

## Chapter review 2

$$\begin{aligned}
 \mathbf{1 \ a} \quad \frac{2}{(r+2)(r+4)} &= \frac{A}{r+2} + \frac{B}{r+4} \\
 2 &= A(r+4) + B(r+2) \\
 2 &= 2A = -2B \\
 \frac{2}{(r+2)(r+4)} &= \frac{1}{r+2} - \frac{1}{r+4}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad \text{Let } f(r) &= \frac{1}{r+2} \\
 \sum_{r=1}^n \frac{2}{(r+2)(r+4)} &= \sum_{r=1}^n \left( \frac{1}{r+2} - \frac{1}{r+4} \right) \\
 &= \sum_{r=1}^n (f(r) - f(r+2)) \\
 &= f(1) + f(2) - f(n+1) - f(n+2) \\
 &= \frac{1}{3} + \frac{1}{4} - \frac{1}{n+3} - \frac{1}{n+4} \\
 &= \frac{7(n+3)(n+4) - 12(n+3) - 12(n+4)}{12(n+3)(n+4)} \\
 &= \frac{7n^2 + 25n}{12(n+3)(n+4)}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{2 \ a} \quad \frac{4}{(4r-1)(4r+3)} &= \frac{A}{4r-1} + \frac{B}{4r+3} \\
 4 &= A(4r+3) + B(4r-1) \\
 4 &= 4A = -4B \\
 \frac{4}{(4r-1)(4r+3)} &= \frac{1}{4r-1} - \frac{1}{4r+3}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad \text{Let } f(r) &= \frac{1}{4r-1} \\
 \sum_{r=1}^n \frac{4}{(4r-1)(4r+3)} &= \sum_{r=1}^n \left( \frac{1}{4r-1} - \frac{1}{4r+3} \right) \\
 &= \sum_{r=1}^n (f(r) - f(r+1)) = f(1) - f(n+1) \\
 &= \frac{1}{3} - \frac{1}{4n+3} \\
 &= \frac{4n}{3(4n+3)}
 \end{aligned}$$

$$\begin{aligned}
 2 \text{ c } & \sum_{r=100}^{200} \frac{4}{(4r-1)(4r+3)} \\
 &= \sum_{r=1}^{200} \frac{4}{(4r-1)(4r+3)} - \sum_{r=1}^{99} \frac{4}{(4r-1)(4r+3)} \\
 &= \frac{800}{3(803)} - \frac{396}{3(399)} = \frac{404}{320397} = 0.00126 \text{ (3 s.f.)}
 \end{aligned}$$

$$\begin{aligned}
 3 \text{ a } & (r+1)^3 - (r-1)^3 \\
 &= r^3 + 3r^2 + 3r + 1 - (r^3 - 3r^2 + 3r - 1) \\
 &= 6r^2 + 2
 \end{aligned}$$

$$\begin{aligned}
 \text{b } r^2 &= \frac{1}{6}((r+1)^3 - (r-1)^3 - 2) \\
 \sum_{r=1}^n (r+1)^3 - (r-1)^3 &= (n+1)^3 + n^3 - 1 \\
 &= 2n^3 + 3n^2 + 3n \\
 \sum_{r=1}^n r^2 &= \frac{1}{6} \left( 2n^3 + 3n^2 + 3n - 2 \sum_{r=1}^n 1 \right) \\
 &= \frac{1}{6} (2n^3 + 3n^2 + 3n - 2n) = \frac{1}{6} n(n+1)(2n+1)
 \end{aligned}$$

$$\begin{aligned}
 4 \quad \frac{4}{(r+1)(r+3)} &= \frac{A}{r+1} + \frac{B}{r+3} \\
 4 &= A(r+3) + B(r+1) \\
 4 &= 2A = -2B
 \end{aligned}$$

$$\text{Let } f(r) = \frac{1}{r+1}$$

$$\begin{aligned}
 \sum_{r=1}^n \frac{4}{(r+1)(r+3)} &= 2 \sum_{r=1}^n \left( \frac{1}{r+1} - \frac{1}{r+3} \right) \\
 &= 2(f(1) + f(2) - f(n+1) - f(n+2)) \\
 &= 2 \left( \frac{1}{2} + \frac{1}{3} - \frac{1}{n+2} - \frac{1}{n+3} \right) = 2 \left( \frac{5}{6} - \frac{1}{n+2} - \frac{1}{n+3} \right) \\
 &= \frac{1}{3} \left( \frac{5(n+2)(n+3) - 6(n+2) - 6(n+3)}{(n+2)(n+3)} \right) \\
 &= \frac{5n^2 + 13n}{3(n+2)(n+3)}
 \end{aligned}$$

$$\begin{aligned}
 5 \quad \sum_{r=1}^n ((r+1)^3 - (r-1)^3) &= (n+1)^3 + n^3 - 1 \\
 \sum_{r=n}^{2n} (r+1)^3 - (r-1)^3 & \\
 &= (2n+1)^3 + (2n)^3 - 1 - (n^3 + (n-1)^3 - 1) \\
 &= 16n^3 + 12n^2 + 6n - (2n^3 - 3n^2 + 3n - 2) \\
 &= 14n^3 + 15n^2 + 3n + 2 \\
 \text{So } a &= 14, b = 15, c = 3 \text{ and } d = 2
 \end{aligned}$$

