RECOGNISING ACHIEVEMENT

GCE

Mathematics (MEI)

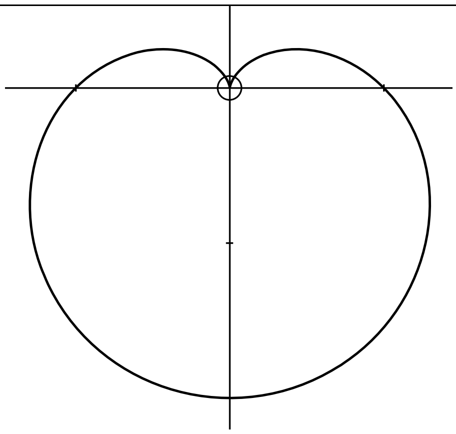
Advanced GCE

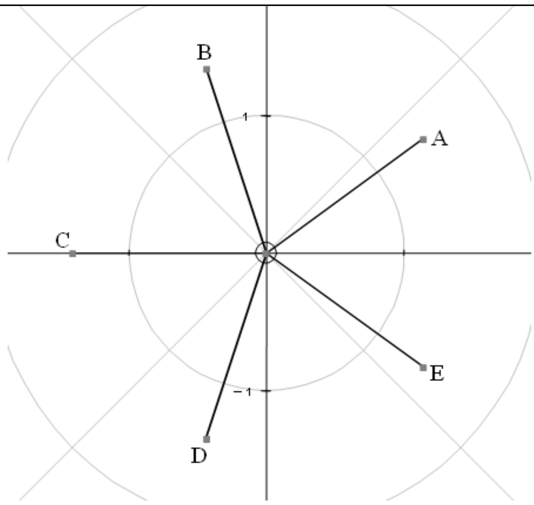
Unit **4756:** Further Methods for Advanced Mathematics

Mark Scheme for June 2011

Oxford Cambridge and RSA Examinations

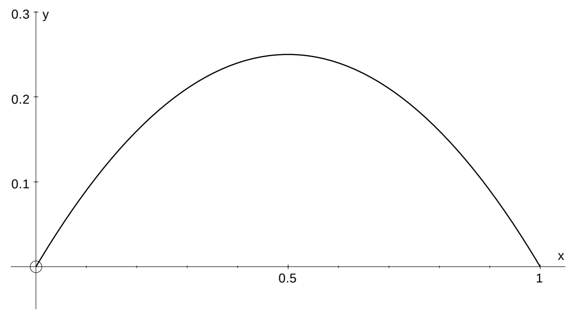
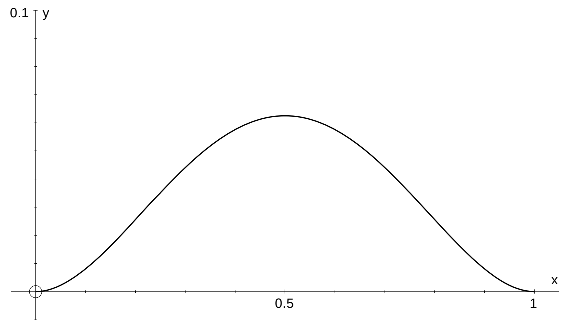
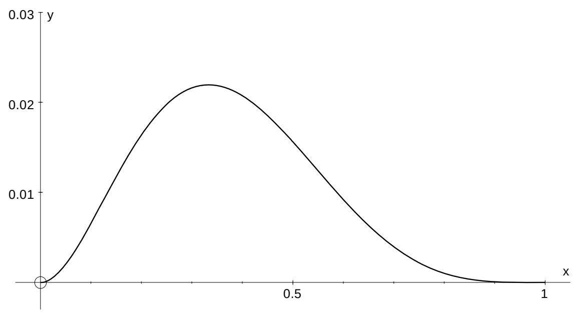
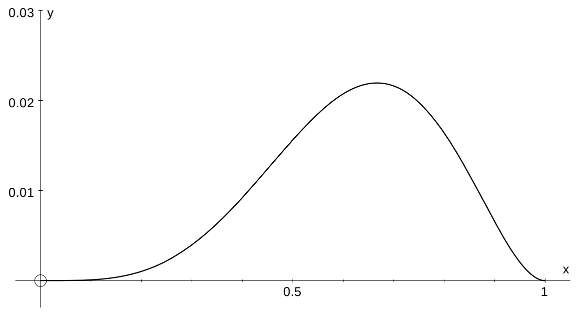
4756 (FP2) Further Methods for Advanced Mathematics

