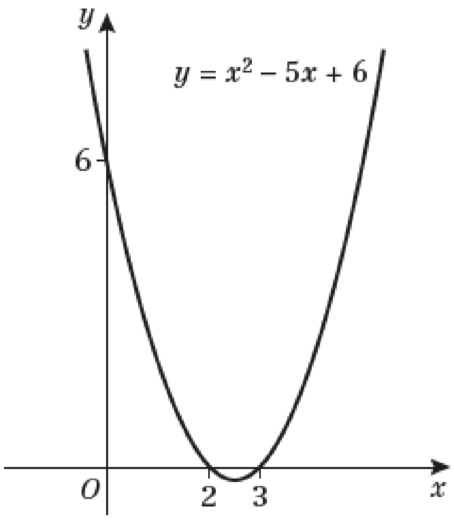
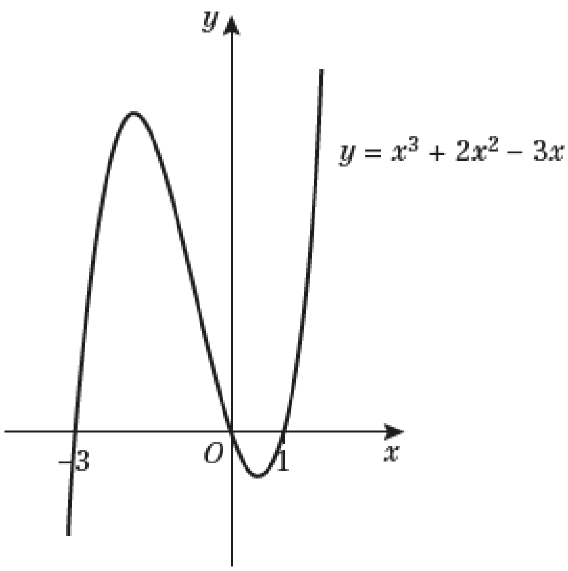
Exercise 1B

1 a $y = x^2 - 5x + 6$
$y = (x-3)(x-2)$ factorising to find when the curve cuts the x-axis
The curve is a quadratic graph with a positive x^2 coefficient, so it is a parabola with a minimum.
The graph crosses the x-axis at (3, 0) and (2, 0) and the y-axis at (0, 6).
So the sketch is:



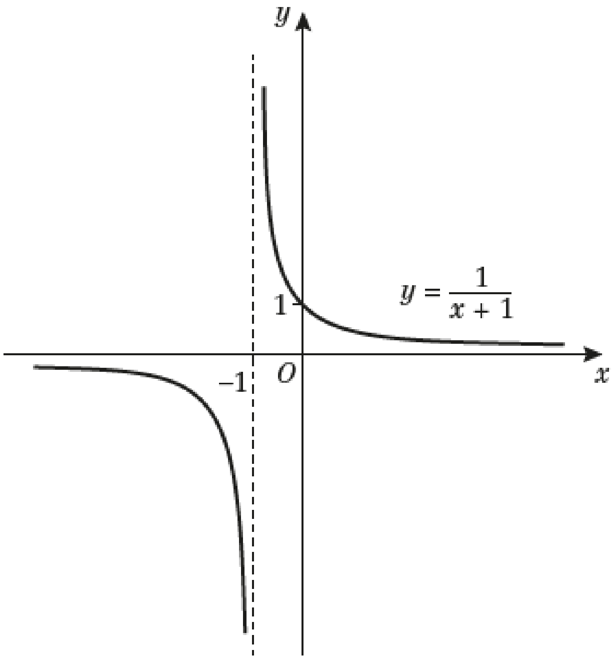
b $y = x^3 + 2x^2 - 3x$
$y = x(x^2 + 2x - 3)$
$y = x(x-1)(x+3)$
The curve is a cubic graph with a positive x^3 coefficient, so as $x \to \infty, y \to \infty$ and as $x \to -\infty, y \to -\infty$ and the graph crosses x-axis at (–3, 0), (0, 0) and (1, 0).
So the sketch is:



1 c $y = \frac{1}{x+1}$

The curve is a reciprocal graph. There is a horizontal asymptote at $y = 0$ (as $x \to \pm\infty, y \to 0$) and a vertical asymptote at $x = -1$ (as $x \to -1, y \to \pm\infty$). The graph crosses the y-axis at (0, 1).

