RECOGNISING ACHIEVEMENT

GCE

Mathematics

Advanced GCE

Unit **4726:** Further Pure Mathematics 2

Mark Scheme for January 2011

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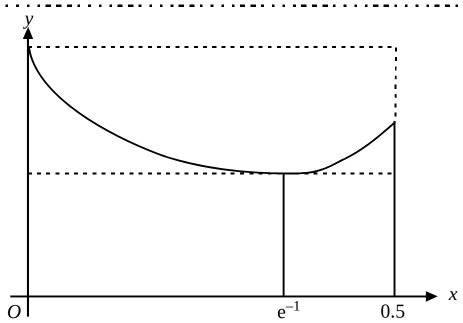
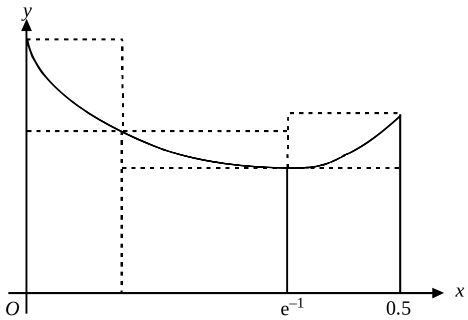
1	$t = \tan\frac{1}{2}x \Rightarrow \mathrm{d}t = \frac{1}{2}\sec^2\frac{1}{2}x\,\mathrm{d}x = \frac{1}{2}(1+t^2)\,\mathrm{d}x$	B1		For correct result **AEF** (may be implied)
	$\int \frac{1}{1+\sin x+\cos x}\,\mathrm{d}x = \int \frac{1}{1+\frac{2t}{1+t^2}+\frac{1-t^2}{1+t^2}} \cdot \frac{2}{1+t^2}\,\mathrm{d}t$	M1 A1		For substituting throughout for x For correct unsimplified t integral
	$= \int \frac{1}{1+t}\,\mathrm{d}t = \ln\lvert 1+t\rvert\ (+c)$	M1		For integrating (even incorrectly) to $a\ln\lvert \mathrm{f}(t)\rvert$. Allow $\lvert\ \rvert$ or $(\ )$
	$= \ln k\left\lvert 1+\tan\frac{1}{2}x\right\rvert\ (+c)$	A1	**5**	For correct x expression k may be numerical, c is not required
			[5]	
2 (i)	$\mathrm{f}(x) = \tanh^{-1}x,\ \mathrm{f}'(x) = \frac{1}{1-x^2},\ \mathrm{f}''(x) = \frac{2x}{(1-x^2)^2}$	M1 A1		For quoting $\mathrm{f}'(x) = \frac{1}{1\pm x^2}$ and attempting to differentiate $\mathrm{f}'(x)$ For $\mathrm{f}''(x)$ correct **www**
	$\mathrm{f}'''(x) =$ $\frac{2(1-x^2)^2 - 2x.2(1-x^2).-2x}{(1-x^2)^4}$ *OR* $\frac{2x.4x}{(1-x^2)^3} + \frac{2}{(1-x^2)^2}$	M1 A1		For using quotient *OR* product rule on $\mathrm{f}''(x)$ For correct unsimplified $\mathrm{f}'''(x)$
	$= \frac{2(1-x^2)^2 + 8x^2(1-x^2)}{(1-x^2)^4}$ *OR* $\frac{8x^2}{(1-x^2)^3} + \frac{2(1-x^2)}{(1-x^2)^3}$			
	$= \frac{2(1+3x^2)}{(1-x^2)^3}$	A1	**5**	For simplified $\mathrm{f}'''(x)$ **www AG**
(ii)	$\mathrm{f}(0) = 0,\ \mathrm{f}'(0) = 1,\ \mathrm{f}''(0) = 0$	B1√		For all values correct (may be implied) f.t. from **(i)**
	$\mathrm{f}'''(0) = 2 \Rightarrow \tanh^{-1}x = x + \frac{1}{3}x^3$	M1 A1	**3**	For evaluating $\mathrm{f}'''(0)$ and using Maclaurin expansion For correct series
			[8]	
3 (i)(a)	Asymptote $y = 0$	B1	**1**	For correct equation (allow x-axis)
(b)	METHOD 1 $y = \frac{5ax}{x^2+a^2} \Rightarrow yx^2 - 5ax + a^2y = 0$	M1 M1		For expressing as a quadratic in x For using $b^2 - 4ac \lesseqgtr 0$
	$b^2 \geqslant 4ac \Rightarrow 25a^2 \geqslant 4a^2y^2 \Rightarrow -\frac{5}{2} \leqslant y \leqslant \frac{5}{2}$	A1 A1	**4**	For $25a^2 - 4a^2y^2$ seen or implied For correct range
	METHOD 2 $y = \frac{5ax}{x^2+a^2} \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-5a\left(x^2-a^2\right)}{\left(x^2+a^2\right)^2}$	M1*		For differentiating y by quotient *OR* product rule
	$\frac{\mathrm{d}y}{\mathrm{d}x} = 0 \Rightarrow x = \pm a \Rightarrow y = \pm\frac{5}{2}$	A1 M1		For correct values of x For finding y values and giving argument for range
	Asymptote, sketch etc $\Rightarrow -\frac{5}{2} \leqslant y \leqslant \frac{5}{2}$	A1 (*dep)		For correct range
(ii)(a)	$y = 0$	B1	**1**	For correct equation (allow x-axis)
(b)	Maximum $\sqrt{\frac{5}{2}}$, minimum $-\sqrt{\frac{5}{2}}$	B1√ B1√	**2**	For correct maximum f.t. from **(i)(b)** For correct minimum f.t. from **(i)(b)** Allow decimals
(c)	$x \geqslant 0$	B1	**1**	For correct set of values (allow in words)
			[9]	

