Solution Bank



Exercise 2A

1 a

$$\frac{1}{2}(r(r+1)-r(r-1))$$

$$=\frac{1}{2}(r^2+r-r^2+r)$$

$$=\frac{1}{2}(2r)$$

$$=r$$

$$=LHS$$

Consider RHS.

Expand and simplify.

b

$$\sum_{r=1}^{n} r = \frac{1}{2} \sum_{r=1}^{n} r(r+1) - \frac{1}{2} \sum_{r=1}^{n} r(r-1)$$

$$r = 1 \qquad \frac{1}{2} \times 1 \times 2 \qquad -\frac{1}{2} \times 1 \times 0$$

$$r = 2 \qquad \frac{1}{2} \times 2 \times 3 \qquad -\frac{1}{2} \times 2 \times 1$$

$$r = 3 \qquad \frac{1}{2} \times 3 \times 4 \qquad -\frac{1}{2} \times 3 \times 2$$
...
$$r = n - 1 \qquad \frac{1}{2} (n-1)(n) \qquad -\frac{1}{2} (n-1)(n-2)$$

$$r = n \qquad \frac{1}{2} n(n+1) \qquad -\frac{1}{2} n(n-1)$$
Hence
$$\sum_{r=1}^{n} r = \frac{1}{2} n(n+1)$$

Use above.

Use method of differences.

When you add, all terms cancel except $\frac{1}{2}n(n+1)$.

Use the information given and equate the summations.

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$$2 \sum_{r=1}^{n} \frac{1}{r(r+1)(r+2)} =$$

$$2 \sum_{r=1}^{n} \frac{1}{r(r+1)(r+2)} = \sum_{r=1}^{n} \frac{1}{2r(r+1)} - \sum_{r=1}^{n} \frac{1}{2(r+1)(r+2)}$$

Put r = 1

$$\frac{1}{2\times1\times2} - \frac{1}{2\times2\times3}$$

Use method of differences.

$$r = 2$$

$$\frac{1}{2 \times 2 \times 3} - \frac{1}{2 \times 3 \times 4}$$

$$\frac{1}{2 \times 3 \times 4} - \frac{1}{2 \times 4 \times 5}$$

All terms cancel except first and last.

$$r = 3$$

$$r = n$$

$$\frac{1}{2n(n+1)} - \frac{1}{2(n+1)(n+2)}$$

Adding the first and last terms we have

$$\sum_{r=1}^{n} \frac{1}{r(r+1)(r+2)} = \frac{1}{4} - \frac{1}{2(n+1)(n+2)}$$

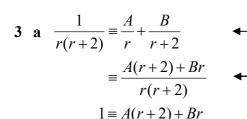
$$= \frac{(n+1)(n+2) - 2}{4(n+1)(n+2)}$$

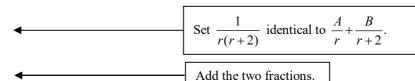
$$= \frac{n^2 + 3n + 2 - 2}{4(n+1)(n+2)}$$

$$= \frac{n(n+3)}{4(n+1)(n+2)}$$

First and last from above.

Simplify.





Put r = 0

Put
$$r = 1$$

 $1 = \frac{1}{2}(3) + B$
 $B = -\frac{1}{2}$

$$\therefore \frac{1}{r(r+2)} = \frac{1}{2r} - \frac{1}{2(r+2)}$$

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3 b
$$\sum_{r=1}^{n} \frac{1}{r(r+2)} = \sum_{r=1}^{n} \frac{1}{2r} - \sum_{r=1}^{n} \frac{1}{2(r+2)}$$

 $r = 1$ $\frac{1}{2 \times 1}$ $-\frac{1}{2 \times 3}$
 $r = 2$ $\frac{1}{2 \times 2}$ $-\frac{1}{2 \times 4}$
 $r = 3$ $\frac{1}{2 \times 3}$ $-\frac{1}{2 \times 5}$
 \vdots
 $r = n-1$ $\frac{1}{2(n-1)}$ $-\frac{1}{2(n+1)}$
 $r = n$ $\frac{1}{2n}$ $-\frac{1}{2(n+2)}$

Use method of differences.