So the sketch is:

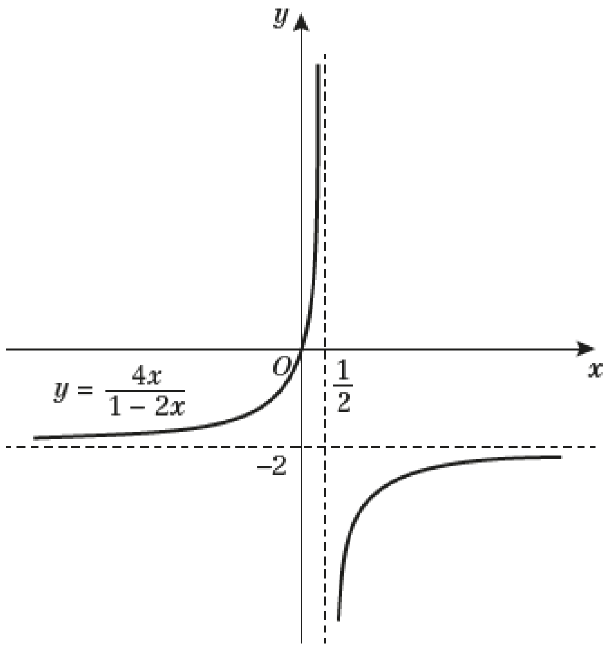


d $y = \frac{4x}{1-2x}$

$y = \frac{4x}{1-2x} = -2\left(1 - \frac{1}{1-2x}\right)$ rearranging to see how the curve behaves as $x \to \infty$

The curve is a reciprocal graph. There is a horizontal asymptote at $y = -2$ (as $x \to \pm\infty, y \to -2$) and a vertical asymptote at $x = \frac{1}{2}$ (as $x \to \frac{1}{2}, y \to \pm\infty$). The graph crosses the axes at (0, 0).

So the sketch is:



2 a $y = x^2 - 2x + 1$

$y = (x-1)(x-1) = (x-1)^2$

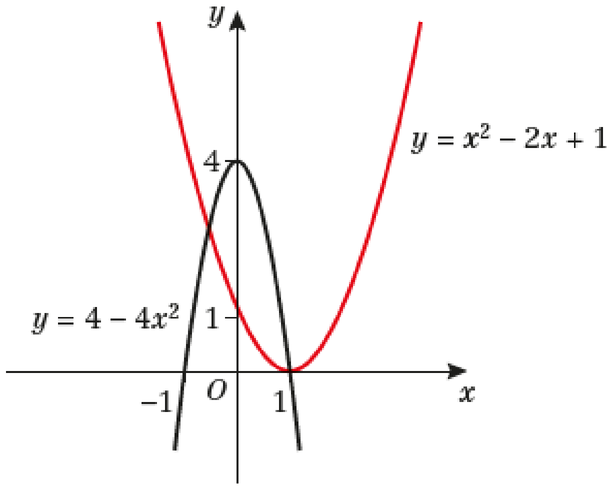
The curve is a quadratic graph with a positive x^2 coefficient, so it is a parabola and it has a minimum at (1, 0). The graph crosses the y-axis at (0, 1).

$y = 4 - 4x^2$

$y = 4(1 - x^2) = -4(x-1)(x+1)$

The curve is a quadratic graph with a negative x^2 coefficient, so it is a parabola and it has a maximum at (0, 4). The graph crosses the x-axis at (–1, 0) and (1, 0).

So the sketch of both curves is:



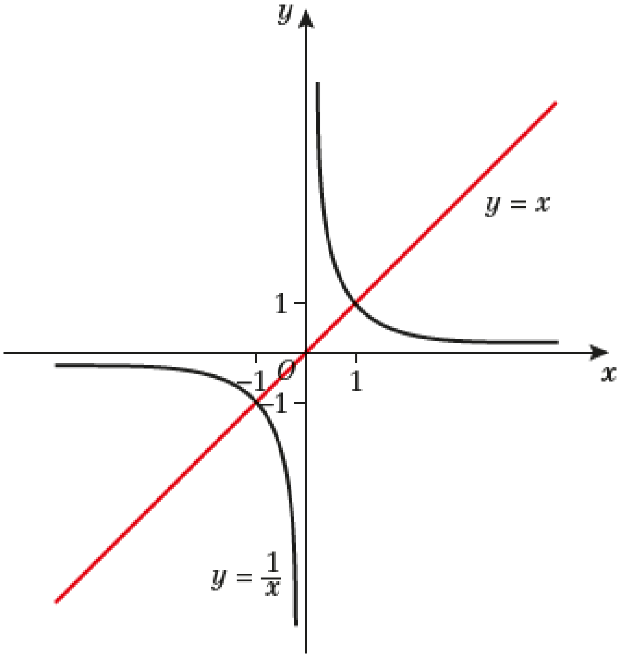
2 b $y = x$

The graph is a straight line with a positive gradient of 1 that passes through (0, 0).

The curve $y = \frac{1}{x}$ has a reciprocal graph.

There is a horizontal asymptote at $y = 0$ (as $x \to \pm\infty, y \to 0$) and a vertical asymptote at $x = 0$ (as $x \to 0, y \to \pm\infty$). The graph does not cut the coordinate axes.