Question	Answer	Marks	Guidance
4 (i)	$8\sinh^4 x \equiv \frac{8}{16}\left(e^x - e^{-x}\right)^4$	B1	$\sinh x = \frac{1}{2}\left(e^x - e^{-x}\right)$ seen or implied
	$\equiv \frac{8}{16}\left(e^{4x} - 4e^{2x} + 6 - 4e^{-2x} + e^{-4x}\right)$	M1	For attempt to expand $(\dots)^4$ by binomial theorem *OR* otherwise
	$\equiv \frac{1}{2}\left(e^{4x} + e^{-4x}\right) - \frac{4}{2}\left(e^{2x} + e^{-2x}\right) + \frac{6}{2}$	M1	For grouping terms for $\cosh 4x$ *or* $\cosh 2x$ *OR* using e^{4x} *or* e^{2x} expressions from RHS
	$\equiv \cosh 4x - 4\cosh 2x + 3$	A1 **4**	For correct expression **AG**
	SR may be done wholly from RHS to LHS	M1 M1 B1 A1	Evidence of factorising required for 2nd M1
(ii)	METHOD 1 $\cosh 4x - 3\cosh 2x + 1 = 0$		
	$\Rightarrow (8\sinh^4 x + 4\cosh 2x - 3) - 3\cosh 2x + 1 = 0$	M1	For using **(i)** and $\cosh 2x \equiv \pm 1 \pm 2\sinh^2 x$
	$\Rightarrow 8\sinh^4 x + 2\sinh^2 x - 1 = 0$	A1	For correct equation
	$\Rightarrow \left(4\sinh^2 x - 1\right)\left(2\sinh^2 x + 1\right) = 0 \Rightarrow \sinh x = \pm\frac{1}{2}$	M1 A1	For solving their quartic for $\sinh x$ For correct $\sinh x$ (ignore other roots)
	$\Rightarrow x = \ln\left(\pm\frac{1}{2} + \frac{1}{2}\sqrt{5}\right) = \pm\ln\left(\frac{1}{2} + \frac{1}{2}\sqrt{5}\right)$	A1√ **5**	For correct answers (and no more) f.t. from their value(s) for $\sinh x$
	SR Similar scheme for $8\cosh^4 x - 14\cosh^2 x + 5 = 0 \Rightarrow \cosh x = \frac{1}{2}\sqrt{5} \Rightarrow x = \pm\ln\left(\frac{1}{2} + \frac{1}{2}\sqrt{5}\right)$		
	METHOD 2 $\cosh 4x - 3\cosh 2x + 1 = 0$		
	$\Rightarrow (2\cosh^2 2x - 1) - 3\cosh 2x + 1 = 0$	M1	For using $\cosh 4x \equiv \pm 2\cosh^2 2x \pm 1$
	$\Rightarrow 2\cosh^2 2x - 3\cosh 2x = 0$	A1	For correct equation
	$\Rightarrow \cosh 2x = \frac{3}{2} \Rightarrow x = \frac{1}{2}\ln\left(\frac{3}{2} \pm \frac{1}{2}\sqrt{5}\right)$ $= \pm\frac{1}{2}\ln\left(\frac{3}{2} + \frac{1}{2}\sqrt{5}\right)$	M1 A1 A1√	For solving for $\cosh 2x$ For correct $\cosh 2x$ (ignore others) For correct answers (and no more) f.t. from value(s) for $\cosh 2x$
	METHOD 3 Put all into exponentials	M1	For changing to $e^{\pm kx}$
	$\Rightarrow e^{4x} - 3e^{2x} + 2 - 3e^{-2x} + e^{-4x} = 0$	A1	For correct equation
	$\Rightarrow \left(e^{4x} + 1\right)\left(e^{4x} - 3e^{2x} + 1\right) = 0$	M1	For solving for e^{2x}
		A1	For correct e^{2x} (ignore others)
	$\Rightarrow e^{2x} = \frac{1}{2}\left(3 \pm \sqrt{5}\right) \Rightarrow x = \frac{1}{2}\ln\left(\frac{3}{2} \pm \frac{1}{2}\sqrt{5}\right)$	A1√	For correct answers (and no more) f.t. from value(s) for e^{2x}
		9	
5 (i)	$x_{n+1} = x_n - \dfrac{x_n^3 - 5x_n + 3}{3x_n^2 - 5} = \dfrac{2x_n^3 - 3}{3x_n^2 - 5}$	M1 A1 A1 **3**	For attempt at N-R formula For correct N-R expression For correct answer (necessary details needed) **AG** Allow omission of suffixes
(ii)	$F'(x) =$ $\dfrac{6x^2\left(3x^2-5\right) - 6x\left(2x^3-3\right)}{\left(3x^2-5\right)^2} = \dfrac{6x\left(x^3-5x+3\right)}{\left(3x^2-5\right)^2}$	M1 M1	For using quotient *OR* product rule to find $F'(x)$ For factorising numerator to show $k\left(x^3 - 5x + 3\right)$
	$F'(\alpha) = \dfrac{6\alpha\left(\alpha^3 - 5\alpha + 3\right)}{\left(3\alpha^2 - 5\right)^2} = 0$ since $\alpha^3 - 5\alpha + 3 = 0$	A1 **3**	For correct explanation of **AG**
(iii)	$x_1 = 2 \Rightarrow 1.85714,\ 1.83479,\ 1.83424,\ 1.83424$ $(\alpha =)\ 1.8342$	B1 B1 B1 **3**	First iterate correct to at least 4 d.p. *OR* $\frac{13}{7}$ For 2 equal iterates to at least 4 d.p. For correct α to 4 d.p.
	SR For starting value leading to another root allow up to B1 B1 B0		Allow answer rounding to 1.8342 **SR** If not N-R, B0 B0 B0
		9	