All terms cancel except
$$\frac{1}{2}, \frac{1}{4}$$

$$\frac{1}{2(n+1)} \text{ and } \frac{1}{2(n+2)}$$

Adding the remaining terms we have

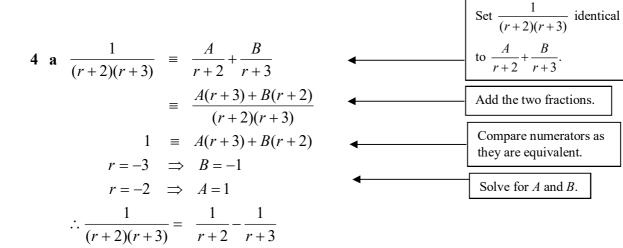
$$\sum_{r=1}^{n} \frac{1}{r(r+2)} = \frac{1}{2} + \frac{1}{4} - \frac{1}{2(n+1)} - \frac{1}{2(n+2)}$$

$$= \frac{2(n+1)(n+2) + (n+1)(n+2) - 2(n+2) - 2(n+1)}{4(n+1)(n+2)}$$

$$= \frac{2n^2 + 6n + 4 + n^2 + 3n + 2 - 2n - 4 - 2n - 2}{4(n+1)(n+2)}$$

$$= \frac{3n^2 + 5n}{4(n+1)(n+2)}$$

$$= \frac{n(3n+5)}{4(n+1)(n+2)}$$



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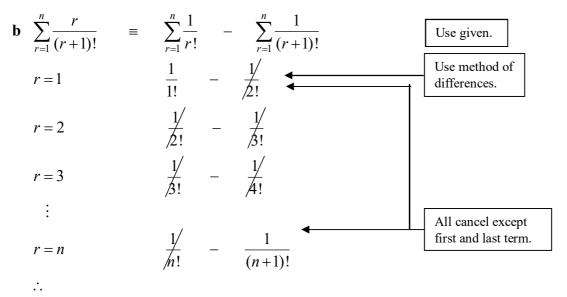
Use the method of differences.

All cancel except first and last.

Adding the remaining terms we have

$$\sum_{r=1}^{n} \frac{1}{(r+2)(r+3)} = \frac{1}{3} - \frac{1}{n+3}$$
$$= \frac{n+3-3}{3(n+3)}$$
$$= \frac{n}{3(n+3)}$$

5 **a**
$$\frac{1}{r!} - \frac{1}{(r+1)!} = \frac{r+1}{(r+1)!} - \frac{1}{(r+1)!} = \frac{r}{(r+1)!}$$



Adding the remaining terms we have

$$\sum_{r=1}^{n} \frac{r}{(r+1)!} = 1 - \frac{1}{(n+1)!}$$

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$$6 \sum_{r=1}^{n} \frac{2r+1}{r^{2}(r+1)^{2}} = \sum_{r=1}^{n} \frac{1}{r^{2}} - \sum_{r=1}^{n} \frac{1}{(r+1)^{2}}$$
Use given.

$$r = 1$$

$$r = 2$$

$$\frac{1}{1} - \frac{1}{2^{2}}$$
Use method of differences.

$$r = 3$$

$$\frac{1}{2^{2}} - \frac{1}{2^{2}}$$
All terms cancel except first and last.

$$\vdots$$

$$r = n$$

$$\frac{1}{n^{2}} - \frac{1}{(n+1)^{2}}$$

So adding the remaining terms we have

$$\sum_{r=1}^{n} \frac{2r+1}{r^2(r+1)^2} = 1 - \frac{1}{(n+1)^2} = \frac{(n+1)^2 - 1}{(n+1)^2}$$
$$= \frac{n^2 + 2n}{(n+1)^2}$$
$$= \frac{n(n+2)}{(n+1)^2}$$

So a = 10, b = 25

Simplify.