So the sketch of both curves is:



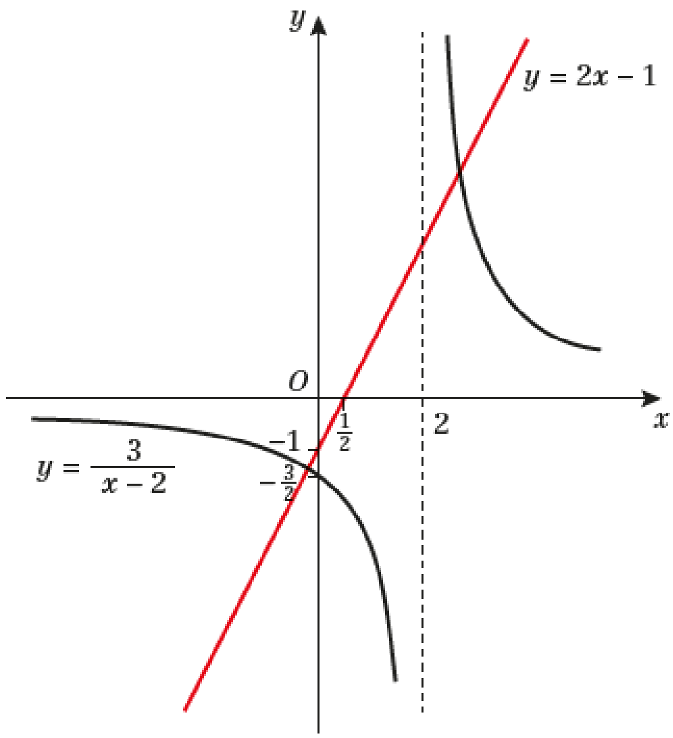
2 c $y = 2x - 1$

The graph is a straight line with a positive gradient of 2 that passes through $(0,-1)$ and $(\frac{1}{2},0)$

$y = \dfrac{3}{x-2}$

The curve is a reciprocal graph. There is a horizontal asymptote at $y = 0$ (as $x \to \pm\infty$, $y \to 0$) and a vertical asymptote at $x = 2$ (as $x \to 2$, $y \to \pm\infty$). The graph crosses the y-axis at $(0,-\frac{3}{2})$.

So the sketch of both curves is:



2 d $y = 4 - 3x$

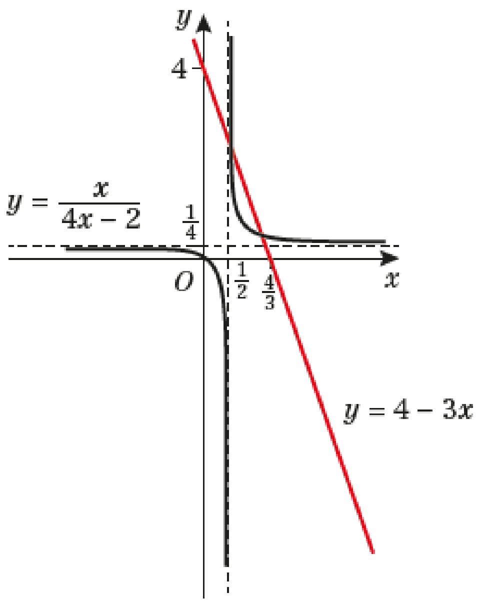
The graph is a straight line with a negative gradient that passes through $(0, 4)$ and $(\frac{4}{3}, 0)$

$$y = \frac{x}{4x-2}$$

$$y = \frac{x}{4x-2} = \frac{1}{4}\left(\frac{4x}{4x-2}\right) = \frac{1}{4}\left(1 + \frac{2}{4x-2}\right)$$ rearranging to see how the curve behaves as $x \to \infty$

The curve is a reciprocal graph. There is a horizontal asymptote at $y = \frac{1}{4}$ (as $x \to \pm\infty$, $y \to \frac{1}{4}$) and a vertical asymptote at $x = \frac{1}{2}$ (as $x \to \frac{1}{2}$, $y \to \pm\infty$). The graph crosses the axes at (0, 0).

So the sketch of both curves is:



3 a The x-coordinate of the point of intersection is found by equating the right-hand side of the two equations.

$$\frac{2}{x+1} = \frac{1}{x-3}$$

$$2(x-3) = x+1$$

$$2x - x = 1 + 6$$

$$\Rightarrow x = 7$$

The y-coordinate of the point of intersection is found by substituting the x-coordinate into either of the two equations.

$$y = \frac{1}{x-3} = \frac{1}{7-3} = \frac{1}{4}$$

Therefore the functions intersect at $(7, \frac{1}{4})$

3 b The x-coordinates of the points of intersection are found by equating the right-hand side of the two equations.

$$x-2=\frac{3x}{x+2}$$
$$(x-2)(x+2)=3x$$
$$x^2-3x-4=0$$
$$(x-4)(x+1)=0$$
$$\Rightarrow x=4,-1$$

The y-coordinates of the points of intersection are found by substituting the x-coordinates into either of the two equations.

