left(\cos \left(\frac{3\pi}{4} \right) + i \sin \left(\frac{3\pi}{4} \right) \right) \right]^8 \\ &= (2\sqrt{2})^8 \left(\cos \left(\frac{24\pi}{4} \right) + i \sin \left(\frac{24\pi}{4} \right) \right) \\ &= (256)(16) (\cos 6\pi + i \sin 6\pi) \\ &= 4096(1 + i(0)) \\ &= 4096 \end{aligned}$$

Therefore, $(-2 + 2i)^8 = 4096$

4 c $(1-i)^6$

If $z = 1 - i$, then



$$r = \sqrt{1^2 + 1^2} = \sqrt{2}$$

$$\theta = \arg z = -\tan^{-1}\left(\frac{1}{1}\right) = -\frac{\pi}{4}$$

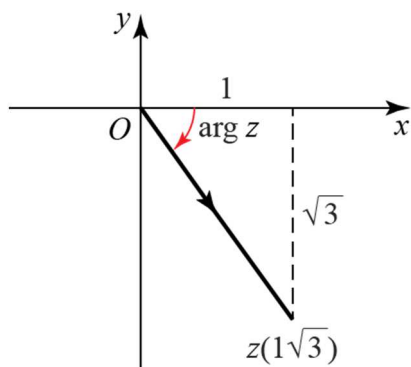
$$\text{So, } 1 - i = \sqrt{2} \left(\cos\left(-\frac{\pi}{4}\right) + i \sin\left(-\frac{\pi}{4}\right) \right)$$

$$\begin{aligned} \therefore (1-i)^6 &= \left[\sqrt{2} \left(\cos\left(-\frac{\pi}{4}\right) + i \sin\left(-\frac{\pi}{4}\right) \right) \right]^6 \\ &= (\sqrt{2})^6 \left(\cos\left(-\frac{6\pi}{4}\right) + i \sin\left(-\frac{6\pi}{4}\right) \right) \\ &= 8 \left(\cos\left(-\frac{3\pi}{2}\right) + i \sin\left(-\frac{3\pi}{2}\right) \right) \\ &= 8(0 + i) \\ &= 8i \end{aligned}$$

Therefore, $(1-i)^6 = 8i$

4 d $(1 - \sqrt{3}i)^6$

If $z = 1 - \sqrt{3}i$, then



$$r = \sqrt{1^2 + (-\sqrt{3})^2} = \sqrt{1+3} = \sqrt{4} = 2$$

$$\theta = \arg z = -\tan^{-1}\left(\frac{\sqrt{3}}{1}\right) = -\frac{\pi}{3}$$

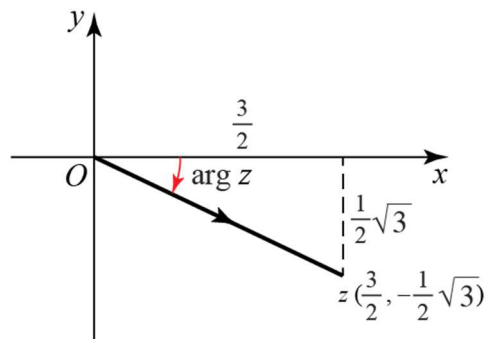
$$\text{So, } 1 - \sqrt{3}i = 2 \left(\cos\left(-\frac{\pi}{3}\right) + i \sin\left(-\frac{\pi}{3}\right) \right)$$

$$\begin{aligned} \therefore (1 - \sqrt{3}i)^6 &= \left[2 \left(\cos\left(-\frac{\pi}{3}\right) + i \sin\left(-\frac{\pi}{3}\right) \right) \right]^6 \\ &= (2)^6 \left(\cos\left(-\frac{6\pi}{3}\right) + i \sin\left(-\frac{6\pi}{3}\right) \right) \\ &= 64 (\cos(-2\pi) + i \sin(-2\pi)) \\ &= 64(1 + i(0)) \\ &= 64 \end{aligned}$$

Therefore, $(1 - \sqrt{3}i)^6 = 64$

$$4 \text{ e } \left(\frac{3}{2} - \frac{1}{2}\sqrt{3}i\right)^9$$

If $z = \frac{3}{2} - \frac{1}{2}\sqrt{3}i$, then



$$r = \sqrt{\left(\frac{3}{2}\right)^2 + \left(-\frac{1}{2}\sqrt{3}\right)^2} = \sqrt{\frac{9}{4} + \frac{3}{4}} = \sqrt{\frac{12}{4}} = \sqrt{3}$$

$$\theta = \arg z = -\tan^{-1}\left(\frac{\frac{1}{2}\sqrt{3}}{\frac{3}{2}}\right) = -\tan^{-1}\frac{\sqrt{3}}{3} = -\frac{\pi}{6}$$