7 **a**
$$\frac{1}{(2r+3)(2r+5)} = \frac{A}{2r+3} + \frac{B}{2r+5}$$

$$1 = A(2r+5) + B(2r+3)$$

$$1 = 2A = -2B$$
Let $f(r) = \frac{1}{2r+3}$

$$\sum_{r=1}^{n} \frac{1}{(2r+3)(2r+5)} = \frac{1}{2} \sum_{r=1}^{n} \left(\frac{1}{2r+3} - \frac{1}{2r+5} \right)$$

$$= \frac{1}{2} \sum_{r=1}^{n} \left(f(r) - f(r+1) \right) = \frac{1}{2} \left(f(1) - f(n+1) \right)$$

$$= \frac{1}{2} \left(\frac{1}{2+3} - \frac{1}{2(n+1)+3} \right) = \frac{1}{2} \left(\frac{2n+5-5}{5(2n+5)} \right)$$

$$= \frac{n}{10n+25}$$

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7 **b**
$$n=1$$
: RHS = $\frac{1}{10+25} = \frac{1}{35} = \frac{1}{5\times7} = \text{LHS}$

Assume true for n = k

$$\sum_{r=1}^{k} \frac{1}{(2r+3)(2r+5)} = \frac{k}{10k+25}$$

$$n = k + 1$$

$$\sum_{r=1}^{k+1} \frac{1}{(2r+3)(2r+5)} = \frac{k}{10k+25} + \frac{1}{(2k+5)(2k+7)}$$

$$= \frac{1}{2k+5} \left(\frac{k}{5} + \frac{1}{2k+7}\right) = \frac{1}{2k+5} \left(\frac{2k^2+7k+5}{5(2k+7)}\right)$$

$$= \frac{k+1}{10(k+1)+25}$$

so true for n = k + 1 if true for n = k

true for n = 1 so true for all natural numbers

$$8 \frac{8}{(3r-2)(3r+4)} = \frac{A}{3r-2} + \frac{B}{3r+4}$$

$$8 = A(3r+4) + B(3r-2)$$

$$8 = 6A = -6B$$
So $A = \frac{4}{3}$ and $B = \frac{-4}{3}$

$$\text{Let } f(r) = \frac{1}{3r-2}$$

$$\sum_{r=1}^{n} \frac{8}{(3r-2)(3r+4)} = \frac{4}{3} \sum_{r=1}^{n} \left(\frac{1}{3r-2} - \frac{1}{3r+4} \right)$$

$$= \frac{4}{3} \sum_{r=1}^{n} \left(f(r) - f(r+2) \right)$$

$$= \frac{4}{3} \left(f(1) + f(2) - f(n+1) - f(n+2) \right)$$

$$= \frac{4}{3} \left(1 + \frac{1}{4} - \frac{1}{3n+1} - \frac{1}{3n+4} \right)$$

$$= \frac{4}{3} \left(\frac{\frac{5}{4}(3n+1)(3n+4) - (3n+1) - (3n+4)}{(3n+1)(3n+4)} \right)$$

$$= \frac{15n^2 + 25n + \frac{20}{3} - 8n - \frac{20}{3}}{(3n+1)(3n+4)} = \frac{n(15n+17)}{(3n+1)(3n+4)}$$

So a = 15 and b = 17.

Solution Bank



9 Let
$$f(r) = (r-1)^2$$

$$\sum_{r=1}^{n} (r+1)^{2} - (r-1)^{2} = f(n+2) + f(n+1) - f(2) - f(1)$$
$$= (n+1)^{2} + n^{2} - 1 = 2n^{2} + 2n = 2n(n+1)$$

So
$$a = 2$$