For $x=4$, $y=x-2=4-2=2$

For $x=-1$, $y=x-2=-1-2=-3$

Therefore the functions intersect at (4, 2) and (–1, –3)

c The x-coordinates of the points of intersection are found by equating the right-hand side of the two equations.

$$x^2-4=\frac{4(x+2)}{x-2}$$
$$(x+2)(x-2)=\frac{4(x+2)}{x-2}$$
$$(x+2)(x-2)^2=4(x+2)$$
$$(x+2)\left((x-2)^2-4\right)=0$$
$$(x+2)(x^2-4x+4-4)=0$$
$$(x+2)(x^2-4x)=0$$
$$(x+2)x(x-4)=0$$
$$\Rightarrow x=-2, 0, 4$$

The y-coordinates of the points of intersection are found by substituting the x-coordinates into either of the two equations.

For $x=-2$, $y=x^2-4=(-2)^2-4=0$

For $x=0$, $y=x^2-4=0^2-4=-4$

For $x=4$, $y=x^2-4=4^2-4=12$

Therefore the functions intersect at (–2, 0), (0, –4) and (4, 12)

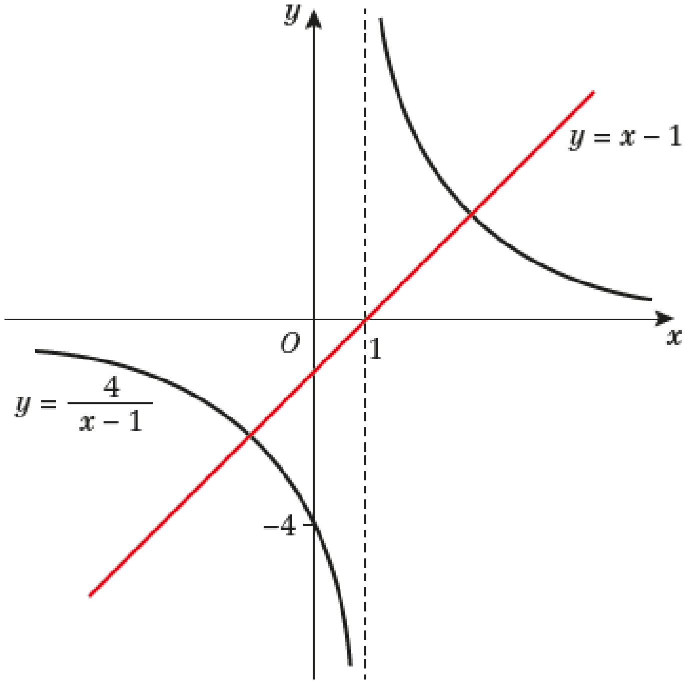
4 a $y = x - 1$

The graph is a straight line with a positive gradient of 1 that passes through (0, –1) and (1, 0)

$y = \dfrac{4}{x-1}$

The curve is a reciprocal graph. There is a horizontal asymptote at $y = 0$ (as $x \to \pm\infty$, $y \to 0$) and a vertical asymptote at $x = 1$ (as $x \to 1$, $y \to \pm\infty$). The graph crosses the y-axis at (0, –4).

So the sketch of both curves is:



b The x-coordinates of the points of intersection are found by equating the right-hand side of the two equations.

$x - 1 = \dfrac{4}{x-1}$

$x^2 - 2x + 1 = 4$

$x^2 - 2x - 3 = 0$

$(x-3)(x+1) = 0$

$\Rightarrow x = -1, 3$

The y-coordinates of the points of intersection are found by substituting the x-coordinates into either of the two equations.

For $x = 3$, $y = x - 1 = 3 - 1 = 2$

For $x = -1$, $y = x - 1 = -1 - 1 = -2$

Therefore the functions intersect at (–1, –2), and (3, 2)

c The solution to the inequality is when the line $y = x - 1$ lies above the curve $y = \dfrac{4}{x-1}$

Using the sketch from part **a** and the points of intersection from part **b** this occurs when $-1 < x < 1$ or $x > 3$. The solution in set notation is $\{x : -1 < x < 1\} \cup \{x : x > 3\}$