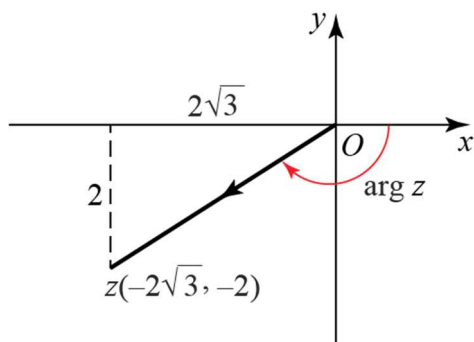
$$\text{So, } \frac{3}{2} - \frac{1}{2}\sqrt{3}i = \sqrt{3}\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right)$$

$$\begin{aligned} \therefore \left(\frac{3}{2} - \frac{1}{2}\sqrt{3}i\right)^9 &= \left[\sqrt{3}\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right)\right]^9 \\ &= (\sqrt{3})^9 \left(\cos\left(-\frac{9\pi}{6}\right) + i\sin\left(-\frac{9\pi}{6}\right)\right) \\ &= 81\sqrt{3}\left(\cos\left(-\frac{3\pi}{2}\right) + i\sin\left(-\frac{3\pi}{2}\right)\right) \\ &= 81\sqrt{3}(0 + i) \\ &= 81\sqrt{3}i \end{aligned}$$

$$\text{Therefore, } \left(\frac{3}{2} - \frac{1}{2}\sqrt{3}i\right)^9 = 81\sqrt{3}i$$

4 f $(-2\sqrt{3} - 2i)^5$

If $z = -2\sqrt{3} - 2i$, then



$$r = \sqrt{(-2\sqrt{3})^2 + (-2)^2} = \sqrt{12 + 4} = \sqrt{16} = 4$$

$$\theta = \arg z = -\pi + \tan^{-1}\left(\frac{2}{2\sqrt{3}}\right) = -\pi + \frac{\pi}{6} = -\frac{5\pi}{6}$$

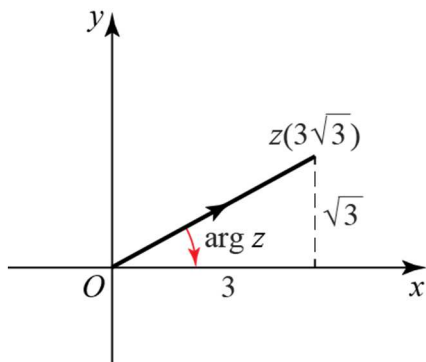
$$\text{So, } -2\sqrt{3} - 2i = 4\left(\cos\left(-\frac{5\pi}{6}\right) + i\sin\left(-\frac{5\pi}{6}\right)\right)$$

$$\begin{aligned} \therefore (-2\sqrt{3} - 2i)^5 &= \left[4\left(\cos\left(-\frac{5\pi}{6}\right) + i\sin\left(-\frac{5\pi}{6}\right)\right)\right]^5 \\ &= 4^5\left(\cos\left(-\frac{25\pi}{6}\right) + i\sin\left(-\frac{25\pi}{6}\right)\right) \\ &= 1024\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right) \\ &= 1024\left(\frac{\sqrt{3}}{2} - \frac{1}{2}i\right) \\ &= 512\sqrt{3} - 512i \end{aligned}$$

$$\text{Therefore, } (-2\sqrt{3} - 2i)^5 = 512\sqrt{3} - 512i$$

5 $(3 + \sqrt{3}i)^5$

If $z = 3 + \sqrt{3}i$, then



$$r = \sqrt{3^2 + (\sqrt{3})^2} = \sqrt{9 + 3} = \sqrt{12} = \sqrt{4 \cdot 3} = 2\sqrt{3}$$

$$\theta = \arg z = \tan^{-1}\left(\frac{\sqrt{3}}{3}\right) = \frac{\pi}{6}$$

$$\text{So, } 3 + \sqrt{3}i = 2\sqrt{3}\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$$

$$\begin{aligned} \therefore (3 + \sqrt{3}i)^5 &= \left[2\sqrt{3}\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)\right]^5 \\ &= (2\sqrt{3})^5\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right) \\ &= 32(9\sqrt{3})\left(-\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) \\ &= 288\sqrt{3}\left(-\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) \\ &= -144\sqrt{3}\sqrt{3} + 144\sqrt{3}i \\ &= -432 + 144\sqrt{3}i \end{aligned}$$