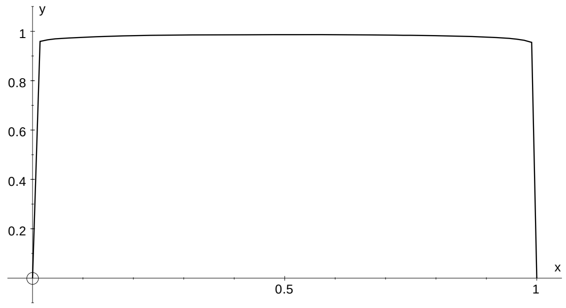
1 (a)(i)		G1 G1 **2**	Correct general shape including symmetry in vertical axis Correct form at O and no extra sections. Dependent on first G1 For an otherwise correct curve with a sharp point at the bottom, award G1G0
(ii)	Area $=\frac{1}{2}a^2\int_0^{2\pi}(1-\sin\theta)^2\,d\theta$ $=\frac{1}{2}a^2\int_0^{2\pi}(1-2\sin\theta+\sin^2\theta)\,d\theta$ $=\frac{1}{2}a^2\int_0^{2\pi}\left(\frac{3}{2}-2\sin\theta-\frac{1}{2}\cos 2\theta\right)d\theta$ $=\frac{1}{2}a^2\left[\frac{3}{2}\theta+2\cos\theta-\frac{1}{4}\sin 2\theta\right]_0^{2\pi}$ $=\frac{3}{2}\pi a^2$	M1 M1 A1 M1 A2 A1 **7**	Integral expression involving $(1-\sin\theta)^2$ Expanding Correct integral expression, incl. limits (which may be implied by later work) Using $\sin^2\theta=\frac{1}{2}-\frac{1}{2}\cos 2\theta$ Correct result of integration. Give A1 for one error Dependent on previous A2
(b)(i)	$\int_{-\frac{1}{2}}^{\frac{1}{2}}\frac{1}{1+4x^2}dx=\frac{1}{4}\int_{-\frac{1}{2}}^{\frac{1}{2}}\frac{1}{\frac{1}{4}+x^2}dx=\frac{1}{4}\left[2\arctan 2x\right]_{-\frac{1}{2}}^{\frac{1}{2}}$ $=\frac{1}{2}\left(\frac{\pi}{4}-\left(-\frac{\pi}{4}\right)\right)$ $=\frac{\pi}{4}$	M1 A1 A1 **3**	arctan alone, or any tan substitution $\frac{1}{4}\times 2$ and $2x$ Evaluated in terms of π
(ii)	$x=\frac{1}{2}\tan\theta$ $\Rightarrow dx=\frac{1}{2}\sec^2\theta\,d\theta$ $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\frac{1}{(\sec^2\theta)^{\frac{3}{2}}}\times\frac{\sec^2\theta}{2}d\theta$ $=\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\frac{1}{2}\cos\theta\,d\theta$ $=\left[\frac{1}{2}\sin\theta\right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}}$ $=\frac{1}{2}\left(\frac{1}{\sqrt{2}}-\left(-\frac{1}{\sqrt{2}}\right)\right)$ $=\frac{1}{\sqrt{2}}$	M1 A1A1 M1 A1ft A1 **6**	Any tan substitution $\frac{1}{(\sec^2\theta)^{\frac{3}{2}}}$, $\frac{\sec^2\theta}{2}$ Integrating $a\cos b\theta$ and using consistent limits. Dependent on M1 above $\frac{a}{b}\sin b\theta$ **18**

2 (a)	$\cos 5\theta + j\sin 5\theta = (\cos\theta + j\sin\theta)^5$ $= c^5 + 5c^4js - 10c^3s^2 - 10c^2js^3 + 5cs^4 + js^5$ $\Rightarrow \cos 5\theta = c^5 - 10c^3s^2 + 5cs^4$ $\sin 5\theta = 5c^4s - 10c^2s^3 + s^5$ $\Rightarrow \tan 5\theta = \dfrac{5c^4s - 10c^2s^3 + s^5}{c^5 - 10c^3s^2 + 5cs^4}$ $= \dfrac{5t - 10t^3 + t^5}{1 - 10t^2 + 5t^4}$ $= \dfrac{t\left(t^4 - 10t^2 + 5\right)}{5t^4 - 10t^2 + 1}$	M1 M1 A1 A1 M1 A1 (ag) **6**	Expanding Separating real and imaginary parts. Dependent on first M1 Alternative: $16c^5 - 20c^3 + 5c$ Alternative: $16s^5 - 20s^3 + 5s$ Using $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ and simplifying
(b)(i)	$\arg(-4\sqrt{2}) = \pi$ $\Rightarrow$ fifth roots have $r = \sqrt{2}$ and $\theta = \dfrac{\pi}{5}$ $\Rightarrow z = \sqrt{2}e^{\frac{1}{5}j\pi}, \sqrt{2}e^{\frac{3}{5}j\pi}, \sqrt{2}e^{j\pi}, \sqrt{2}e^{\frac{7}{5}j\pi}, \sqrt{2}e^{\frac{9}{5}j\pi}$	B1 B1 M1 A1 **4**	 No credit for arguments in degrees Adding (or subtracting) $\dfrac{2\pi}{5}$ All correct. Allow $-\pi \le \theta < \pi$
(ii)		G1 G1 **2**	Points at vertices of "regular" pentagon, with one on negative real axis Points correctly labelled
(iii)	$\arg(w) = \dfrac{1}{2}\left(\dfrac{\pi}{5} + \dfrac{3\pi}{5}\right) = \dfrac{2\pi}{5}$ $\lvert w\rvert = \sqrt{2}\cos\dfrac{\pi}{5}$	B1 M1 A1ft **3**	 Attempting to find length F.t. (positive) r from (i)
(iv)	$w = \sqrt{2}\cos\dfrac{\pi}{5}\, e^{\frac{2}{5}\pi j} \Rightarrow w^n = \left(\sqrt{2}\cos\dfrac{\pi}{5}\right)^n e^{\frac{2}{5}\pi nj}$ which is real if $\sin\dfrac{2\pi n}{5} = 0 \Rightarrow n = 5$ $\lvert w^5\rvert = \left(\sqrt{2}\cos\dfrac{\pi}{5}\right)^5$ $\Rightarrow a = 2^{\frac{5}{2}}\cos^5\dfrac{\pi}{5}$	B1 M1 A1 **3**	 Attempting the nth power of his modulus in (iii), or attempting the modulus of the nth power here Accept 1.96 or better **18**