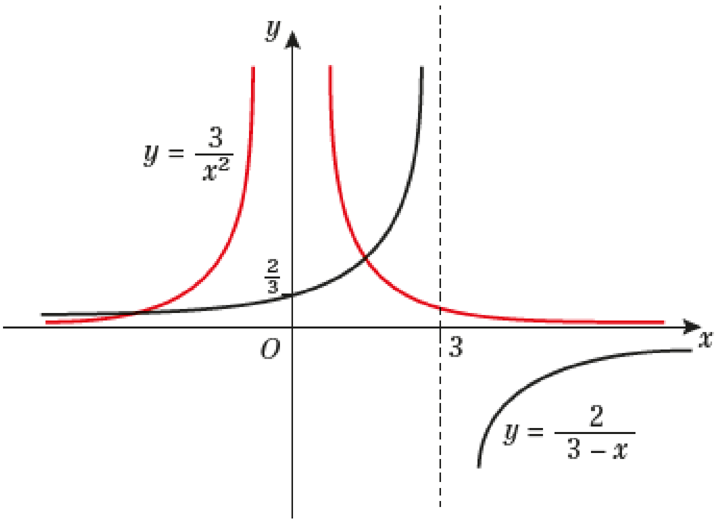
5 a $y = \mathrm{f}(x) = \dfrac{3}{x^2}$

This curve is always positive ($y > 0$), with a horizontal asymptote at $y = 0$ (as $x \to \pm\infty$, $y \to 0$) and a vertical asymptote at $x = 0$ (as $x \to 0$, $y \to \infty$). The graph does not cut the coordinate axes.

$y = \mathrm{g}(x) = \dfrac{2}{3 - x}$

The curve is a reciprocal graph. There is a horizontal asymptote at $y = 0$ (as $x \to \pm\infty$, $y \to 0$) and a vertical asymptote at $x = 3$ (as $x \to 3$, $y \to \pm\infty$). The graph crosses the y-axis at $(0, \frac{2}{3})$.

So the sketch of both curves is:



b The x-coordinates of the points of intersection are found by equating the right-hand side of the two equations.

$\dfrac{3}{x^2} = \dfrac{2}{3 - x}$

$3(3 - x) = 2x^2$

$2x^2 + 3x - 9 = 0$

$(2x - 3)(x + 3) = 0$

$\Rightarrow x = -3, \dfrac{3}{2}$

For $x = \dfrac{3}{2}$, $y = \dfrac{3}{x^2} = \dfrac{3}{\left(\frac{3}{2}\right)^2} = \dfrac{4}{3}$

For $x = -3$, $y = \dfrac{3}{x^2} = \dfrac{3}{(-3)^2} = \dfrac{1}{3}$

Therefore the points of intersection are $(-3, \frac{1}{3})$ and $(\frac{3}{2}, \frac{4}{3})$

5 c The solution to the inequality is when the curve $y=\frac{3}{x^2}$ lies above the curve $y=\frac{2}{3-x}$

Using the sketch from part **a** and the points of intersection from part **b** this occurs when

$-3<x<\frac{3}{2}$ or $x>3$

So the solution in set notation is $\{x: -3<x<\frac{3}{2}\}\cup\{x: x>3\}$

6 a $y=\frac{3x}{2-x}$

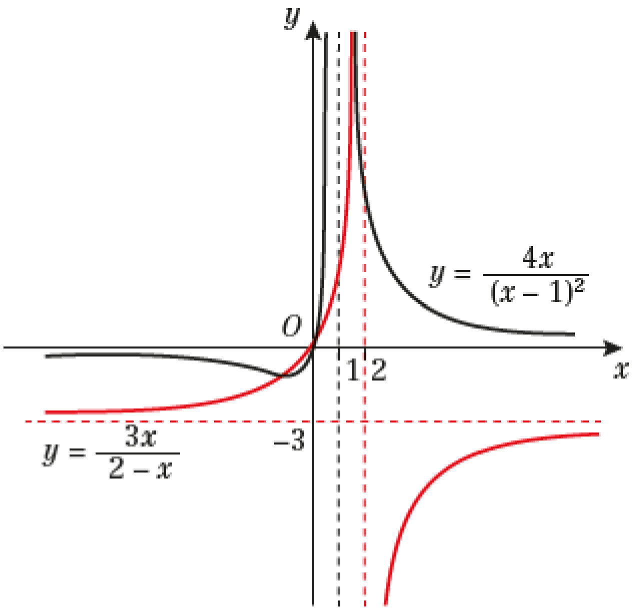
$y=\frac{3x}{2-x}=-3\left(1-\frac{2}{2-x}\right)$ rearranging to see how the curve behaves as $x\to\infty$

The curve is a reciprocal graph. There is a horizontal asymptote at $y=-3$ (as $x\to\pm\infty, y\to -3$) and a vertical asymptote at $x=2$ (as $x\to 2, y\to\pm\infty$). The graph crosses the axes at (0, 0).

$y=\frac{4x}{(x-1)^2}$

There is a horizontal asymptote at $y=0$ (as $x\to\pm\infty, y\to 0$) and a vertical asymptote at $x = 1$ (as $x\to 1, y\to\infty$). The graph crosses the axes at (0, 0). Note also that as the denominator is always positive (for $x\neq 1$) then if $x > 0$, then $y > 0$; and if $x < 0$, then $y < 0$.