6	(i)	$y = x^x \Rightarrow \ln y = x \ln x \Rightarrow \dfrac{1}{y}\dfrac{dy}{dx} = 1 + \ln x$	M1		For differentiating $\ln y$ *OR* $x \ln x$ w.r.t. x
		$\dfrac{dy}{dx} = x^x(1+\ln x) = 0 \Rightarrow \ln x = -1 \Rightarrow x = e^{-1}$	A1		For $(1+\ln x)$ seen or implied
			A1	3	For correct x-value from fully correct working **AG**
	(ii)	$A > 0.2 \times 0.5^{0.5} + 0.2 \times 0.7^{0.7} + 0.1 \times 0.9^{0.9}$	M1		For areas of 3 lower rectangles
		$\Rightarrow A > 0.3881(858) > 0.388$	A1	2	For lower bound rounding to **AG**
	(iii)	$A < 0.2 \times 0.7^{0.7} + 0.2 \times 0.9^{0.9} + 0.1 \times 1^1$	M1		For areas of 3 upper rectangles
		$\Rightarrow A < 0.4377(177) < 0.438$	A1	2	For upper bound rounding to 0.438
	(iv)		M1		Consider rectangle of height $f(e^{-1})$
			A1		Use at least 1 lower rectangle, height $f(e^{-1})$
			B1	3	Use at least 1 upper rectangle, height $f(0)$
					**SR** If more than one rectangle is used for either bound, they must be shown correctly
				10	
7	(i)	$\cos 3\theta = \cos(-3\theta)$ *OR* $\cos\theta = \cos(-\theta)$ for all θ	M1		For a correct procedure for symmetry related to the equation *OR* to $\cos 3\theta$
		$\Rightarrow$ equation is unchanged, so symmetrical about $\theta = 0$	A1	2	For correct explanation relating to equation **AG**
	(ii)	$r = 0 \Rightarrow \cos 3\theta = -1$	M1		For obtaining equation for tangents
		$\Rightarrow \theta = \pm\frac{1}{3}\pi, \pi$	A1		A1 for any 2 values
			A1	3	A1 for all, no extras (ignore outside range)
	(iii)	$\int_{-\frac{1}{3}\pi}^{\frac{1}{3}\pi} \frac{1}{2}(1+\cos 3\theta)^2 \, (d\theta)$	B1		For correct integral with limits soi (limits may be $\left[0, \frac{1}{3}\pi\right]$ at any stage)
		$= \frac{1}{2}\int_{-\frac{1}{3}\pi}^{\frac{1}{3}\pi} 1 + 2\cos 3\theta + \cos^2 3\theta \, d\theta$	M1*		For multiplying out, giving at least 2 terms
		$= \frac{1}{2}\int_{-\frac{1}{3}\pi}^{\frac{1}{3}\pi} 1 + 2\cos 3\theta + \frac{1}{2}(1+\cos 6\theta) \, d\theta$	M1		For integration to $A\theta + B\sin 3\theta + C\sin 6\theta$ **AEF**
		$= \frac{1}{2}\left[\theta + \frac{2}{3}\sin 3\theta + \left(\frac{1}{2}\theta + \frac{1}{12}\sin 6\theta\right)\right]_{-\frac{1}{3}\pi}^{\frac{1}{3}\pi}$	M1 (*dep)		For completing integration and substituting their limits into terms in $\frac{\cos}{\sin} n\theta$
		$= \frac{1}{2}\pi$	A1	5	For correct area **www**
				10	