3 (i)	det(**M**) = 1(16 − 12) + 1(20 − 18) + k(10 − 12) = 6 − 2k ⇒ no inverse if k = 3 $\mathbf{M}^{-1} = \frac{1}{6-2k}\begin{pmatrix} 4 & 4+2k & -6-4k \\ -2 & 4-3k & 5k-6 \\ -2 & -5 & 9 \end{pmatrix}$	M1 A1 A1 M1 A1 M1 A1 **7**	Obtaining det(**M**) in terms of k Accept $k \neq 3$ after correct determinant Evaluating at least four cofactors (including one involving k) Six signed cofactors correct (including one involving k) Transposing and dividing by det(**M**). Dependent on previous M1M1
(ii)	$\begin{pmatrix} 1 & -1 & 3 \\ 5 & 4 & 6 \\ 3 & 2 & 4 \end{pmatrix}\begin{pmatrix} -3 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ 3 \\ 1 \end{pmatrix}$	M1 A1 **2**	Setting k = 3 and multiplying
(iii)	$\begin{pmatrix} -3 \\ 3 \\ 1 \end{pmatrix}$ is an eigenvector corresponding to an eigenvalue of 1	B1 B1 **2**	For credit here, 2/2 scored in (ii) Accept "invariant point"
(iv)	$3x + 6y = 1 - 2t$, $x + 2y = 2$, $5x + 10y = -4t$ (or $9x + 18z = 4t + 1$, $5x + 10z = 2t$, $x + 2z = -1$) (or $9y - 9z = 1 - 5t$, $5y - 5z = -3t$, $2y - 2z = 3$) For solutions, $1 - 2t = 3 \times 2$ $\Rightarrow t = -\frac{5}{2}$ $x = \lambda$, $y = 1 - \frac{1}{2}\lambda$, $z = -\frac{1}{2} - \frac{1}{2}\lambda$ Straight line	M1 A1 M1 A1 M1 A1 B1 **7**	Eliminating one variable in two different ways Two correct equations Validly obtaining a value of t Obtaining general solution by setting one unknown = λ and finding other two in terms of λ (accept unknown instead of λ) Accept "sheaf". Independent of all previous marks **18**