So the sketch of both curves is:



6 **b** The x-coordinates of the points of intersection are found by equating the right-hand side of the two equations.

$$\frac{3x}{2-x}=\frac{4x}{(x-1)^2}$$

$$3x(x-1)^2=4x(2-x)$$

$$x(3x^2-6x+3-8+4x)=0$$

$$x(3x^2-2x-5)=0$$

$$x(3x-5)(x+1)=0$$

$$\Rightarrow x=-1,\ 0,\ \tfrac{5}{3}$$

For $x=-1$, $y=\dfrac{3x}{2-x}=\dfrac{3\times-1}{2-(-1)}=-1$

For $x=0$, $y=0$

For $x=\dfrac{5}{3}$, $y=\dfrac{3x}{2-x}=\dfrac{3\times\frac{5}{3}}{2-\frac{5}{3}}=15$

Therefore the points of intersection are $(-1,-1)$, $(0, 0)$ and $(\frac{5}{3}, 15)$

c The solution to the inequality is when the curve $y=\dfrac{4x}{(x-1)^2}$ lies on or above the curve $y=\dfrac{3x}{2-x}$

Using the sketch from part **a** and the points of intersection from part **b** this occurs when

$$x\leqslant-1 \text{ or } 0\leqslant x<1 \text{ or } 1<x\leqslant\tfrac{5}{3} \text{ or } x\geqslant 2$$

Note that there are four intervals as the inequality is not defined when $x = 1$, and as the inequality is less than or equal to ($\leqslant$), the values of x at the points of intersection are included in the solution set (i.e. when $x=-1,\ 0$ or $\frac{5}{3}$).

7 a $y = x - 2$

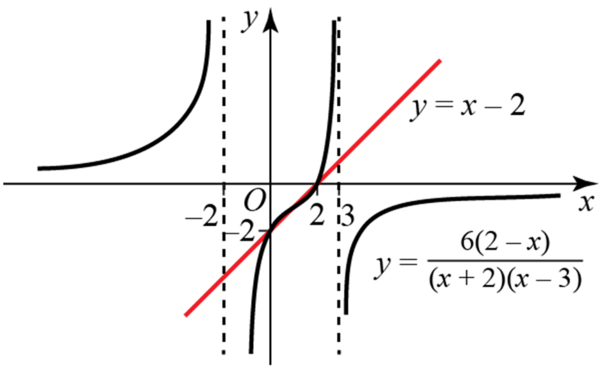
The graph is a straight line with a positive gradient of 1 that passes through (0, –2) and (2, 0)

$$y = \frac{6(2-x)}{(x+2)(x-3)}$$

The graph crosses the y-axis at (0, –2) and the x–axis at (2, 0). There are vertical asymptotes at $x = 3$ and $x = -2$. There is a horizontal asymptote at $y = 0$ (as $x \to \pm\infty$, $y \to 0$).

Note the regions where y is positive, and where it is negative: for $x > 3$, $y < 0$; for $2 < x < 3$, $y > 0$; for $-2 < x < 2$, $y < 0$; for $x < -2$, $y > 0$.

So the sketch of both curves is:



b The x-coordinates of the points of intersection are found by equating the right-hand side of the two equations.

$$x - 2 = \frac{6(2-x)}{(x+2)(x-3)}$$

$$(x-2)(x+2)(x-3) = -6(x-2)$$

$$(x-2)(x^2 - x - 6 + 6) = 0$$

$$(x-2)x(x-1) = 0$$

$$\Rightarrow x = 0,\ 1,\ 2$$

For $x = 2$, $y = x - 2 = 2 - 2 = 0$

For $x = 0$, $y = x - 2 = 0 - 2 = -2$

For $x = 1$, $y = x - 2 = 1 - 2 = -1$

Therefore the points of intersection are (0, –2), (1, –1) and (2, 0)

c The solution is when the line $y = x - 2$ lies on or below the curve $y = \dfrac{6(2-x)}{(x+2)(x-3)}$

Using the sketch from part **a** and the points of intersection from part **b** this occurs when

