Chapter review 1

1 To ensure multiplication by a positive quantity, multiply both sides by $(x-2)^2 x^2$

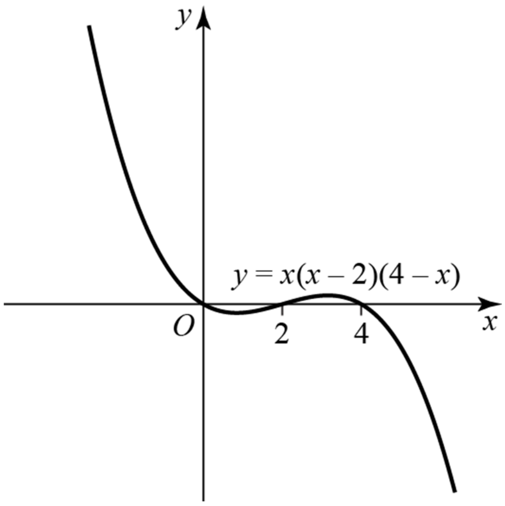
$$\frac{1}{\cancel{(x-2)}} \times (x-2)^{\cancel{2}} x^2 \leqslant \frac{2}{\cancel{x}} \times (x-2)^2 x^{\cancel{2}}$$

$$x(x-2)\big(x-2(x-2)\big) \leqslant 0$$

$$x(x-2)(4-x) \leqslant 0$$

So the critical values are $x = 0, 2$ or 4

The curve $y = x(x-2)(4-x)$ is a cubic graph with negative x^3 coefficient, so the curve starts in the top left and ends in the bottom right and passes through $(0,0), (2,0)$ and $(4,0)$.



The solution corresponds to the section of the graph that is on or below the x-axis.
So the solution is $0 \leqslant x \leqslant 2$ or $x \geqslant 4$

2 To ensure multiplication by a positive quantity, multiply both sides by $(x+2)^2$

$$\frac{2x^2-2}{(x+2)}\times(x+2)^2>4(x+2)^2$$

$$(x+2)\big((2x^2-2)-4(x+2)\big)>0$$

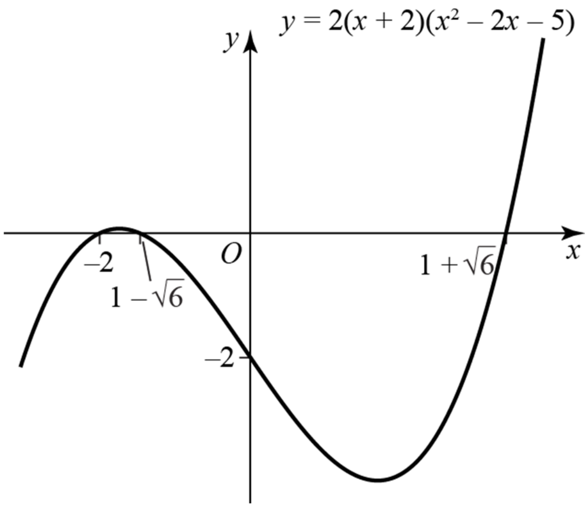
$$(x+2)(2x^2-4x-10)>0$$

$$2(x+2)(x^2-2x-5)>0$$

So using the quadratic formula the critical values are $x=-2$ or $\frac{2\pm\sqrt{24}}{2}$

This simplifies to $x=-2$ or $1\pm\sqrt{6}$

The curve $y=2(x+2)(x^2-2x-5)$ is a cubic graph with positive x^3 coefficient, so the curve starts in the bottom left and ends in the top right and passes through $(-2,0),(1-\sqrt{6},0)$ and $(1+\sqrt{6},0)$.



The solution corresponds to the section of the graph that is above the x-axis.