8 (i)	METHOD 1 $\sinh(\cosh^{-1}2) =$ $\sinh\beta = \sqrt{\cosh^2\beta - 1} = \sqrt{2^2 - 1} = \sqrt{3}$	M1 A1	 **2**	For appropriate use of $\sinh^2\theta = \cosh^2\theta - 1$ For correct verification to **AG**
	METHOD 2 $\sinh^{-1}\sqrt{3} = \ln\left(\sqrt{3}+2\right)$, $\cosh^{-1}2 = \ln\left(2+\sqrt{3}\right)$ $\Rightarrow \sinh(\cosh^{-1}2) = \sqrt{3}$	M1 A1		For attempted use of ln forms of $\sinh^{-1}x$ and $\cosh^{-1}x$ For both ln expressions seen
	METHOD 3 $\cosh^{-1}2 = \ln\left(2+\sqrt{3}\right)$ $\sinh\left(\cosh^{-1}2\right) = \frac{1}{2}\left(e^{\ln\left(2+\sqrt{3}\right)} - e^{-\ln\left(2+\sqrt{3}\right)}\right)$ $= \frac{1}{2}\left(2+\sqrt{3} - \left(2-\sqrt{3}\right)\right) = \sqrt{3}$	M1 A1		For use of ln form of $\cosh^{-1}x$ and definition of $\sinh x$ For correct verification to **AG** **SR** Other similar methods may be used Note that $\ln\left(2+\sqrt{3}\right) = -\ln\left(2-\sqrt{3}\right)$
(ii)	$I_n = \int_0^\beta \cosh^n x\,dx$ $= \left[\sinh x.\cosh^{n-1}x\right]_0^\beta - \int_0^\beta \sinh^2 x.(n-1)\cosh^{n-2}x\,dx$ $= \sinh\beta.\cosh^{n-1}\beta - (n-1)\int_0^\beta \left(\cosh^2 x - 1\right)\cosh^{n-2}x\,dx$ $= 2^{n-1}\sqrt{3} - (n-1)\left(I_n - I_{n-2}\right)$ $\Rightarrow nI_n = 2^{n-1}\sqrt{3} + (n-1)I_{n-2}$	M1* A1 M1 (*dep) A1 B1 A1	 **6**	For attempt to integrate $\cosh x.\cosh^{n-1}x$ by parts For correct first stage of integration (ignore limits) For using $\sinh^2 x = \cosh^2 x - 1$ For $(n-1)\left(I_n - I_{n-2}\right)$ correct For $2^{n-1}\sqrt{3}$ correct at any stage For correct result **AG**
(iii)	$I_1 = \int_0^\beta \cosh x\,dx = \sinh\beta = \sqrt{3}$ $I_3 = \frac{1}{3}\left(2^2\sqrt{3} + 2\sqrt{3}\right) = 2\sqrt{3}$ $I_5 = \frac{1}{5}\left(2^4\sqrt{3} + 8\sqrt{3}\right) = \frac{24}{5}\sqrt{3}$	B1 M1 A1 A1	 **4**	For correct value For using **(ii)** with $n = 3$ *OR* $n = 5$ For $I_3 = \frac{1}{3}\left(2^2\sqrt{3} + 2I_1\right)$ *OR* $I_5 = \frac{1}{5}\left(2^4\sqrt{3} + 4I_3\right)$ For correct value
			12	

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