$$x < -2 \text{ or } 0 \leqslant x \leqslant 1 \text{ or } 2 \leqslant x < 3$$

8 a $y = \frac{1}{x}$

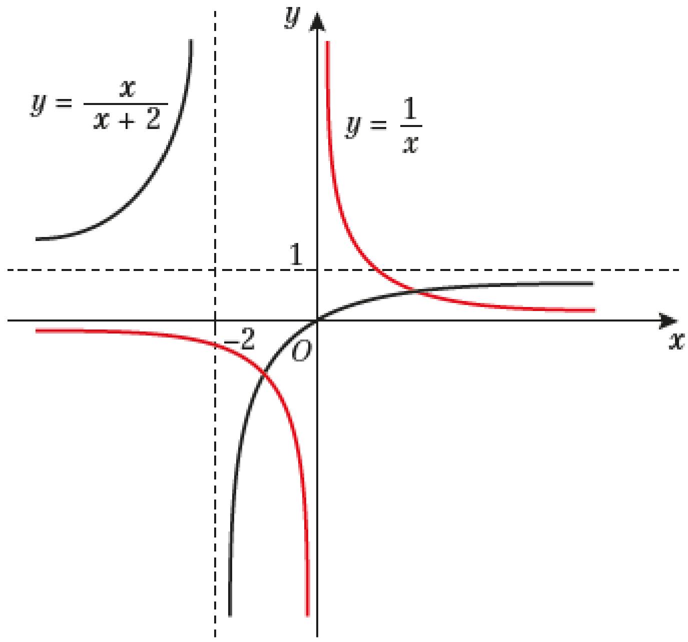
The curve is a reciprocal graph. There is a horizontal asymptote at $y = 0$ (as $x \to \pm\infty, y \to 0$) and a vertical asymptote at $x = 0$ (as $x \to 0, y \to \pm\infty$). The graph does not cross the axes.

$y = \frac{x}{x+2}$

$y = \frac{x}{x+2} = \frac{x+2-2}{x+2} = 1 - \frac{2}{x+2}$ rearranging to see how the curve behaves as $x \to \infty$

The curve is a reciprocal graph. There is a horizontal asymptote at $y = 1$ (as $x \to \pm\infty, y \to 1$) and a vertical asymptote at $x = -2$ (as $x \to -2, y \to \pm\infty$). The graph crosses the axes at (0, 0).

So the sketch of both curves is:



b The x-coordinates of the points of intersection are found by equating the right-hand side of the two equations.

$\frac{1}{x} = \frac{x}{x+2}$

$x + 2 = x^2$

$x^2 - x - 2 = 0$

$(x-2)(x+1) = 0$

$\Rightarrow x = -1, 2$

For $x = 2$, $y = \frac{1}{x} = \frac{1}{2}$

For $x = -1$, $y = \frac{1}{x} = -1$

Therefore the points of intersection are $(-1, -1)$ and $(2, \frac{1}{2})$

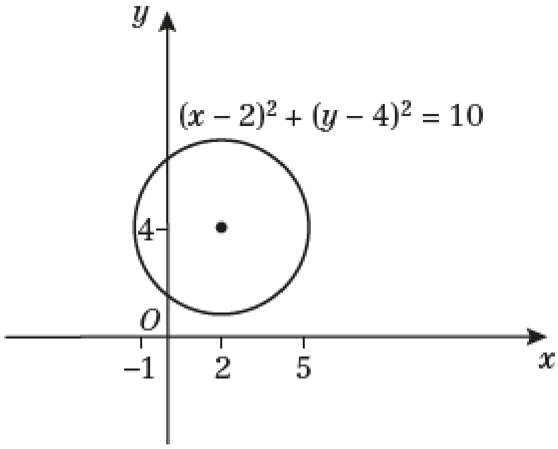
c The solution is when the curve $y = \frac{1}{x}$ lies above the curve $y = \frac{x}{x+2}$

Using the sketch from part **a** and the points of intersection from part **b** this occurs when

$-2 < x < -1$ or $0 < x < 2$

Challenge

a The circle has its centre at (2, 4). The radius of the circle is $\sqrt{10}$.
When $y = 0, (x-2)^2 + (-4)^2 = 10 \Rightarrow (x-2)^2 = -6$. There are no real solutions, so the circle does not intersect the x-axis.
When $x = 0, (-2)^2 + (y-4)^2 = 10 \Rightarrow (y-4)^2 = 6 \Rightarrow y = 4 \pm \sqrt{6}$.
So the circle intersects the y-axis at $\left(0, 4-\sqrt{6}\right)$ and $\left(0, 4+\sqrt{6}\right)$
So the sketch is:



Challenge

b The x-coordinates of the points of intersection are found by substituting the equation for y into the equation of the circle.

$$(x-2)^2+\left(\frac{4x-5}{x-2}-4\right)^2=10$$

$$(x-2)^2+\left(\frac{4x-5-4(x-2)}{x-2}\right)^2=10$$

$$(x-2)^2+\left(\frac{4x-5-4x+8}{x-2}\right)^2=10$$

$$(x-2)^2+\left(\frac{3}{x-2}\right)^2=10$$

$$(x-2)^4+9=10(x-2)^2$$

$$(x-2)^4-10(x-2)^2+9=0$$

$$\left((x-2)^2-9\right)\left((x-2)^2-1\right)=0$$

$$(x^2-4x-5)(x^2-4x+3)=0$$

$$(x-5)(x+1)(x-3)(x-1)=0$$

$$\Rightarrow x=-1,\ 1,\ 3,\ 5$$

For $x=-1$, $y=\frac{4x-5}{x-2}=\frac{4\times-1-5}{-1-2}=3$

For $x=1$, $y=\frac{4x-5}{x-2}=\frac{4\times1-5}{1-2}=1$

For $x=3$, $y=\frac{4x-5}{x-2}=\frac{4\times3-5}{3-2}=7$

For $x=5$, $y=\frac{4x-5}{x-2}=\frac{4\times5-5}{5-2}=5$

Therefore the points of intersection are (–1, 3), (1, 1), (3, 7) and (5, 5)