So the solution is $-2<x<1-\sqrt{6}$ or $x>1+\sqrt{6}$

3 To ensure multiplication by a positive quantity, multiply both sides by $(x-2)^2$

$$\frac{2x^2-3x+4}{(x-2)}\times(x-2)^2<(4x-2)(x-2)^2$$

$$(x-2)\left((2x^2-3x+4)-(4x-2)(x-2)\right)<0$$

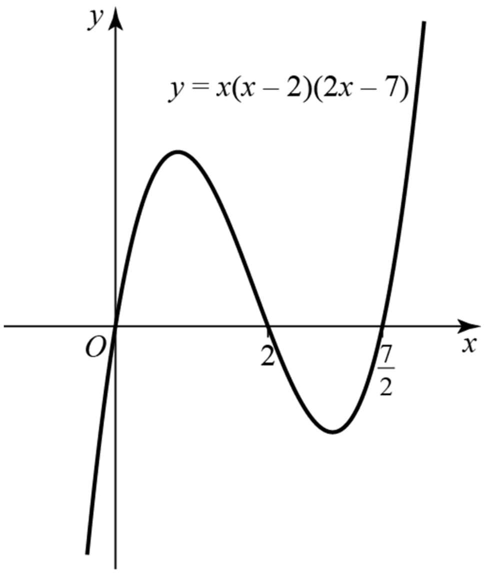
$$(x-2)(-2x^2+7x)<0$$

$$x(x-2)(7-2x)<0$$

$x(x-2)(2x-7)>0$ multiplying by -1 so changing the direction of the inequality

So the critical values are $x=0,\ 2$ or $\frac{7}{2}$

The curve $y=x(x-2)(2x-7)$ is a cubic graph with positive x^3 coefficient, so the curve starts in the bottom left and ends in the top right and passes through $(0,0), (2,0)$ and $(\frac{7}{2},0)$.



The solution corresponds to the section of the graph that is above the x-axis.
So the solution is $0<x<2$ or $x>\frac{7}{2}$

4 To ensure multiplication by a positive quantity, multiply both sides by $(2x-3)^2(x-3)^2$

$$\frac{x+1}{(2x-3)}\times(2x-3)^2(x-3)^2<\frac{1}{(x-3)}\times(2x-3)^2(x-3)^2$$

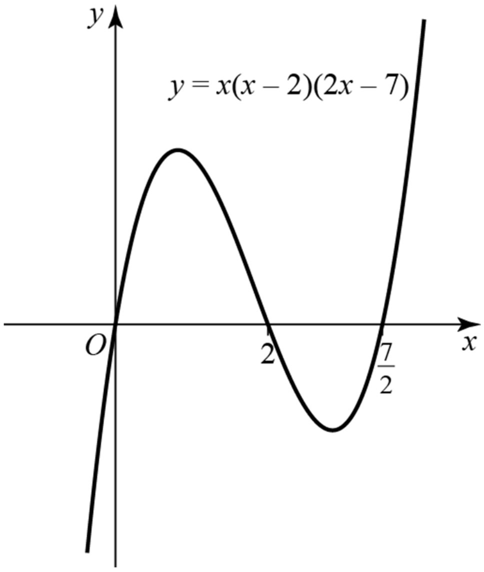
$$(2x-3)(x-3)\big((x+1)(x-3)-(2x-3)\big)<0$$

$$(2x-3)(x-3)(x^2-4x)<0$$

$$x(2x-3)(x-3)(x-4)<0$$

So the critical values are $x = 0, \frac{3}{2}, 3$ or 4

The curve $y=x(2x-3)(x-3)(x-4)$ is a quartic graph with positive x^4 coefficient, so the curve starts in the top left and ends in the top right and passes through $(0,0), (\frac{3}{2},0), (3,0)$ and $(4,0)$.



The solution corresponds to the section of the graph that is below the x-axis.
So the solution in set notation is $\{x: 0<x<\frac{3}{2}\}\cup\{x: 3<x<4\}$

5 To ensure multiplication by a positive quantity, multiply both sides by $(x-1)^2$

$$\frac{(x+3)(x+9)}{\cancel{(x-1)}}\times(x-1)^{\cancel{2}} > (3x-5)(x-1)^2$$

$$(x-1)\big((x+3)(x+9)-(3x-5)(x-1)\big)>0$$

$$(x-1)((x^2+12x+27)-(3x^2-8x+5)>0$$

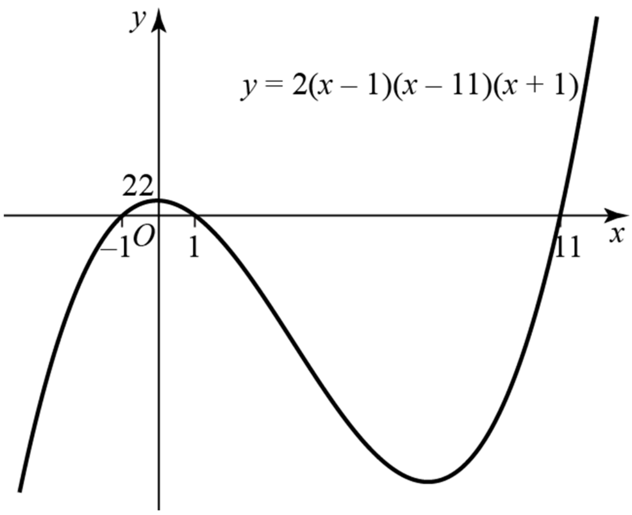
$$(x-1)(-2x^2+20x+22)>0$$

$$2(x-1)(x^2-10x-11)<0$$ multiplying by -1 so changing the direction of the inequality

$$2(x-1)(x-11)(x+1)<0$$

So the critical values are $x=-1$, 1 or 11

The curve $y=2(x-1)(x-11)(x+1)$ is a cubic graph with positive x^3 coefficient, so the curve starts in the bottom left and ends in the top right and passes through $(-1,0)$, $(1,0)$ and $(11,0)$.



