Exercise 2A

1 a

$\frac{1}{2}(r(r+1)-r(r-1))$

$=\frac{1}{2}(r^2+r-r^2+r)$

$=\frac{1}{2}(2r)$

$=r$

$=\text{LHS}$

Consider RHS.

Expand and simplify.

b

$$\sum_{r=1}^{n} r = \frac{1}{2}\sum_{r=1}^{n} r(r+1) - \frac{1}{2}\sum_{r=1}^{n} r(r-1)$$

Use above.

$r=1$	$\frac{1}{2}\times 1\times 2$	$-\frac{1}{2}\times 1\times 0$
$r=2$	$\frac{1}{2}\times 2\times 3$	$-\frac{1}{2}\times 2\times 1$
$r=3$	$\frac{1}{2}\times 3\times 4$	$-\frac{1}{2}\times 3\times 2$
…	…	
$r=n-1$	$\frac{1}{2}(n-1)(n)$	$-\frac{1}{2}(n-1)(n-2)$
$r=n$	$\frac{1}{2}n(n+1)$	$-\frac{1}{2}n(n-1)$

Use method of differences.

When you add, all terms cancel except $\frac{1}{2}n(n+1)$.

Hence $\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$

Use the information given and equate the summations.

2 $\sum_{r=1}^{n} \frac{1}{r(r+1)(r+2)} = \sum_{r=1}^{n} \frac{1}{2r(r+1)} - \sum_{r=1}^{n} \frac{1}{2(r+1)(r+2)}$ ←

Use method of differences.

Put $r = 1$ $\quad \frac{1}{2\times1\times2} - \frac{1}{2\times2\times3}$

$r = 2$ $\quad \frac{1}{2\times2\times3} - \frac{1}{2\times3\times4}$

All terms cancel except first and last.

$r = 3$ $\quad \frac{1}{2\times3\times4} - \frac{1}{2\times4\times5}$

$\vdots$

$r = n$ $\quad \frac{1}{2n(n+1)} - \frac{1}{2(n+1)(n+2)}$

Adding the first and last terms we have

$$\sum_{r=1}^{n} \frac{1}{r(r+1)(r+2)} = \frac{1}{4} - \frac{1}{2(n+1)(n+2)}$$

First and last from above.

$$= \frac{(n+1)(n+2)-2}{4(n+1)(n+2)}$$

Simplify.

$$= \frac{n^2+3n+2-2}{4(n+1)(n+2)}$$

$$= \frac{n(n+3)}{4(n+1)(n+2)}$$

3 a $\frac{1}{r(r+2)} \equiv \frac{A}{r} + \frac{B}{r+2}$ ← Set $\frac{1}{r(r+2)}$ identical to $\frac{A}{r} + \frac{B}{r+2}$.

$\equiv \frac{A(r+2)+Br}{r(r+2)}$ ← Add the two fractions.

$1 \equiv A(r+2) + Br$

Put $r = 0$

$1 = 2A$

$A = \frac{1}{2}$

Put $r = 1$

$1 = \frac{1}{2}(3) + B$

$B = -\frac{1}{2}$

$\therefore \frac{1}{r(r+2)} = \frac{1}{2r} - \frac{1}{2(r+2)}$

3 b $\displaystyle\sum_{r=1}^{n}\frac{1}{r(r+2)}=\sum_{r=1}^{n}\frac{1}{2r}-\sum_{r=1}^{n}\frac{1}{2(r+2)}$

Use method of differences.