Therefore, $(3 + \sqrt{3}i)^5 = -432 + 144\sqrt{3}i$

6

$$w = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$$

$$\begin{aligned} w^4 &= 2^4\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)^4 = 16\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right) = 16\left(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right) \\ &= -8 + 8i\sqrt{3} \end{aligned}$$

7

$$z = \sqrt{3} \left(\cos \frac{3\pi}{4} - i \sin \frac{3\pi}{4} \right)$$

$$\begin{aligned} z^6 &= (\sqrt{3})^6 \left(\cos \frac{-3\pi}{4} + i \sin \frac{-3\pi}{4} \right)^6 = 27 \left(\cos \frac{-18\pi}{4} + i \sin \frac{-18\pi}{4} \right) \\ &= 27 \left(\cos \frac{-\pi}{2} + i \sin \frac{-\pi}{2} \right) = -27i \end{aligned}$$

- 8 a** Let $z = 1 + i\sqrt{3}$ and $w = 1 - i\sqrt{3}$ first of all note that $|z| = |w| = \sqrt{1+3} = 2$ so it follows that we can write $z = 2e^{i\theta}$ and $w = 2e^{i\varphi}$ with $\tan \theta = \sqrt{3}$ and $\tan \varphi = -\sqrt{3}$ so that $\theta = \frac{\pi}{3}$ and $\varphi = -\frac{\pi}{3}$ then it follows that

$$\frac{1+i\sqrt{3}}{1-i\sqrt{3}} = \frac{z}{w} = \frac{2e^{\frac{\pi i}{3}}}{2e^{-\frac{\pi i}{3}}} = e^{\frac{2\pi i}{3}}$$

- b** We have

$$\left(\frac{1+i\sqrt{3}}{1-i\sqrt{3}} \right)^n = e^{\frac{2n\pi i}{3}} = \cos \frac{2n\pi}{3} + i \sin \frac{2n\pi}{3}$$

This is real precisely when $\frac{2n\pi}{3} = k\pi$ for some integer k so the smallest value of n for which this is real is $n = \frac{3}{2}$ and we have $\cos \frac{2\pi}{3} = -1$ so this is not admissible, the next smallest value for

which this is real is $n = 3$ in which case $\left(\frac{1+i\sqrt{3}}{1-i\sqrt{3}} \right)^3 = \cos 2\pi = 1$ which is positive hence this is the smallest value.

- 9** We start by noting that we can write

$$a + bi = r(\cos \theta + i \sin \theta) = re^{i\theta}$$

$$a - bi = r(\cos \theta - i \sin \theta) = re^{-i\theta}$$

Then it follows that

$$\begin{aligned} (a + bi)^n + (a - bi)^n &= r^n e^{in\theta} + r^n e^{-in\theta} = r^n (e^{in\theta} + e^{-in\theta}) \\ &= r^n (\cos n\theta + i \sin n\theta + \cos n\theta - i \sin n\theta) = 2r^n \cos n\theta \end{aligned}$$

which is real.

Challenge

Given $n \in \mathbb{R}^+$, we have:

$$(r(\cos\theta + i \sin\theta))^{-n} = \frac{1}{(r(\cos\theta + i \sin\theta))^n} = \frac{1}{r^n (\cos n\theta + i \sin n\theta)}$$

by de Moivre's theorem for positive integer exponents.

$$= \frac{1}{r^n (\cos n\theta + i \sin n\theta)} \times \frac{\cos n\theta - i \sin n\theta}{\cos n\theta - i \sin n\theta}$$

$$= \frac{\cos n\theta - i \sin n\theta}{r^n (\cos^2 n\theta - i^2 \sin^2 n\theta)} = \frac{\cos n\theta - i \sin n\theta}{r^n (\cos^2 n\theta + \sin^2 n\theta)}$$

$$= r^{-n} (\cos n\theta - i \sin n\theta) = r^{-n} (\cos(-n\theta) + i \sin(-n\theta))$$