The solution corresponds to the section of the graph that is below the x-axis.
So the solution in set notation is $\{x: x<-1\}\cup\{x: 1<x<11\}$

6 a $y = 2x + 2$

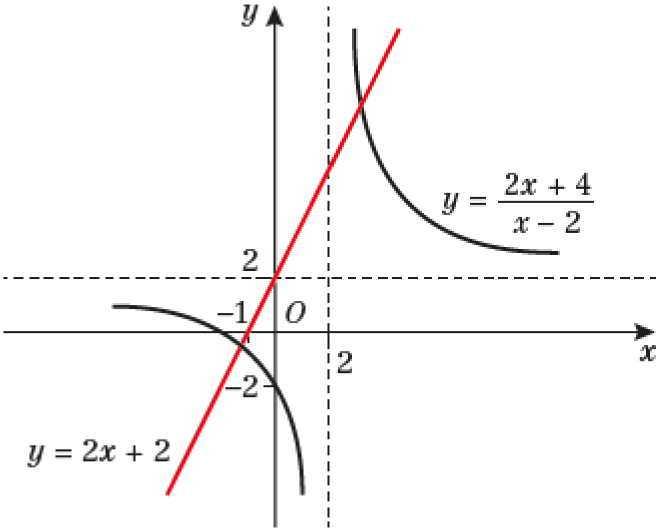
The graph is a straight line with a positive gradient that passes through $(-1, 0)$ and $(0, 2)$

$y = \dfrac{2x+4}{x-2}$

$y = 2\left(\dfrac{x+2}{x-2}\right) = 2\left(\dfrac{x-2+4}{x-2}\right) = 2\left(1+\dfrac{4}{x-2}\right)$ rearranging to see how the curve behaves as $x \to \infty$

The curve is a reciprocal graph. There is a horizontal asymptote at $y = 2$ (as $x \to \pm\infty, y \to 2$) and a vertical asymptote at $x = 2$ (as $x \to 2, y \to \pm\infty$). The graph crosses the y-axis at (0, –2).

So the sketch of both curves is:



b The x-coordinates of the points of intersection are found by equating the right-hand side of the two equations.

$2x + 2 = \dfrac{2x+4}{x-2}$

$(x+1)(x-2) = x + 2$

$x^2 - x - 2 = x + 2$

$x^2 - 2x - 4 = 0$

$\Rightarrow x = \dfrac{2 \pm \sqrt{20}}{2} = 1 \pm \sqrt{5}$ using the quadratic formula

The solution to the inequality is when the line $y = 2x + 2$ lies above the curve $y = \dfrac{2x+4}{x-2}$

Using the sketch from part **a** and the x-coordinates of the points of intersection this occurs when

$1 - \sqrt{5} < x < 2$ or $x > 1 + \sqrt{5}$

7 a $y = 2 - 4x$

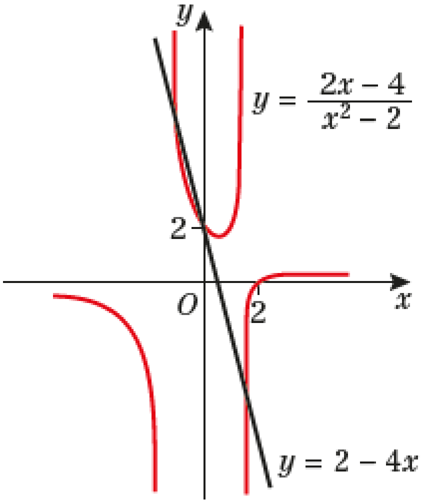
The graph is a straight line with a negative gradient that passes through (0, 2) and $(\frac{1}{2}, 0)$

$$y = \frac{2x-4}{x^2-2}$$

The graph crosses the y-axis at (0, 2) and the x-axis at (2, 0). There are vertical asymptotes at $x = \sqrt{2}$ and $x = -\sqrt{2}$. There is a horizontal asymptote at $y = 0$ (as $x \to \pm\infty$, $y \to 0$).