$r=1 \quad \frac{1}{2\times1}-\frac{1}{2\times3}$

$r=2 \quad \frac{1}{2\times2}-\frac{1}{2\times4}$

$r=3 \quad \frac{1}{2\times3}-\frac{1}{2\times5}$

$\vdots$

$r=n-1 \quad \frac{1}{2(n-1)}-\frac{1}{2(n+1)}$

$r=n \quad \frac{1}{2n}-\frac{1}{2(n+2)}$

All terms cancel except $\frac{1}{2},\frac{1}{4}$, $\frac{1}{2(n+1)}$ and $\frac{1}{2(n+2)}$

Adding the remaining terms we have

$$\sum_{r=1}^{n}\frac{1}{r(r+2)}=\frac{1}{2}+\frac{1}{4}-\frac{1}{2(n+1)}-\frac{1}{2(n+2)}$$

$$=\frac{2(n+1)(n+2)+(n+1)(n+2)-2(n+2)-2(n+1)}{4(n+1)(n+2)}$$

$$=\frac{2n^2+6n+4+n^2+3n+2-2n-4-2n-2}{4(n+1)(n+2)}$$

$$=\frac{3n^2+5n}{4(n+1)(n+2)}$$

$$=\frac{n(3n+5)}{4(n+1)(n+2)}$$

4 a $\frac{1}{(r+2)(r+3)}\equiv\frac{A}{r+2}+\frac{B}{r+3}$ ← Set $\frac{1}{(r+2)(r+3)}$ identical to $\frac{A}{r+2}+\frac{B}{r+3}$.

$\equiv\frac{A(r+3)+B(r+2)}{(r+2)(r+3)}$ ← Add the two fractions.

$1\equiv A(r+3)+B(r+2)$ ← Compare numerators as they are equivalent.

$r=-3 \Rightarrow B=-1$

$r=-2 \Rightarrow A=1$ ← Solve for A and B.

$\therefore \frac{1}{(r+2)(r+3)}=\frac{1}{r+2}-\frac{1}{r+3}$

4 b $\sum_{r=1}^{n}\frac{1}{(r+2)(r+3)} \equiv \sum_{r=1}^{n}\frac{1}{(r+2)} - \sum_{r=1}^{n}\frac{1}{(r+3)}$

Use the method of differences.

$r=1 \qquad \frac{1}{3} - \frac{1}{4}$

$r=2 \qquad \frac{1}{4} - \frac{1}{5}$

All cancel except first and last.

$r=3 \qquad \frac{1}{5} - \frac{1}{6}$

$\vdots$

$r=n \qquad \frac{1}{n+2} - \frac{1}{n+3}$

Adding the remaining terms we have

$$\sum_{r=1}^{n}\frac{1}{(r+2)(r+3)} = \frac{1}{3} - \frac{1}{n+3}$$
$$= \frac{n+3-3}{3(n+3)}$$
$$= \frac{n}{3(n+3)}$$

5 a $\frac{1}{r!} - \frac{1}{(r+1)!} = \frac{r+1}{(r+1)!} - \frac{1}{(r+1)!} = \frac{r}{(r+1)!}$

b $\sum_{r=1}^{n}\frac{r}{(r+1)!} \equiv \sum_{r=1}^{n}\frac{1}{r!} - \sum_{r=1}^{n}\frac{1}{(r+1)!}$

Use given.

$r=1 \qquad \frac{1}{1!} - \frac{1}{2!}$

Use method of differences.

$r=2 \qquad \frac{1}{2!} - \frac{1}{3!}$

$r=3 \qquad \frac{1}{3!} - \frac{1}{4!}$

$\vdots$

$r=n \qquad \frac{1}{n!} - \frac{1}{(n+1)!}$

All cancel except first and last term.

$\therefore$

Adding the remaining terms we have

$$\sum_{r=1}^{n}\frac{r}{(r+1)!} = 1 - \frac{1}{(n+1)!}$$

6 $$\sum_{r=1}^{n}\frac{2r+1}{r^2(r+1)^2} = \sum_{r=1}^{n}\frac{1}{r^2} - \sum_{r=1}^{n}\frac{1}{(r+1)^2}$$

Use given.

$r=1$: $\frac{1}{1} - \frac{1}{2^2}$

$r=2$: $\frac{1}{2^2} - \frac{1}{3^2}$

$r=3$: $\frac{1}{3^2} - \frac{1}{4^2}$

$\vdots$

$r=n$: $\frac{1}{n^2} - \frac{1}{(n+1)^2}$

Use method of differences.

All terms cancel except first and last.