4 (i)	$\cosh y = x \Rightarrow x = \frac{1}{2}\left(e^{y} + e^{-y}\right)$	B1	Using correct exponential definition
	$\Rightarrow 2x = e^{y} + e^{-y}$		
	$\Rightarrow \left(e^{y}\right)^{2} - 2xe^{y} + 1 = 0$	M1	Obtaining quadratic in e^{y}
	$\Rightarrow e^{y} = \dfrac{2x \pm \sqrt{4x^{2}-4}}{2} = x \pm \sqrt{x^{2}-1}$	M1 A1	Solving quadratic $x \pm \sqrt{x^{2}-1}$
	$\Rightarrow y = \ln(x \pm \sqrt{x^{2}-1})$		
	$\left(x+\sqrt{x^{2}-1}\right)\left(x-\sqrt{x^{2}-1}\right) = 1$	M1	Validly attempting to justify ± in printed answer
	$\Rightarrow y = \pm\ln(x+\sqrt{x^{2}-1})$	A1 (ag)	
	arcosh(x) = $\ln(x+\sqrt{x^{2}-1})$ because this is the principal value	B1	Reference to arcosh as a function, or correctly to domains/ranges
		7	
(ii)	$\displaystyle\int_{\frac{4}{5}}^{1} \frac{1}{\sqrt{25x^{2}-16}}\,dx = \frac{1}{5}\int_{\frac{4}{5}}^{1} \frac{1}{\sqrt{x^{2}-\frac{16}{25}}}\,dx$		
	$= \frac{1}{5}\left[\text{arcosh}\left(\frac{5x}{4}\right)\right]_{\frac{4}{5}}^{1}$	M1 A1A1	arcosh alone, or any cosh substitution $\frac{1}{5}$, $\frac{5x}{4}$
	$= \frac{1}{5}\left(\text{arcosh}\left(\frac{5}{4}\right) - \text{arcosh}(1)\right)$		
	$= \frac{1}{5}\ln\left(\frac{5}{4} + \sqrt{\left(\frac{5}{4}\right)^{2} - 1}\right) - 0$	M1	Substituting limits and using (i) correctly at any stage (or using limits of u in logarithmic form). Dep. on first M1
	$= \frac{1}{5}\ln 2$	A1	
	OR $= \frac{1}{5}\left[\ln\left(x+\sqrt{x^{2}-\frac{16}{25}}\right)\right]_{\frac{4}{5}}^{1}$ M		$\ln\left(kx+\sqrt{k^{2}x^{2}+\ldots}\right)$ Give M1 for $\ln\left(k_{1}x+\sqrt{k_{2}^{2}x^{2}+\ldots}\right)$
	A1A		$\frac{1}{5}$, $\ln\left(x+\sqrt{x^{2}-\frac{16}{25}}\right)$ o.e.
	$= \frac{1}{5}\ln\frac{8}{5} - \frac{1}{5}\ln\frac{4}{5}$		
	$= \frac{1}{5}\ln 2$ A		
		5	
(iii)	5 cosh x − cosh 2x = 3		
	$\Rightarrow$ 5 cosh x − (2 $\cosh^{2}x$ − 1) = 3	M1	Attempting to express cosh 2x in terms of cosh x
	$\Rightarrow$ 2 $\cosh^{2}x$ − 5 cosh x + 2 = 0		
	$\Rightarrow$ (2 cosh x − 1)(cosh x − 2) = 0	M1	Solving quadratic to obtain at least one real value of cosh x
	$\Rightarrow$ cosh x = $\frac{1}{2}$ (rejected)	A1	Or factor 2 cosh x − 1
	or cosh x = 2	A1	
	$\Rightarrow x = \ln\left(2+\sqrt{3}\right)$	A1ft	F.t. cosh x = k, k > 1
	$x = -\ln\left(2+\sqrt{3}\right)$ or $\ln\left(2-\sqrt{3}\right)$	A1ft **6**	F.t. other value. Max. A1A0 if additional real values quoted **18**

5 (i)	**(A)** $m = 1$, $n = 1$  	G1	Negative parabola from (0,0) to (1,0), symmetrical about $x = 0.5$
	(B) $m = 2$, $n = 2$  	G1	Bell-shape from (0,0) to (1,0), symmetrical about $x = 0.5$; flat ends, and obviously different to (A)
	(C) $m = 2$, $n = 4$  	G1	Skewed curve from (0,0) to (1,0), maximum to left of $x = 0.5$
	(D) $m = 4$, $n = 2$  	G1 **4**	Skewed curve from (0,0) to (1,0), maximum to right of $x = 0.5$
(ii)	When $m = n$, the curve is symmetrical Exchanging m and n reflects the curve	B1 B1 **2**	
(iii)	If $m > n$, the maximum is to the right of $x = 0.5$ As m increases relative to n, the maximum point moves further to the right $y = x^m(1-x)^n \Rightarrow \frac{dy}{dx} = mx^{m-1}(1-x)^n - nx^m(1-x)^{n-1}$ $= x^{m-1}(1-x)^{n-1}\left[m(1-x) - nx\right]$ $\frac{dy}{dx} = 0 \Rightarrow$ maximum at $x = \frac{m}{m+n}$	B1 B1 M1 A1 M1 A1 **6**	o.e. Give B1B0 if the idea is correct but vaguely expressed Using product rule Any correct form Setting derivative = 0 and solving to find a value of x other than 0 or 1

(iv)	$y'(0) = 0$ provided $m > 1$ $y'(1) = 0$ provided $n > 1$	B1 B1 **2**	
(v)	For large m and n, the curve approaches the x-axis $\Rightarrow \int_0^1 x^m (1-x)^n dx \to 0$ as $m, n \to \infty$	B1 B1 **2**	Comment on shape Independent
(vi)	e.g. $m = 0.01$, $n = 0.01$  The curve tends to $y = 1$	M1 A1 **2**	Evidence of investigation s.o.i. Accept "three sides of (unit) square" **18**