Note the regions where y is negative: $2x - 4 < 0$ for $x < 2$ and $x^2 - 2 < 0$ for $-\sqrt{2} < x < \sqrt{2}$ so $y < 0$ for $x < -\sqrt{2}$ and $\sqrt{2} < x < 2$

So the sketch of both curves is:



b The x-coordinates of the points of intersection are found by equating the right-hand side of the two equations.

$$2 - 4x = \frac{2x-4}{x^2-2}$$

$$(1-2x)(x^2-2) = x - 2$$

$$-2x^3 + x^2 + 4x - 2 = x - 2$$

$$2x^3 - x^2 - 3x = 0$$

$$x(2x^2 - x - 3) = 0$$

$$\Rightarrow x = 0, \frac{1 \pm \sqrt{1+24}}{4}$$ using the quadratic formula

$$\Rightarrow x = -1, 0 \text{ or } \tfrac{3}{2}$$

The solution to the inequality is when the line $y = 2 - 4x$ lies below the curve $y = \frac{2x-4}{x^2-2}$

Using the sketch from part **a** and the x-coordinates of the points of intersection this occurs when

$$-\sqrt{2} < x < -1 \text{ or } 0 < x < \sqrt{2} \text{ or } x > \tfrac{3}{2}$$

8 a $y = \dfrac{x-2}{3x-1}$

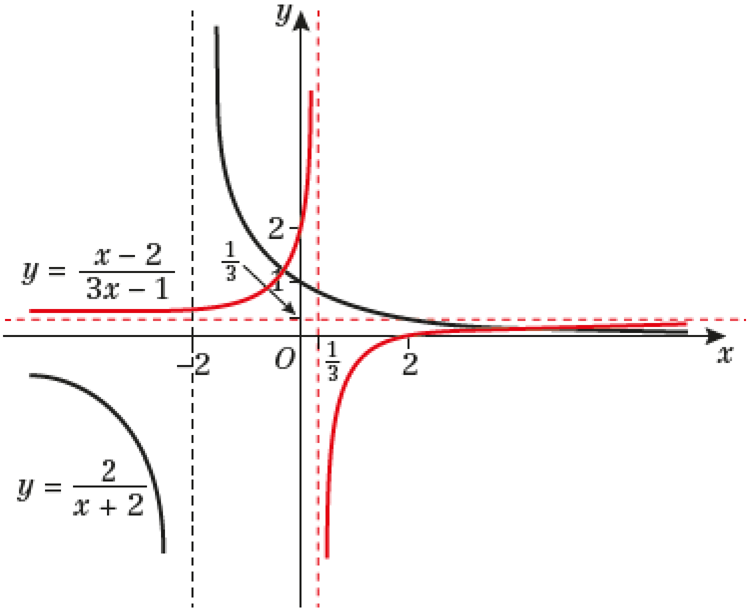
$y = \dfrac{x-2}{3x-1} = \dfrac{1}{3}\left(\dfrac{3x-1-5}{3x-1}\right) = \dfrac{1}{3}\left(1 - \dfrac{5}{3x-1}\right)$ rearranging to see how the curve behaves as $x \to \infty$

The curve is a reciprocal graph. There is a horizontal asymptote at $y = \frac{1}{3}$ (as $x \to \pm\infty$, $y \to \frac{1}{3}$) and a vertical asymptote at $x = \frac{1}{3}$ (as $x \to \frac{1}{3}$, $y \to \pm\infty$). The graph crosses the axes at (0, 2) and (2, 0).

$y = \dfrac{2}{x+2}$

The curve is a reciprocal graph. There is a horizontal asymptote at $y = 0$ (as $x \to \pm\infty$, $y \to 0$) and a vertical asymptote at $x = -2$ (as $x \to -2$, $y \to \pm\infty$). The graph crosses the axes at (0, 1).
When $x < -2$, $y < 0$ and when $x > -2$, $y > 0$.

So the sketch of both curves is:



b The x-coordinates of the points of intersection are found by equating the right-hand side of the two equations.

$\dfrac{x-2}{3x-1} = \dfrac{2}{x+2}$

$(x-2)(x+2) = 2(3x-1)$

$x^2 - 4 = 6x - 2$

$x^2 - 6x - 2 = 0$

$\Rightarrow x = \dfrac{6 \pm \sqrt{36+8}}{2}$ using the quadratic formula

$\Rightarrow x = 3 \pm \sqrt{11}$

The solution to the inequality is when the curve $y = \dfrac{x-2}{3x-1}$ lies below the curve $y = \dfrac{2}{x+2}$

Using the sketch from part **a** and the x-coordinates of the points of intersection this occurs when