$$\begin{aligned}
 6 \quad \text{a} \quad \frac{3}{(3r+1)(3r+4)} &= \frac{A}{3r+1} + \frac{B}{3r+4} \\
 3 &= A(3r+4) + B(3r+1) \\
 3 &= 3A - 3B \\
 \text{Let } f(r) &= \frac{1}{3r+1} \\
 \sum_{r=1}^n \frac{3}{(3r+1)(3r+4)} &= \sum_{r=1}^n \left( \frac{1}{3r+1} - \frac{1}{3r+4} \right) \\
 &= \sum_{r=1}^n (f(r) - f(r+1)) = f(1) - f(n+1) \\
 &= \frac{1}{4} - \frac{1}{3n+4} = \frac{3n}{12n+16} \\
 \text{So } a &= 3, b = 12 \text{ and } c = 16
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad \sum_{r=n}^{2n} \frac{3}{(3r+1)(3r+4)} &= \frac{6n}{24n+16} - \frac{3(n-1)}{12(n-1)+16} \\
 &= \frac{3}{4} \left( \frac{n(3n+1) - (n-1)(3n+2)}{(3n+1)(3n+2)} \right) \\
 &= \frac{3}{4} \left( \frac{2n+2}{(3n+1)(3n+2)} \right) = \frac{3(n+1)}{2(3n+1)(3n+2)}
 \end{aligned}$$

$$\begin{aligned}
 7 \quad \frac{2r+1}{r(r+1)} &= \frac{1}{r} + \frac{1}{r+1}, f(r) = \frac{1}{r} \\
 \sum_{r=1}^n \frac{2r+1}{r(r+1)} &= f(1) + f(2) + f(2) + f(3) + \dots + \\
 &f(n-1) + f(n) + f(n) + f(n+1) \\
 \text{Terms don't cancel}
 \end{aligned}$$

$$8 \quad \frac{1}{r(r+2)} = \frac{A}{r} + \frac{B}{r+2}$$

$$1 = A(r+2) + Br$$

$$1 = 2A = -2B$$

$$\text{Let } f(r) = \frac{1}{r}$$

$$\begin{aligned} \sum_{r=1}^n \frac{1}{r(r+2)} &= \frac{1}{2} \sum_{r=1}^n \left( \frac{1}{r} - \frac{1}{r+2} \right) \\ &= \frac{1}{2} (f(1) + f(2) - f(n+1) - f(n+2)) \\ &= \frac{1}{2} \left( 1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2} \right) = \frac{3}{4} - \frac{1}{2} \frac{2n+3}{(n+1)(n+2)} \\ &= \frac{3}{4} - \frac{1}{2} \frac{2n+3}{(n+1)(n+2)} \end{aligned}$$

So  $a = 2$  and  $b = 3$

$$9 \quad \text{a} \quad \frac{4}{(2r+1)(2r+5)} = \frac{A}{2r+1} + \frac{B}{2r+5}$$

$$4 = A(2r+5) + B(2r+1)$$

$$4 = 4A = -4B$$

So  $A = 1$  and  $B = -1$

$$\frac{4}{(2r+1)(2r+5)} = \frac{1}{(2r+1)} - \frac{1}{(2r+5)}$$

$$\text{b} \quad \text{Let } f(r) = \frac{1}{2r+1}$$

$$\sum_{r=16}^{25} \frac{4}{r(2r+5)} = \sum_{r=16}^{25} \left( \frac{1}{2r+1} - \frac{1}{2r+5} \right)$$

$$= f(16) - f(18) + f(17) - f(19) + f(18) - \dots$$

$$+ f(25) - f(27)$$

$$= f(16) + f(17) - f(26) - f(27)$$

$$= \frac{1}{33} + \frac{1}{35} - \frac{1}{53} - \frac{1}{55} = 0.0218 \text{ (4 d.p.)}$$

## Challenge

$$\mathbf{a} \quad \ln\left(1 + \frac{1}{r+2}\right) = \ln\left(\frac{r+3}{r+2}\right) = \ln(r+3) - \ln(r+2)$$

$$\text{Let } f(r) = \ln(r+2)$$

$$\sum_{r=1}^n \ln\left(1 + \frac{1}{r+2}\right) = f(n+1) - f(1)$$

$$= \ln(n+3) - \ln 3 = \ln \frac{n+3}{3}$$

$$n = 30 : \sum_{r=1}^{30} \ln\left(1 + \frac{1}{r+2}\right) = \ln 11$$

$$\text{So } k = 11$$

$$\mathbf{b} \quad \frac{18}{r(r+3)} = \frac{A}{r} + \frac{B}{r+3}$$

$$18 = A(r+3) + Br$$

$$18 = 3A - 3B$$

$$\text{Let } f(r) = \frac{1}{r}$$

$$\sum_{r=1}^n \frac{18}{r(r+3)} = 6 \sum_{r=1}^n \left( \frac{1}{r} - \frac{1}{r+3} \right)$$

$$= 6(f(1) + f(2) + f(3) - f(n+1) - f(n+2) - f(n+3))$$

$$= 6\left(1 + \frac{1}{2} + \frac{1}{3} - \frac{1}{n+1} - \frac{1}{n+2} - \frac{1}{n+3}\right)$$

$$= \frac{11(n+1)(n+2)(n+3)}{(n+1)(n+2)(n+3)}$$

$$- 6 \frac{(n+1)(n+2) + (n+1)(n+3) + (n+2)(n+3)}{(n+1)(n+2)(n+3)}$$

$$= \frac{11n^3 + 48n^2 + 49n}{(n+1)(n+2)(n+3)}$$

$$\text{So } a = 11, b = 48 \text{ and } c = 49$$