So adding the remaining terms we have

$$\sum_{r=1}^{n}\frac{2r+1}{r^2(r+1)^2} = 1 - \frac{1}{(n+1)^2} = \frac{(n+1)^2-1}{(n+1)^2}$$

$$= \frac{n^2+2n}{(n+1)^2}$$

$$= \frac{n(n+2)}{(n+1)^2}$$

Simplify.

7 a $$\frac{1}{(2r+3)(2r+5)} = \frac{A}{2r+3} + \frac{B}{2r+5}$$

$1 = A(2r+5) + B(2r+3)$

$1 = 2A = -2B$

Let $f(r) = \frac{1}{2r+3}$

$$\sum_{r=1}^{n}\frac{1}{(2r+3)(2r+5)} = \frac{1}{2}\sum_{r=1}^{n}\left(\frac{1}{2r+3} - \frac{1}{2r+5}\right)$$

$$= \frac{1}{2}\sum_{r=1}^{n}\left(f(r) - f(r+1)\right) = \frac{1}{2}\left(f(1) - f(n+1)\right)$$

$$= \frac{1}{2}\left(\frac{1}{2+3} - \frac{1}{2(n+1)+3}\right) = \frac{1}{2}\left(\frac{2n+5-5}{5(2n+5)}\right)$$

$$= \frac{n}{10n+25}$$

So $a=10$, $b=25$

7 b $n=1$: RHS $=\dfrac{1}{10+25}=\dfrac{1}{35}=\dfrac{1}{5\times 7}=$LHS

Assume true for $n=k$

$$\sum_{r=1}^{k}\frac{1}{(2r+3)(2r+5)}=\frac{k}{10k+25}$$

$n=k+1$:

$$\sum_{r=1}^{k+1}\frac{1}{(2r+3)(2r+5)}=\frac{k}{10k+25}+\frac{1}{(2k+5)(2k+7)}$$

$$=\frac{1}{2k+5}\left(\frac{k}{5}+\frac{1}{2k+7}\right)=\frac{1}{2k+5}\left(\frac{2k^2+7k+5}{5(2k+7)}\right)$$

$$=\frac{k+1}{10(k+1)+25}$$

so true for $n=k+1$ if true for $n=k$

true for $n=1$ so true for all natural numbers

8 $$\frac{8}{(3r-2)(3r+4)}=\frac{A}{3r-2}+\frac{B}{3r+4}$$

$8=A(3r+4)+B(3r-2)$

$8=6A=-6B$

So $A=\dfrac{4}{3}$ and $B=\dfrac{-4}{3}$

Let $\mathrm{f}(r)=\dfrac{1}{3r-2}$

$$\sum_{r=1}^{n}\frac{8}{(3r-2)(3r+4)}=\frac{4}{3}\sum_{r=1}^{n}\left(\frac{1}{3r-2}-\frac{1}{3r+4}\right)$$

$$=\frac{4}{3}\sum_{r=1}^{n}\left(\mathrm{f}(r)-\mathrm{f}(r+2)\right)$$

$$=\frac{4}{3}(\mathrm{f}(1)+\mathrm{f}(2)-\mathrm{f}(n+1)-\mathrm{f}(n+2))$$

$$=\frac{4}{3}\left(1+\frac{1}{4}-\frac{1}{3n+1}-\frac{1}{3n+4}\right)$$

$$=\frac{4}{3}\left(\frac{\frac{5}{4}(3n+1)(3n+4)-(3n+1)-(3n+4)}{(3n+1)(3n+4)}\right) \quad \mathbf{1}$$

$$=\frac{15n^2+25n+\frac{20}{3}-8n-\frac{20}{3}}{(3n+1)(3n+4)}=\frac{n(15n+17)}{(3n+1)(3n+4)}$$

So $a=15$ and $b=17$.

9 Let $f(r) = (r-1)^2$

$$\sum_{r=1}^{n} (r+1)^2 - (r-1)^2 = f(n+2) + f(n+1) - f(2) - f(1)$$

$$= (n+1)^2 + n^2 - 1 = 2n^2 + 2n = 2n(n+1)$$

So $a = 2$