$-2 < x < 3 - \sqrt{11}$ or $\frac{1}{3} < x < 3 + \sqrt{11}$

9 a $y = \frac{x+1}{x-2}$

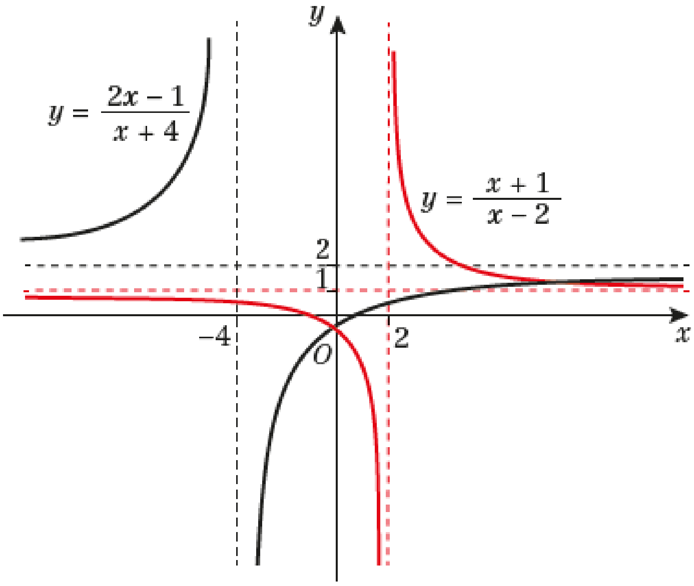
$y = \frac{x+1}{x-2} = \frac{x-2+3}{x-2} = 1 + \frac{3}{x-2}$ rearranging to see how the curve behaves as $x \to \infty$

The curve is a reciprocal graph. There is a horizontal asymptote at $y = 1$ (as $x \to \pm\infty$, $y \to 1$) and a vertical asymptote at $x = 2$ (as $x \to 2$, $y \to \pm\infty$). The graph crosses the axes at $(0, -\frac{1}{2})$ and $(-1, 0)$.

$y = \frac{2x-1}{x+4}$

$y = \frac{2x-1}{x+4} = 2\left(\frac{x+4-\frac{9}{2}}{x+4}\right) = 2\left(1 - \frac{9}{2x+8}\right)$ rearranging to see how the curve behaves as $x \to \infty$

The curve is a reciprocal graph. There is a horizontal asymptote at $y = 2$ (as $x \to \pm\infty$, $y \to 2$) and a vertical asymptote at $x = -4$ (as $x \to -4$, $y \to \pm\infty$). The graph crosses the axes at $(0, -\frac{1}{4})$ and $(\frac{1}{2}, 0)$. So the sketch of both curves is:



b The x-coordinates of the points of intersection are found by equating the right-hand side of the two equations.

$\frac{x+1}{x-2} = \frac{2x-1}{x+4}$

$(x+1)(x+4) = (2x-1)(x-2)$

$x^2 + 5x + 4 = 2x^2 - 5x + 2$

$x^2 - 10x - 2 = 0$

$\Rightarrow x = \frac{10 \pm \sqrt{100+8}}{2}$ using the quadratic formula

$\Rightarrow x = 5 \pm \sqrt{27}$

$\Rightarrow x = 5 \pm 3\sqrt{3}$

The solution to the inequality is when the curve $y = \frac{x+1}{x-2}$ lies below the curve $y = \frac{2x-1}{x+4}$

Using the sketch from part **a** and the x-coordinates of the points of intersection this occurs when

$x < -4$ or $5 - 3\sqrt{3} < x < 2$ or $x > 5 + 3\sqrt{3}$

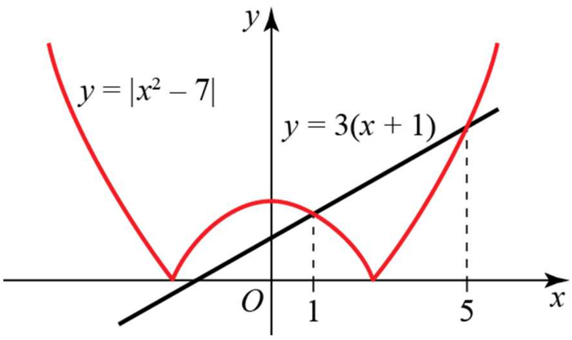
10 $y = |x^2 - 7|$

This is based on a parabola that is part reflected in the x-axis. It has a local maximum at (0, 7) and it touches the the x-axis at $(-\sqrt{7}, 0)$ and $(\sqrt{7}, 0)$

$y = 3(x+1)$

The graph is a straight line with a positive gradient that passes through (–1, 0) and (0, 3).

So the sketch of both curves is:



The critical values are given by

$$x^2 - 7 = 3x + 3$$
$$x^2 - 3x - 10 = 0$$
$$(x-5)(x+2) = 0$$
$$x = -2, 5$$

Or

$$-(x^2 - 7) = 3x + 3$$
$$x^2 + 3x - 4 = 0$$
$$(x+4)(x-1) = 0$$
$$x = -4, 1$$

From the sketch, the only valid critical values are $x = 1$ and $x = 5$.