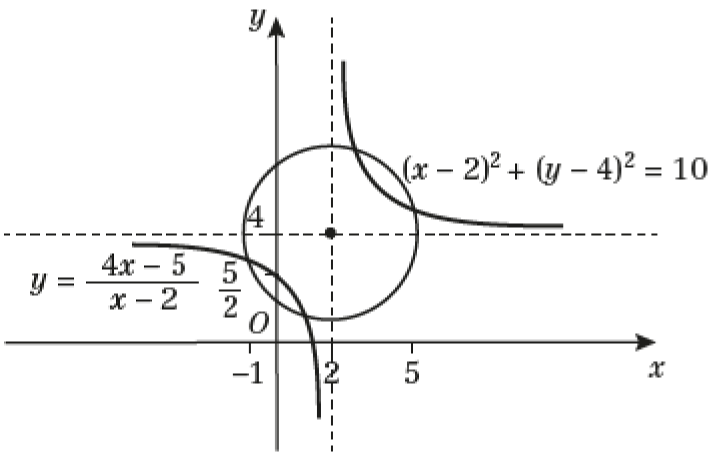
Challenge

c $y = \dfrac{4x-5}{x-2}$

$y = \dfrac{4x-5}{x-2} = 4\left(\dfrac{x-\frac{5}{4}}{x-2}\right) = 4\left(\dfrac{x-2+\frac{3}{4}}{x-2}\right) = 4\left(1+\dfrac{3}{4(x-2)}\right)$

The curve is a reciprocal graph. There is a horizontal asymptote at $y = 4$ (as $x \to \pm\infty$, $y \to 4$) and a vertical asymptote at $x = 2$ (as $x \to 2$, $y \to \pm\infty$). The graph crosses the axes at $(\frac{5}{4}, 0)$ and $(0, \frac{5}{2})$.

So the sketch of both curves is:



Challenge

d The inequality holds when the curve $y = \dfrac{4x-5}{x-2}$ lies within the circle.

Using the sketch from part **c** and the points of intersection from part **b** this occurs when

$-1 < x < 1$ or $3 < x < 5$

Alternatively, the problem can be tackled algebraically by solving

$$(x-2)^2 + \left(\frac{4x-5}{x-2} - 4\right)^2 < 10$$

$$(x-2)^2 + \left(\frac{4x-5+4(x-2)}{x-2}\right)^2 < 10$$

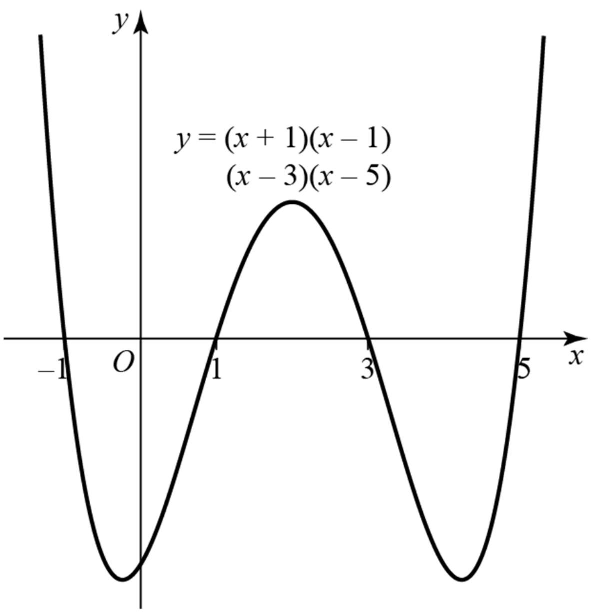
$$(x-2)^2 + \left(\frac{3}{x-2}\right)^2 < 10$$

Multiply both sides by $(x-2)^2$ and following the same algebraic steps as part **b** gives

$(x-5)(x+1)(x-3)(x-1) < 0$

So the critical values are $x = -1,\ 1,\ 3$ or 5

The curve $y = (x+1)(x-1)(x-3)(x-5)$ is a quartic graph with positive x^4 coefficient, so the curve starts in the top left and ends in the top right and passes through $(-1, 0), (1, 0), (3, 0)$ and $(5, 0)$. A sketch of the curve is



The solution to corresponds to the section of the graph that is below the x-axis.
So the solution is $-1 < x < 1$ or $3 < x < 5$