The solution is when the line is above the curve
So the solution is $1 < x < 5$

11 Rearranging and simplifying gives

$$\frac{x^2}{|x|+6} < 1$$

$x^2 < |x| + 6$ because $|x| + 6$ is always positive

$x^2 - 6 < |x|$

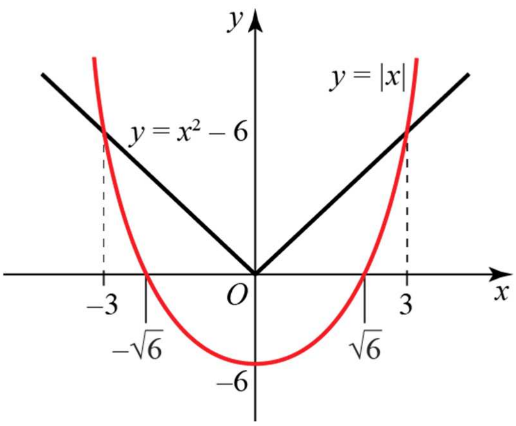
The critical values are given by

$$x^2 - 6 = x$$
$$x^2 - x - 6 = 0$$
$$(x-3)(x+2) = 0$$
$$x = -2, 3$$

Or

$$x^2 - 6 = -x$$
$$x^2 + x - 6 = 0$$
$$(x-2)(x+3) = 0$$
$$x = -3, 2$$

Sketching both curves gives:



From the sketch, the intersections are $> \left|\sqrt{6}\right|$, so the valid critical values are $x = -3$ and $x = 3$.

The solution is when the curve is below $y = |x|$
So the solution is $-3 < x < 3$

12 The critical values are given by

$x-1=6x-1$

$x=0$

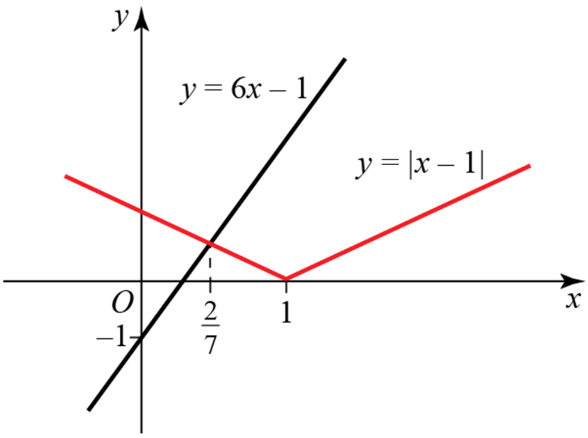
Or

$-(x-1)=6x-1$

$7x=2$

$x=\frac{2}{7}$

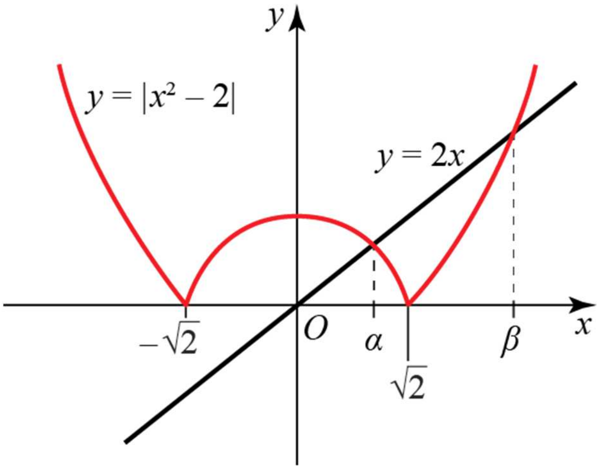
Sketching both curves gives:



From the sketch, the only valid critical values is $x=\frac{2}{7}$

The solution is when the v-shaped curve is above the line
So the solution is $x<\frac{2}{7}$

13 Sketching $y = |x^2 - 2|$ and $y = 2x$ gives



The critical values are given by

$x^2 - 2 = 2x$

$x^2 - 2x - 2 = 0$

$x = \frac{2 \pm \sqrt{4+8}}{2} = 1 \pm \sqrt{3}$ using the quadratic formula

Or

$-(x^2 - 2) = 2x$

$x^2 + 2x - 2 = 0$

$x = \frac{-2 \pm \sqrt{4+8}}{2} = -1 \pm \sqrt{3}$ using the quadratic formula

From the sketch, valid critical values are greater than 0, so the two valid critical values are $\alpha = \sqrt{3} - 1$ and $\beta = 1 + \sqrt{3}$

The solution is when the line is below the curve

So the solution is $x < \sqrt{3} - 1$ or $x > 1 + \sqrt{3}$

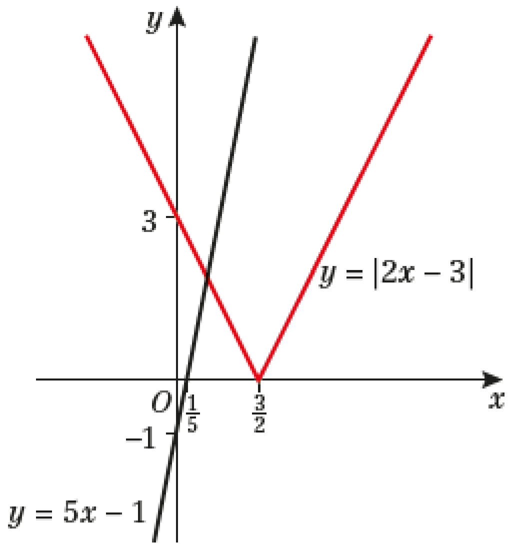
14 a $y = 5x - 1$

The graph is a straight line with a positive gradient that passes through (0, –1) and $(\frac{1}{5}, 0)$

$y = |2x - 3|$

This function has a v-shaped graph that touches the x-axis at $(\frac{3}{2}, 0)$ and crosses that y-axis at (0, 3). The graph has a positive gradient for $x > \frac{3}{2}$ and a negative gradient for $x < \frac{3}{2}$

Sketching both curves gives:



b The critical values are given by

$2x - 3 = 5x - 1$

$x = -\frac{2}{3}$

From the sketch, this is not a valid critical value.

Or

$-(2x - 3) = 5x - 1$

$x = \frac{4}{7}$

The solution is when the v-shaped curve is below the line

So the solution is $x > \frac{4}{7}$

15 a Consider, in turn, when the argument of the modulus function is positive and is negative.

$$2x^2 + x - 6 = 6 - 3x$$
$$2x^2 + 4x - 12 = 0$$
$$2(x^2 + 2x - 6) = 0$$
$$x = \frac{-2 \pm \sqrt{28}}{2} \quad \text{using the quadratic formula}$$
$$x = -1 \pm \sqrt{7}$$

$$-(2x^2 + x - 6) = 6 - 3x$$
$$2x^2 - 2x = 0$$
$$2x(x-1) = 0$$
$$x = 0 \text{ or } 1$$

Solution: $x = -1-\sqrt{7},\ 0,\ 1,\ -1+\sqrt{7}$

b $y = 2x^2 + x - 6 = (2x-3)(x+2)$

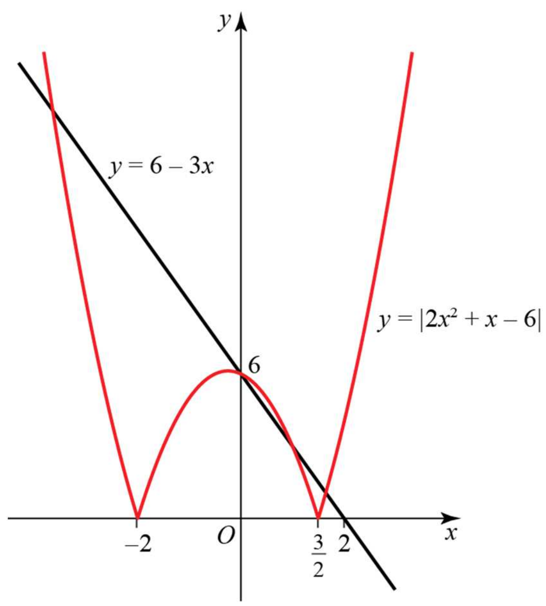
The curve is a quadratic graph with a positive x^2 coefficient, so it is a parabola with a minimum below the x-axis. The graph crosses the x-axis at $(-2, 0)$ and $(\frac{3}{2}, 0)$ and the y-axis at $(0, -6)$.

So the graph of $y = |2x^2 + x - 6|$ will be the graph of $y = 2x^2 + x - 6$ but with the section of the latter curve that is below the x-axis reflected in the x-axis. The graph crosses the x-axis at $(-2, 0)$ and $(\frac{3}{2}, 0)$ and the y-axis at $(0, 6)$.

$y = 6 - 3x$

The graph is a straight line with a negative gradient that passes through $(0, 6)$ and $(2, 0)$.

So a sketch of both curves is:



c The solution is when the curve is above the line

So the solution is $x < -1-\sqrt{7}$ or $0 < x < 1$ or $x > \sqrt{7} - 1$

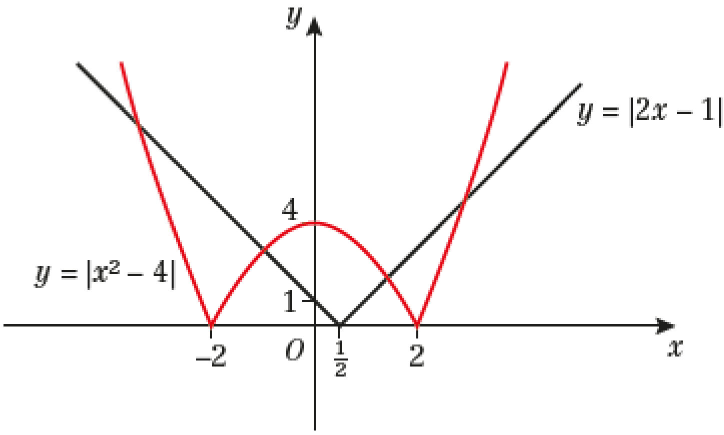
16 a $y = |x^2 - 4|$

The graph of $y = |x^2 - 4|$ will be the graph of $y = x^2 - 4$ but with the section of the latter curve that is below the x-axis reflected in the x-axis. The graph crosses the x-axis at $(-2, 0)$ and $(2, 0)$ and the y-axis at (0, 4), where there is a local maximum

$y = |2x - 1|$

This function has a v-shaped graph that touches the x-axis at $(\frac{1}{2}, 0)$ and crosses that y-axis at (0, 1). The graph has a positive gradient for $x > \frac{1}{2}$ and a negative gradient for $x < \frac{1}{2}$

So a sketch of both curves is:



b Consider, in turn, when the argument of the modulus function is positive and is negative.

$$x^2 - 4 = 2x - 1$$
$$x^2 - 2x - 3 = 0$$
$$(x-3)(x+1) = 0$$
$$x = -1, 3$$

$$-(x^2 - 4) = 2x - 1$$
$$x^2 + 2x - 5 = 0$$
$$x = \frac{-2 \pm \sqrt{24}}{2} \quad \text{using the quadratic formula}$$
$$x = -1 \pm \sqrt{6}$$

Solution: $x = -1 - \sqrt{6},\ -1,\ -1 + \sqrt{6},\ 3$

c The solution is when the v-shaped graph is below the curve
So the solution is $x < -1 - \sqrt{6}$ or $-1 < x < -1 + \sqrt{6}$ or $x > 3$

Challenge

Solving $|x^2 - 5x + 2| = |x - 3|$

$x^2 - 5x + 2 = x - 3$

$x^2 - 6x + 5 = 0$

$(x - 5)(x - 1) = 0$

$x = 1, 5$

Or

$-(x^2 - 5x + 2) = x - 3$

$x^2 - 4x - 1 = 0$

$x = \frac{4 \pm \sqrt{20}}{2}$ using the quadratic formula

$x = 2 \pm \sqrt{5}$

Solution: $x = 2 - \sqrt{5},\ 1,\ 2 + \sqrt{5},\ 5$

$y = x^2 - 5x + 2$

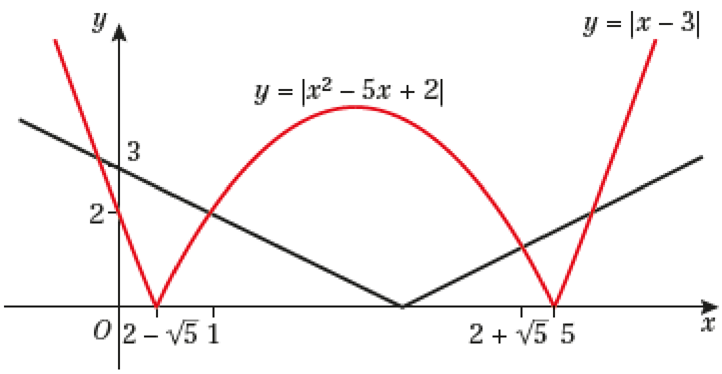
The curve is a quadratic graph with a positive x^2 coefficient. It cuts the y-axis at (0, 2). It cuts the x-axis when $x = \frac{1}{2}(5 \pm \sqrt{17})$.

So the graph of $y = |x^2 - 5x + 2|$ will be the graph of $y = x^2 - 5x + 2$ but with the section of the latter curve that is below the x-axis reflected in the x-axis.

$y = |x - 3|$

This function has a v-shaped graph that touches the x-axis at $(3, 0)$ and crosses that y-axis at (0, 3). The graph has a positive gradient for $x > 3$ and a negative gradient for $x < 3$

So a sketch of both curves is:



The solution is when the curve is above the v-shaped graph

So the solution in set notation is $\{x: x < 2 - \sqrt{5}\} \cup \{x: 1 < x < 2 + \sqrt{5}\} \cup \{x: x > 5\}$